

F-theoretic vs microscopic description of a conformal $\mathcal{N} = 2$ SYM theory

Marco Billò,^{a,b} Laurent Gallot,^c Alberto Lerda^d and Igor Pesando^a

^a*Dipartimento di Fisica Teorica, Università di Torino, and I.N.F.N. - sezione di Torino, Via P. Giuria 1, I-10125 Torino, Italy*

^b*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030, U.S.A.*

^c*LAPTH, Université de Savoie, CNRS, 9, Chemin de Bellevue, 74941 Annecy le Vieux Cedex, France*

^d*Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale, and I.N.F.N. - Gruppo Collegato di Alessandria - sezione di Torino, Viale T. Michel 11, I-15121 Alessandria, Italy*

E-mail: billò@to.infn.it, laurent.gallot@lapp.in2p3.fr, lerda@to.infn.it, ipesando@to.infn.it

ABSTRACT: The F-theory background of four D7 branes in a type I' orientifold was conjectured to be described by the Seiberg-Witten curve for the superconformal $SU(2)$ gauge theory with four flavors. This relation was explained by considering in this background a probe D3 brane, which supports this theory with $SU(2)$ realized as $Sp(1)$. Here we explicitly compute the non-perturbative corrections to the D7/D3 system in type I' due to D-instantons. This computation provides both the quartic effective action on the D7 branes and the quadratic effective action on the D3 brane; the latter agrees with the F-theoretic prediction. The action obtained in this way is related to the one derived from the usual instanton calculus à la Nekrasov (or from its AGT realization in terms of Liouville conformal blocks) by means of a non-perturbative redefinition of the coupling constant. We also point out an intriguing relation between the four-dimensional theory on the probe D3 brane with $SO(8)$ flavor symmetry and the eight-dimensional dynamics on the D7 branes. On the latter, $SO(8)$ represents a gauge group and the flavor masses correspond to the vacuum expectation values of an adjoint scalar field m : what we find is that the *exact* effective coupling in four dimensions is obtained from its *perturbative* part by taking into account in its mass dependence the full quantum dynamics of the field m in eight dimensions.

KEYWORDS: D-branes, Supersymmetric gauge theory, F-Theory, Supersymmetric Effective Theories

ARXIV EPRINT: [1008.5240](https://arxiv.org/abs/1008.5240)

Contents

1	Introduction and motivations	2
2	F-theory and the D7/D3 system in type I'	4
2.1	The local case	5
2.2	F-theoretic description and D3 brane probes	6
2.2.1	The SW curve for the D3 gauge theory	8
3	D-instanton corrections	11
3.1	Instanton moduli spectrum	11
3.2	BRST structure of moduli space	13
3.3	Moduli integration via localization and instanton partition function	15
3.4	Non-perturbative prepotentials	18
4	Eight-dimensional effective prepotential and chiral ring	20
5	Four-dimensional effective prepotential	23
6	An intriguing relation	25
7	Comparison with the SU(2) instanton calculus	26
7.1	Instanton calculus	27
7.2	The Nekrasov prepotential from the AGT realization	28
8	Summary and conclusions	30
A	Notations and theta-function conventions	31
B	Flavor invariants	32
B.1	Triality extension of $SL(2, \mathbb{Z})$	33
C	Details on the D-instanton computation	34
D	Details on the AGT correspondence	36
E	Decoupling limits to $N_f = 3, 2, 1, 0$	37

1 Introduction and motivations

Phenomenologically viable string models based on consistent D-brane configurations have attracted a lot of attention in the last years [1]–[3]. In such constructions it is necessary to take into account possible non-perturbative corrections due D-instantons and (wrapped) euclidean branes; for a review see, for example, [4]. Some of these instantonic branes reproduce gauge instantons [5]–[8], other provide inherently stringy (or “exotic”) instanton effects; these latter can be responsible of important terms in the effective action which would be perturbatively forbidden [9]–[11]. Recently there has been much progress in the explicit computation of such contributions, both ordinary and exotic, at least in supersymmetric cases (again, see [4] and references therein).

Another framework where phenomenological models with highly desirable features can be set up, and where in particular Grand Unified Theories occur naturally and consistently [12]–[14], is represented by F-theory compactifications [15] (for reviews see, for instance, [16, 17]). F-theory gives a *non-perturbative* geometric description of type IIB backgrounds containing D7 branes and orientifold planes; it somehow resums the non-perturbative corrections arising from certain instantonic branes. Understanding in detail how this resummation takes place would improve our knowledge of the relation between the two descriptions. This could be useful for a better comprehension of further non-perturbative effects in F-theory through a lift of their type IIB counterparts, a subject that is recently¹ receiving quite some attention [19]–[24].

In this paper we work out a simple, yet non-trivial, example where we are able to compute the D-instanton effects in the IIB description and show that they reconstruct the F-theory curve. This example was considered by A. Sen in [25], and is given by the compactification of F-theory on the orbifold limit of an elliptically fibered K3 surface, for which the complex structure modulus τ of the fiber is constant. This background was shown to correspond to the so-called type I' theory, which is T-dual to type I theory compactified on a 2-torus T_2 , and thus possesses one O7 plane at each of the four fixed points of T_2 with four D7 branes on top of it. Focusing on the vicinity of one orientifold fixed-plane, and allowing the four D7 branes to move out of it, Sen conjectured that the corresponding F-theory background should be described by the Seiberg-Witten (SW) curve [26] for the $\mathcal{N} = 2$ superconformal Yang-Mills theory with gauge group $SU(2)$ and $N_f = 4$ flavors [27]. This relation was later explained by T. Banks, M. Douglas and N. Seiberg [28] by considering a D3 brane in this background, which indeed supports such a four-dimensional gauge theory on its world-volume, with $SU(2)$ realized as $Sp(1)$. The $SO(8)$ flavor symmetry of the $N_f = 4$ theory is nothing else but the gauge group on the D7 branes.

Here we explicitly compute the non-perturbative corrections to the D7/D3 system in type I' due to D-instantons. This requires to identify the spectrum of moduli, i.e. of excitations of the strings with at least one end-point on the D-instantons, and the moduli action that arises from disks with at least part of their boundary on a D-instanton, which was already discussed in [29]. To obtain the non-perturbative effects it is necessary to

¹For an earlier discussion in an $\mathcal{N} = 2$ context see [18].

integrate over the moduli; we explicitly perform this integration by applying the by-now standard techniques based on the BRST structure of the moduli spectrum and action and its deformation by means of suitable RR backgrounds [30]–[34]. This induces a complete localization of the integral, similarly to what happens in supersymmetric instanton calculus in field theory [35]–[40]. These techniques have been recently applied to similar brane setups, such as the D7 system in type I' [31, 33], and the D7/D3 system on T^4/\mathbb{Z}_2 [34].

D-instanton effects induce corrections to both the quartic effective action on the D7 branes and the quadratic effective action on the D3 brane. We use the prescription proposed in [34] to disentangle the two contributions. The non-perturbative action on the D7's turns out to be exactly the same of the D7 system in type I' theory considered in [31]. The non-perturbative effective coupling on the D3 brane agrees with the one extracted from the SW curve, that is with the F-theoretic prediction.

One interesting question is the precise relation between the eight-dimensional quartic effective action on the D7 branes and the F-theory curve. This problem was already addressed in the past using the duality of certain F-theory compactifications, including the one corresponding to type I' theory, to heterotic models. Despite some interesting results [41, 42], this relation is not yet totally clear. Since in our case the F-theory curve is nothing else but the SW curve encoding the effective theory on the D3 probe, another way to state the above question is: what is the relation between the non-perturbative effective actions on the D7 branes and on the D3 brane? More generally, how are the quantum dynamics on the D7's and that on the D3 related to each other?

We do not have a full answer to this question, but we uncover an intriguing relation that goes as follows. On the D7's, the $SO(8)$ flavor symmetry represents a gauge group and the flavor masses m_i correspond to the vacuum expectation values of an adjoint scalar field $m(X)$: we find that the *exact* effective four-dimensional coupling is obtained from its *perturbative* part by taking into account in its mass dependence the eight-dimensional quantum dynamics of the field m , and in particular the so-called “chiral ring” formed by the correlators $\langle \text{Tr } m^{2l} \rangle$. In this way the F-theory geometry is related explicitly to these eight-dimensional quantities. It would be very interesting to investigate whether such sort of perturbative propagation of the full quantum dynamics on a brane stack (in our case, the D7's) to another one (in our case the D3 probe) takes place also in other systems.²

The theory that, in our example, lives on the D3 brane is a four-dimensional conformal $\mathcal{N} = 2$ theory. This class of theories have recently attracted much attention in relation to the so-called AGT conjecture put forward by L. F. Alday, D. Gaiotto and Y. Tachikawa [44]. This conjecture relates the effective actions obtained from usual instanton calculus à la Nekrasov [35] to suitable correlators of the Liouville theory in two dimensions [45] (see also [46]–[48]). In particular, the non-perturbative action for the $SU(2)$ theory with $N_f = 4$ can be extracted from the 4-point functions on the sphere. It is interesting to compare these results to what we get in the D7/D3 system in type I' , where the conformal theory is realized as an $Sp(1)$ gauge theory. It turns out that the effective action derived from the Nekrasov's

²Recently, D3 probes in F-theory have been considered in [20, 43]. In particular in [43] it is pointed out that the interplay between the eight-dimensional theory on the D7 and the four-dimensional theory on the probe, and the interpretation of parameters in the latter as adjoint fields on the former, plays a crucial rôle.

prescription for the $SU(2)$, $N_f = 4$ case or, more efficiently, from its AGT realization in terms of Liouville conformal blocks does not coincide, at first sight, with our results, nor with the SW curve proposed in [27]. There is a discrepancy already at the massless level: in this case from the SW curve (and from our microscopic computation) we see that the tree-level coupling receives no corrections. When computed following Nekrasov’s prescription for $SU(2)$, or by the AGT method, instead, the coupling gets non-perturbatively modified.³ This suggests that a redefinition of the coupling constant is needed in order to compare the two approaches; after such a redefinition is performed, remarkably the two methods are reconciled and the two results agree completely also in the massive case.

The paper is subdivided into several sections as we now describe. In section 2 we introduce the model and its F-theory description through the SW curve; from this curve we extract the instanton expansion of the effective coupling. In section 3 we describe the microscopic computation of D-instanton contributions in our model. The resulting effective action on the D7 branes is discussed in section 4, while in section 5 we write the D-instanton induced effective action on the D3 brane, which is in full agreement with the F-theoretic description. In section 6 we put forward our conjecture about the exact effective coupling on the D3 being determined by its perturbative part plus the eight-dimensional dynamics of the mass parameters. Finally, in section 7 we discuss how our results compare to Nekrasov instanton calculus (or its AGT realization) for the $SU(2)$ theory with $N_f = 4$, and in section 8 we present our conclusions. In the appendices some technical details and the extension of some results to asymptotically free cases with $N_f < 4$ are given.

2 F-theory and the D7/D3 system in type I'

In F-theory compactifications over an elliptically fibered manifold [15], the complex structure modulus τ of the fiber corresponds to the varying axio-dilaton profile of a suitable type IIB compactification on the base manifold. In [25] A. Sen studied F-theory on an elliptically fibered K3 surface, which is conjectured to be dual to heterotic string compactified on a two-dimensional torus. He considered the particular case in which the K3 is at the orbifold limit in moduli space where it is described by the following curve in Weierstrass form

$$y^2 = x^3 - \frac{1}{4}G_2(z)x - \frac{1}{4}G_3(z) . \tag{2.1}$$

Here z is the coordinate on the base of the fibration and

$$G_2(z) \propto Q^2(z), \quad G_3(z) \propto Q^3(z), \quad Q(z) = \prod_{I=1}^4 (z - f_I), \tag{2.2}$$

with f_I constants. The absolute modular invariant of this curve

$$J = \frac{G_2^3}{G_3^3 - 27 G_2^2} \tag{2.3}$$

³A discrepancy in this sense was already noticed at the two-instanton level in [49, 50], where the direct integration over the moduli was performed without resorting to localization techniques.

is z -independent, and so is its complex structure modulus τ which can be determined from J by inverting the relation

$$J = \left(\frac{\vartheta_2^8(\tau) + \vartheta_3^8(\tau) + \vartheta_4^8(\tau)}{24\eta^8(\tau)} \right)^3 \tag{2.4}$$

where the ϑ_a 's are the Jacobi theta-functions and η is the Dedekind function. By studying the metric on the base space, it can be seen that the latter has the geometry of a torus orbifold of the type T_2/\mathbb{Z}_2 , with \mathbb{Z}_2 acting as parity reflection along T_2 , in which the f_I 's appearing in (2.2) correspond to the points of T_2 that are fixed under the \mathbb{Z}_2 action.

This specific F-theory background can be identified with the so-called type I' theory, namely type IIB compactified on a torus T_2 and modded out by

$$\Omega = \omega (-1)^{F_L} I_2, \tag{2.5}$$

where ω is the world-sheet parity reversal, F_L is the left-moving space-time fermion number and I_2 the inversion on T_2 . This is the T-dual version of type I theory compactified on T_2 , and possesses four O7 orientifold planes located at the points of T_2 that are fixed under I_2 (see figure 1a). Each orientifold plane carries (-4) units of 7-brane charge, which need to be neutralized by putting 16 D7 branes transverse to T_2 . If we place them in groups of 4 over each orientifold plane, the tadpole cancellation becomes local and the axio-dilaton is constant over T_2 . From now on we take a local perspective and focus on one of the orientifold fixed planes (say, the one at $z = f_1$) and its associated stack of 4 D7 branes.

2.1 The local case

The action of the orientifold projection Ω is such that each group of 4 D7 branes supports an eight-dimensional theory with gauge group $SO(8)$. Indeed, the massless degrees of freedom of the 7/7 open strings build up an eight-dimensional chiral superfield in the adjoint representation of $SO(8)$, whose first few terms are

$$M(X, \Theta) = m(X) + \sqrt{2}\Theta \sigma(X) + \frac{1}{2}\Theta\gamma^{MN}\Theta f_{MN}(X) + \dots, \tag{2.6}$$

where f_{MN} is the field-strength, σ is the gaugino, m is a complex scalar, and (X, Θ) are the eight-dimensional super-coordinates.

If the D7 branes are moved away from the orientifold plane, i.e. if we give a diagonal vacuum expectation value to the scalar field m , the charges no longer cancel locally, and correspondingly the solution of the equation of motion for τ displays logarithmic singularities at the orientifold and D7 brane locations. Let us parametrize the region near the orientifold fixed point with a coordinate⁴ w and let the D7 branes and their images be located at $w = \pm m_i/\sqrt{2}$ with $i = 1, \dots, 4$ (see figure 1b). This corresponds to choose the following vacuum expectation values⁵

$$\langle m \rangle = \text{diag} \left(m_1/\sqrt{2}, \dots, m_4/\sqrt{2}, -m_1/\sqrt{2}, \dots - m_4/\sqrt{2} \right). \tag{2.7}$$

⁴With respect to the global coordinate z used above, $w \propto (z - f_1)$.

⁵Notice that m is a complex field in the adjoint of $SO(8)$, i.e. it is a complex antisymmetric matrix. If m were real, its eigenvalues m_i would be imaginary.

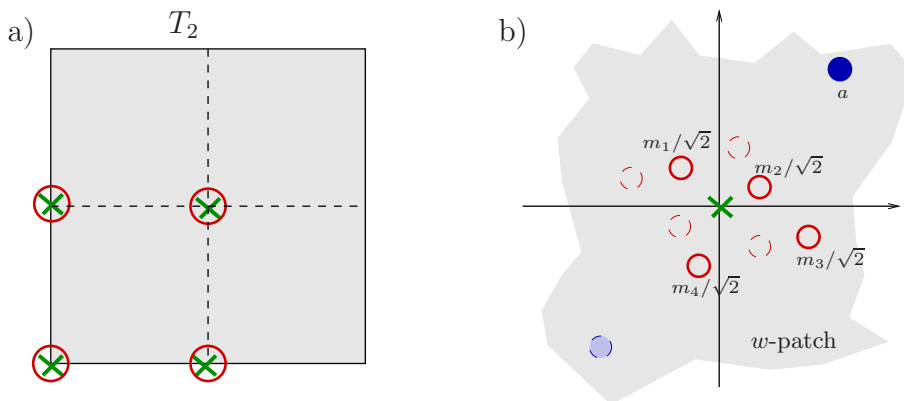


Figure 1. a) The initial set-up: 4 D7 branes (denoted by the red circles) are placed in the transverse torus T_2 at each of the O7 fixed planes (denoted by the green crosses). b) In the local limit near one of the fixed planes, the D7 brane displacements with respect to the original position are given by $m_i/\sqrt{2}$ (the image branes, denoted by dashed circles, sit at $-m_i/\sqrt{2}$). A D3 brane probe (depicted as a blue dot) is placed in a (its image being in $-a$).

With this choice the axio-dilaton becomes

$$2\pi i \tau(w) = 2\pi i \tau_0 + \left\{ \sum_{i=1}^4 \left[\ln(w - m_i/\sqrt{2}) + \ln(w + m_i/\sqrt{2}) \right] - 8 \ln w \right\} \quad (2.8)$$

where τ_0 is the “bare” coupling. This solution does not make sense everywhere, since $\text{Im } \tau$ (proportional to the inverse string coupling) becomes negative close to the orientifold location ($w = 0$). As is well known, this problem is cured by non-perturbative corrections.

2.2 F-theoretic description and D3 brane probes

In [25] A. Sen proposed an F-theoretic interpretation of the local set-up introduced above in which the exact axio-dilaton profile is given by the complex structure modulus τ of the auxiliary Seiberg-Witten (SW) curve that describes the quantum moduli space of the $\mathcal{N} = 2$ SYM theory in four dimensions with gauge group $SU(2)$ and $N_f = 4$ fundamental flavors [27]. The appearance of this seemingly unrelated four-dimensional gauge theory was explained in [28] by considering D3 brane probes in the orientifold background described in the previous section. As is well known, the orientifold projection that leads to an orthogonal gauge group on the D7 branes gives rise to a symplectic gauge group on the D3’s. A single D3 and its image at an orientifold fixed point support a $Sp(1) \sim SU(2)$ gauge group. The degrees of freedom of the 3/3 strings fill up a four-dimensional $\mathcal{N} = 2$ chiral multiplet in the adjoint representation of $Sp(1)$, namely

$$\Phi(x, \theta) = \phi(x) + \sqrt{2} \theta \Lambda(x) + \frac{1}{2} \theta \sigma^{\mu\nu} \theta F_{\mu\nu}(x) + \dots, \quad (2.9)$$

where we denoted the four-dimensional super-coordinates as (x, θ) . Notice that in this orientifold background the 3/3 spectrum contains also a neutral hypermultiplet.⁶

⁶With N D3 branes, this hypermultiplet transforms in the antisymmetric representation of $Sp(N)$.

The open strings stretching between the D3's and the D7's contain precisely the degrees of freedom of four fundamental hypermultiplets. From the D3 point of view, the $SO(8)$ Chan-Paton group on the D7 branes represents the global flavor symmetry group, while the D7 brane positions m_i appear as masses for the hypers, since the 3/7 strings become stretched. It turns out that the neutral hypermultiplet from the 3/3 sector is also completely decoupled from the flavored matter, and thus it will be ignored in our subsequent analysis.

Moving the probe D3 brane away from the orientifold fixed point corresponds to explore the Coulomb moduli space of its world-volume theory. In the D3 brane action the effective coupling in front of the quadratic gauge Lagrangian for the chiral multiplet in the Cartan direction is exactly given by the axio-dilaton field, that is, we have the identification

$$\tau = \frac{\theta_{\text{YM}}}{\pi} + i \frac{8\pi}{g_{\text{YM}}^2}, \tag{2.10}$$

where θ_{YM} is the theta-angle and g_{YM} the gauge coupling. This coupling receives perturbative corrections at the one-loop level only, and for N_f flavors it takes the form⁷

$$2\pi i \tau_{\text{pert}} = 2\pi i \tau_0 + \left\{ \sum_{i=1}^{N_f} \left[\ln \frac{a - m_i/\sqrt{2}}{\Lambda} + \ln \frac{a + m_i/\sqrt{2}}{\Lambda} \right] - 8 \ln \frac{a}{\Lambda} - 2 \ln 16 \right\}, \tag{2.11}$$

where a parametrizes the vacuum expectation value of the adjoint $Sp(1)$ scalar field, namely

$$\langle \phi \rangle = \text{diag}(a, -a). \tag{2.12}$$

For $N_f = 4$ this expression is in agreement (a part from the finite renormalization term $2 \ln 16$ on which we will come back later) with eq. (2.8), obtained by solving the field equations for the axio-dilaton sourced by the D7 branes and O7 planes. In this comparison we see that a represents the position of the probe D3 brane and hence corresponds to the coordinate w . For large a , the following expansion holds

$$\begin{aligned} 2\pi i \tau_{\text{pert}} &= 2\pi i \tau_0 + \left\{ (2N_f - 8) \ln \frac{a}{\Lambda} - 2 \ln 16 - \sum_{l=1}^{\infty} \frac{1}{2^l l} \frac{\sum_i m_i^{2l}}{a^{2l}} \right\} \\ &= 2\pi i \tau_0 + \left\{ (2N_f - 8) \ln \frac{a}{\Lambda} - 2 \ln 16 - \sum_{l=1}^{\infty} \frac{1}{2^l l} \frac{\text{Tr} \langle m \rangle^{2l}}{a^{2l}} \right\}, \end{aligned} \tag{2.13}$$

where in the second line we used eq. (2.7). This expansion will turn out to be extremely useful in the following.

As is well known, from this field theory point of view it is possible to go beyond the perturbative results and derive the *exact* effective coupling on the moduli space of the D3 gauge theory from the appropriate SW curve. This coupling is then the exact axio-dilaton configuration for our set-up, i.e. it represents the F-theoretic solution.

⁷In the $N_f = 4$ case, the scale Λ cancels out.

2.2.1 The SW curve for the D3 gauge theory

The SW curve for the SU(2) SYM theory with $N_f = 4$ massive fundamental flavors was proposed in [27] to be given by a torus described by the equation

$$y^2 = P_3(x), \tag{2.14}$$

where the cubic polynomial $P_3(x)$ is

$$P_3(x) = W_1 W_2 W_3 + A \left[W_1 T_1 (e_2 - e_3) + W_2 T_2 (e_3 - e_1) + W_3 T_3 (e_1 - e_2) \right] - A^2 N. \tag{2.15}$$

Here, for $\ell = 1, 2, 3$, we have introduced

$$W_\ell = x - e_\ell \tilde{u} - e_\ell^2 R, \tag{2.16}$$

while R , T_ℓ and N are invariants of the flavor group SO(8) that are, respectively, quadratic, quartic and sextic in the masses m_i (see appendix B for details). The three quartic invariants T_ℓ satisfy the relation $T_1 + T_2 + T_3 = 0$, and hence we can use as independent quantities T_1 and the Pfaffian

$$\text{Pfm} \equiv m_1 m_2 m_3 m_4, \tag{2.17}$$

since

$$T_2 = -\frac{1}{2} (T_1 + \text{Pfm}), \quad T_3 = -\frac{1}{2} (T_1 - \text{Pfm}). \tag{2.18}$$

The quantities e_ℓ and A are functions of the “bare” coupling τ_0 , namely

$$e_1 = \frac{\vartheta_3^4(\tau_0) + \vartheta_4^4(\tau_0)}{3}, \quad e_2 = \frac{-\vartheta_2^4(\tau_0) - \vartheta_3^4(\tau_0)}{3}, \quad e_3 = \frac{\vartheta_2^4(\tau_0) - \vartheta_4^4(\tau_0)}{3}, \tag{2.19}$$

and

$$A = (e_1 - e_2)(e_2 - e_3)(e_3 - e_1) = 16 \eta^{12}(\tau_0). \tag{2.20}$$

Notice that all these functions can be expanded in power series with respect to

$$q = e^{\pi i \tau_0}, \tag{2.21}$$

which represents the weight of a gauge instanton configuration. Finally, the parameter \tilde{u} appearing in (2.16) is given by

$$\tilde{u} = u - \frac{1}{2} e_1 R, \tag{2.22}$$

with u being the SU(2) invariant

$$u \equiv \langle \text{Tr} \phi^2 \rangle \tag{2.23}$$

that parametrizes the Coulomb moduli space. In the semiclassical regime, i.e. when u is large, we have

$$u = \text{Tr} \langle \phi \rangle^2 = 2a^2, \tag{2.24}$$

where the second equality follows from (2.12).

Notice that in the massless case ($m_i = 0$) the SW curve (2.14) reduces to the equation of a torus with

$$\tau = \tau_0. \tag{2.25}$$

In the massive case, instead, the complex structure τ associated to the curve (2.14) differs from the “bare” coupling τ_0 . It represents the exact coupling in the low-energy effective theory on the D3 probe and hence the exact axio-dilaton background associated to the D7 brane configuration. We shall now extract from the SW curve the instanton expansion of τ in the region of large a .

By a suitable shift in x , the SW curve (2.14) can be put in Weierstrass form

$$y^2 = x^3 - \frac{g_2}{4}x - \frac{g_3}{4} \tag{2.26}$$

where the coefficients g_2 and g_3 can be determined explicitly in terms of u , of the mass invariants and of q . The absolute modular invariant of the curve

$$j = \frac{g_2^3}{g_2^3 - 27g_3^2} \tag{2.27}$$

has then an explicit expression in terms of the same parameters. This expression is such that j diverges as $q \rightarrow 0$; indeed

$$j = \frac{1}{1728q^2} \frac{u^4}{\prod_i (u - m_i^2)} + O(q^{-1}) . \tag{2.28}$$

The modular invariant j is related to the complex structure modulus of the torus by the expression given in (2.4), which we rewrite here for convenience:

$$j = \left(\frac{\vartheta_2^8(\tau) + \vartheta_3^8(\tau) + \vartheta_4^8(\tau)}{24\eta^8(\tau)} \right)^3 . \tag{2.29}$$

Inverting this relation yields the effective coupling as a function of j , namely

$$2\pi i \tau = \ln \left(\frac{1}{1728} \frac{1}{j} \right) + \frac{31}{72} \frac{1}{j} + \frac{13157}{82944} \frac{1}{j^2} + \dots . \tag{2.30}$$

Then, using the expansion of j given in (2.28), we can organize the effective coupling as a series in q

$$2\pi i \tau = 2\pi i \tau_0 + \sum_k f_k(m, u) q^k \tag{2.31}$$

in which each coefficient f_k can in turn be expanded for large u .

The instanton expansion is then obtained by rewriting, in the semi-classical regime, the modulus u in terms of the vacuum expectation value a . To this end, let us recall that the periods of the holomorphic form dx/y on the Weierstrass torus have an explicit expression in terms of j . In the region where j is large, one period has the following closed-form expression

$$\omega_1 = (48g_2)^{-\frac{1}{4}} F \left(\frac{1}{12}, \frac{5}{12}, 1; \frac{1}{j} \right) , \tag{2.32}$$

in terms of the hypergeometric function F , and is related to the period a of the SW meromorphic form by

$$\frac{\partial a}{\partial u} = \omega_1 . \tag{2.33}$$

Indeed, it is possible to expand ω_1 , and hence a , as a double series in q and in $1/u$, obtaining

$$a(u) = \sqrt{\frac{u}{2}} \left(1 - 4 \frac{\text{Pfm}}{u^2} q + \dots \right). \quad (2.34)$$

Inverting this expansion, one gets the expression of $u(a)$ as a sum of instanton contributions, valid for large a , namely

$$u(a) = 2a^2 + 4 \frac{\text{Pfm}}{a^2} q + O(q^2) \quad (2.35)$$

in which the leading term is in agreement with the classical result (2.24). Inserting this expansion into eq. (2.31), we obtain⁸

$$\begin{aligned} 2\pi i \tau = & 2\pi i \tau_0 - \frac{R}{a^2} + \frac{1}{a^4} \left[\frac{-R^2 + 6T_1}{4} + 12 \text{Pfm} q + 6(R^2 + 6T_1) q^2 + 48 \text{Pfm} q^3 + \dots \right] \\ & + \frac{1}{a^6} \left[\frac{-6N - R^3 + 15RT_1}{12} - 60(2N + RT_1) q^2 - 320 \text{Pfm} R q^3 + \dots \right] \\ & + \frac{1}{a^8} \left[\frac{2(\text{Pfm})^2 - 16NR - R^4 + 28R^2T_1 - 36T_1^2}{32} + \frac{105}{2} (\text{Pfm})^2 q^2 \right. \\ & \left. + 280 \text{Pfm} (R^2 + 6T_1) q^3 + \dots \right] + \dots \end{aligned} \quad (2.36)$$

Using the relations (B.3) and (B.7) given in appendix B, we can rewrite this expression as follows

$$\begin{aligned} 2\pi i \tau = & 2\pi i \tau_0 - \frac{1}{2} \frac{\sum_i m_i^2}{a^2} + \frac{1}{a^4} \left[-\frac{1}{8} \sum_i m_i^4 + 12 \text{Pfm} q + 6 \sum_{i<j} m_i^2 m_j^2 q^2 + 48 \text{Pfm} q^3 + \dots \right] \\ & + \frac{1}{a^6} \left[-\frac{1}{24} \sum_i m_i^6 - 30 \sum_{i<j<k} m_i^2 m_j^2 m_k^2 q^2 - 160 \text{Pfm} \sum_i m_i^2 q^3 + \dots \right] \\ & + \frac{1}{a^8} \left[-\frac{1}{64} \sum_i m_i^8 + \frac{105}{2} (\text{Pfm})^2 q^2 + 280 \text{Pfm} \sum_{i<j} m_i^2 m_j^2 q^3 + \dots \right] + \dots \end{aligned} \quad (2.37)$$

The effective coupling τ encoded in the SW curve contains a perturbative part in perfect agreement⁹ with eq. (2.13) for $N_f = 4$ plus a series of non-perturbative corrections δ_l , and thus takes the form

$$2\pi i \tau = 2\pi i \tau_0 - \sum_{l=1}^{\infty} \frac{1}{2l} \frac{\text{Tr} \langle m \rangle^{2l} + \delta_l}{a^{2l}}, \quad (2.38)$$

⁸The expansions are rather cumbersome, so we limit ourselves to order q^3 and to order $1/a^8$. This already provides a very strong check against the result of the explicit instanton computations presented in the following sections.

⁹A part from the finite renormalization constant $2 \ln 16$.

where the explicit expressions for δ_l can be deduced from eq. (2.37). In particular, the first few corrections are

$$\begin{aligned}
 \delta_1 &= 0, \\
 \delta_2 &= -48 \text{Pfm } q - 24 \sum_{i < j} m_i^2 m_j^2 q^2 - 192 \text{Pfm } q^3 + \dots, \\
 \delta_3 &= 180 \sum_{i < j < k} m_i^2 m_j^2 m_k^2 q^2 + 960 \text{Pfm } \sum_i m_i^2 q^3 + \dots, \\
 \delta_4 &= -420 (\text{Pfm})^2 q^2 - 2240 \text{Pfm } \sum_{i < j} m_i^2 m_j^2 q^3 + \dots.
 \end{aligned}
 \tag{2.39}$$

In the following we will provide a microscopic derivation of this result by introducing stacks of D-instantons in our orientifold model.

3 D-instanton corrections

We now discuss the non-perturbative effects obtained by adding k D-instantons to the D3/D7 brane system described so far.¹⁰ The D(-1)'s describe ordinary gauge instantons in the four-dimensional $\mathcal{N} = 2$ Sp(1) gauge theory with $N_f = 4$ realized on the world-volume of the D3 branes, while they represent exotic instantons with respect to the eight-dimensional $\mathcal{N} = 2$ SO(8) gauge theory realized on the D7 branes, as discussed in [31, 33, 51]. In the following we describe the moduli spectrum and its BRST structure and compute the D-instanton partition function using localization methods.

3.1 Instanton moduli spectrum

The open strings with at least one end-point on the D(-1)'s account for the instanton collective coordinates (or moduli) and can be distinguished into three classes corresponding to open strings with both end-points on the D-instantons (the (-1)/(-1) sector), and to open strings with one end-point on the D-instantons and the other either on the D3 branes or on the D7 branes (the (-1)/3 sector and the (-1)/7 sector respectively). This system has already been considered and described in [29], but for completeness we now briefly recall its main features, taking advantage of the analysis presented in [34] for a very similar set-up.

(-1)/(-1) sector. The string excitations in this sector carry Chan-Paton (CP) factors that are $(k \times k)$ matrices in a definite representation of SO(k) selected by the orientifold projection Ω . They also organize in representations of the Lorentz group which in our case is

$$\text{SO}(4) \times \widehat{\text{SO}}(4) \times \text{SO}(2) \simeq \text{SU}(2)_+ \times \text{SU}(2)_- \times \widehat{\text{SU}}(2)_+ \times \widehat{\text{SU}}(2)_- \times \text{U}(1)
 \tag{3.1}$$

as dictated by the presence of the D7 and D3 branes.

¹⁰According to the terminology used in the previous section this actually corresponds to adding k half-D(-1) branes plus their orientifold images.

In the NS sector the two bosonic moduli along the D7 transverse directions, with ADHM dimension of $(\text{length})^{-1}$, are odd under Ω and thus transform in the anti-symmetric representation of $\text{SO}(k)$. They can be organized in complex conjugate moduli χ and $\bar{\chi}$ that are singlets of the four $\text{SU}(2)$'s in (3.1) and carry charge +1 and -1 under the $\text{U}(1)$ factor. The eight bosonic moduli along the D7 longitudinal directions, with ADHM dimension of $(\text{length})^{+1}$, are instead even under Ω and hence transform in the symmetric representation of $\text{SO}(k)$. They form two vectors for each of the two $\text{SO}(4)$ factors in (3.1) that we denote, respectively, a^μ ($\mu = 0, \dots, 3$) and a^m ($m = 4, \dots, 7$). The diagonal components $\text{Tr } a^\mu$ define the (center of mass) position of the D-instantons in the four-dimensional space that is longitudinal to both the D7's and the D3's, and thus they can be identified with the four-dimensional space-time coordinates x^μ . The other four diagonal components $\text{Tr } a^m$ represent the position in the remaining four longitudinal directions of the D7 branes, and together with the x 's they can be identified with the eight-dimensional coordinates X used in the previous section.

A similar analysis can be performed in the Ramond sector of the $(-1)/(-1)$ strings. Here we have sixteen fermionic moduli which we can group into four sets $M^{\dot{\alpha}a}$, $M^{\alpha\dot{a}}$, $N^{\alpha a}$ and $N^{\dot{\alpha}\dot{a}}$ with $\alpha, \dot{\alpha}, a$ and \dot{a} labelling the spinor representations of the four $\text{SU}(2)$'s in (3.1). We have denoted by M 's and N 's the components with positive and negative $\text{SO}(2)$ chiralities, respectively, that correspond to eigenvalues +1 and -1 under Ω so that the M 's, with ADHM dimension of $(\text{length})^{+1/2}$, are in the symmetric representation of $\text{SO}(k)$, while the N 's, with dimension of $(\text{length})^{-3/2}$, are in the anti-symmetric representation of $\text{SO}(k)$.

$(-1)/3$ sector. The CP factors of the open strings connecting the D-instantons with the D3 brane and its orientifold image are $(k \times 2)$ matrices transforming in the fundamental representation of both $\text{SO}(k)$ and $\text{Sp}(1)$. The open strings with opposite orientation, stretching between the D3's and the D(-1)'s, carry CP factors that are $(2 \times k)$ matrices but they should not be counted as independent because of the identification enforced by the orientifold projection.

The first four directions of the $(-1)/3$ strings have mixed Dirichlet-Neumann boundary conditions. In the NS sector we therefore find two bosonic moduli, with ADHM dimension of $(\text{length})^1$, in the spinor representation of $\text{SU}(2)_+$ which we denote by w_α . However, if we want to exhibit also the $\text{SO}(k)$ and $\text{Sp}(1)$ indices, we should write $w_{\alpha RU}$ (with $R = 1, \dots, k$ and $U = 1, 2$). For the opposite string orientation the bosonic excitations are $\bar{w}^{\alpha UR}$, but the following orientifold identification holds [29]:

$$\bar{w}^{\alpha UR} = \epsilon^{\alpha\beta} \epsilon^{UV} \delta^{RS} w_{\beta SV} . \tag{3.2}$$

In the R sector we find four fermionic moduli, with dimension of $(\text{length})^{1/2}$, which we denote as μ_a and $\mu_{\dot{a}}$ since they are space-time scalars but spinors of the internal Lorentz group. The fermionic moduli $\bar{\mu}^a$ and $\bar{\mu}^{\dot{a}}$ arising from open strings with opposite orientation are subject to orientifold identifications similar to that displayed in (3.2).

$(-1)/7$ sector. In this sector the open string excitations carry CP factors that are $(k \times 8)$ matrices transforming in the fundamental representation of both $\text{SO}(k)$ and $\text{SO}(8)$. Again

the orientifold projection enforces identifications with the excitations of open strings with opposite orientations, which therefore should not be counted as independent. Since the $(-1)/7$ strings have eight directions with mixed Dirichlet-Neumann boundary conditions, the physical states are only in the R sector where we find one fermionic modulus μ' with ADHM dimension of $(\text{length})^{+1/2}$. To exhibit also the $SO(k)$ and $SO(8)$ indices we should write μ'_{RI} (with $R = 1, \dots, k$ and $I = 1, \dots, 8$). The physical excitations of the $7/(-1)$ strings, whose CP factors are $(8 \times k)$ matrices, correspond to the modulus $\bar{\mu}'$ which is subject to the following orientifold identification

$$\bar{\mu}'^{RI} = \delta^{RS} \delta^{IJ} \mu'_{SJ} . \tag{3.3}$$

The absence of physical bosonic moduli in this sector is typical of exotic instanton configurations.

3.2 BRST structure of moduli space

The moduli space we have described above admits a BRST structure that will play a crucial rôle for the localization of the integral over the D-instantons.

The BRST charge Q can be obtained by choosing any component of the supersymmetry charges preserved on the brane system. The supersymmetry charges are invariant under $SO(k)$, $Sp(1)$ and $SO(8)$, but transform as a spinor of the $SO(4) \times \widehat{SO}(4)$ subgroup of the Lorentz group (3.1), so that the choice of Q breaks it to the $SU(2)^3$ subgroup which preserves this spinor. In our case we take

$$SU(2)_1 \times SU(2)_2 \times SU(2)_3 \equiv SU(2)_- \times \widehat{SU}(2)_- \times \text{diag} \left[SU(2)_+ \times \widehat{SU}(2)_+ \right] , \tag{3.4}$$

which corresponds to identify the spinor indices α and a of the first and third $SU(2)$'s in (3.1). After this identification is made, the fermionic moduli $M^{\dot{\alpha}a}$ and $M^{a\dot{\alpha}}$ can be renamed as $M^{\mu=\dot{\alpha}\alpha}$ and $M^{m=a\dot{a}}$, and paired with a^μ and a^m into BRST doublets such that $Qa^\mu = M^\mu$ and $Qa^m = M^m$. Similarly, in the fermionic sector of the $(-1)/(-1)$ strings the singlet component $N = \epsilon_{\alpha a} N^{\alpha a}$ has the right transformation properties to qualify as BRST partner of $\bar{\chi}$, i.e. $Q\bar{\chi} = N$. Likewise, in the $(-1)/3$ sector the fermionic moduli $\mu_{\alpha=a}$ can be paired with the bosons w_α and form another BRST doublet such that $Qw_\alpha = \mu_\alpha$.

The remaining fields $N^c = (\tau^c)_{\alpha a} N^{\alpha a}$ (τ^c being the three Pauli matrices), $N^{\dot{\alpha}\dot{a}}$, $\mu_{\dot{a}}$ and μ' are unpaired, and should be supplemented with auxiliary fields having identical transformation properties. Denoting such fields D^c , $D^{\dot{\alpha}\dot{a}}$, $h_{\dot{a}}$ and h' , respectively, we therefore have $QN^c = D^c$, $QN^{\dot{\alpha}\dot{a}} = D^{\dot{\alpha}\dot{a}}$ and so on.

In the usual instanton theory the auxiliary fields collect the D- and F-terms of the gauge theory on the D(-1)'s, and the corresponding D- and F-flatness conditions are the ADHM constraints on the instanton moduli space (see for example [8, 37, 40] for details). In our case we have an extension of this construction to a sort of generalized “exotic” instanton moduli space. More precisely, the seven auxiliary moduli D^c and $D^{\dot{\alpha}\dot{a}}$, of scaling dimension $(\text{length})^{-2}$ linearize the quartic interactions among the a^μ 's and the a^m 's, and in the explicit string realization correspond to vertex operators that are bi-linear in the fermionic string coordinates [8, 31]. In particular, the triplet D^c disentangles the quartic

sector	(Ψ_0, Ψ_1)	$SO(k) \times Sp(1) \times SO(8)$	$SU(2)^3$
D(-1)/D(-1)	(a^μ, M^μ)	$(\square, \mathbf{1}, \mathbf{1})$	$(\mathbf{2}, \mathbf{1}, \mathbf{2})$
	(a^m, M^m)	$(\square, \mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{2}, \mathbf{2})$
	$(N^{\dot{a}\dot{a}}, D^{\dot{a}\dot{a}})$	$(\boxplus, \mathbf{1}, \mathbf{1})$	$(\mathbf{2}, \mathbf{2}, \mathbf{1})$
	(N^c, D^c)	$(\boxplus, \mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1}, \mathbf{3})$
	$(\bar{\chi}, \eta)$	$(\boxplus, \mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1}, \mathbf{1})$
	χ	$(\boxplus, \mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1}, \mathbf{1})$
D(-1)/D3	(w_α, μ_α)	$(\square, \square, \mathbf{1})$	$(\mathbf{1}, \mathbf{1}, \mathbf{2})$
	$(\mu_{\dot{a}}, h_{\dot{a}})$	$(\square, \square, \mathbf{1})$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})$
D(-1)/D7	(μ', h')	$(\square, \mathbf{1}, \square)$	$(\mathbf{1}, \mathbf{1}, \mathbf{1})$

Table 1. Spectrum of moduli for the D7/D3/D(-1) system in type I', arranged in BRST doublets (Ψ_0, Ψ_1) such that $Q\Psi_0 = \Psi_1$. The last column displays the representations of the various moduli under the three $SU(2)$'s defined in (3.4).

interactions of the a^μ 's and the a^m 's among themselves, while the quartet $D^{\dot{a}\dot{a}}$ decouples the quartic interactions between the a^μ 's and the a^m 's. Finally, the dimensionless auxiliary moduli $h_{\dot{a}}$ disentangle the quartic interactions between a^m and w_α , while h' completes the BRST multiplet in the $(-1)/7$ sector. In the end only one modulus, namely χ , remains unpaired and therefore $Q\chi = 0$. All this is summarized in table 1.

Since the scaling dimension of the BRST charge is $(\text{length})^{-1/2}$, the dimensions of the components (Ψ_0, Ψ_1) of any BRST doublet are of the form $(\text{length})^\Delta$ and $(\text{length})^{\Delta-1/2}$. Thus, recalling that a fermionic variable and its differential have opposite dimensions, the measure on the instanton moduli space

$$d\mathcal{M}_k \equiv d\chi \prod_{(\Psi_0, \Psi_1)} d\Psi_0 d\Psi_1 \quad (3.5)$$

has the total dimension

$$(\text{length})^{-\frac{1}{2}k(k-1) + \frac{1}{2}n_b - \frac{1}{2}n_f} . \quad (3.6)$$

Here, the first term in the exponent accounts for the unpaired modulus χ in the anti-symmetric representation of $SO(k)$, while n_b (n_f) denotes the number of BRST multiplets whose lowest components Ψ_0 are bosonic (fermionic). From table 1 it is not difficult to verify that

$$n_b = \frac{9}{2}k^2 + \frac{15}{2}k \quad \text{and} \quad n_f = \frac{7}{2}k^2 + \frac{17}{2}k . \quad (3.7)$$

Hence, eq. (3.6) implies that the full measure $d\mathcal{M}_k$ is dimensionless, in agreement with the conformality of the theories on both the D3's and the D7's.

For all the moduli we have listed above, including the auxiliary ones, it is possible to write vertex operators of conformal dimension one and use them to obtain the moduli action $\mathcal{S}(\mathcal{M}_k)$ by computing disk amplitudes along the lines discussed in [7, 8, 30, 31]. The

result is the moduli action derived in [29] (see in particular eq. (3.3) of that paper). A first generalization that will be necessary in the following is the introduction of the interaction terms between the moduli and the $\text{Sp}(1)$ adjoint scalar ϕ of the 3/3 sector or the $\text{SO}(8)$ adjoint scalar m of the 7/7 sector. As explained for example in [30], such interaction terms can be derived from mixed disk amplitudes with a portion of the boundary on the D3's or the D7's where a scalar field ϕ or m can be emitted. The result of these computations is the action $\mathcal{S}(\mathcal{M}_k; \phi, m)$. It turns out that this can be obtained from $\mathcal{S}(\mathcal{M}_k)$ with a simple effective rule corresponding to a shift of the χ moduli. More precisely, we have to make the following replacements¹¹

$$\begin{aligned} \epsilon^{UV} \chi^{RS} &\rightarrow \epsilon^{UV} \chi^{RS} + \phi^{UV} \delta^{RS} , \\ \delta^{IJ} \chi^{RS} &\rightarrow \delta^{IJ} \chi^{RS} + i m^{IJ} \delta^{RS} . \end{aligned} \tag{3.8}$$

Furthermore, one can prove that this action is BRST exact, namely

$$\mathcal{S}(\mathcal{M}_k; \phi, m) = Q \Xi \tag{3.9}$$

for a suitably defined fermion Ξ and that Q is nilpotent up to infinitesimal transformations of $\text{SO}(k)$, $\text{Sp}(1)$ and $\text{SO}(8)$ parameterized respectively by χ , ϕ and $i m$. This means that on any modulus we have

$$Q^2 \bullet = [T_{\text{SO}(k)}(\chi) + T_{\text{Sp}(1)}(\phi) + T_{\text{SO}(8)}(i m)] \bullet , \tag{3.10}$$

where the T 's act in the specific $\text{SO}(k)$, $\text{Sp}(1)$ and $\text{SO}(8)$ representations given in table 1.

For our later purposes it is enough to consider the Cartan directions of the various groups. We label the Cartan components of the $\text{SO}(k)$ parameters of Q by $\vec{\chi} = \{\chi_r\}$ with $r = 1, \dots, \text{rank SO}(k)$, those of $\text{Sp}(1)$ by $\vec{\phi} = \{a\}$, and those of $\text{SO}(8)$ by $\vec{m} = \{m_i/\sqrt{2}\}$ with $i = 1, \dots, 4$. The latter two choices are in agreement with the choices and normalizations used in the previous section, see in particular eqs. (2.12) and (2.7). Using these ingredients, one can easily see that Q^2 in (3.10) corresponds to infinitesimal Cartan actions which can be diagonalized in any representation by going to the basis provided by the weights. Indeed, denoting respectively by $\vec{\pi}$, $\vec{\sigma}$ and $\vec{\rho}$ the weights of the $\text{SO}(k)$, $\text{Sp}(1)$ and $\text{SO}(8)$ representations under which a given modulus transforms, we can rewrite (3.10) as

$$Q^2 \bullet = [\vec{\chi} \cdot \vec{\pi} + i \vec{\phi} \cdot \vec{\sigma} + i \vec{m} \cdot \vec{\rho}] \bullet . \tag{3.11}$$

To fully localize the integral over moduli space it is necessary to use a BRST charge that is equivariant with respect to all symmetries of the model. In our case these include also the residual Lorentz symmetry $\text{SU}(2)^3$ defined in (3.4), besides the $\text{SO}(k)$, $\text{Sp}(1)$ and $\text{SO}(8)$ symmetries considered so far.

3.3 Moduli integration via localization and instanton partition function

As explained for example in [35]–[40], the fully equivariant cohomology can be obtained by deforming the moduli action with parameters associated to “rotations” along the Cartan

¹¹As noted in footnote 5, in our conventions it is the field im which is in the adjoint of $\text{SO}(8)$.

directions of the residual Lorentz group. For our model, these rotations can be parameterized by $\vec{\epsilon} = \{\epsilon_A\}$ with $A = 1, \dots, 4$, subject to the constraint

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0. \quad (3.12)$$

Although only three out of the ϵ 's are independent variables, in all calculations it is convenient to use all four of them and impose the relation (3.12) only at the very end. From the string point of view the ϵ -deformation can be obtained by switching on a RR background on the D7 brane world-volume corresponding to graviphoton field strengths of the type

$$\mathcal{F}_{\mu\nu} \sim \begin{pmatrix} 0 & +\epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +\epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{F}_{mn} \sim \begin{pmatrix} 0 & +\epsilon_3 & 0 & 0 \\ -\epsilon_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & +\epsilon_4 \\ 0 & 0 & -\epsilon_4 & 0 \end{pmatrix}. \quad (3.13)$$

This RR background induces new ϵ -dependent terms in the moduli action (3.9) which then gets replaced by a deformed action $\mathcal{S}(\mathcal{M}_k; \phi, m, \epsilon)$. The ϵ -dependent terms can be explicitly derived by computing mixed open/closed string amplitudes on disks with insertions of the moduli vertex operators on the boundary and of the vertex operators representing the graviphoton field strengths \mathcal{F} in the interior, as shown in [30]–[32] for similar systems. The net result of these computations amounts to replace the old BRST charge Q with a deformed charge \tilde{Q} satisfying the following relation

$$\tilde{Q}^2 \bullet = [\vec{\chi} \cdot \vec{\pi} + i \vec{\phi} \cdot \vec{\sigma} + i \vec{m} \cdot \vec{\rho} + \vec{\epsilon} \cdot \vec{\gamma}] \bullet, \quad (3.14)$$

which is a direct generalization of (3.11). Here we have taken the deformation parameters ϵ_A to be of dimensions $(\text{length})^{-1}$ and have denoted by $\vec{\gamma}$ the weights of the representation of the residual Lorentz group (3.4) under which a given modulus transforms. Using the rules explained in [34], one can show that

$$\vec{\epsilon} \cdot \vec{\gamma} = (\epsilon_1 - \epsilon_2)\gamma_1 + (\epsilon_3 - \epsilon_4)\gamma_2 + (\epsilon_1 + \epsilon_2)\gamma_3, \quad (3.15)$$

where γ_ℓ are the weights of the three $SU(2)$'s, namely $\gamma_\ell = 0$ for a modulus in the $\mathbf{1}$, $\gamma_\ell = \pm \frac{1}{2}$ for a modulus in the $\mathbf{2}$ and so on. The explicit values of $\vec{\epsilon} \cdot \vec{\gamma}$ for all BRST doublets of moduli are collected in table 2.

The ϵ -deformed moduli action is still BRST exact (with respect to the new BRST charge \tilde{Q}), namely

$$\mathcal{S}(\mathcal{M}_k; \phi, m, \epsilon) = \tilde{Q} \tilde{\Xi} \quad (3.16)$$

for a suitable fermion $\tilde{\Xi}$. This deformed BRST structure allows to perform rescalings of various moduli and show that the k -instanton partition function

$$\mathcal{Z}_k \equiv \mathcal{N}_k \int d\mathcal{M}_k e^{-\mathcal{S}(\mathcal{M}_k; \phi, m, \epsilon)} \quad (3.17)$$

(with \mathcal{N}_k a normalization factor) can be computed exactly in the semiclassical approximation. Indeed, the complete localization of the integral over moduli space around isolated

sector	(Ψ_0, Ψ_1)	$\vec{\epsilon} \cdot \vec{\gamma}$
D(-1)/D(-1)	(a^μ, M^μ)	$\pm\epsilon_1, \pm\epsilon_2$
	(a^m, M^m)	$\pm\epsilon_3, \pm\epsilon_4$
	$(N^{\dot{a}\dot{a}}, D^{\dot{a}\dot{a}})$	$\pm(\epsilon_2 + \epsilon_3), \pm(\epsilon_1 + \epsilon_3)$
	(N^c, D^c)	$0, \pm(\epsilon_1 + \epsilon_2)$
	$(\bar{\chi}, \eta)$	0
D(-1)/D3	(w_α, μ_α)	$\pm\frac{1}{2}(\epsilon_1 + \epsilon_2)$
	$(\mu_{\dot{a}}, h_{\dot{a}})$	$\pm\frac{1}{2}(\epsilon_3 - \epsilon_4)$
D(-1)/D7	(μ', h')	0

Table 2. The last column displays the values of $\vec{\epsilon} \cdot \vec{\gamma}$ for the various BRST doublets.

fixed points implies that \mathcal{Z}_k is given by the (super)-determinant of \tilde{Q}^2 evaluated at the fixed points of \tilde{Q} [35]–[40]. As we already mentioned, the moduli χ and $\bar{\chi}$ appear very asymmetrically in the BRST formalism: χ parametrizes the $SO(k)$ rotations, while $\bar{\chi}$ falls into one of the BRST doublets. Moreover, the contribution of the $(\bar{\chi}, \eta)$ multiplet to the super-determinant cancels against an identical contribution coming from the neutral component in (N^c, D^c) with identical transformation properties and opposite statistics. Taking this into account, the super-determinant of \tilde{Q}^2 takes a simple product form in terms of the \tilde{Q}^2 -eigenvalues given in (3.14) and the k -instanton partition function (3.17) can be expressed by the localization formula

$$\mathcal{Z}_k = \mathcal{N}_k \int \prod_{r=1}^{\text{rank } SO(k)} \left(\frac{d\chi_r}{2\pi i} \right) \Delta(\chi) \frac{P_{\square}(\chi)}{P_{\square}(\chi)} P_{\square}(\chi). \quad (3.18)$$

Here $\Delta(\chi)$ is the Vandermonde determinant representing the Jacobian factor for the diagonalization of the integration variables χ . The factor $P_{\square}(\chi)$ arises from the integration over the BRST doublets containing N^c and $N^{\dot{a}\dot{a}}$ which transform in the \square representation of $SO(k)$. It is given by

$$P_{\square}(\chi) = (-s_1 s_2 s_3)^{r_{\square}} \prod_{\ell=1}^3 \left\{ \prod_{\vec{\pi} \in \square^+} [(\vec{\chi} \cdot \vec{\pi})^2 - s_{\ell}^2] \right\}, \quad (3.19)$$

where we have introduced the following combinations

$$s_1 = \epsilon_2 + \epsilon_3, \quad s_2 = \epsilon_1 + \epsilon_3, \quad s_3 = \epsilon_1 + \epsilon_2, \quad (3.20)$$

and denoted by r_{\square} the number of null weights in the anti-symmetric representation of $SO(k)$ (namely, the rank of the group) and by \square^+ the set of its positive weights.

The factor $P_{\square\square}(\chi)$ in the denominator of (3.18) arises from the integration over the BRST doublets containing a^μ and a^m , and is given by

$$P_{\square\square}(\chi) = (\epsilon_1\epsilon_2\epsilon_3\epsilon_4)^{r_{\square\square}} \prod_{A=1}^4 \left\{ \prod_{\vec{\pi} \in \square\square^+} [(\vec{\chi} \cdot \vec{\pi})^2 - \epsilon_A^2] \right\}, \quad (3.21)$$

where again $r_{\square\square}$ is the number of null weights in the symmetric representation of $\text{SO}(k)$ and $\square\square^+$ is the set of its positive weights.

Finally, the last factor $P_{\square}(\chi)$ in (3.18) collects the contributions from the integration over the BRST doublets containing the moduli with only one end-point on the D-instantons, namely the μ' 's from the $(-1)/7$ sector, and the w_α 's and $\mu_{\dot{a}}$'s from the $(-1)/3$ sector. It reads

$$P_{\square}(\chi) = \left(\frac{\text{Pf}m}{4} \frac{a^2 + \left(\frac{\epsilon_3 - \epsilon_4}{2}\right)^2}{a^2 + \left(\frac{\epsilon_1 + \epsilon_2}{2}\right)^2} \right)^{r_{\square}} \prod_{\vec{\pi} \in \square^+} \left\{ \prod_{i=1}^4 \left[(\vec{\chi} \cdot \vec{\pi})^2 + \frac{m_i^2}{2} \right] \right. \\ \left. \times \frac{\left[(\vec{\chi} \cdot \vec{\pi})^2 - \left(ia + \frac{\epsilon_3 - \epsilon_4}{2}\right)^2 \right] \left[(\vec{\chi} \cdot \vec{\pi})^2 - \left(ia - \frac{\epsilon_3 - \epsilon_4}{2}\right)^2 \right]}{\left[(\vec{\chi} \cdot \vec{\pi})^2 - \left(ia + \frac{\epsilon_1 + \epsilon_2}{2}\right)^2 \right] \left[(\vec{\chi} \cdot \vec{\pi})^2 - \left(ia - \frac{\epsilon_1 + \epsilon_2}{2}\right)^2 \right]} \right\}, \quad (3.22)$$

where r_{\square} is the number of null weights in the fundamental representation of $\text{SO}(k)$ and \square^+ the set of its positive weights.

More explicit expressions for $\Delta(\chi)$, $P_{\square\square}(\chi)$, $P_{\square\square}(\chi)$ and $P_{\square}(\chi)$ are given in appendix C. Here we just remark that the integrand in (3.18) is a rational function and that the integrals over χ_r have to be computed according to the prescription of [52] as contour integrals after giving the deformation parameters ϵ_A 's a positive imaginary part such that

$$\text{Im } \epsilon_1 > \text{Im } \epsilon_2 > \dots > \text{Im } \epsilon_4 > \text{Im } \frac{\epsilon_1}{2} > \dots > \text{Im } \frac{\epsilon_4}{2}. \quad (3.23)$$

In this way the calculation of the D-instanton partition function \mathcal{Z}_k is reduced to the sum of the residues of a rational function with simple poles.

3.4 Non-perturbative prepotentials

In order to derive the non-perturbative contributions to the world-volume effective actions of the space-filling branes we have to sum over all instanton numbers and consider the “grand-canonical” instanton partition function

$$Z = \sum_{k=0}^{\infty} \mathcal{Z}_k q^k, \quad (3.24)$$

where we have conventionally set $\mathcal{Z}_0 = 1$. Then, we have to switch off the ϵ -deformations. However, in doing this we have to pay attention to a couple of points. In fact, as is clear from (3.17), the integrals appearing in \mathcal{Z}_k run over *all* moduli, including also the “center of mass” super-coordinates X and Θ . In presence of the ϵ -deformations it is rather easy to see that the integration over this eight-dimensional super-space yields a volume factor

growing as $1/(\epsilon_1\epsilon_2\epsilon_3\epsilon_4)$ in the limit of small ϵ_A 's. Therefore, to obtain the integral over the centered moduli and hence the contributions to the brane effective actions, this volume factor has to be removed before turning off the ϵ -deformations. In addition, we have to take into account the fact that the k -th order in the q -expansion receives contributions not only from genuine k -instanton configurations but also from disconnected ones, corresponding to copies of instantons of lower numbers k_ℓ such that $\sum_\ell k_\ell = k$. To isolate the connected components we have to take the logarithm of Z .

The singularity structure of $\log Z$ with respect to the ϵ -deformations turns out to be the following:

$$\log Z = \frac{1}{\epsilon_1\epsilon_2\epsilon_3\epsilon_4} \widehat{\mathcal{F}}_{\text{n.p.}}^{(8)} + \frac{1}{\epsilon_1\epsilon_2} \widehat{\mathcal{F}}_{\text{n.p.}}^{(4)}. \quad (3.25)$$

Here, $\widehat{\mathcal{F}}_{\text{n.p.}}^{(8)}$ and $\widehat{\mathcal{F}}_{\text{n.p.}}^{(4)}$ are expressions which are regular when the deformations are switched off; the subscript refers to the non-perturbative nature of these quantities, while the superscripts signal their eight-dimensional, respectively four-dimensional, nature. Since the ϵ_A 's have dimension of $(\text{length})^{-1}$, $\widehat{\mathcal{F}}_{\text{n.p.}}^{(8)}$ must have mass-dimension four, while $\widehat{\mathcal{F}}_{\text{n.p.}}^{(4)}$ must have mass-dimension two. Indeed, $\widehat{\mathcal{F}}_{\text{n.p.}}^{(8)}$ is a quartic expression in m_i and ϵ_A , without any dependence on a . Therefore we can conclude that $\widehat{\mathcal{F}}_{\text{n.p.}}^{(8)}$ represents the non-perturbative quartic prepotential for the D7 brane world-volume theory. On the other hand, $\widehat{\mathcal{F}}_{\text{n.p.}}^{(4)}$ is quadratic in ϵ_A , a and m_i ; since $1/(\epsilon_1\epsilon_2)$ represents the regulated (super)volume in the first four directions, we can conclude that $\widehat{\mathcal{F}}_{\text{n.p.}}^{(4)}$ is the non-perturbative quadratic prepotential for the D3 brane world-volume theory.¹²

The prepotentials $\widehat{\mathcal{F}}_{\text{n.p.}}^{(8)}$ and $\widehat{\mathcal{F}}_{\text{n.p.}}^{(4)}$ contain ‘‘gravitational’’ corrections encoded in the deformation parameters ϵ_A which, as we have described in section 3.3, arise from the RR closed string sector. In this paper we are not interested in this dependence, and we limit ourselves to the prepotentials

$$\mathcal{F}_{\text{n.p.}}^{(8)} = \lim_{\epsilon_A \rightarrow 0} \widehat{\mathcal{F}}_{\text{n.p.}}^{(8)}, \quad \mathcal{F}_{\text{n.p.}}^{(4)} = \lim_{\epsilon_A \rightarrow 0} \widehat{\mathcal{F}}_{\text{n.p.}}^{(4)} \quad (3.26)$$

which depend on the gauge theory parameters only.

It should not be a surprise that the D-instanton partition function Z in presence of D7 and D3 branes contains both an eight-dimensional and a four-dimensional information. Indeed, depending on the point of view, the D(-1)'s can be considered alternatively either as (exotic) instantons for the eight-dimensional theory on the D7's or as (ordinary) instantons for the four-dimensional theory on the D3's. In the following sections we will consider in detail the non-perturbative prepotentials (3.26), give their explicit expressions for the first instanton numbers (up to $k = 5$), analyze their properties and discuss their relations with the F-theory perspective and the SW curve we discussed in the previous section.

¹²Note that terms in $\widehat{\mathcal{F}}_{\text{n.p.}}^{(8)}$ of the form $\epsilon_3\epsilon_4 f$ could in principle be interpreted also as terms in $\widehat{\mathcal{F}}_{\text{n.p.}}^{(4)}$ proportional to f . We fix this kind of ambiguity by requiring that all terms assigned to $\widehat{\mathcal{F}}_{\text{n.p.}}^{(8)}$ have the structure and symmetry properties that are appropriate for a correct eight-dimensional interpretation as a prepotential.

4 Eight-dimensional effective prepotential and chiral ring

The non-perturbative prepotential for the eight-dimensional gauge theory on the D7 branes can be computed using formulae (3.25) and (3.26). Up to instanton order $k = 5$, we find

$$\begin{aligned} \mathcal{F}_{\text{n.p.}}^{(8)} = & -2 \text{Pf}m \left(q + \frac{4}{3}q^3 + \frac{6}{5}q^5 + \dots \right) \\ & - \frac{1}{2} \sum_{i < j} m_i^2 m_j^2 q^2 - \frac{1}{8} \left(\sum_i m_i^4 + 4 \sum_{i < j} m_i^2 m_j^2 \right) q^4 + \dots \end{aligned} \quad (4.1)$$

This prepotential is independent on the vacuum expectation value a of the D3 adjoint multiplet, and coincides exactly¹³ with the result found in [31] for the D7/D(-1) system in type I'. Using the quadratic and quartic SO(8) invariants defined in appendix B, we can rewrite the prepotential (4.1) as follows

$$\begin{aligned} \mathcal{F}_{\text{n.p.}}^{(8)} = & 2T_1 \left(q - \frac{3}{2}q^2 + \frac{4}{3}q^3 - \frac{3}{4}q^4 + \frac{6}{5}q^5 + \dots \right) \\ & + 4T_2 \left(q + \frac{4}{3}q^3 + \frac{6}{5}q^5 + \dots \right) - R^2 \left(\frac{1}{2}q^2 + \frac{3}{4}q^4 + \dots \right). \end{aligned} \quad (4.2)$$

This non-perturbative part has to be added to the tree-level prepotential

$$\mathcal{F}_{\text{tree}}^{(8)} = \frac{\pi i \tau_0}{4!} \text{Tr} m^4 = -\frac{1}{4} T_1 \ln q + \frac{1}{24} R^2 \ln q, \quad (4.3)$$

and to the perturbative 1-loop contribution which, up to terms that depend on the complex structure U of the torus transverse to the D7 branes and that we do not write for brevity, is given by [51]

$$\mathcal{F}_{1\text{-loop}}^{(8)} = \frac{1}{32} \ln \left(\frac{\text{Im} \tau_0}{2} \right) (\text{Tr} m^2)^2 = \frac{1}{8} \ln \left(\frac{\text{Im} \tau_0}{2} \right) R^2. \quad (4.4)$$

As shown in [31], the above results up to order $k = 5$ are consistent with heterotic/type I' duality [29, 41, 53, 54] and may be extended to all orders in q . Indeed, the complete prepotential may be written as

$$\begin{aligned} \mathcal{F}^{(8)} = & \mathcal{F}_{\text{tree}}^{(8)} + \mathcal{F}_{1\text{-loop}}^{(8)} + \mathcal{F}_{\text{n.p.}}^{(8)} \\ = & -\frac{1}{4} T_1 \ln \left(\frac{\kappa(q)}{16} \right) - \frac{1}{4} T_2 \ln (1 - \kappa(q)) + \frac{1}{8} R^2 \ln \left(\frac{\text{Im} \tau_0}{2} \eta^4(\tau_0) \right), \end{aligned} \quad (4.5)$$

where

$$\kappa(q) = \frac{\vartheta_2^4(\tau_0)}{\vartheta_3^4(\tau_0)}. \quad (4.6)$$

¹³Notice that our conventions are changed with respect to [31] in that we now have $q = \exp(\pi i \tau_0)$ to agree with the conventions of [26], while there we wrote $q = \exp(2\pi i \tau)$. Moreover, we choose the overall normalization of the instanton partition function in a way which corresponds to sending $q \rightarrow -q$, i.e. to changing the sign in front of the Pfaffian terms. Finally, the field m we use here corresponds to the field ϕ of [31]; however, because of our conventions in eq. (2.7), the eigenvalues ϕ_i of the latter correspond to $-im_i/\sqrt{2}$.

Expanding the ϑ -functions and the Dedekind function in powers of q , one can easily check the agreement with the perturbative and non-perturbative expressions reported above. Up to τ_0 -independent terms that can be presumably absorbed by a (finite) renormalization of the 1-loop contribution (4.4) which was not considered in [31], we can further rewrite the eight-dimensional prepotential as follows

$$\mathcal{F}^{(8)} = -\frac{1}{4} T_1 \ln(\kappa(q)) - \frac{1}{4} T_2 \ln(1 - \kappa(q)) + \frac{1}{8} R^2 \ln(\text{Im } \tau_0 \eta^4(\tau_0)) . \quad (4.7)$$

Notice that the single-trace quartic structure present in T_1 , whose tree-level coupling is $\pi i \tau_0$ (see eq. (4.3)), has an effective coupling given by

$$\ln \kappa(q) = \ln q + \ln 16 - 8q + 12q^2 - \frac{32}{3}q^3 + 6q^4 - \frac{48}{5}q^5 + \dots . \quad (4.8)$$

This single-trace structure is conformal at the perturbative level but from (4.8) we see that it receives both a finite 1-loop renormalization (the $\ln 16$ term above) and a series of instanton contributions. In section 7 we will find again this same non-perturbative redefinition (or more precisely its inverse) in the four-dimensional $SU(2)$ SYM theory living on the D3 branes.

From the prepotential $\mathcal{F}^{(8)}$ we obtain the eight-dimensional $SO(8)$ effective action on the D7 branes by promoting the diagonal vacuum expectation values m_i to the full chiral superfield $M(X, \Theta)$ given in (2.6) and integrating over the eight-dimensional chiral superspace, namely

$$S^{(8)} = \frac{1}{(2\pi)^4} \int d^8 X d^8 \Theta \mathcal{F}^{(8)}(M) + \text{c.c.} . \quad (4.9)$$

Among various other terms, this effective action contains quartic couplings of the type $\text{Tr}(t_8 f^4)$ with t_8 being the totally anti-symmetric eight-index tensor appearing in various string amplitudes [55] and f being the field strength, that under the heterotic/type I' duality exactly match the heterotic results [31].

As we discussed in section 2.2, the mathematical description of the non-perturbative effects in the eight-dimensional type I' theory should be identical to the non-perturbative description of the four-dimensional $\mathcal{N} = 2$ SYM theory with gauge group $SU(2)$ and $N_f = 4$ (massive) flavors. For such a theory, which possesses a global $SO(8)$ flavor symmetry, it was argued in [26] that the full symmetry of the effective action is $SL(2, \mathbb{Z})$ whose generators act as standard modular transformations on the tree-level gauge coupling accompanied by triality transformations on the flavor $SO(8)$ representations. Therefore, following the arguments of [25], we expect that such $SL(2, \mathbb{Z})$ be a symmetry of the eight-dimensional effective action (4.9), in which now $SO(8)$ is the gauge group. We now show that this is indeed the case. For simplicity we assume that the background axionic value be vanishing, i.e. we take $\text{Re } \tau_0 = 0$.

Let us first consider the transformation $\widehat{\mathcal{S}}$ defined as

$$\widehat{\mathcal{S}} : \tau_0 \longrightarrow -\frac{1}{\tau_0} , \quad T_1 \longleftrightarrow T_2 . \quad (4.10)$$

This is the combination of the \mathcal{S} modular transformation on the coupling constant τ_0 with the $\text{SO}(8)$ triality transformation that exchanges the vector representation (associated to T_1) with one of the spinorial representations (associated to T_2) leaving fixed the other (associated to $T_3 = -T_1 - T_2$) as well as the quadratic invariant R . Using the modular properties of the Jacobi ϑ -functions (see eq. (A.5)), it is easy to see that

$$\kappa(q) \longleftrightarrow 1 - \kappa(q), \tag{4.11}$$

and hence that the first two terms of the prepotential (4.7) are invariant under $\widehat{\mathcal{S}}$. Furthermore, for the case at hand we have

$$\text{Im } \tau_0 \eta^4(\tau_0) \longrightarrow -\frac{\tau_0}{\bar{\tau}_0} \text{Im } \tau_0 \eta^4(\tau_0) = \text{Im } \tau_0 \eta^4(\tau_0), \tag{4.12}$$

so that also the R^2 term in (4.7) is invariant under $\widehat{\mathcal{S}}$.

Now let us consider the transformation $\widehat{\mathcal{T}}$ defined as

$$\widehat{\mathcal{T}} : \tau_0 \longrightarrow \tau_0 + 1, \quad T_2 \longleftrightarrow T_3 = -T_1 - T_2. \tag{4.13}$$

This is the combination of the \mathcal{T} transformation acting on τ_0 with the triality transformation that exchanges the two spinor representations of $\text{SO}(8)$, leaving fixed the vector representation (and the quadratic invariant R). From the modular properties under \mathcal{T} given in (A.6), it is easy to check that

$$\kappa(q) \longrightarrow e^{\pi i} \frac{\kappa(q)}{1 - \kappa(q)}, \quad 1 - \kappa(q) \longrightarrow \frac{1}{1 - \kappa(q)}, \quad \text{Im } \tau_0 \eta^4(\tau_0) \longrightarrow e^{\frac{\pi i}{3}} \text{Im } \tau_0 \eta^4(\tau_0), \tag{4.14}$$

and hence that the prepotential (4.7) transforms under $\widehat{\mathcal{T}}$ as follows

$$\mathcal{F}^{(8)} \longrightarrow \mathcal{F}^{(8)} - \frac{\pi i}{4} T_1 + \frac{\pi i}{24} R^2 = \mathcal{F}^{(8)} + \frac{\pi i}{4!} \text{Tr } m^4. \tag{4.15}$$

Even if the prepotential is not invariant, the change produced in the effective action is unobservable because

$$\begin{aligned} \delta S^{(8)} &= \frac{1}{(2\pi)^4} \int d^8 X d^8 \Theta \delta \mathcal{F}^{(8)}(M) + \text{c.c.} \\ &= \frac{\pi i}{4!(2\pi)^4} \int d^8 X d^8 \Theta \left[\text{Tr } M^4 - \text{Tr } \overline{M}^4 \right] = 2\pi i c_4, \end{aligned} \tag{4.16}$$

where in the last step c_4 is the topological index of order four which is an integer number. Since the effective action shifts by an integer multiple of $2\pi i$, we can conclude¹⁴ that the theory is invariant also under $\widehat{\mathcal{T}}$. Thus, the symmetry group of the quantum $\text{SO}(8)$ gauge theory living on the D7 branes is the one generated by $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{T}}$, which, as shown in appendix B.1, is isomorphic to $\text{SL}(2, \mathbb{Z})$ in agreement with our expectations.

¹⁴This is the same argument used to show that the classical tree-level action following from (4.3) is invariant under $\tau_0 \rightarrow \tau_0 + 1$.

Besides the non-perturbative effective action and its symmetry properties, another useful set of information on the eight-dimensional theory is provided by the quantum vacuum expectation values of the composite operators $\text{Tr } m^{2l}$, forming the so-called “chiral ring” of the model. These have been computed in [33] via localization techniques with suitable insertions in the instanton partition function for a generic number of D7 branes. When this number is four, i.e. for our $\text{SO}(8)$ gauge theory, the results for the first few values of l are¹⁵

$$\begin{aligned}
 \langle \text{Tr } m^2 \rangle &= \text{Tr } \langle m \rangle^2, \\
 \langle \text{Tr } m^4 \rangle &= \text{Tr } \langle m \rangle^4 - 48 \text{Pfm } q - 24 \sum_{i < j} m_i^2 m_j^2 q^2 - 192 \text{Pfm } q^3 + \dots, \\
 \langle \text{Tr } m^6 \rangle &= \text{Tr } \langle m \rangle^6 + 180 \sum_{i < j < k} m_i^2 m_j^2 m_k^2 q^2 + 960 \text{Pfm } \sum_i m_i^2 q^3 + \dots, \\
 \langle \text{Tr } m^8 \rangle &= \text{Tr } \langle m \rangle^8 - 420 (\text{Pfm})^2 q^2 - 2240 \text{Pfm } \sum_{i < j} m_i^2 m_j^2 q^3 + \dots.
 \end{aligned} \tag{4.17}$$

It is interesting to observe that the non-perturbative part of these expressions exactly coincides with the non-perturbative corrections for the coupling constant of the four-dimensional $\text{SU}(2)$ gauge theory with $N_f = 4$ as reported in eq. (2.39). We will elaborate more on this relation in section 6.

5 Four-dimensional effective prepotential

The non-perturbative prepotential for the four-dimensional $\text{Sp}(1)$ gauge theory on the D3 branes can be computed by extracting the sub-leading $1/(\epsilon_1 \epsilon_2)$ divergence in the deformed D-instanton partition function using formulae (3.25) and (3.26). The result of these calculations, which are technically similar to those reported in [34], can be written as follows

$$\mathcal{F}_{\text{n.p.}}^{(4)} = \sum_k \mathcal{F}_k q^k, \tag{5.1}$$

where the first five instanton contributions are explicitly given by

$$\mathcal{F}_1 = \frac{2 \text{Pfm}}{a^2}, \tag{5.2}$$

$$\mathcal{F}_2 = \frac{5 (\text{Pfm})^2}{4 a^6} - \frac{3 \sum_{i < j < k} m_i^2 m_j^2 m_k^2}{2 a^4} + \frac{\sum_{i < j} m_i^2 m_j^2}{a^2}, \tag{5.3}$$

$$\mathcal{F}_3 = \text{Pfm} \left\{ \frac{3 (\text{Pfm})^2}{a^{10}} - \frac{14 \sum_{i < j < k} m_i^2 m_j^2 m_k^2}{3 a^8} + \frac{20 \sum_{i < j} m_i^2 m_j^2}{3 a^6} - \frac{8 \sum_i m_i^2}{a^4} + \frac{8}{a^2} \right\}, \tag{5.4}$$

¹⁵With respect to [33], our present conventions are such that their $\text{Tr } \phi^{2l}$ is mapped to $(-1)^l \text{Tr } m^{2l}$ and their q goes into our $(-q)$.

$$\begin{aligned}
 \mathcal{F}_4 = & \frac{1469(\text{Pfm})^4}{128 a^{14}} - \frac{715(\text{Pfm})^2 \sum_{i<j<k} m_i^2 m_j^2 m_k^2}{32 a^{12}} \\
 & + \frac{153 \sum_{i<j<k} m_i^4 m_j^4 m_k^4 + 1332(\text{Pfm})^2 \sum_{i<j} m_i^2 m_j^2}{32 a^{10}} \\
 & - \frac{63 \sum_i m_i^2 \sum_{j<k \neq i} m_j^4 m_k^4 + 588 (\text{Pfm})^2 \sum_i m_i^2}{8 a^8} \\
 & + \frac{5 \sum_{i<j} m_i^4 m_j^4 + 100 \sum_i m_i^4 \sum_{j<k \neq i} m_j^2 m_k^2 + 960 (\text{Pfm})^2}{8 a^6} \\
 & - \frac{3 \sum_{i \neq j} m_i^4 m_j^2 + 36 \sum_{i<j<k} m_i^2 m_j^2 m_k^2}{2 a^4} + \frac{\sum_i m_i^4 + 4 \sum_{i<j} m_i^2 m_j^2}{2 a^2}, \tag{5.5}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{F}_5 = & \text{Pfm} \left\{ \frac{4471 (\text{Pfm})^4}{80 a^{18}} - \frac{525 (\text{Pfm})^2 \sum_{i<j<k} m_i^2 m_j^2 m_k^2}{4 a^{16}} \right. \\
 & + \frac{1131 \sum_{i<j<k} m_i^4 m_j^4 m_k^4 + 5980 (\text{Pfm})^2 \sum_{i<j} m_i^2 m_j^2}{20 a^{14}} \\
 & - \frac{121 \sum_i m_i^2 \sum_{j<k \neq i} m_j^4 m_k^4 + 660 (\text{Pfm})^2 \sum_i m_i^2}{a^{12}} \\
 & + \frac{207 \sum_{i<j} m_i^4 m_j^4 + 1260 \sum_i m_i^4 \sum_{j<k \neq i} m_j^2 m_k^2 + 7020 (\text{Pfm})^2}{5 a^{10}} \\
 & - \frac{84 \sum_{i \neq j} m_i^4 m_j^2 + 504 \sum_{i<j<k} m_i^2 m_j^2 m_k^2}{a^8} + \frac{28 \sum_i m_i^4 + 160 \sum_{i<j} m_i^2 m_j^2}{a^6} \\
 & \left. - \frac{48 \sum_i m_i^2}{a^4} + \frac{12}{a^2} \right\}. \tag{5.6}
 \end{aligned}$$

To the above non-perturbative prepotential we must add the tree-level term

$$\mathcal{F}_{\text{tree}}^{(4)} = \frac{\pi i \tau_0}{2!} \text{Tr} \phi^2 = a^2 \ln q, \tag{5.7}$$

where in the last step we replaced the adjoint field ϕ with its vacuum expectation value according to (2.12), and the perturbative 1-loop contribution given by (see for example [56])

$$\begin{aligned}
 \mathcal{F}_{1\text{-loop}}^{(4)} = & \frac{1}{2} \sum_i \left[(a + m_i/\sqrt{2})^2 \ln \left(\frac{a + m_i/\sqrt{2}}{\Lambda} \right) + (a - m_i/\sqrt{2})^2 \ln \left(\frac{a - m_i/\sqrt{2}}{\Lambda} \right) \right] \\
 & - 4 a^2 \ln \left(\frac{a}{\Lambda} \right). \tag{5.8}
 \end{aligned}$$

Note that for $N_f = 4$ the only dependence on the UV cut-off Λ is in terms proportional to $\sum_i m_i^2 \log(a/\Lambda)$ which, however, do not contribute to the effective action.

The total prepotential

$$\mathcal{F}^{(4)} = \mathcal{F}_{\text{tree}}^{(4)} + \mathcal{F}_{1\text{-loop}}^{(4)} + \mathcal{F}_{\text{n.p.}}^{(4)} \tag{5.9}$$

determines the effective coupling τ of the four-dimensional gauge theory on the D3 branes according to the relation

$$2\pi i \tau = \frac{\partial^2 \mathcal{F}^{(4)}}{\partial a^2}. \tag{5.10}$$

If we organize the resulting expression as an expansion in inverse powers of a , we obtain

$$\begin{aligned}
 2\pi i \tau = & 2 \ln q - \frac{1}{2} \frac{\sum_i m_i^2}{a^2} + \frac{1}{a^4} \left[-\frac{1}{8} \sum_i m_i^4 + 12 \text{Pfm} q + 6 \sum_{i < j} m_i^2 m_j^2 q^2 + 48 \text{Pfm} q^3 + \dots \right] \\
 & + \frac{1}{a^6} \left[-\frac{1}{24} \sum_i m_i^6 - 30 \sum_{i < j < k} m_i^2 m_j^2 m_k^2 q^2 - 160 \text{Pfm} \sum_i m_i^2 q^3 + \dots \right] \\
 & + \frac{1}{a^8} \left[-\frac{1}{64} \sum_i m_i^8 + \frac{105}{2} (\text{Pfm})^2 q^2 + 280 \text{Pfm} \sum_{i < j} m_i^2 m_j^2 q^3 + \dots \right] + \dots \quad (5.11)
 \end{aligned}$$

This completely agrees with the effective coupling reported in eq. (2.37) and derived from the SW curve for the $SU(2)$ $N_f = 4$ theory [26]. Thus, we can conclude that the non-perturbative features predicted by the SW curve for this theory can be interpreted as D-instanton effects in the D7/D3 system of type I' . Furthermore, this agreement puts on a very solid ground the prescription described in eq. (3.25), which, as a matter of fact, was successfully tested already in [34] in a different context by exploiting the heterotic/type I' duality.

6 An intriguing relation

By comparing the non-perturbative terms of the effective coupling (5.11) with the vacuum expectation values (4.17) of the operators $\text{Tr} m^{2l}$ forming the chiral ring of the eight-dimensional theory living on the D7 branes, we find that (up to the available orders in q and in $1/a$) our results take the following very intriguing and suggestive form

$$2\pi i \tau = 2\pi i \tau_0 - \sum_{l=1}^{\infty} \frac{1}{2l} \frac{\langle \text{Tr} m^{2l} \rangle}{a^{2l}}. \quad (6.1)$$

In other words, the *non-perturbative corrections* to the effective coupling on the D3's are obtained from its *perturbative* expression (2.13) after promoting the mass parameters m_i , which classically correspond to the vacuum expectation values of the $SO(8)$ adjoint field m on the D7 branes according to eq. (2.7), to a bona-fide quantum field of which the appropriate expectation values have to be taken. We conjecture¹⁶ that this relation actually extends to all orders in q and in $1/a^2$, and in figure 2 we give a graphical representation of it. The effective gauge coupling τ on the probe D3 brane (the blue dot) receives perturbative contributions. They arise from loops of open string states suspended between the D3 and the D7 branes (red circles), their images (dotted red circles) and the orientifold (green cross). These diagrams have a dual interpretation as tree-level closed string exchanges which are depicted as straight lines (the lines to the image D7 branes are not shown to avoid clutter). If we take into account the full quantum dynamics, including the non-perturbative effects, of the D7 fields whose eigenvalues correspond to the D7 positions (this procedure is graphically indicated by the circular red-shaded area including the D7

¹⁶The authors of [33] have pushed their computations up to $\langle \text{Tr} m^{18} \rangle$ and to instanton number $k = 5$, finding that the agreement with the effective four-dimensional coupling τ persists [57].

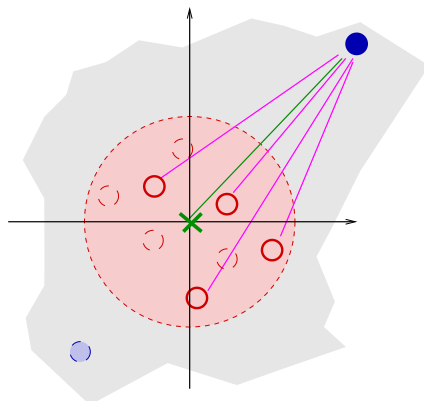


Figure 2. An ideographic representation of the relation we discuss in this section (see the main text for a discussion of its content).

branes) and we still communicate this information to the probe D3 brane in a perturbative fashion, we get the *exact* effective coupling, i.e.

$$\tau_{\text{pert}} \rightarrow \tau \quad \text{if} \quad \text{Tr} \langle m \rangle^{2l} \rightarrow \langle \text{Tr} m^{2l} \rangle . \quad (6.2)$$

With simple formal manipulations we can rewrite the relation (6.1) as follows:

$$a \frac{\partial}{\partial a} (2\pi i \tau) = \left\langle \text{Tr} \frac{m^2}{a^2 - m^2} \right\rangle , \quad (6.3)$$

from which, after using (5.10) in the left-hand side, we obtain

$$\frac{\partial^2}{\partial a^2} \left(a \frac{\partial}{\partial a} \mathcal{F}^{(4)} - 2\mathcal{F}^{(4)} \right) = \left\langle \text{Tr} \frac{m^2}{a^2 - m^2} \right\rangle . \quad (6.4)$$

Note that $(a \partial_a \mathcal{F}^{(4)} - 2\mathcal{F}^{(4)})$ is the same combination that appears in the Matone relation [58] for the pure SU(2) SYM theory, and hence we can regard eq. (6.4) as a sort of generalization of this relation to the SU(2) theory with four massive flavors.

In conclusion, it can be said that knowing the chiral ring in the eight-dimensional theory is equivalent to knowing the four-dimensional prepotential. Conversely the eight-dimensional prepotential can be read off directly from the four-dimensional one by means of

$$\mathcal{F}_{\text{n.p.}}^{(4)} = -\frac{1}{a^2} \left(q \frac{\partial}{\partial q} \mathcal{F}_{\text{n.p.}}^{(8)} \right) + \mathcal{O} \left(\frac{1}{a^4} \right) . \quad (6.5)$$

This equation is a consequence of the particular relation $\frac{1}{4!} \langle \text{Tr} m^4 \rangle = q \frac{\partial}{\partial q} \mathcal{F}_{\text{n.p.}}^{(8)}$ explained in [33], and of eqs. (6.1) and (5.10).

7 Comparison with the SU(2) instanton calculus

In this last section we pursue the comparison of our results with the outcome of the standard supersymmetric instanton calculus for the SU(2) theory with N_f flavors.

7.1 Instanton calculus

The computation of instanton effects in field theories with $\mathcal{N} = 2$ supersymmetry, $SU(N)$ gauge group and N_f fundamental hypermultiplets was performed by N. Nekrasov exploiting localization techniques [35, 38]. In fact, it is very easy to extract from his work explicit expressions for the prepotential, at least up to three instantons; for instance, the result for the $SU(2)$ gauge theory with $N_f = 3$ is reported here in appendix E (see in particular eq. (E.3)). This result fully agrees, up to irrelevant constant rescalings of the mass parameters and of the dynamical scale, with the limit in which we decouple one flavor from the four-dimensional prepotential (5.9) obtained for the D3/D7 system in type I'. This agreement represents another very reassuring check of our procedure.

Although originally proposed for asymptotically free theories, and thus limited to $N_f < 4$ in the case of the $SU(2)$ gauge theory, Nekrasov's prescription can formally be extended without any problem also to the conformal theory with $N_f = 4$. We will describe the outcome of Nekrasov's prescription and elaborate on its relation with our results (and the SW approach) in the next subsection by pushing our analysis up to five instantons. To do so we resort, as a very efficient computational tool, to the recently discovered AGT connection that allows to obtain the instanton partition function à la Nekrasov for the $SU(2)$ $N_f = 4$ gauge theory from the conformal blocks of a two-dimensional Liouville conformal field theory [44].

Before giving some details on this relation, let us anticipate the key points of our findings. Denoting by τ_{uv} the tree-level coupling in this approach, and by

$$x = e^{\pi i \tau_{uv}} \tag{7.1}$$

the corresponding instanton expansion parameter, one finds that in the *massless* case the *effective* coupling τ receives instanton corrections and takes the form

$$\pi i \tau = \ln x - \ln 16 + \frac{1}{2} x + \frac{13}{64} x^2 + \frac{23}{192} x^3 + \frac{2701}{32768} x^4 + \frac{5057}{81920} x^5 + \dots \tag{7.2}$$

This is to be contrasted with what happens in our treatment and in the SW description, where for zero masses the effective coupling coincides with the tree-level coupling τ_0 , i.e.

$$\pi i \tau = \ln q \tag{7.3}$$

as is clear from eq. (5.11) when $m_i = 0$. If one assumes that the quantity that should agree in any description of the same quantum field theory is the effective coupling, then eqs. (7.2) and (7.3) imply

$$\ln q = \ln x - \ln 16 + \frac{1}{2} x + \frac{13}{64} x^2 + \frac{23}{192} x^3 + \frac{2701}{32768} x^4 + \frac{5057}{81920} x^5 + \dots \tag{7.4}$$

One can check that this is the inverse of the expansion given in (4.8), so that we are led to identify x with $\kappa(q)$ and write

$$x = \frac{\vartheta_2^4(\tau_0)}{\vartheta_3^4(\tau_0)}, \tag{7.5}$$

which coincides with the closed-form expression proposed in [44] (see also [46]–[48], and [59] where that expression was first proposed). As a pure matter-of-fact remark, it is interesting to notice that the *tree-level* coupling $\ln x$ utilized in Nekrasov’s approach corresponds to the *effective* coupling $\ln \kappa(q)$ that appears in front of the $\text{SO}(8)$ invariant structure T_1 in the eight-dimensional action on the D7 branes (see eqs. (4.6) and (4.7)). This is another signal of a non-trivial interplay between quantum effects in four dimensions on the D3’s and in eight dimensions on the D7’s, in addition to the one described in section 6.

Turning to the massive case, one finds that the Nekrasov/AGT prepotential for the $\text{SU}(2)$ $N_f = 4$ theory contains, at each order in its instanton expansion, several terms which do not appear in our findings. However, if we start from our complete prepotential (5.9) and substitute our expansion parameter q in terms of x according to eq. (7.4), namely

$$q = \frac{x}{16} \left(1 + \frac{1}{2}x + \frac{21}{64}x^2 + \frac{31}{128}x^3 + \frac{6257}{32768}x^4 + \dots \right), \quad (7.6)$$

we obtain an exact matching with the Nekrasov/AGT prepotential (up to constant terms which do not contribute to the effective action). Let us now show this in some more detail.

7.2 The Nekrasov prepotential from the AGT realization

In [44] a remarkable relation between the deformed instanton partition function for $\mathcal{N} = 2$ $\text{SU}(2)$ gauge theories in four dimensions and the correlation functions of the Liouville theory in two dimensions has been discovered. In this correspondence the central charge c of the Liouville theory and the deformation parameters ϵ_1 and ϵ_2 of the instanton partition functions are related as follows

$$c = 1 + 6Q^2 = 1 + 6 \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2}, \quad (7.7)$$

while the logarithm of the conformal block of four Liouville operators [45] is related to the non-perturbative prepotential $F_{\text{n.p.}}$ of the $\mathcal{N} = 2$ $\text{SU}(2)$ theory with four massive hypermultiplets as computed in Nekrasov’s approach [35]–[39]. More precisely we have

$$F_{\text{n.p.}}^{(4)} = - \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \left\{ \epsilon_1 \epsilon_2 \ln \left[(1-x)^{2\mu_1 \mu_3} B_{\Delta_1 \Delta_2; \Delta_3 \Delta_4}^\Delta(x) \right] \right\}. \quad (7.8)$$

Here $B_{\Delta_1 \Delta_2; \Delta_3 \Delta_4}^\Delta(x)$ is the conformal block of four Liouville primary fields with conformal dimensions Δ_i , located at ∞ , 1, x and 0, and factorized in the channel $(\Delta_1 - \Delta_2) \sim (\Delta_3 - \Delta_4)$ with an intermediate state of conformal dimension Δ . These dimensions can be parameterized as follows

$$\Delta = \frac{Q^2}{4} - \frac{a^2}{\epsilon_1 \epsilon_2}, \quad \Delta_i = \frac{Q^2}{4} - \mu_i^2, \quad (7.9)$$

where a is the vacuum expectation value of the $\text{SU}(2)$ adjoint scalar field, and the μ_i ’s are related to the masses m_i of the fundamental hypermultiplets according to

$$\begin{aligned} \mu_1 + \mu_2 + \frac{Q}{2} &= \frac{m_1}{\sqrt{2\epsilon_1 \epsilon_2}}, & \mu_1 - \mu_2 + \frac{Q}{2} &= \frac{m_2}{\sqrt{2\epsilon_1 \epsilon_2}}, \\ \mu_3 + \mu_4 + \frac{Q}{2} &= \frac{m_3}{\sqrt{2\epsilon_1 \epsilon_2}}, & \mu_3 - \mu_4 + \frac{Q}{2} &= \frac{m_4}{\sqrt{2\epsilon_1 \epsilon_2}}. \end{aligned} \quad (7.10)$$

The dressing factor $(1-x)^{2\mu_1\mu_3}$ has been inserted to decouple a U(1) factor [44] and get the instantonic partition function for SU(2) and not for U(2). Note that in the literature there are different expressions for such a factor, containing also terms that depend on the background charge Q . However, these terms are not relevant in the limit $\epsilon_1, \epsilon_2 \rightarrow 0$ we are considering here.¹⁷

Using the expression of the conformal blocks [45] (see also [46]–[48]) and expanding in powers of x , we have

$$F_{\text{n.p.}}^{(4)} = \sum_k F_k x^k, \quad (7.11)$$

where the first few coefficients are

$$F_1 = \frac{\text{Pfm}}{8a^2} + \frac{\sum_{i<j} m_i m_j}{4} + \frac{a^2}{2}, \quad (7.12)$$

$$F_2 = \frac{5(\text{Pfm})^2}{1024a^6} - \frac{3\sum_{i<j<k} m_i^2 m_j^2 m_k^2}{512a^4} + \frac{\sum_{i<j} m_i^2 m_j^2 + 16\text{Pfm}}{256a^2} + \frac{\sum_i m_i^2 + 16\sum_{i<j} m_i m_j}{128} + \frac{13a^2}{64}, \quad (7.13)$$

$$F_3 = \frac{3(\text{Pfm})^3}{4096a^{10}} - \frac{7\text{Pfm}\sum_{i<j<k} m_i^2 m_j^2 m_k^2}{6144a^8} + \frac{5\text{Pfm}\sum_{i<j} m_i^2 m_j^2 + 15(\text{Pfm})^2}{3072a^6} - \frac{3\sum_{i<j<k} m_i^2 m_j^2 m_k^2 + \text{Pfm}\sum_i m_i^2}{512a^4} + \frac{\sum_{i<j} m_i^2 m_j^2 + 11\text{Pfm}}{256a^2} + \frac{3\sum_i m_i^2 + 32\sum_{i<j} m_i m_j}{384} + \frac{23a^2}{192}. \quad (7.14)$$

In appendix D we report also the explicit expressions for F_4 and F_5 (see eqs. (D.5) and (D.6)), as well as some other technical details on this approach.

To this non-perturbative prepotential we must add the tree-level contribution

$$F_{\text{tree}}^{(4)} = a^2 \ln x, \quad (7.15)$$

and the x -independent 1-loop part

$$F_{1\text{-loop}}^{(4)} = \frac{1}{2} \sum_i \left[(a + m_i/\sqrt{2})^2 \ln \left(\frac{a + m_i/\sqrt{2}}{\Lambda} \right) + (a - m_i/\sqrt{2})^2 \ln \left(\frac{a - m_i/\sqrt{2}}{\Lambda} \right) \right] - 4a^2 \ln \left(\frac{2a}{\Lambda} \right), \quad (7.16)$$

which coincides with (5.8) except for a small difference in the last term where the effect of a finite renormalization pointed out in [50] has been taken into account.

The total prepotential

$$F^{(4)} = F_{\text{tree}}^{(4)} + F_{1\text{-loop}}^{(4)} + F_{\text{n.p.}}^{(4)} \quad (7.17)$$

¹⁷These Q -dependent pieces are relevant instead if also gravitational corrections to the gauge theory are considered, i.e. if the deformations parameters ϵ_1 and ϵ_2 are not switched off.

does not agree, at first sight, with the prepotential $\mathcal{F}^{(4)}$ we have found in section 5 (see eqs. (5.1)–(5.9)). Indeed, one can easily see that in the massless case ($m_i = 0$) the non-perturbative part $\mathcal{F}_{\text{n.p.}}^{(4)}$ is vanishing while $F_{\text{n.p.}}^{(4)}$ is not, and that the numerical coefficients as well as the various structures are different in the two expressions. However, one can explicitly check that *all* a -dependent structures of $\mathcal{F}^{(4)}$ and $F^{(4)}$ are exactly mapped into each other if q and x are related as in (7.6) (or equivalently as in (7.4)). Let us show this in some cases. Consider for example the terms in $F^{(4)}$ that are proportional to a^2 , namely

$$a^2 \left(\ln x - \ln 16 + \frac{1}{2}x + \frac{13}{64}x^2 + \frac{23}{192}x^3 + \dots \right). \quad (7.18)$$

Upon using (7.4), they simply become $a^2 \ln q$, that is exactly the a^2 term of $\mathcal{F}^{(4)}$ given in (5.7). Now consider the terms in $F^{(4)}$ that are proportional to $1/a^2$, namely

$$\frac{1}{a^2} \left\{ -\frac{\sum_i m_i^4}{48} + \frac{\text{Pfm}}{8}x + \frac{\sum_{i<j} m_i^2 m_j^2 + 16 \text{Pfm}}{256}x^2 + \frac{\sum_{i<j} m_i^2 m_j^2 + 11 \text{Pfm}}{256}x^3 + \dots \right\} \quad (7.19)$$

where the first x -independent contribution arises from $F_{1\text{-loop}}^{(4)}$. If we use the relation (7.6), we can rewrite (7.19) as

$$\frac{1}{a^2} \left\{ -\frac{\sum_i m_i^4}{48} + 2(\text{Pfm})q + \left(\sum_{i<j} m_i^2 m_j^2 \right) q^2 + 8(\text{Pfm})q^3 + \dots \right\}, \quad (7.20)$$

which exactly coincides with the part of $\mathcal{F}^{(4)}$ proportional to $1/a^2$ obtained in section 5. Likewise one can check that the agreement persists in all other a -dependent terms, and up to five instantons using the expressions reported in appendix D.

The only mismatch is in the constant a -independent part, which however does not contribute to the four-dimensional effective action obtained from the prepotential by promoting a to the full-fledged chiral superfield Φ and integrating over the chiral four-dimensional superspace. Thus, this mismatch is irrelevant. We also note that while the non-perturbative prepotential $\mathcal{F}_{\text{n.p.}}$ obtained from the D7/D3/D(-1) system is fully invariant under the SO(8) flavor symmetry acting on the massive multiplets, the prepotential $F_{\text{n.p.}}$ derived above contains the structure $\sum_{i<j} m_i m_j$ which is not an SO(8) invariant. Since this structure appears only in the a -independent part, it does not contribute to the effective action, which indeed is SO(8) invariant. It is interesting also to notice that all terms proportional to $\sum_{i<j} m_i m_j$ could be removed from the prepotential by changing the dressing factor in the AGT relation (7.8), and using $(1-x)^{-\frac{1}{2}(\mu_1^2 - \mu_2^2 + \mu_3^2 - \mu_4^2)}$ in place of $(1-x)^{2\mu_1\mu_3}$. Finally, the a -independent terms proportional to $\sum_i m_i^2$ that were absent in $\mathcal{F}_{\text{n.p.}}$ could be reabsorbed with a suitable redefinition of the UV cut-off appearing in the 1-loop part of the prepotential. As we have already remarked, all these ambiguities in the constant piece are not significant and do not influence the effective action of the four-dimensional gauge fields.

8 Summary and conclusions

We have considered the local system composed of four D7 branes and a D3 brane near one O7 fixed plane in type I' theory on a T_2 torus. We have computed by means of localization

techniques the non-perturbative corrections due to D-instantons to the effective actions on the D7 branes (for which they correspond to exotic instantons) and on the D3 brane. On the D3, which supports a $\text{Sp}(1)$, $N_f = 4$ theory, they represent gauge instantons. The effective coupling on the D3 coincides with the exact axio-dilaton field, and indeed we find that it agrees with the F-theoretic prediction put forward long ago by Sen [25]. We thus show by an explicit microscopic computation how F-theory resums D-instanton effects. Our results are consistent with the outcome of usual instanton calculus à la Nekrasov [35] for the $\text{SU}(2)$, $N_f = 4$ conformal theory, or with its AGT reformulation in terms of 2d Liouville blocks [44], upon a non-perturbative redefinition of the tree-level coupling.

We point out a very interesting relation between the effective coupling τ on the D3 (that is, the modular parameter of the F-theory curve) and the eight-dimensional dynamics on the D7-branes, and specifically the “chiral ring” correlators $\langle \text{Tr } m^{2l} \rangle$, where $m(X)$ is the eight-dimensional field in the adjoint of $\text{SO}(8)$ whose eigenvalues m_i appear as mass parameters on the D3. Indeed, we find that the exact expression for τ is obtained from its perturbative part replacing the occurrences of the masses with the corresponding chiral ring correlators in eight dimensions.

Acknowledgments

We thank R. Argurio, L. Ferro and especially M. Frau, F. Fucito, J.F. Morales and R. Poghossian for several useful discussions. M. B. thanks the K.I.T.P, Santa Barbara, and the organizers of the workshop “Strings at the LHC and in the Early Universe” for hospitality during the initial stage of this work. This research was supported in part by the National Science Foundation under Grant No. NSF PHY05-51164.

A Notations and theta-function conventions

Couplings. We use the following names for the various quadratic gauge couplings:

- τ_0 is the tree-level coupling in the D3/D7 system in type I' , i.e. the asymptotic value of the axio-dilaton field. In this set-up the gauge theory on the D3 is realized as a $\text{Sp}(1)$ theory with four flavors.
- τ_{uv} is the tree-level coupling in the usual field-theoretical description of the $\text{SU}(2)$ theory with $N_f = 4$ flavors.
- τ is the exact *effective* coupling, which depends on the tree-level coupling, the vacuum expectation value a of the adjoint scalar parametrizing the Coulomb moduli space and on the mass parameters. This coupling must agree in the two descriptions.

Theta-functions. We adopt the following definition for the Jacobi ϑ -functions:

$$\vartheta_{[b]}^a(v|\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau (n - \frac{a}{2})^2} e^{2\pi i (n - \frac{a}{2})(v - \frac{b}{2})}, \quad (\text{A.1})$$

and the usual naming conventions

$$\vartheta_1(\tau) = \vartheta_{[1]}^1(0|\tau), \quad \vartheta_2(\tau) = \vartheta_{[0]}^1(0|\tau), \quad \vartheta_3(\tau) = \vartheta_{[0]}^0(0|\tau), \quad \vartheta_4(\tau) = \vartheta_{[1]}^0(0|\tau). \quad (\text{A.2})$$

The Dedekind η -function is defined as

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i \tau n}). \quad (\text{A.3})$$

The following properties are relevant for us:

$$\vartheta_3^4(\tau) - \vartheta_4^4(\tau) - \vartheta_2^4(\tau) = 0, \quad \vartheta_2(\tau) \vartheta_3(\tau) \vartheta_4(\tau) = 2\eta^3(\tau). \quad (\text{A.4})$$

We will also use the transformation properties under the modular group generators, which are

$$S : \begin{cases} \vartheta_{[b]}^a(v|\tau| - 1/\tau) = \sqrt{-i\tau} e^{\frac{\pi i}{2}(ab + \frac{2v^2}{\tau})} \vartheta_{[-a]}^b(v|\tau), \\ \eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau), \end{cases} \quad (\text{A.5})$$

and

$$T : \begin{cases} \vartheta_{[b]}^a(v|\tau + 1) = e^{-\frac{\pi i}{4}a(a-2)} \vartheta_{[a+b]}^a(v|\tau), \\ \eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau). \end{cases} \quad (\text{A.6})$$

B Flavor invariants

The parameters m_i correspond to the components along the Cartan directions of Spin(8) of the vacuum expectation values of the D7/D7 scalar field $m(x)$ introduced in eq. (2.7). In a given representation \mathcal{R} , the element $m^i H_i$ of the algebra is diagonal in the basis of weights \vec{w}_A ($A = 1, \dots, \dim \mathcal{R}$), with eigenvalues $\vec{m} \cdot \vec{w}_A$.

Spin(8) has an S_3 group of external automorphisms (“trinality”) that permutes its three 8-dimensional representations: the vector (v), chiral spinor (s) and antichiral spinor (c), remapping the weights of each of these representations in those of another one. This S_3 can be generated by any two exchanges, say

$$g_1 : s \leftrightarrow c, \quad g_2 : v \leftrightarrow c. \quad (\text{B.1})$$

Given the form of the weights of these representations, we can view these triality operations as acting on the Cartan parameters m_i , as follows:

$$\begin{aligned} g_1 : \{m_1, m_2, m_3, m_4\} &\rightarrow \{m_1, m_2, m_3, -m_4\}, \\ g_2 : \{m_1, m_2, m_3, m_4\} &\rightarrow \frac{1}{2} \{m_1 + m_2 + m_3 - m_4, m_1 + m_2 - m_3 + m_4, \\ &\quad m_1 - m_2 + m_3 + m_4, -m_1 + m_2 + m_3 + m_4\}. \end{aligned} \quad (\text{B.2})$$

The SW curve for the SU(2) theory with $N_f = 4$ flavors, described in section 2.2.1, is written in terms of SO(8)-invariant expressions of order 2, 4 and 6 in the masses m_i [27]. The quadratic invariant is

$$R = \frac{1}{2} \sum_i m_i^2 = \frac{1}{2} \text{Tr } m^2, \quad (\text{B.3})$$

where the second equality follows from (2.7). This expression is also invariant under triality, as one can see by applying to it the generators (B.2).

The three independent quartic SO(8) invariants can be chosen to be R^2 and

$$\begin{aligned} T_1 &= \frac{1}{12} \sum_{i<j} m_i^2 m_j^2 - \frac{1}{24} \sum_i m_i^4 = \frac{1}{24} (\text{Tr } m^2)^2 - \frac{1}{6} \text{Tr } m^4, \\ T_2 &= -\frac{1}{24} \sum_{i<j} m_i^2 m_j^2 + \frac{1}{48} \sum_i m_i^4 - \frac{1}{2} \text{Pfm} = -\frac{1}{48} (\text{Tr } m^2)^2 + \frac{1}{12} \text{Tr } m^4 - \frac{1}{2} \text{Pfm}. \end{aligned} \tag{B.4}$$

The triality generators act on the invariants T_ℓ as follows:

$$g_1 : T_2 \longleftrightarrow T_3 = -T_1 - T_2 \quad \text{and} \quad g_2 : T_1 \longleftrightarrow T_2. \tag{B.5}$$

In other words, the triality group S_3 permutes the T_ℓ invariants.

The four sextic SO(8) invariants can be chosen to be R^3 , RT_ℓ and

$$N = \frac{3}{16} \sum_{i<j<k} m_i^2 m_j^2 m_k^2 - \frac{1}{96} \sum_{i \neq j} m_i^2 m_j^4 + \frac{1}{96} \sum_i m_i^6 \tag{B.6}$$

which is invariant also under triality.

The following combinations are relevant for us:

$$\begin{aligned} R^2 + 6 T_1 &= \sum_{i<j} m_i^2 m_j^2, \\ R^2 - 6 T_1 &= \frac{1}{2} \sum_i m_i^4, \\ 2 N + R T_1 &= \frac{1}{2} \sum_{i<j<k} m_i^2 m_j^2 m_k^2, \\ 6 N + R^3 - 15 R T_1 &= \frac{1}{2} \sum_i m_i^6, \\ 2(\text{Pfm})^2 - 16 R N - R^4 + 28 R^2 T_1 - 36 T_1^2 &= -\frac{1}{2} \sum_i m_i^8. \end{aligned} \tag{B.7}$$

B.1 Triality extension of $\text{SL}(2, \mathbb{Z})$

Consider an $\text{SL}(2, \mathbb{Z})$ group generated by \mathcal{S} and \mathcal{T} subject to the relations

$$\mathcal{S}^2 = 1, \quad (\mathcal{ST})^3 = 1. \tag{B.8}$$

In our case this group is the modular group acting on the bare coupling τ_0 . Now consider also the permutation group S_3 generated by g_1 and g_2 , subject to the relations

$$g_1^2 = 1, \quad g_2^2 = 1, \quad (g_2 g_1)^3 = 1. \tag{B.9}$$

In our case this group is the triality group of SO(8), we have described above.

Let us then construct the semi-direct product of $SL(2, \mathbb{Z})$ and S_3 , which is generated by the following combined actions:

$$\widehat{T} = (\mathcal{T}, g_1), \quad \widehat{S} = (\mathcal{S}, g_2). \quad (\text{B.10})$$

These two generators obey

$$\widehat{S}^2 = 1, \quad (\widehat{S}\widehat{T})^3 = 1; \quad (\text{B.11})$$

therefore this group is again isomorphic to $SL(2, \mathbb{Z})$.

C Details on the D-instanton computation

The expressions of the factors $P_{\square}(\chi)$, $P_{\square\square}(\chi)$ and $P_{\square}(\chi)$ appearing in the integrand of the formula (3.18) for the instanton partition function are given respectively in eqs. (3.19), (3.21) and (3.22) in terms of the weights of the relevant representations of the $SO(k)$ instantonic symmetry group. Let us recall the form of these weight vectors.

Weight sets of $SO(2n + 1)$. This group has rank n . If we denote by \vec{e}_i the versors in the \mathbb{R}^n weight space, then

- the set of the $2n + 1$ weights $\vec{\pi}$ of the vector representation is given by

$$\pm \vec{e}_i, \quad \vec{0} \quad \text{with multiplicity } 1; \quad (\text{C.1})$$

- the set of $n(2n + 1)$ weights $\vec{\rho}$ of the adjoint representation (corresponding to the two-index antisymmetric tensor) is the following:

$$\pm \vec{e}_i \pm \vec{e}_j \quad (i < j), \quad \pm \vec{e}_i, \quad \vec{0} \quad \text{with multiplicity } n; \quad (\text{C.2})$$

- the $(n + 1)(2n + 1)$ weights of the two-index symmetric tensor¹⁸ are

$$\pm \vec{e}_i \pm \vec{e}_j \quad (i < j), \quad \pm \vec{e}_i, \quad \pm 2\vec{e}_i, \quad \vec{0} \quad \text{with multiplicity } n + 1. \quad (\text{C.3})$$

Weight sets of $SO(2n)$. This group has rank n . If we denote by \vec{e}_i the versors in the \mathbb{R}^n weight space, then

- the set of the $2n$ weights $\vec{\pi}$ of the vector representation is given by

$$\pm \vec{e}_i; \quad (\text{C.4})$$

- the set of $n(2n - 1)$ weights $\vec{\rho}$ of the adjoint representation (corresponding to the two-index antisymmetric tensor) is the following:

$$\pm \vec{e}_i \pm \vec{e}_j \quad (i < j), \quad \vec{0} \quad \text{with multiplicity } n; \quad (\text{C.5})$$

¹⁸In fact, this is not an irreducible representation: it decomposes into the $(n + 1)(2n + 1) - 1$ traceless symmetric tensor plus a singlet. One of the $\vec{0}$ weights corresponds to the singlet.

- the $n(2n + 1)$ weights of the two-index symmetric tensor¹⁹ are

$$\pm \vec{e}_i \pm \vec{e}_j \quad (i < j), \quad \pm 2\vec{e}_i, \quad \vec{0} \quad \text{with multiplicity } n. \quad (\text{C.6})$$

Inserting these expressions into eq. (3.19) we have

$$P_{\square} = \begin{cases} (-s_1 s_2 s_3)^n \prod_{l=1}^3 \prod_{I>J=1}^n [(\chi_I - \chi_J)^2 - s_l^2] [(\chi_I + \chi_J)^2 - s_l^2] & \text{for } k = 2n, \\ (-s_1 s_2 s_3)^n \prod_{l=1}^3 \prod_{I>J=1}^n [(\chi_I - \chi_J)^2 - s_l^2] [(\chi_I + \chi_J)^2 - s_l^2] \\ \quad \times \prod_{K=1}^n (\chi_K^2 - s_l^2) & \text{for } k = 2n + 1. \end{cases} \quad (\text{C.7})$$

From eq. (3.21) we get instead

$$P_{\square} = \begin{cases} (\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4)^n \prod_{A=1}^4 \prod_{I>J=1}^n [(\chi_I - \chi_J)^2 - \epsilon_A^2] [(\chi_I + \chi_J)^2 - \epsilon_A^2] \\ \quad \times \prod_{K=1}^n (4\chi_K^2 - \epsilon_A^2) & \text{for } k = 2n, \\ (\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4)^{n+1} \prod_{A=1}^4 \prod_{I>J=1}^n [(\chi_I - \chi_J)^2 - \epsilon_A^2] [(\chi_I + \chi_J)^2 - \epsilon_A^2] \\ \quad \times \prod_{K=1}^n (4\chi_K^2 - \epsilon_A^2)(\chi_K^2 - \epsilon_A^2) & \text{for } k = 2n + 1. \end{cases} \quad (\text{C.8})$$

From eq. (3.22) we find

$$P_{\square} = \prod_{I=1}^n \prod_{i=1}^4 \left(\chi_I^2 + \frac{m_i^2}{2} \right) \frac{[\chi_I^2 - (ia + \frac{\epsilon_3 - \epsilon_4}{2})^2] [\chi_I^2 - (ia - \frac{\epsilon_3 - \epsilon_4}{2})^2]}{[\chi_I^2 - (ia + \frac{\epsilon_1 + \epsilon_2}{2})^2] [\chi_I^2 - (ia - \frac{\epsilon_1 + \epsilon_2}{2})^2]} \quad \text{for } k = 2n, \quad (\text{C.9})$$

while for k odd we get simply an extra null-vector contribution:

$$P_{\square} = \frac{\text{Pf}m}{4} \frac{a^2 - (\frac{\epsilon_3 - \epsilon_4}{2})^2}{a^2 - (\frac{\epsilon_1 + \epsilon_2}{2})^2} \times \text{r.h.s. of eq. (C.9)} \quad \text{for } k = 2n + 1. \quad (\text{C.10})$$

Finally, the Vandermonde determinant $\Delta(\chi)$ is explicitly given by

$$\Delta = \begin{cases} \prod_{I>J=1}^n (\chi_I^2 - \chi_J^2)^2 & \text{for } k = 2n, \\ \prod_{I>J=1}^n (\chi_I^2 - \chi_J^2)^2 \prod_{K=1}^n \chi_K^2 & \text{for } k = 2n + 1. \end{cases} \quad (\text{C.11})$$

¹⁹Again, this is not an irreducible representation, since it contains a singlet.

D Details on the AGT correspondence

The four-point conformal block $B_{\Delta_1\Delta_2;\Delta_3\Delta_4}^\Delta(x)$ appearing in the AGT relation (7.8) can be written [45] as a formal series according to

$$B_{\Delta_1\Delta_2;\Delta_3\Delta_4}^\Delta(x) = \sum_{|Y|=|Y'|} x^{|Y|} \gamma_{\Delta_1\Delta_2\Delta}(Y) Q_\Delta^{-1}(Y, Y') \gamma_{\Delta_3\Delta_4\Delta}(Y'). \quad (\text{D.1})$$

Here $Y = \{n_1 \geq n_2 \geq \dots \geq n_l > 0\}$ is a partition, $|Y| = n_1 + n_2 + \dots + n_l$ is its order, $Q_\Delta(Y, Y')$ is the correlator of two descendants of a primary state of dimension Δ , namely

$$Q_\Delta(Y, Y') = \langle \Delta | \mathcal{L}_Y \mathcal{L}_{-Y'} | \Delta \rangle, \quad (\text{D.2})$$

where

$$\mathcal{L}_{-Y} | \Delta \rangle = L_{-n_1} L_{-n_2} \dots L_{-n_l} | \Delta \rangle \quad (\text{D.3})$$

with L_n being the Virasoro generators, and finally $\gamma_{\Delta_1\Delta_2\Delta}(Y)$ is given by

$$\gamma_{\Delta_1\Delta_2\Delta}(Y) = \prod_l \left(\Delta + n_l \Delta_1 - \Delta_2 + \sum_{r < l} n_r \right). \quad (\text{D.4})$$

For further details, see for example [46].

The explicit evaluation of $Q_\Delta(Y, Y')$ and its inverse is a straightforward but tedious task. To perform the algebra we used the open-source program REDUCE and, exploiting the AGT relation (7.8), were able to compute²⁰ the non-perturbative prepotential $F_{\text{n.p.}}^{(4)}$ up to the fifth order in x . The first three coefficients F_1 , F_2 and F_3 were reported in the main text in eqs. (7.12), (7.13) and (7.14) respectively. Here we give the expressions for the fourth and fifth coefficients. We have

$$\begin{aligned} F_4 = & \frac{1469(\text{Pfm})^4}{8388608 a^{14}} - \frac{715(\text{Pfm})^2 \sum_{i < j < k} m_i^2 m_j^2 m_k^2}{2097152 a^{12}} \\ & + \frac{153 \sum_{i < j < k} m_i^4 m_j^4 m_k^4 + 1332(\text{Pfm})^2 \sum_{i < j} m_i^2 m_j^2 + 2304(\text{Pfm})^3}{2097152 a^{10}} \\ & - \frac{63 \sum_i m_i^2 \sum_{j < k \neq i} m_j^4 m_k^4 + 588 (\text{Pfm})^2 \sum_i m_i^2 + 896 \text{Pfm} \sum_{i < j < k} m_i^2 m_j^2 m_k^2}{524288 a^8} \\ & + \frac{5 \sum_{i < j} m_i^4 m_j^4 + 100 \sum_i m_i^4 \sum_{j < k \neq i} m_j^2 m_k^2 + 3280 (\text{Pfm})^2 + 1280 \text{Pfm} \sum_{i < j} m_i^2 m_j^2}{524288 a^6} \\ & - \frac{3 \sum_{i \neq j} m_i^4 m_j^2 + 732 \sum_{i < j < k} m_i^2 m_j^2 m_k^2 + 384 \text{Pfm} \sum_i m_i^2}{131072 a^4} \\ & + \frac{\sum_i m_i^4 + 468 \sum_{i < j} m_i^2 m_j^2 + 4352 \text{Pfm}}{131072 a^2} \\ & + \frac{233 \sum_i m_i^2 + 2048 \sum_{i < j} m_i m_j}{32768} + \frac{2701 a^2}{32768} \end{aligned} \quad (\text{D.5})$$

²⁰We found convenient to compute the conformal blocks in the particular case $\epsilon_1 + \epsilon_2 = 0$, i.e. for $Q = 0$ and $c = 1$; this has no implications on the result, since according to eq. (7.8) in the computation of $F_{\text{n.p.}}^{(4)}$ only the limit in which the $\epsilon_{1,2}$ parameters vanish matter. Here we would like comment that, despite the fact that the only unitary $c = 1$ conformal theory is the free one, the conformal block we computed is not the one associated with the free theory. This happens because the conformal block is derived using the Virasoro algebra only, and thus the conformal dimension of the intermediate state is arbitrary, which would not happen in the free theory.

and

$$\begin{aligned}
 F_5 = & \frac{4471(\text{Pfm})^5}{83886080 a^{18}} - \frac{525(\text{Pfm})^3 \sum_{i<j<k} m_i^2 m_j^2 m_k^2}{4194304 a^{16}} \\
 & + \frac{1131 \text{Pfm} \sum_{i<j<k} m_i^4 m_j^4 m_k^4 + 5980(\text{Pfm})^3 \sum_{i<j} m_i^2 m_j^2 + 7345(\text{Pfm})^4}{20971520 a^{14}} \\
 & - \frac{121 \text{Pfm} \sum_i m_i^2 \sum_{j<k \neq i} m_j^4 m_k^4 + 715(\text{Pfm})^2 \sum_{i<j<k} m_i^2 m_j^2 m_k^2 + 660(\text{Pfm})^3 \sum_i m_i^2}{1048576 a^{12}} \\
 & + \frac{1}{5242880 a^{10}} \left[765 \sum_{i<j<k} m_i^4 m_j^4 m_k^4 + 1260 \text{Pfm} \sum_i m_i^4 \sum_{j<k \neq i} m_j^2 m_k^2 \right. \\
 & \quad \left. + 207 \text{Pfm} \sum_{i<j} m_i^4 m_j^4 + 6660(\text{Pfm})^2 \sum_{i<j} m_i^2 m_j^2 + 13680(\text{Pfm})^3 \right] \\
 & - \frac{1}{262144 a^8} \left[63 \sum_i m_i^2 \sum_{j<k \neq i} m_j^4 m_k^4 + 644 \text{Pfm} \sum_{i<j<k} m_i^2 m_j^2 m_k^2 \right. \\
 & \quad \left. + 21 \text{Pfm} \sum_{i \neq j} m_i^4 m_j^2 + 588(\text{Pfm})^2 \sum_i m_i^2 \right] \\
 & + \frac{1}{262144 a^6} \left[5 \sum_{i<j} m_i^4 m_j^4 + 100 \sum_i m_i^4 \sum_{j<k \neq i} m_j^2 m_k^2 + 7 \text{Pfm} \sum_i m_i^4 \right. \\
 & \quad \left. + 780 \text{Pfm} \sum_{i<j} m_i^2 m_j^2 + 2000(\text{Pfm})^2 \right] \\
 & - \frac{3 \sum_i m_i^4 \sum_{j \neq i} m_j^2 + 348 \sum_{i<j<k} m_i^2 m_j^2 m_k^2 + 225 \text{Pfm} \sum_i m_i^2}{65536 a^4} \\
 & + \frac{\sum_i m_i^4 + 212 \sum_{i<j} m_i^2 m_j^2 + 1787 \text{Pfm}}{65536 a^2} \\
 & + \frac{525 \sum_i m_i^2 + 4096 \sum_{i<j} m_i m_j}{81920} + \frac{5057 a^2}{81920} . \tag{D.6}
 \end{aligned}$$

E Decoupling limits to $N_f = 3, 2, 1, 0$

From the prepotential we derived in eqs. (5.2)–(5.6) for the $N_f = 4$ case, it is possible to deduce the prepotential for the asymptotically free theories with $N_f = 3, 2, 1, 0$ by decoupling in turn the massive fundamental hypermultiplets. We will now compare the results obtained in this way with those following from Nekrasov’s prescriptions for the instanton calculus, which were originally given just for these asymptotically free cases. We will find full agreement in all terms contributing to the effective action. This enhances our confidence in our results, and shows that the difference with Nekrasov’s formulas (or, equivalently, with the AGT methods) discussed in section 7 only occurs in the conformal case $N_f = 4$.

Let us start decoupling one flavor and get the $N_f = 3$ theory by taking the limit

$$m_4 \rightarrow \infty \quad \text{with } \Lambda \equiv q m_4 \text{ fixed} . \tag{E.1}$$

From eqs. (5.2)–(5.4) we find

$$\begin{aligned} \mathcal{F}_{\text{n.p.}}^{(4)} = & \frac{2 m_1 m_2 m_3}{a^2} \Lambda + \left\{ \frac{5(m_1 m_2 m_3)^2}{4 a^6} - \frac{3 \sum_{i<j=1}^3 m_i^2 m_j^2}{2 a^4} + \frac{\sum_{i=1}^3 m_i^2}{a^2} + \frac{1}{4} \right\} \Lambda^2 \\ & + m_1 m_2 m_3 \left\{ \frac{3(m_1 m_2 m_3)^2}{a^{10}} - \frac{14 \sum_{i<j=1}^3 m_i^2 m_j^2}{3 a^8} + \frac{20 \sum_{i=1}^3 m_i^2}{3 a^6} - \frac{8}{a^4} \right\} \Lambda^3 + O(\Lambda^4). \end{aligned} \quad (\text{E.2})$$

On the other hand, from Nekrasov’s paper [38], using eq. (3.16) and the relevant definitions given in the previous pages therein, we can extract the following expression of the SU(2) prepotential for $N_f = 3$ flavors with masses M_i :

$$\begin{aligned} F_{\text{n.p.}} = & \frac{M_1 M_2 M_3 + a^2(M_1 + M_2 + M_3)}{2 a^2} \hat{\Lambda} \\ & + \frac{5 M_1^2 M_2^2 M_3^2 - 3 a^2 \sum_{i<j=1}^3 M_i^2 M_j^2 + a^4 \sum_{i=1}^3 M_i^2 + a^6}{64 a^6} \hat{\Lambda}^2 \\ & + M_1 M_2 M_3 \frac{9 M_1^2 M_2^2 M_3^2 - 7 a^2 \sum_{i<j=1}^3 M_i^2 M_j^2 + 5 a^4 \sum_{i=1}^3 M_i^2 - 3 a^6}{192 a^{10}} \hat{\Lambda}^3 + O(\hat{\Lambda}^4). \end{aligned} \quad (\text{E.3})$$

The two expressions (E.2) and (E.3) match after taking into account a different normalization of the masses and of the dynamical scale, namely

$$m_i = \sqrt{2} M_i, \quad \Lambda = \frac{\hat{\Lambda}}{8\sqrt{2}}. \quad (\text{E.4})$$

The only difference is the constant term proportional to $(M_1 + M_2 + M_3)$ in the first line of (E.3), which however does not contribute to the effective action upon integration over the chiral superspace, and thus is irrelevant.

The agreement persists for lower values of N_f , upon taking further decoupling limits. For instance, the pure SU(2) case ($N_f = 0$) can be reached from our $N_f = 4$ results in eqs. (5.2)–(5.4) by sending

$$m_{1,2,3,4} \rightarrow \infty, \quad \text{with } \Lambda^4 \equiv q \text{Pfm fixed.} \quad (\text{E.5})$$

The resulting prepotential reads

$$\mathcal{F} = 2 \frac{\Lambda^4}{a^2} + \frac{5 \Lambda^8}{4 a^6} + 3 \frac{\Lambda^{12}}{a^{10}} + \frac{1469 \Lambda^{16}}{128 a^{14}} + \frac{4471 \Lambda^{20}}{80 a^{18}} + \dots \quad (\text{E.6})$$

and coincides with Nekrasov’s result if we set $\Lambda^4 = \hat{\Lambda}^4/4$. The above results are perfectly consistent also with the instanton expansion of the effective coupling derived from the SW curves [26, 27] for SU(2) gauge theories with $N_f = 0, 1, 2, 3$.

Also the relation (6.1) between the exact gauge coupling τ on the D3 branes and the eight-dimensional chiral ring for the conformal $N_f = 4$ case can be extended to the non-conformal cases with lower N_f . We can check this by exploiting the results of [33], where the first chiral ring expectation values $\langle \text{Tr } m^{2l} \rangle$ have been given in the generic case of N_f D7-branes supporting an $\text{SO}(2N_f)$ group. Therefore we can compare these expressions to

the effective coupling derived from the quadratic prepotentials we just described. Let us consider as an example the $N_f = 3$ case. Deriving the prepotential (E.2) with respect to a or, equivalently, taking the limit (E.1) on the non-perturbative effective coupling (5.11), we find

$$\begin{aligned}
 2\pi i \tau_{\text{n.p.}} = & \frac{1}{a^4} \left(12 m_1 m_2 m_3 \Lambda + 6 \sum_{i=1}^3 m_i^2 \Lambda^2 + \dots \right) \\
 & - \frac{1}{a^6} \left(30 \sum_{i<j=1}^3 m_i^2 m_j^2 \Lambda^2 + 160 m_1 m_2 m_3 \Lambda^3 + \dots \right) \\
 & + \frac{1}{a^8} \left(\frac{105}{2} (m_1 m_2 m_3)^2 \Lambda^2 + 280 m_1 m_2 m_3 \sum_{i=1}^3 m_i^2 \Lambda^3 + \dots \right) + \dots .
 \end{aligned} \tag{E.7}$$

On the other hand, from [33] we can deduce, in our conventions, the following results for $N_f = 3$

$$\begin{aligned}
 \langle \text{Tr } m^2 \rangle &= \text{Tr } \langle m \rangle^2 , \\
 \langle \text{Tr } m^4 \rangle &= \text{Tr } \langle m \rangle^4 - 48 m_1 m_2 m_3 \Lambda - 24 \sum_{i=1}^3 m_i^2 \Lambda^2 + \dots , \\
 \langle \text{Tr } m^6 \rangle &= \text{Tr } \langle m \rangle^6 + 180 \sum_{i<j=1}^3 m_i^2 m_j^2 \Lambda^2 + 960 m_1 m_2 m_3 \Lambda^3 + \dots , \\
 \langle \text{Tr } m^8 \rangle &= \text{Tr } \langle m \rangle^8 - 420 (m_1 m_2 m_3)^2 \Lambda^2 - 2240 m_1 m_2 m_3 \sum_{i=1}^3 m_i^2 \Lambda^3 + \dots .
 \end{aligned} \tag{E.8}$$

We therefore see that also in this asymptotically free case the lowest orders in Λ and in $1/a$ are compatible with the relation

$$2\pi i \tau_{\text{n.p.}} = - \sum_{l=1}^{\infty} \frac{1}{2l} \frac{\langle \text{Tr } m^{2l} \rangle|_{\text{n.p.}}}{a^{2l}} , \tag{E.9}$$

from which eq. (6.1) easily follows. The full effective coupling τ can thus be obtained from its perturbative expression (2.13) upon taking into account the eight-dimensional quantum dynamics of the mass parameters m_i .

The chiral ring is mathematically defined for any number N_f of D7 branes but, as emphasized in [33], only for $N_f > 4$ (resp. $N_f = 4$) it can be associated to an eight-dimensional $\text{SO}(2N_f)$ gauge theory with negative (resp. vanishing) β function for which a weak coupling analysis is reliable. Our analysis in the cases $N_f = 4$ and $N_f < 4$ shows that the chiral ring for those values of N_f can be given a sensible meaning in the four-dimensional theory.

Open Access. This article is distributed under the terms of the Creative Commons Attribution Noncommercial License which permits any noncommercial use, distribution, and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] R. Blumenhagen, M. Cvetič, P. Langacker and G. Shiu, *Toward realistic intersecting D-brane models*, *Ann. Rev. Nucl. Part. Sci.* **55** (2005) 71 [[hep-th/0502005](#)] [[SPIRES](#)].
- [2] R. Blumenhagen, B. Kors, D. Lust, and S. Stieberger, *Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes*, *Phys. Rept.* **445** (2007) 1 [[hep-th/0610327](#)] [[SPIRES](#)].
- [3] F. Marchesano, *Progress in D-brane model building*, *Fortsch. Phys.* **55** (2007) 491 [[hep-th/0702094](#)] [[SPIRES](#)].
- [4] R. Blumenhagen, M. Cvetič, S. Kachru and T. Weigand, *D-Brane Instantons in Type II Orientifolds*, *Ann. Rev. Nucl. Part. Sci.* **59** (2009) 269 [[arXiv:0902.3251](#)] [[SPIRES](#)].
- [5] E. Witten, *Small Instantons in String Theory*, *Nucl. Phys. B* **460** (1996) 541 [[hep-th/9511030](#)] [[SPIRES](#)].
- [6] M.R. Douglas, *Branes within branes*, [[hep-th/9512077](#)] [[SPIRES](#)].
- [7] M.B. Green and M. Gutperle, *D-instanton induced interactions on a D3-brane*, *JHEP* **02** (2000) 014 [[hep-th/0002011](#)] [[SPIRES](#)].
- [8] M. Billò et al., *Classical gauge instantons from open strings*, *JHEP* **02** (2003) 045 [[hep-th/0211250](#)] [[SPIRES](#)].
- [9] R. Blumenhagen, M. Cvetič and T. Weigand, *Spacetime instanton corrections in 4D string vacua - the seesaw mechanism for D-brane models*, *Nucl. Phys. B* **771** (2007) 113 [[hep-th/0609191](#)] [[SPIRES](#)].
- [10] L.E. Ibáñez and A.M. Uranga, *Neutrino Majorana masses from string theory instanton effects*, *JHEP* **03** (2007) 052 [[hep-th/0609213](#)] [[SPIRES](#)].
- [11] B. Florea, S. Kachru, J. McGreevy and N. Saulina, *Stringy Instantons and Quiver Gauge Theories*, *JHEP* **05** (2007) 024 [[hep-th/0610003](#)] [[SPIRES](#)].
- [12] R. Donagi and M. Wijnholt, *Model Building with F-theory*, [[arXiv:0802.2969](#)] [[SPIRES](#)].
- [13] C. Beasley, J.J. Heckman and C. Vafa, *GUTs and Exceptional Branes in F-theory - I*, *JHEP* **01** (2009) 058 [[arXiv:0802.3391](#)] [[SPIRES](#)].
- [14] C. Beasley, J.J. Heckman and C. Vafa, *GUTs and Exceptional Branes in F-theory - II: Experimental Predictions*, *JHEP* **01** (2009) 059 [[arXiv:0806.0102](#)] [[SPIRES](#)].
- [15] C. Vafa, *Evidence for F-theory*, *Nucl. Phys. B* **469** (1996) 403 [[hep-th/9602022](#)] [[SPIRES](#)].
- [16] F. Denef, *Les Houches Lectures on Constructing String Vacua*, [[arXiv:0803.1194](#)] [[SPIRES](#)].
- [17] J.J. Heckman, *Particle Physics Implications of F-theory*, [[arXiv:1001.0577](#)] [[SPIRES](#)].
- [18] P. Berglund and P. Mayr, *Non-perturbative superpotentials in F-theory and string duality*, [[hep-th/0504058](#)] [[SPIRES](#)].
- [19] R. Blumenhagen, A. Collinucci and B. Jurke, *On Instanton Effects in F-theory*, *JHEP* **08** (2010) 079 [[arXiv:1002.1894](#)] [[SPIRES](#)].
- [20] M. Cvetič, I. Garcia-Etxebarria and J. Halverson, *Global F-theory Models: Instantons and Gauge Dynamics*, [[arXiv:1003.5337](#)] [[SPIRES](#)].
- [21] M. Alim et al., *Hints for Off-Shell Mirror Symmetry in type-II/F-theory Compactifications*, *Nucl. Phys. B* **841** (2010) 303 [[arXiv:0909.1842](#)] [[SPIRES](#)].

- [22] T.W. Grimm, T.-W. Ha, A. Klemm and D. Klevers, *Computing Brane and Flux Superpotentials in F-theory Compactifications*, *JHEP* **04** (2010) 015 [[arXiv:0909.2025](#)] [[SPIRES](#)].
- [23] T.W. Grimm, T.-W. Ha, A. Klemm and D. Klevers, *Five-Brane Superpotentials and Heterotic/F-theory Duality*, *Nucl. Phys. B* **838** (2010) 458 [[arXiv:0912.3250](#)] [[SPIRES](#)].
- [24] H. Jockers, P. Mayr and J. Walcher, *On $N = 1$ 4d Effective Couplings for F-theory and Heterotic Vacua*, [[arXiv:0912.3265](#)] [[SPIRES](#)].
- [25] A. Sen, *F-theory and Orientifolds*, *Nucl. Phys. B* **475** (1996) 562 [[hep-th/9605150](#)] [[SPIRES](#)].
- [26] N. Seiberg and E. Witten, *Monopole Condensation, and Confinement in $N = 2$ Supersymmetric Yang-Mills Theory*, *Nucl. Phys. B* **426** (1994) 19 [[hep-th/9407087](#)] [[SPIRES](#)].
- [27] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys. B* **431** (1994) 484 [[hep-th/9408099](#)] [[SPIRES](#)].
- [28] T. Banks, M.R. Douglas and N. Seiberg, *Probing F-theory with branes*, *Phys. Lett. B* **387** (1996) 278 [[hep-th/9605199](#)] [[SPIRES](#)].
- [29] E. Gava, K.S. Narain and M.H. Sarmadi, *Instantons in $N = 2$ $Sp(N)$ superconformal gauge theories and the AdS/CFT correspondence*, *Nucl. Phys. B* **569** (2000) 183 [[hep-th/9908125](#)] [[SPIRES](#)].
- [30] M. Billò, M. Frau, F. Fucito and A. Lerda, *Instanton calculus in RR background and the topological string*, *JHEP* **11** (2006) 012 [[hep-th/0606013](#)] [[SPIRES](#)].
- [31] M. Billò et al., *Exotic instanton counting and heterotic/type-I' duality*, *JHEP* **07** (2009) 092 [[arXiv:0905.4586](#)] [[SPIRES](#)].
- [32] K. Ito, H. Nakajima, T. Saka and S. Sasaki, *$N = 2$ Instanton Effective Action in Omega-background and D3/D(-1)-brane System in RR Background*, [[arXiv:1009.1212](#)] [[SPIRES](#)].
- [33] F. Fucito, J.F. Morales and R. Poghossian, *Exotic prepotentials from $D(-1)D7$ dynamics*, *JHEP* **10** (2009) 041 [[arXiv:0906.3802](#)] [[SPIRES](#)].
- [34] M. Billò et al., *Stringy instanton corrections to $N = 2$ gauge couplings*, *JHEP* **05** (2010) 107 [[arXiv:1002.4322](#)] [[SPIRES](#)].
- [35] N.A. Nekrasov, *Seiberg-Witten Prepotential From Instanton Counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [[hep-th/0206161](#)] [[SPIRES](#)].
- [36] R. Flume and R. Poghossian, *An algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential*, *Int. J. Mod. Phys. A* **18** (2003) 2541 [[hep-th/0208176](#)] [[SPIRES](#)].
- [37] U. Bruzzo, F. Fucito, J.F. Morales and A. Tanzini, *Multi-instanton calculus and equivariant cohomology*, *JHEP* **05** (2003) 054 [[hep-th/0211108](#)] [[SPIRES](#)].
- [38] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, [[hep-th/0306211](#)] [[SPIRES](#)].
- [39] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, [[hep-th/0306238](#)] [[SPIRES](#)].
- [40] U. Bruzzo and F. Fucito, *Superlocalization formulas and supersymmetric Yang-Mills theories*, *Nucl. Phys. B* **678** (2004) 638 [[math-ph/0310036](#)] [[SPIRES](#)].

- [41] W. Lerche and S. Stieberger, *Prepotential, mirror map and F-theory on K3*, *Adv. Theor. Math. Phys.* **2** (1998) 1105 [[hep-th/9804176](#)] [[SPIRES](#)].
- [42] W. Lerche, S. Stieberger and N.P. Warner, *Quartic gauge couplings from K3 geometry*, *Adv. Theor. Math. Phys.* **3** (1999) 1575 [[hep-th/9811228](#)] [[SPIRES](#)].
- [43] J.J. Heckman and C. Vafa, *An Exceptional Sector for F-theory GUTs*, [[arXiv:1006.5459](#)] [[SPIRES](#)].
- [44] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, *Lett. Math. Phys.* **91** (2010) 167 [[arXiv:0906.3219](#)] [[SPIRES](#)].
- [45] A.B. Zamolodchikov, *Conformal symmetry in two-dimensions: an explicit recurrence formula for the conformal partial wave amplitude*, *Commun. Math. Phys.* **96** (1984) 419 [[SPIRES](#)].
- [46] A. Marshakov, A. Mironov and A. Morozov, *On Combinatorial Expansions of Conformal Blocks*, *Theor. Math. Phys.* **164** (2010) 831 [[arXiv:0907.3946](#)] [[SPIRES](#)].
- [47] A. Marshakov, A. Mironov and A. Morozov, *Zamolodchikov asymptotic formula and instanton expansion in $N = 2$ SUSY $N_f = 2N_c$ QCD*, *JHEP* **11** (2009) 048 [[arXiv:0909.3338](#)] [[SPIRES](#)].
- [48] R. Poghossian, *Recursion relations in CFT and $N = 2$ SYM theory*, *JHEP* **12** (2009) 038 [[arXiv:0909.3412](#)] [[SPIRES](#)].
- [49] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton calculus in $N = 2$ supersymmetric gauge theory. II: Coupling to matter*, *Phys. Rev. D* **54** (1996) 7832 [[hep-th/9607202](#)] [[SPIRES](#)].
- [50] N. Dorey, V.V. Khoze and M.P. Mattis, *On $N = 2$ supersymmetric QCD with 4 flavors*, *Nucl. Phys. B* **492** (1997) 607 [[hep-th/9611016](#)] [[SPIRES](#)].
- [51] M. Billò, M. Frau, L. Gallot, A. Lerda and I. Pesando, *Classical solutions for exotic instantons?*, *JHEP* **03** (2009) 056 [[arXiv:0901.1666](#)] [[SPIRES](#)].
- [52] G.W. Moore, N. Nekrasov and S. Shatashvili, *D-particle bound states and generalized instantons*, *Commun. Math. Phys.* **209** (2000) 77 [[hep-th/9803265](#)] [[SPIRES](#)].
- [53] M. Gutperle, *Heterotic/type-I duality, D-instantons and a $N = 2$ AdS/CFT correspondence*, *Phys. Rev. D* **60** (1999) 126001 [[hep-th/9905173](#)] [[SPIRES](#)].
- [54] E. Kiritsis, N.A. Obers and B. Pioline, *Heterotic/type-II triality and instantons on K3*, *JHEP* **01** (2000) 029 [[hep-th/0001083](#)] [[SPIRES](#)].
- [55] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies and Phenomenology*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge U.K. (1987).
- [56] E. D'Hoker and D.H. Phong, *Lectures on supersymmetric Yang-Mills theory and integrable systems*, [[hep-th/9912271](#)] [[SPIRES](#)].
- [57] F. Fucito, J.F. Morales, and R. Poghossian, private communication.
- [58] M. Matone, *Instantons and recursion relations in $N = 2$ SUSY gauge theory*, *Phys. Lett. B* **357** (1995) 342 [[hep-th/9506102](#)] [[SPIRES](#)].
- [59] T.W. Grimm, A. Klemm, M. Mariño and M. Weiss, *Direct integration of the topological string*, *JHEP* **08** (2007) 058 [[hep-th/0702187](#)] [[SPIRES](#)].