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Exotic instanton counting and heterotic/type I′ duality

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Abstract: We compute the partition function for the exotic instanton system corresponding to D-instantons on D7 branes in Type I′ theory. We exploit the BRST structure of the moduli action and its deformation by RR background to fully localize the integration. The resulting prepotential describes non-perturbative corrections to the quartic couplings of the gauge field F living on the D7's. The results match perfectly those obtained in the dual heterotic theory from a protected 1-loop computation, thus providing a non-trivial test of the duality itself.

Keywords: Superstrings and Heterotic Strings, D-branes, String Duality

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JHEP07(2009)092

Contents

JHEP07(2009)092

1 Introduction and motivations

In recent years, the possibility of acquiring some control over space-time non-perturbative effects has been a unifying theme behind many developments in String Theory.

Much progress in this direction has been realized by exploiting the web of dualities relating the five 10-dimensional string theories and the 11-dimensional M-theory through operations that map classical or perturbative statements in one model to non-perturbative statements in its dual. One of the most notable examples of such relations is the heterotic/type I duality which has been tested by checking the stable spectra on both sides [1],

and by studying the BPS-saturated quartic couplings F^4 for the gauge field and their gravitational counterparts $F^2\mathcal{R}^2$ or $\mathcal{R}^4$ [2].[1] In this context, these protected quartic interactions are completely captured by a 1-loop computation on the heterotic side, while on the type I side they receive both perturbative and non-perturbative contributions.

In this paper we will consider a set-up in which the heterotic theory is compactified on a 2-torus $\mathcal{T}_2$ with Wilson lines breaking the gauge group to $\big[\mathrm{SO}(8)\big]^4$. In the dual theory, called type I′, the gauge degrees of freedom are supported by stacks of D7-branes, while the non-perturbative contributions arise by adding D(–1)-branes, also called D-instantons. Using the recent advances in the instanton calculus in string theory (for a review see [16]) together with localization techniques [17, 18], we will extract from the integration over the D-instanton moduli the quartic type I′ interactions for the gauge fields and their gravitational corrections, and check explicitly (up to instanton number $k = 5$) the agreement with the heterotic expressions. Although the structure of the type I′ contributions has already been investigated in the literature [8][12], and checks of the heterotic results have been performed against the F-theory background that should represent the non-perturbative completion of the type I′ model [19], we think that our calculations provide the first case in which the agreement is verified by a direct explicit evaluation of non-perturbative corrections in the "microscopic" theory.

The heterotic/type I′ duality is not the only motivation for the computation presented here: in fact, it can be regarded also as a prototypical instance of integration over the moduli space of exotic or stringy multi-instantons. Let us explain what we mean by this. The construction of "brane-world" models in which four-dimensional gauge and matter theories live on the world-volume of suitable D-brane stacks has assumed a prominent rôle for possible phenomenological applications of string theory. In this context, non-perturbative contributions to the effective action for the gauge/matter degrees of freedom can arise from instantonic branes, that is from branes that are point-like in the four non-compact space-time directions. Instantonic branes which in the internal space coincide with the D-branes that support the gauge theory correspond to the usual gauge instanton configurations [20][23]. From the CFT point of view, open strings suspended between instantonic and gauge branes have four directions with mixed Neumann-Dirichlet (ND) boundary conditions, and possess massless excitations in the Neveu-Schwarz sector corresponding to the moduli which describe the size and gauge orientation of field-theoretical instanton solutions.

On the other hand, instantonic branes which do not coincide with the gauge branes in the internal directions are usually referred to as exotic or stringy instantons. Much interest in their properties was sparkled by the realization that they can generate terms in the effective action which are forbidden in perturbation theory but are necessary for phenomenological applications, such as neutrino Majorana mass terms or certain Yukawa couplings in GUT models (see ref. [16] and references therein). From the CFT point of view, mixed open strings have extra twisted directions besides the four ND space-time directions. As a consequence, the bosonic moduli corresponding to the size are missing and certain fermionic zero-modes become difficult to saturate. These unwanted zero-modes

[1] For earlier calculations of higher order couplings in the heterotic theory see refs. [13].

JHEP07(2009)092

must be either lifted [24][26] or removed by appropriate projections [27][29] in order to get non-vanishing contributions.

The extension of the instanton calculus to the exotic cases is therefore of great relevance. In some set-ups with $\mathcal{N} = 1$ supersymmetry it has been shown that novel interactions terms in the effective superpotential can arise from sectors with a specific instanton number [16]. With $\mathcal{N} = 2$ supersymmetry, instead, one expects contributions from all sectors, in analogy with what happens for ordinary gauge instantons in four dimensions. In this case, in fact, using the exact Seiberg-Witten solution of $\mathcal{N} = 2$ super Yang-Mills (SYM) theories [30], one can show that the effective prepotential receives contributions from all instantons. A few years ago [18], such a prediction was finally checked against the direct evaluation of the non-perturbative effects at all instanton numbers in the microscopic SYM theory. This remarkable computation was made possible by a BRST-invariant reformulation of the instanton moduli action, the introduction of suitable deformations and the use of localization techniques [17, 18].[2] In ref. [36], this procedure was reproduced in a stringy way using systems of D3/D(−1)-branes. In that context, the localization deformations arise from interactions with a Ramond-Ramond (RR) closed string graviphoton background.

Here we extend this approach to systems of D7/D(−1)-branes in the type I′ theory. This extension is not a priori obvious, given the very different structure of the moduli space, but actually, as we will see, it carries over in a rather natural way, and in the end it allows us to explicitly perform the integration over the instanton moduli and check the predictions from the heterotic string. As discussed in detail in ref. [37], the D7/D(−1) brane systems display the typical features of the exotic instantons in that they have "more than four" ND directions (eight, in fact) and lack the bosonic charged moduli related to the size. The gauge theory living on the eight-dimensional world-volume of the D7-branes has a quartic action for the gauge fields that is described by a prepotential function, analogously to the quadratic action for the $\mathcal{N} = 2$ SYM theories in four dimensions. This prepotential receives non-perturbative contributions from all numbers of D-instantons, and here we show how to compute them relying on the BRST structure of the instanton action, the introduction of deformations from the RR sector and the use of localization techniques. Given the similarities of the moduli spectra, the techniques used in this case should be useful also for the treatment of exotic instanton contributions in four-dimensional theories.

The structure of this paper is as follows: in the next section we briefly review the results expected from the heterotic/type I′ duality for the non-perturbative contributions to the quartic couplings. In section 3 we describe the BRST structure of the instanton moduli action, which we deform by introducing a RR background in section 4. Then, in section 5 we discuss the rescalings that lead to the localization of the moduli integrals that are explicitly evaluated in section 6 up to instanton number $k = 5$. In the last two sections we collect our results and present our conclusions. Finally, some technical details on the conventions, on the interactions with the RR background and on the evaluation of the moduli integrals are contained in three appendices.

[2] See refs. [31][35] for further applications and generalizations.

JHEP07(2009)092

2 Heterotic results and duality to type I′

In order to be self-contained, we begin by briefly reviewing the heterotic results on the quartic effective action for the system we want to consider, and the philosophy of the stringy instanton calculus that we will apply on the type I′ side.

2.1 Heterotic vs type I′ results for the quartic effective action

Let us consider a toroidal compactification of the SO(32) heterotic string. Differently from what happens for the uncompactified case, the gauge quartic terms F^4 and their gravitational counterparts $F^2\mathcal{R}^2$ and $\mathcal{R}^4$ are not completely fixed by supersymmetry and anomaly cancellation, but still are sensitive only to the BPS sector of the theory and, as such, enjoy non-renormalization properties [7]. Thus, these quartic couplings are the natural terms to consider in order to test the duality map between the heterotic string and the type I theory.

On the heterotic side, the quartic terms are exact at one loop and have been computed in various toroidal compactifications with non-trivial Wilson lines. Here we consider a compactification on a 2-torus $\mathcal{T}_2$ with Wilson lines that break the gauge group SO(32) down to $\big[\mathrm{SO}(8)\big]^4$. This case presents some interesting peculiarities since, besides the single-trace and double-trace quartic invariants, the group SO(8) possesses a third independent invariant of order four: the Pfaffian. As a consequence, the algebraic structure of the quartic effective action is richer. The part containing the simple- and double-trace terms was computed in refs. [8, 10, 12], while the Pfaffian part was considered in ref. [11]. In our normalizations, and denoting by $T_{\rm h}$ and $U_{\rm h}$, respectively, the (complexified) Kähler modulus and the complex structure of the 2-torus $\mathcal{T}_2$, the quartic effective couplings read

$$\begin{aligned}\frac{t_8\,\mathrm{Tr}\,F^4}{4}\,\log\left|\frac{\eta(4T_{\rm h})}{\eta(2T_{\rm h})}\right|^4 \;+\; \frac{t_8\,(\mathrm{Tr}\,F^2)^2}{16}\,\log\left(\mathrm{Im}\,T_{\rm h}\,\mathrm{Im}\,U_{\rm h}\frac{|\eta(2T_{\rm h})|^8\,|\eta(U_{\rm h})|^4}{|\eta(4T_{\rm h})|^4}\right)\\ +\;2\,t_8\,\mathrm{Pf}\,F\,\log\left|\frac{\eta(T_{\rm h}+1/2)}{\eta(T_{\rm h})}\right|^4\end{aligned} \tag{2.1}$$

where η is the Dedekind function and t_8 is the eight-index tensor arising in various string amplitudes [38] (see appendix A.2 for more details). More precisely, the notation $t_8\,\mathrm{Tr}\,F^4$ stands for

$$\begin{aligned}t_8\mathrm{Tr}F^4 &\equiv \frac{1}{2^4}\,t_8^{\mu_1\mu_2\cdots\mu_7\mu_8}\,\mathrm{Tr}\left(F_{\mu_1\mu_2}\cdots F_{\mu_7\mu_8}\right) \\ &= \mathrm{Tr}\left(F_{\mu\nu}F^{\nu\rho}F^{\lambda\mu}F_{\rho\lambda}+\frac{1}{2}\,F_{\mu\nu}F^{\rho\nu}F_{\rho\lambda}F^{\mu\lambda}-\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}F_{\rho\lambda}F^{\rho\lambda}-\frac{1}{8}\,F_{\mu\nu}F_{\rho\lambda}F^{\mu\nu}F^{\rho\lambda}\right),\end{aligned} \tag{2.2}$$

with a similar expression for $t_8\,(\mathrm{Tr}\,F^2)^2$, while the loose notation $t_8\,\mathrm{Pf}\,F$ actually means

$$t_8\,\mathrm{Pf}\,F \equiv \frac{1}{2^8}\,t_8^{\mu_1\mu_2\cdots\mu_7\mu_8}\,\epsilon_{a_1a_2\cdots a_7a_8}\,F^{a_1a_2}_{\mu_1\mu_2}\cdots F^{a_7a_8}_{\mu_7\mu_8} \tag{2.3}$$

with a_i's being indices of the fundamental representation of SO(8). It is interesting to observe that the coupling functions appearing in front of all the three gauge-invariant

JHEP07(2009)092

structures in (2.1) are invariant under the modular[3] subgroup $\Gamma_0(4)$ acting on $T_{\rm h}$ which is preserved by the insertion of the Wilson lines.

In this model there are also quartic interactions involving the space-time curvature two-form $\mathcal{R}$. They correspond (schematically) to the following structures

$$t_8 \,\mathrm{Tr}\, F^2 \,\mathrm{Tr}\, \mathcal{R}^2\,, \qquad t_8 \,\mathrm{Tr}\, \mathcal{R}^4\,, \qquad t_8 \left(\mathrm{Tr}\, \mathcal{R}^2\right)^2 \tag{2.4}$$

and, like the pure gauge terms, they are also captured exactly by a 1-loop heterotic computation. From the results contained for example in refs. [8, 12] one can deduce that (up to an overall convention dependent coefficient) such gravitational terms are

$$t_8 \left[\mathrm{Tr}\, \mathcal{R}^4 + \frac{1}{4}\left(\mathrm{Tr}\, \mathcal{R}^2\right)^2 - 16\,\mathrm{Tr}\, F^2\,\mathrm{Tr}\, \mathcal{R}^2\right] \log\left(\mathrm{Im}\, T_{\rm h}\, \mathrm{Im}\, U_{\rm h}\, |\eta(2T_{\rm h})|^4\, |\eta(U_{\rm h})|^4\right)\,. \tag{2.5}$$

Let us now focus on the $T_{\rm h}$ dependence and introduce the parameter

$$q_{\rm h} = \mathrm{e}^{2\pi \mathrm{i}\, T_{\rm h}}\,. \tag{2.6}$$

Then, the quartic gauge couplings (2.1) can be rewritten as

$$t_8\,\mathrm{Tr}\, F^4 \left\{\left(\frac{\pi \mathrm{i} T_{\rm h}}{12} - \frac{1}{2}\sum_{k=1}^{\infty}\left(d_k q_{\rm h}^{4k} - d_k q_{\rm h}^{2k}\right)\right) + \text{c.c.}\right\} \tag{2.7a}$$

$$+\, t_8 (\mathrm{Tr}\, F^2)^2 \left\{\frac{1}{16}\log\left(\mathrm{Im}\, T_{\rm h} \mathrm{Im}\, U_{\rm h} |\eta(U_{\rm h})|^4\right) + \frac{1}{8}\left(\sum_{k=1}^{\infty}\left(d_k q_{\rm h}^{4k} - 2 d_k q_{\rm h}^{2k}\right) + \text{c.c.}\right)\right\} \tag{2.7b}$$

$$+\, 8 t_8 \mathrm{Pf}\, F \left(\sum_{k=1}^{\infty} d_{2k-1} q_{\rm h}^{2k-1} + \text{c.c}\right) \tag{2.7c}$$

where the coefficients d_k are given by the sum of the inverse of the divisors of k:

$$d_k = \sum_{\ell | k} \frac{1}{\ell}\,. \tag{2.8}$$

Likewise, the gravitational couplings (2.5) become

$$t_8 \left[\mathrm{Tr}\, \mathcal{R}^4 + \frac{1}{4}\left(\mathrm{Tr}\, \mathcal{R}^2\right)^2 - 16\,\mathrm{Tr}\, F^2\,\mathrm{Tr}\, \mathcal{R}^2\right] \times \tag{2.9}$$

$$\times \left\{\left(\frac{\pi \mathrm{i} T_{\rm h}}{3} + \frac{1}{2}\log\left(\mathrm{Im}\, T_{\rm h}\, \mathrm{Im}\, U_{\rm h}\, |\eta(U_{\rm h})|^4\right) - 2\sum_{k=1}^{\infty} d_k q_{\rm h}^{2k}\right) \;+\; \text{c.c.}\right\}.$$

Written in this form, the quartic terms admit a direct interpretation in the dual type I′ theory. To see this, let us first recall that the type I′ theory is obtained from the type IIB string compactified on $\mathcal{T}_2$ with the orientifold projection

$$\Omega = \omega\,(-1)^{F_L}\,\mathcal{I}_2 \tag{2.10}$$

[3]The subgroup $\Gamma_0(4) \subset \mathrm{SL}(2,\mathbb{Z})$ is generated by t and st^4s, if $t : T_{\rm h} \to T_{\rm h} + 1$ and $s : T_{\rm h} \to -1/T_{\rm h}$ are the usual $\mathrm{SL}(2,\mathbb{Z})$ generators.

JHEP07(2009)092

where ω is the world-sheet parity, F_L is the left-moving target-space fermion number, and $\mathcal{I}_2$ is the inversion along the two directions of $\mathcal{T}_2$. The resulting theory is an unoriented string model with sixteen supercharges. The action of Ω has four fixed-points on $\mathcal{T}_2$ where four O7-planes are placed. A local cancellation of the RR tadpoles produced by these O7-planes requires to place at each fixed-point eight D7-branes or, equivalently, four D7 branes plus their orientifold images. Focusing on only one of the fixed-points, we therefore have a gauge theory with group SO(8) and $\mathcal{N} = 2$ supersymmetry in eight dimensions.

The type I′ model is dual to the $\big[\mathrm{SO}(8)\big]^4$ heterotic string on $\mathcal{T}_2$. In particular, the duality map relates the complexified Kähler modulus of the torus T_h on the heterotic side and the axion-dilaton field τ on the type I′ side, while the complex structure remains the same:

$$T_\mathrm{h} \leftrightarrow \tau \equiv C_0 + \frac{\mathrm{i}}{g_s} \ , \qquad U_\mathrm{h} \leftrightarrow U \ , \tag{2.11}$$

where g_s and C_0 are, respectively, the string coupling constant and the scalar of the RR sector. Thus, on the type I′ side, upon the replacement of q_h with

$$q \equiv \mathrm{e}^{2\pi\mathrm{i}\tau} \ . \tag{2.12}$$

we should retrieve exactly the results of eq.s (2.7) and (2.9), namely

$$t_8 \mathrm{Tr}\, F^4 \left\{ \left(\frac{\pi\mathrm{i}\tau}{12} + \frac{1}{2}q^2 + \frac{1}{4}q^4 + \dots \right) + \mathrm{c.c.} \right\} \tag{2.13a}$$

$$+ t_8 (\mathrm{Tr}\, F^2)^2 \left\{ \left(\frac{1}{32} \log \left(\mathrm{Im}\,\tau \mathrm{Im}\, U |\eta(U)|^4 \right) - \frac{1}{4}q^2 - \frac{1}{4}q^4 + \dots \right) + \mathrm{c.c.} \right\} \tag{2.13b}$$

$$+ 8 t_8 \mathrm{Pf}\, F \left\{ \left(q + \frac{4}{3}q^3 + \frac{6}{5}q^5 + \dots \right) + \mathrm{c.c} \right\} \ . \tag{2.13c}$$

Thus, for the single trace structure, from eq. (2.13a) we expect to find a tree-level term proportional to $\mathrm{Im}\,\tau = 1/g_s$ plus a series of non-perturbative contributions weighted by powers of q which, as we will see, are due to D-instantons. For the double trace structure we identify in eq. (2.13b) a term proportional to $\log(\mathrm{Im}\,\tau) = -\log g_s$ that arises at 1-loop, plus a series of D-instanton contributions. The Pfaffian structure, instead, gets only non-perturbative contributions with odd instanton number, as we see from eq. (2.13b). Finally, the quartic gravitational couplings of type I′ have a tree-level term proportional to $\mathrm{Im}\,\tau$, a 1-loop term proportional to $\log(\mathrm{Im}\,\tau)$ and a series of non-perturbative contributions with even instanton number, as indicated in the second line of eq. (2.9).

In the literature, the heterotic results we described above have been compared [8, 12] with F-theory compactified on K3, which has been argued [19] to represent a geometrized non-perturbative version of the type I′ model. Our aim is instead to compare them with a direct computation of non-perturbative D-instanton effects in the type I′ string theory. The general philosophy behind such a computation is briefly summarized in the next subsection.

2.2 D-instanton contributions to the quartic effective action in type I′

When k D(–1)-branes are added to the D7-branes, new open string sectors appear, corresponding to open strings with at least one endpoint attached to the D(–1)'s. The excita-

JHEP07(2009)092

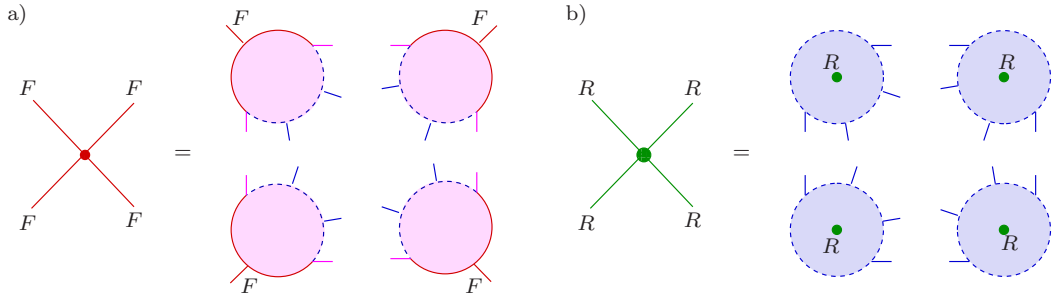

Figure 1. a) A quartic interaction vertex for the gauge field F can be induced by mixed disks having part of their boundary attached to the D-instantons and carrying the insertion of a vertex for F and of moduli vertices. The above diagram is connected by the integration over the moduli. b) Instanton disks with an insertion of a closed vertex can produce curvature interactions through the moduli integration.

tions of such strings carry no momentum; we call them moduli,[4] and collectively denote them as $\mathcal{M}_{(k)}$. The action for k D-instantons has a classical part, $\mathcal{S}_{\rm cl} = -2\pi i\tau\, k$, and a moduli-dependent part, $\mathcal{S}(\mathcal{M}_{(k)})$, arising from disk diagrams with at least a portion of their boundary attached to the D(–1)'s. There are also mixed disk diagrams describing the interactions between the moduli and the gauge fields living on the D7-branes which are encoded in a (chiral) superfield Φ. By including also these diagrams, the moduli action $\mathcal{S}(\mathcal{M}_{(k)})$ is promoted to $\mathcal{S}(\mathcal{M}_{(k)}, \Phi)$.

Non-perturbative contributions to the effective action for Φ arise upon integration over the moduli of the exponentiated field-dependent action of the D-instantons [22, 23]:

$$\sum_k \mathrm{e}^{-\mathcal{S}_{\rm cl}} \int d\mathcal{M}_{(k)}\, \mathrm{e}^{-\mathcal{S}(\mathcal{M}_{(k)},\Phi)} = \sum_k q^k \int d\mathcal{M}_{(k)}\, \mathrm{e}^{-\mathcal{S}(\mathcal{M}_{(k)},\Phi)} \ . \tag{2.14}$$

In particular, the integration over the moduli of those mixed disks that are sources for some components of Φ, can produce new effective couplings, as represented in figure 1a for the quartic interaction among four gauge field strengths F.

As already noted in the literature (see for instance refs. [5–8, 10, 12]), these D-instanton induced couplings have potentially the right structure to agree with the heterotic results reported in eq. (2.7). Indeed, the sum over the number k of D-instantons is weighted by q^k and the dimensionality of the moduli measure $d\mathcal{M}_{(k)}$ implies that the effective contributions must be quartic in the gauge fields for all k [37].

To turn the schematic expression (2.14) into a real computational tool, it is necessary to precisely identify the moduli $\mathcal{M}_{(k)}$, compute their field-dependent action $\mathcal{S}(\mathcal{M}_{(k)}, \Phi)$ and explicitly perform the matrix integrals. The latter task is far from being trivial. The

[4] If one considers systems of D3/D(–1)-branes (or more generally D($p+4$)/Dp-branes), the moduli excitations in the k D-instanton sectors are in full correspondence with the moduli of the classical instanton solutions with instanton number k, as encoded in the ADHM construction. The name "moduli" continues to be used in more general situations where the correspondence with classical solutions might be less immediate.

JHEP07(2009)092

2-instanton case was already considered in ref. [10] where it was argued that the correct gauge-invariant structures $\mathrm{Tr}\, F^4$ and $(\mathrm{Tr}\, F^2)^2$ are obtained from the integration over the moduli $\mathcal{M}_{(2)}$ with a relative coefficient in agreement with eq. (2.7). However, to reach more solid conclusions, an analysis at higher values of k is necessary.

In this case, the only viable route to get explicit results is a generalization of the methods that were successfully applied for the instanton calculus in $\mathcal{N} = 2$ SYM theories in four dimensions. This requires to exploit the particular algebraic structure and the supersymmetry of the moduli action $\mathcal{S}(\mathcal{M}_{(k)}, \Phi)$ and write it as a Q-exact expression with respect to a suitable BRST charge Q, in such a way that the localization techniques [17, 18] can be applied. These involve the introduction of deformations of $\mathcal{S}(\mathcal{M}_{(k)}, \Phi)$ which, while not altering the final result, may drastically simplify the computation. The needed deformations, which could be introduced *ad hoc* from a purely mathematical point of view, arise naturally from mixed disk diagrams describing the interaction of the moduli $\mathcal{M}_{(k)}$ with closed string graviphoton backgrounds from the RR sector of the theory. Treating the RR field-strengths as constant parameters to be put to zero at the end of the computation allows to write explicit contour integral expressions which, in principle, can be evaluated for any k, and from which the quartic effective action for the gauge fields can be extracted. If we consider the RR fields as genuine, dynamical graviphotons sitting in the same supermultiplet $\mathcal{W}$ of the curvature two-form $\mathcal{R}$, we can generalize eq. (2.14) and use a field-dependent moduli action $\mathcal{S}(\mathcal{M}_{(k)}, \Phi, \mathcal{W})$ that contains also gravitational terms. Then, the corresponding D-instanton partition functions will yield also the $\mathrm{Tr}\, F^2\, \mathrm{Tr}\, \mathcal{R}^2$ and $\mathrm{Tr}\, \mathcal{R}^4$ interactions (see for example figure 1b) which from the heterotic side are given in eq. (2.5).

This procedure will be described in great detail in the following sections.

3 The D7/D(–1) system and its BRST structure

We now discuss the main features of the D7/D(–1) system in the type I′ theory, both at the perturbative and the non-perturbative level.

3.1 The perturbative sectors

As we have already explained, the world-volume theory on the eight D7-branes located at one of the orientifold fixed points of the type I′ string model is an eight-dimensional gauge theory with sixteen supercharges and gauge group SO(8). Its bosonic action contains, besides the usual Yang-Mills term, also terms of higher order in the field strength F and its covariant derivatives. Among them, a crucial rôle for our purposes is played by the conformally invariant tree-level quartic terms

$$S_{(4)} = -\frac{1}{96\pi^3 g_s}\int d^8x\; t_8 \mathrm{Tr} F^4 - \frac{\mathrm{i}\, C_0}{192\pi^3}\int \mathrm{Tr}\left(F\wedge F\wedge F\wedge F\right) . \tag{3.1}$$

Introducing the chiral superfield

$$\Phi(x,\theta) = \phi(x) + \sqrt{2}\,\theta\Lambda(x) + \frac{1}{2}\,\theta\gamma^{\mu\nu}\theta\, F_{\mu\nu}(x) + \dots\,, \tag{3.2}$$

where Λ is the gaugino and ϕ a complex scalar, the quartic action (3.1) can be conveniently rewritten as

$$S_{(4)} = \frac{1}{(2\pi)^4}\int d^8x\, d^8\theta\ \left[\frac{\mathrm{i}\pi\tau}{12}\,\mathrm{Tr}\,\Phi^4\right] +\ \text{c.c.}\,, \tag{3.3}$$

where τ is the axion-dilaton combination appearing in eq. (2.11).

Other quartic terms are produced at 1-loop. Indeed, as shown for example in section 4.2 of ref. [37], the annulus and Möbius diagrams for this brane system yield the following (divergent) contribution[5]

$$-\frac{1}{128\pi^4}\int d^8x\ t_8\big(\mathrm{Tr}\,F^2\big)^2\left[\int_0^\infty \frac{dt}{2t}\,\Gamma(t)\right] \tag{3.4}$$

where $\Gamma(t)$ represents the sum over the winding modes in the two compact transverse directions, given by

$$\Gamma(t) = \sum_{(r_1,r_2)\in\mathbb{Z}^2} \mathrm{e}^{-2\pi t\,\frac{|r_1+r_2U|^2\mathrm{Im}T}{\mathrm{Im}U}} \tag{3.5}$$

with U and T being, respectively, the complex and Kähler structures of the 2-torus $\mathcal{T}_2$. The integral over the modular loop parameter t can be computed using the regularization procedure introduced in ref. [39] (and reviewed for example in appendix A of ref. [40]) with the result

$$\begin{aligned}\int_0^\infty \frac{dt}{2t}\,\Gamma(t) &= -\frac{1}{2}\,\log\big(\alpha'\mu^2\big) - \frac{1}{2}\,\log\left(\frac{\mathrm{Im}U\,|\eta(U)|^4}{\mathrm{Im}T}\right)\\ &= -\frac{1}{2}\,\log\left(\frac{\mu^2}{M_{\rm P}^2}\right) + \Delta^{1-\text{loop}}\,.\end{aligned} \tag{3.6}$$

Here μ is a low-energy scale that regularizes the IR divergence due to the massless open string states circulating in the loop, while $M_{\rm P}$ is the eight-dimensional Planck mass

$$M_{\rm P}^2 = \frac{\mathrm{Im}T}{\alpha'\,g_s} \tag{3.7}$$

which serves as UV cut-off in the field theory. Finally, $\Delta^{1-\text{loop}}$ represents the (finite) threshold corrections given by

$$\Delta^{1-\text{loop}} = -\frac{1}{2}\,\log\big(\mathrm{Im}\tau\,\mathrm{Im}U\,|\eta(U)|^4\big)\,. \tag{3.8}$$

From these results, we therefore find the following 1-loop term in the effective action

$$\begin{aligned}S_{(4)}^{1-\text{loop}} &= \frac{1}{256\pi^4}\int d^8x\ \log\big(\mathrm{Im}\tau\,\mathrm{Im}U\,|\eta(U)|^4\big)\,t_8\big(\mathrm{Tr}\,F^2\big)^2\\ &= \frac{1}{(2\pi)^4}\int d^8x\,d^8\theta\ \left[\frac{1}{32}\,\log\big(\mathrm{Im}\tau\,\mathrm{Im}U\,|\eta(U)|^4\big)\big(\mathrm{Tr}\,\Phi^2\big)^2\right] +\ \text{c.c.}\end{aligned} \tag{3.9}$$

which has to be added to the tree-level contribution (3.3). Due to $\mathcal{N}=2$ supersymmetry, there are no higher-loop quartic terms in the effective action.

[5] Note that for eight D7-branes there is no contribution to $\mathrm{Tr}\,F^4$ at 1-loop.

JHEP07(2009)092

3.2 The non-perturbative sectors

As discussed in ref. [37], the non-perturbative sectors of this theory can be described by adding k D(–1)-branes in the same fixed point where the D7's are located. The D(–1)-branes are sources for the RR scalar C_0; thus, considering the Wess-Zumino part of the D7 action (3.1), it follows that k D-instantons correspond to a gauge field configuration with fourth Chern number

$$c_{(4)} = \frac{1}{4!\,(2\pi)^4} \int \mathrm{Tr}\left(F \wedge F \wedge F \wedge F\right) = k \ . \tag{3.10}$$

Moreover, this gauge field configuration must be such that its classical quartic action reduces to k times the D-instanton action [37], *i.e.* $S_{(4)} = -2\pi\mathrm{i}\,\tau\,k$.

The physical excitations of the open strings with at least one end-point on the D-instantons account for the moduli $\mathcal{M}_{(k)}$ of such instanton-like configurations. The neutral sector, corresponding to D(–1)/D(–1) open strings, comprises the moduli that do not transform under the gauge group and includes, in an ADHM inspired notation, the vector a_μ and the scalar χ (plus its conjugate $\bar\chi$) in the Neveu-Schwarz sector, and the chiral and anti-chiral fermions M^α and $\lambda_{\dot\alpha}$ in the Ramond sector. The bosonic moduli have canonical dimensions of $(\text{length})^{-1}$, while the fermionic ones have canonical dimensions of $(\text{length})^{-\frac{3}{2}}$.

All these neutral moduli are $k \times k$ matrices, but the consistency with the orientifold projection on the D7-branes requires that χ, $\bar\chi$ and $\lambda_{\dot\alpha}$ transform in the anti-symmetric (or adjoint) representation of SO(k), while a_μ and M^α must be in the symmetric one. The diagonal parts of a_μ and M^α represent the bosonic and fermionic Goldstone modes of the (super)translations of the D7-branes world-volume that are broken by the D-instantons and thus can be identified with the bosonic and fermionic coordinates x_μ and θ^α of the eight-dimensional superspace. More precisely, we have

$$x_\mu = (2\pi\alpha')\,\mathrm{tr}\big(a_\mu\big) \quad , \quad \theta^\alpha = (2\pi\alpha')\,\mathrm{tr}\big(M^\alpha\big)\,, \tag{3.11}$$

where the factors of α' have been introduced to give x_μ and θ^α the appropriate dimensions.

The open strings stretching between the D-instantons and the D7-branes account for the charged moduli, which transform in the fundamental representations of both SO(8) and SO(k). The D7/D(–1) open strings have eight ND directions and thus, as discussed for example in ref. [37], it is not possible to find bosonic excitations that satisfy the physical state conditions. The absence of charged bosonic moduli is the hallmark of the "exotic" instanton configurations, and has to be contrasted with what happens in the D3/D(–1) systems where, instead, physical bosonic moduli, related to the gauge instanton size, exist. On the other hand, the fermionic Ramond sector of the D7/D(–1) system is not empty and contains physical moduli, denoted as μ and $\bar\mu$ depending on the orientation. They are, respectively, $k \times N$ and $N \times k$ matrices (with $N = 8$ in our specific case). Since the orientifold parity (2.10) exchanges the two orientations, in the Type I′ theory μ and $\bar\mu$ are not independent of each other but are related according to $\bar\mu = -\,{}^{\mathrm{t}}\mu$.

For all the physical moduli $\mathcal{M}_{(k)}$ listed above, it is possible to write vertex operators of conformal dimension 1 and use them to obtain the moduli action $\mathcal{S}(\mathcal{M}_{(k)}) \equiv \mathcal{S}$ by

computing disk amplitudes[6] along the lines discussed in refs. [22, 23, 36]. As a result one finds [37]

$$\mathcal{S} = \mathcal{S}_{\text{cubic}} + \mathcal{S}_{\text{quartic}} + \mathcal{S}_{\text{mixed}} \tag{3.12}$$

where[7]

$$\mathcal{S}_{\text{cubic}} = \frac{1}{g_0^2}\,\text{tr}\left\{\text{i}\,\lambda_{\dot\alpha}\gamma_\mu^{\dot\alpha\beta}[a^\mu, M_\beta] - \frac{\text{i}}{\sqrt{2}}\,\lambda_{\dot\alpha}[\chi, \lambda^{\dot\alpha}] - \frac{\text{i}}{\sqrt{2}}\,M^\alpha[\bar\chi, M_\alpha]\right\}, \tag{3.13a}$$

$$\mathcal{S}_{\text{quartic}} = \frac{1}{g_0^2}\,\text{tr}\left\{-\frac{1}{4}\,[a_\mu, a_\nu]^2 - [a_\mu, \bar\chi]\,[a^\mu, \chi] + \frac{1}{2}\,[\bar\chi, \chi]^2\right\}, \tag{3.13b}$$

$$\mathcal{S}_{\text{mixed}} = \frac{1}{g_0^2}\text{tr}\left\{-\text{i}\sqrt{2}\,{}^{\text{t}}\mu\,\chi\,\mu\right\}, \tag{3.13c}$$

with g_0 being the Yang-Mills coupling constant in zero dimensions:

$$g_0^2 = \frac{g_s}{4\pi^3\alpha'^2}\ . \tag{3.14}$$

Indeed, the total action (3.12) can also be derived by dimensionally reducing the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with fundamental matter from ten to zero dimensions.

The quartic interactions $[a_\mu, a_\nu]^2$ appearing in (3.13b) can be disentangled by introducing seven auxiliary fields D_m $(m = 1, \ldots, 7)$ and replacing $\mathcal{S}_{\text{quartic}}$ with

$$\mathcal{S}'_{\text{quartic}} = \frac{1}{g_0^2}\,\text{tr}\left\{\frac{1}{2}D_m D^m - \frac{1}{2}D_m(\tau^m)_{\mu\nu}\,[a^\mu, a^\nu] - [a_\mu, \bar\chi]\,[a^\mu, \chi] + \frac{1}{2}\,[\bar\chi, \chi]^2\right\}\ . \tag{3.15}$$

Here $(\tau^m)_{\mu\nu}$ are the γ-matrices of SO(7) (related to the octonionic structure constants as shown in appendix A) implying that the eight-dimensional indices $\mu, \nu, \ldots$ are interpreted as spinorial indices of SO(7). The resulting moduli action is similar in structure to the one considered in ref. [17] in the analysis of the so-called Yang-Mills integrals in $d = 10$. By eliminating D_m through the field equation

$$D_m = \frac{1}{2}(\tau^m)_{\mu\nu}\,[a^\mu, a^\nu]\ , \tag{3.16}$$

and by exploiting the properties of the τ^m matrices, one can easily see that $\mathcal{S}'_{\text{quartic}}$ is equivalent to the initial action (3.13b).

Let us now reorganize the moduli in an "octonionic" form (i.e. in representations of SO(7)) by relabeling some of them as follows:

$$M_\alpha \to M_\mu \equiv (M_m, -M_8)\,, \quad \lambda_{\dot\alpha} \to (\lambda_m, \eta) \equiv (\lambda_m, \lambda_8)\ . \tag{3.17}$$

In other words, the chiral moduli M^α are assembled into a spinor of SO(7), while the anti-chiral moduli $\lambda_{\dot\alpha}$ are split into a vector and a scalar of SO(7). Then, by using the explicit

[6] In general one should include in the moduli action also 1-loop instantonic amplitudes without insertions, as argued in [41, 42], see also [40, 43]. However, in the present case such 1-loop contributions vanish for $N = 8$ due to a cancellation between annuli and Möbius diagrams, as shown in eq. (4.14) of [37].

[7] Here we use slightly different conventions for the μ's as compared to ref. [37].

JHEP07(2009)092

form of the γ^μ matrices given in appendix A, we can rewrite the cubic action (3.13a) as

$$\mathcal{S}'_{\rm cubic} = \frac{1}{g_0^2}\,{\rm tr}\left\{\lambda_m(\tau^m)_{\mu\nu}\,[a^\mu, M^\nu] + \eta\,[a_\mu, M^\mu] - \frac{\rm i}{\sqrt{2}}\,M_\mu\,[\bar\chi, M^\mu]\right. \tag{3.18}$$

$$\left. -\frac{\rm i}{\sqrt{2}}\,\eta\,[\chi,\eta] - \frac{\rm i}{\sqrt{2}}\,\lambda_m\,[\chi,\lambda^m]\right\}. \tag{3.19}$$

It is also convenient to replace the mixed action (3.13c) with

$$\mathcal{S}'_{\rm mixed} = \frac{1}{g_0^2}{\rm tr}\left\{{}^{\rm t}w\,w - {\rm i}\sqrt{2}\,{}^{\rm t}\mu\,\chi\,\mu\right\} \tag{3.20}$$

where w is an auxiliary field in the fundamental representations of SO(k) and SO(8) which does not interact with any other modulus. Even if this auxiliary field looks trivial, it is nevertheless useful to introduce it for reasons which will become clear in a moment.[8]

The total action

$$\mathcal{S}' = \mathcal{S}'_{\rm cubic} + \mathcal{S}'_{\rm quartic} + \mathcal{S}'_{\rm mixed} \tag{3.21}$$

is invariant under transformations of the D-instanton group SO(k), of the gauge group SO(8) and of the auxiliary group SO(7). It is also invariant under the following fermionic BRST transformations

$$\begin{aligned}
Qa^\mu &= M^\mu\,, & QM^\mu &= {\rm i}\sqrt{2}\,[\chi, a^\mu]\,,\\
Q\lambda_m &= D_m\,, & QD_m &= {\rm i}\sqrt{2}\,[\chi,\lambda_m]\,,\\
Q\bar\chi &= -{\rm i}\sqrt{2}\eta\,, & Q\eta &= -\,[\chi,\bar\chi]\,, & Q\chi &= 0\,,\\
Q\mu &= w\,, & Qw &= {\rm i}\sqrt{2}\,\chi\,\mu\,.
\end{aligned} \tag{3.22}$$

The BRST charge Q is one of the supersymmetries that are preserved both by the D-instantons and by the D7-branes; more precisely, after using (3.17), one can see that Q is the component of the anti-chiral supercharge $Q_{\dot\alpha}$ corresponding to $\dot\alpha = 8$ (see eq. (B.8)). The BRST charge is nilpotent up to an (infinitesimal) SO(k) rotation parameterized by ${\rm i}\sqrt{2}\,\chi$. Indeed, on the moduli transforming either in the symmetric or in the anti-symmetric representation of SO(k), such as a^μ or λ_m respectively, we have[9]

$$Q^2\,\bullet = {\rm i}\sqrt{2}\,[\chi, \bullet]\,, \tag{3.23}$$

while on the moduli transforming in the fundamental representation of SO(k), like μ or w, we have

$$Q^2\,\bullet = {\rm i}\sqrt{2}\,\chi\,\bullet\,. \tag{3.24}$$

[8] We remark that just like the physical moduli, also the auxiliary fields, including w, can be given an explicit string description in terms of vertex operators with conformal dimension 1, see appendix B and refs. [23, 36] for details.

[9] Independently of its symmetry properties, any $k\times k$ matrix M^{IJ} transforms under an SO(k) rotation R as $M^{IJ} \to R^I{}_K R^J{}_L M^{KL} = R^I{}_K M^{KL} ({}^tR)_L{}^J = (RMR^{-1})^{IJ}$. If $R = \exp(A)$, with A an antisymmetric matrix whose elements parameterize the rotation, to first order we have $\delta M = [A, M]$.

JHEP07(2009)092

The two BRST actions (3.23) and (3.24) can be combined into a single formula by writing

$$Q^2 \bullet = T_{\mathrm{SO}(k)}\big(\mathrm{i}\sqrt{2}\,\chi\big)\bullet \tag{3.25}$$

where $T_{\mathrm{SO}(k)}\big(\mathrm{i}\sqrt{2}\chi\big)$ denotes an infinitesimal rotation of $\mathrm{SO}(k)$, parameterized by $\mathrm{i}\sqrt{2}\chi$, in the appropriate representation of the modulus on which it acts.

By exploiting the above properties and using the gauge-invariance under $\mathrm{SO}(k)$, one can easily show that the total moduli action (3.21) is Q-exact; indeed

$$\mathcal{S}' = Q\,\Xi \tag{3.26}$$

with the "gauge fermion" given by

$$\Xi = \frac{1}{g_0^2}\,\mathrm{tr}\,\Big\{\frac{1}{2}D_m\lambda^m - \frac{1}{2}\lambda_m(\tau^m)_{\mu\nu}\,[a^\mu, a^\nu] + \frac{\mathrm{i}}{\sqrt{2}}\bar\chi\,[a_\mu, M^\mu] - \frac{1}{2}\eta\,[\chi,\bar\chi] + {}^{\mathrm{t}}\mu\, w\Big\}\ . \tag{3.27}$$

This property will play a crucial rôle in discussing the localization of the integral on the instanton moduli space, as we will see in section 5.

Let us now discuss the interactions among the instanton moduli and the gauge fields propagating on the world-volume of the D7-branes, which we have combined into the superfield (3.2). Such interactions can be easily obtained by computing mixed disk amplitudes involving both vertex operators for moduli and vertex operators for dynamical fields, as discussed in detail in refs. [23, 36] for the analogous D(–1)/D3 systems. In the present case the result is

$$\frac{1}{g_0^2}\mathrm{tr}\,\Big\{\mathrm{i}\sqrt{2}\,{}^{\mathrm{t}}\mu\,\mu\,\Phi(x,\theta)\Big\} \tag{3.28}$$

which has to be added to the moduli action (3.21). For our later purposes it is enough to focus on the dependence on the vacuum expectation value

$$\phi = \langle\Phi(x,\theta)\rangle\,, \tag{3.29}$$

and hence we will consider the following modified mixed action

$$\mathcal{S}'_{\mathrm{mixed}}(\phi) = \mathcal{S}'_{\mathrm{mixed}} + \frac{1}{g_0^2}\mathrm{tr}\,\Big\{\mathrm{i}\sqrt{2}\,{}^{\mathrm{t}}\mu\,\mu\,\phi\Big\}\ . \tag{3.30}$$

Then the total moduli action becomes

$$\mathcal{S}'(\phi) = \mathcal{S}'_{\mathrm{cubic}} + \mathcal{S}'_{\mathrm{quartic}} + \mathcal{S}'_{\mathrm{mixed}}(\phi)\ . \tag{3.31}$$

It is not difficult to realize that the above ϕ-dependent terms can be obtained by deforming the action of the BRST charge Q on the auxiliary field w and replacing the last equation of (3.22) by

$$Qw = \mathrm{i}\sqrt{2}\,\chi\,\mu - \mathrm{i}\sqrt{2}\,\mu\,\phi \tag{3.32}$$

with all the rest, including the gauge fermion (3.27), unchanged. Notice that with the deformation (3.32) the BRST charge becomes nilpotent not only up to infinitesimal rotations

JHEP07(2009)092

	SO(k)	SO(8)	SO(7)	dimensions
a^μ	symm	**1**	$\mathbf{8}_s$	$(\text{length})^{-1}$
M^μ	symm	**1**	$\mathbf{8}_s$	$(\text{length})^{-3/2}$
D_m	adj	**1**	**7**	$(\text{length})^{-2}$
λ_m	adj	**1**	**7**	$(\text{length})^{-3/2}$
$\bar{\chi}$	adj	**1**	**1**	$(\text{length})^{-1}$
η	adj	**1**	**1**	$(\text{length})^{-3/2}$
χ	adj	**1**	**1**	$(\text{length})^{-1}$
μ	**k**	$\mathbf{8}_v$	**1**	$(\text{length})^{-3/2}$
w	**k**	$\mathbf{8}_v$	**1**	$(\text{length})^{-2}$

Table 1. Transformation properties and scaling dimensions of the moduli in the D(–1)/D7 system.

of SO(k), but also up to infinitesimal rotations of the gauge group SO(8), parameterized respectively by $\mathrm{i}\sqrt{2}\,\chi$ and $-\mathrm{i}\sqrt{2}\,\phi$. Thus, eq. (3.25) gets replaced by

$$Q^2 \bullet = T_{\mathrm{SO}(k)}\big(\mathrm{i}\sqrt{2}\,\chi\big)\,\bullet - T_{\mathrm{SO}(8)}\big(\mathrm{i}\sqrt{2}\,\phi\big)\,\bullet \ . \tag{3.33}$$

Clearly, $T_{\mathrm{SO}(8)}\big(\mathrm{i}\sqrt{2}\,\phi\big)$ is non-trivial only on μ and w, which are the only charged moduli transforming under the gauge group SO(8). Finally, using (3.33) one can easily show that

$$\mathcal{S}'_{\mathrm{mixed}}(\phi) = \frac{1}{g_0^2}\,\mathrm{tr}\left\{{}^{\mathrm{t}}w\,w + {}^{\mathrm{t}}\mu\,Q^2\,\mu\right\}, \tag{3.34}$$

where Q^2 is represented as an $(8k\times 8k)$ matrix acting in the tensor product of the vector representations of SO(k) and SO(8) that are the representations under which the μ's transform.

We conclude our description of the D7/D(–1) system of Type I′ by summarizing in table 1 the transformation properties of the various moduli under SO(k), SO(8) and SO(7), as well as their scaling dimensions.

4 Deformation by a RR background

In the previous section we have exhibited the BRST structure of the moduli action for the D(–1)/D7 system of Type I′ and found a BRST charge Q that is nilpotent on quantities invariant under the D-instanton group SO(k) and the D7 group SO(8), as shown in (3.33). However, since the moduli action is also invariant under the auxiliary group SO(7), it is natural to consider an SO(7)-equivariant cohomology [17], using a deformed BRST charge that squares to zero up to an infinitesimal SO(7) transformation as well. Such a deformation is the analogue of the ϵ-deformation introduced in ref. [18] to derive the non-perturbative contributions to the prepotential of $\mathcal{N} = 2$ super Yang-Mills theories in four dimensions

JHEP07(2009)092

using localization techniques for the integral over the instanton moduli space (see for example refs. [31] [35]). As shown in ref. [36], the ϵ-deformation has a natural interpretation in the string realization of the instanton calculus since it can be obtained from the interactions of the open strings of the D(–1)/D3 system with the 3-form field strength of the Ramond-Ramond (RR) closed string sector representing a constant (self-dual) graviphoton background. In this section we are going to show that also the SO(7) deformation can be obtained by turning on a constant RR background in the D(–1)/D7 system.

To this aim, let us consider a RR 3-form field strength of type $F_{\mu\nu z}$, i.e. with two indices along the 8-dimensional world-volume of the D7 branes and one holomorphic index in the internal torus T_2. It is not difficult to realize that such a field strength survives the orientifold projection (2.10), since $F_{\mu\nu z}$ is even under the world-sheet parity ω (like any other RR 3-form field strength), odd under $(-1)^{F_L}$ (like any field of the RR sector) and odd under the inversion $\mathcal{I}_2$ (like any field with one index in the internal torus). From now on we denote $F_{\mu\nu z}$ simply as $\mathcal{F}_{\mu\nu}$ and choose it to describe a rotation of SO(7) in the spinor representation[10] namely we take

$$\mathcal{F}_{\mu\nu} = \frac{1}{2}\, f_{mn} (\tau^{mn})_{\mu\nu} \tag{4.1}$$

where $\tau^{mn} = \frac{1}{2}\,[\tau^m, \tau^n]$ and f_{mn} are the twenty-one parameters specifying the SO(7) rotation.

The effects on the moduli action of this RR background can be derived by computing mixed open/closed string amplitudes on disks with insertions of the moduli vertex operators on the boundary, and of the vertex operators representing $\mathcal{F}$ in the interior. A few details are given in appendix B for completeness, but we refer to ref. [36, 44] for a systematic analysis and a thorough discussion of this method. In the present case the result of the evaluation of such mixed amplitudes leads to new couplings in the moduli action which can be accounted by replacing the cubic and quartic terms, given in (3.19) and (3.15), as follows

$$\mathcal{S}'_{\rm cubic} \to \mathcal{S}'_{\rm cubic}(\mathcal{F}) = \mathcal{S}'_{\rm cubic} + \frac{1}{g_0^2}\,{\rm tr}\left\{-\frac{1}{2}\, f^{mn}\,\lambda_m\lambda_n\right\}\,, \tag{4.2}$$

$$\mathcal{S}'_{\rm quartic} \to \mathcal{S}'_{\rm quartic}(\mathcal{F}) = \mathcal{S}'_{\rm quartic} + \frac{1}{g_0^2}\,{\rm tr}\left\{\frac{\rm i}{2\sqrt{2}}\,[a_\mu, \bar\chi]\,\mathcal{F}^{\mu\nu} a_\nu\right\}\,. \tag{4.3}$$

Thus, when the RR background (4.1) is turned on, the moduli action becomes

$$\mathcal{S}'(\mathcal{F},\phi) = \mathcal{S}'_{\rm cubic}(\mathcal{F}) + \mathcal{S}'_{\rm quartic}(\mathcal{F}) + \mathcal{S}'_{\rm mixed}(\phi) \tag{4.4}$$

with the last term given in (3.34). This new action is still BRST exact, but with respect to a modified BRST charge Q'. Indeed, taking

$$\begin{aligned}
Q' a^\mu &= M^\mu\,, & Q' M^\mu &= {\rm i}\sqrt{2}\,[\chi, a^\mu] - \frac{1}{2}\,\mathcal{F}^{\mu\nu}\,a_\nu\,,\\
Q'\lambda_m &= D_m\,, & Q' D_m &= {\rm i}\sqrt{2}\,[\chi,\lambda_m] + f_{mn}\,\lambda^n\,,\\
Q'\bar\chi &= -{\rm i}\sqrt{2}\eta\,, & Q'\eta &= -\,[\chi,\bar\chi]\,, & Q'\chi &= 0\,,\\
Q'\mu &= w\,, & Q' w &= {\rm i}\sqrt{2}\,\chi\,\mu - {\rm i}\sqrt{2}\,\mu\,\phi\,,
\end{aligned} \tag{4.5}$$

[10] In the notation of appendix A this means that we only turn on the components $\mathcal{F}^{\mathbf{21}}_{\mu\nu}$, see eq. (A.30).

one can check that

$$\mathcal{S}'(\mathcal{F},\phi) = Q'\,\Xi \tag{4.6}$$

where the gauge fermion Ξ is the one defined in (3.27). The deformed BRST charge Q' is nilpotent up to (infinitesimal) transformations of all the symmetry groups of the system, including the rotations of SO(7) under which the moduli carrying indices of type $m, n, \ldots$ (like λ_m) transform in the vector representation and the moduli carrying indices of type $\mu, \nu, \ldots$ (like a^μ) transform in the spinor representation. Indeed, from (4.5) one can easily show that

$$Q'^2\,\bullet = T_{\mathrm{SO}(k)}\big(\mathrm{i}\sqrt{2}\,\chi\big)\,\bullet - T_{\mathrm{SO}(8)}\big(\mathrm{i}\sqrt{2}\,\phi\big)\,\bullet + T_{\mathrm{SO}(7)}\big(\mathcal{F}\big)\,\bullet\ . \tag{4.7}$$

As discussed in ref. [17], in view of the explicit evaluation of the integral over the instanton moduli space using localization methods, it is useful to further deform the above action. Proceeding in strict analogy with ref. [36], we turn on also the component of the RR 3-form field-strength with an anti-holomorphic index, i.e. $F_{\mu\nu\bar{z}} \equiv \bar{\mathcal{F}}_{\mu\nu}$, and then compute mixed disk amplitudes with $\bar{\mathcal{F}}$ insertions to obtain the couplings with the instanton moduli. Choosing

$$\bar{\mathcal{F}}_{\mu\nu} = \frac{1}{2}\,\bar{f}_{mn}(\tau^{mn})_{\mu\nu} \tag{4.8}$$

one finds the following new terms

$$\frac{1}{g_0^2}\,\mathrm{tr}\left\{\frac{\mathrm{i}}{2\sqrt{2}}\,[a_\mu,\chi]\,\bar{\mathcal{F}}^{\mu\nu}a_\nu + \frac{1}{8}\,\bar{\mathcal{F}}^{\mu\nu}a_\nu\,\mathcal{F}_{\mu\rho}a^\rho + \frac{1}{4}\bar{\mathcal{F}}_{\mu\nu}M^\mu M^\nu\right\} \tag{4.9}$$

which have to be added to the moduli action (4.4). Notice that the anti-holomorphic RR background $\bar{\mathcal{F}}$ produces quadratic "mass" terms for the moduli a^μ and its fermionic partners M^μ.

Another class of deformations which we will use in the following is obtained by adding to $\mathcal{F}$ a vector component (see eq. (A.30)), namely by taking the holomorphic RR polarization tensor to be given by

$$\mathcal{F}_{\mu\nu} = \frac{1}{2}f_{mn}(\tau^{mn})_{\mu\nu} + h_m(\tau^m)_{\mu\nu}\ . \tag{4.10}$$

In this way one gets the following new couplings in the moduli action

$$\frac{1}{g_0^2}\,\mathrm{tr}\left\{h^m\lambda_m\,\eta + \frac{\mathrm{i}}{\sqrt{2}}\,h^m D_m\bar{\chi}\right\}\ . \tag{4.11}$$

It is important to observe that both the $\bar{\mathcal{F}}$ terms (4.9) and the h terms (4.11) can be incorporated in the BRST structure of the moduli action by deforming the gauge fermion Ξ and replacing it according to

$$\Xi \ \rightarrow\ \ \Xi' = \Xi - \frac{1}{g_0^2}\,\mathrm{tr}\left\{\frac{\mathrm{i}}{\sqrt{2}}\,h^m\lambda_m\bar{\chi} + \frac{1}{4}\,\bar{\mathcal{F}}^{\mu\nu}a_\nu M_\mu\right\}\ . \tag{4.12}$$

Then, the full instanton moduli action in the presence of a RR background given by (4.10) and (4.8) and of a vacuum expectation value ϕ for the adjoint scalar of the gauge multiplet, is given by

$$\mathcal{S}'(\mathcal{F},\bar{\mathcal{F}},\phi) = Q'\,\Xi'\ . \tag{4.13}$$

JHEP07(2009)092

We will take advantage of the BRST exactness of the moduli action in the following section when we will discuss the integral over the instanton moduli space.

5 Rescalings and localization

Our next goal is to compute the instanton partition function for the D(–1)/D7 system using the deformed moduli action derived in the previous section, in order to extract from it the non-perturbative contributions to the effective action of the SO(8) gauge theory. To do so, it is convenient to first introduce ADHM-like variables by means of the following replacements

$$a^\mu \to a'^\mu = \frac{a^\mu}{g_0}\,, \quad M^\mu \to M'^\mu = \frac{M^\mu}{g_0}\,, \quad \mu \to \mu' = \frac{\mu}{g_0}\,, \quad w \to w' = \frac{w}{g_0}\,, \tag{5.1}$$

in such a way that a'^μ has dimension of (length), M'^μ and μ' have dimensions of $(\text{length})^{1/2}$, and w' is dimensionless. Then we define the partition function at instanton number k as the following integral:[11]

$$Z_k = \mathcal{N}_k \int \{da'^\mu\, dM'^\mu\, dD_m\, d\lambda_m\, d\bar\chi\, d\eta\, d\chi\, d\mu'\, dw'\}\ \mathrm{e}^{-\mathcal{S}'(\mathcal{F},\bar{\mathcal{F}},\phi)} \tag{5.2}$$

where $\mathcal{N}_k$ is a suitable (dimensionless) normalization factor, and $\mathcal{S}'(\mathcal{F},\bar{\mathcal{F}},\phi)$ is the moduli action obtained from eq. (4.13) upon using the rescalings (5.1).

The charged moduli w' and μ' appear only quadratically in the action $S'(\mathcal{F},\bar{\mathcal{F}},\phi)$ (see eq. (3.34)) and can be easily integrated, yielding[12]

$$\int \{d\mu'\, dw'\}\ \mathrm{e}^{-\mathrm{tr}\left({}^{\mathrm{t}}w\, w + {}^{\mathrm{t}}\mu' Q'^2 \mu'\right)} \sim \mathrm{Pf}_{(\mathbf{k},\mathbf{8}_v,\mathbf{1})}\big(Q'^2\big) \tag{5.3}$$

where the labels on the Pfaffian specify the representations on which Q'^2 acts. For $k = 1$ no χ's are present and the integral over w' and μ' produces just $\mathrm{Pf}_{(\mathbf{1},\mathbf{8}_v,\mathbf{1})}\big(Q'^2\big) \sim \mathrm{Pf}\,\big(\phi\big)$.

Absorbing all numerical factors into the overall normalization, we can rewrite the partition function (5.2) as

$$Z_k = \mathcal{N}_k \int \big\{da'^\mu\, dM'^\mu\, dD_m\, d\lambda_m\, d\bar\chi\, d\eta\, d\chi\big\}\ \mathrm{e}^{-\mathcal{S}'(\mathcal{F},\bar{\mathcal{F}})}\ \mathrm{Pf}_{(\mathbf{k},\mathbf{8}_v,\mathbf{1})}\big(Q'^2\big) \tag{5.4}$$

where

$$\begin{aligned}\mathcal{S}'(\mathcal{F},\bar{\mathcal{F}}) = \mathrm{tr}\Big\{ & \lambda_m(\tau^m)_{\mu\nu}\left[a'^\mu, M'^\nu\right] + \eta\left[a'_\mu, M'^\mu\right] - \frac{\mathrm{i}}{\sqrt{2}}\, M'_\mu\left[\bar\chi, M'^\mu\right] \\ & + \frac{1}{2} D_m(\tau^m)_{\mu\nu}\left[a'^\mu, a'^\nu\right] - \left[a'_\mu, \bar\chi\right]\left[a'^\mu, \chi\right] \\ & + \frac{\mathrm{i}}{2\sqrt{2}}\left[a'_\mu, \bar\chi\right]\mathcal{F}^{\mu\nu} a'_\nu - \frac{\mathrm{i}}{\sqrt{2} g_0^2}\,\eta\,[\chi,\eta] + \frac{1}{2g_0^2}\,[\bar\chi,\chi]^2\end{aligned} \tag{5.5}$$

[11] Here, for simplicity, we do not include the exponential of (minus) the classical instanton action, $\mathrm{e}^{2\pi\mathrm{i}\tau k}$; we will restore these factors later on.

[12] Notice that on the μ''s the action Q and Q' coincide.

$$
\begin{aligned}
&+\frac{1}{2g_0^2}\, D_m D^m - \frac{1}{2g_0^2}\,\lambda_m \left(\mathrm{i}\sqrt{2}\,[\chi,\lambda^m] + f^{mn}\lambda_n\right) \\
&+\frac{1}{4}\, a'_\mu\, \bar{\mathcal{F}}^{\mu\nu}\left(\mathrm{i}\sqrt{2}\,[\chi,a'_\nu] - \frac{1}{2}\mathcal{F}_{\nu\rho}\, a'^{\rho}\right) + \frac{1}{4}\bar{\mathcal{F}}_{\mu\nu} M'^{\mu} M'^{\nu} \\
&+\frac{1}{g_0^2}\, h^m \left(\lambda_m\,\eta + \frac{\mathrm{i}}{\sqrt{2}}\, D_m\bar{\chi}\right)\Bigg\}\ .
\end{aligned}
$$

As customary in this type of manipulations [18], we treat the variables χ and $\bar{\chi}$ as independent of each other and, in particular, according to our conventions, we take them to be purely imaginary and real respectively. Then, we evaluate the integral (5.4) in the semi-classical approximation, which due to the BRST structure of the instanton action turns out to be exact. To proceed it is convenient to perform the following change of integration variables

$$
\begin{aligned}
a'^{\mu} &\to \frac{1}{x}\, a'^{\mu}, \qquad & M'^{\mu} &\to \frac{1}{x}\, M'^{\mu}\,, \\
D_m &\to x^2\, D_m\ , \qquad & \lambda_m &\to x^2\,\lambda_m\,, \\
\bar{\chi} &\to y\,\bar{\chi}\ , \qquad & \eta &\to y\,\eta\,,
\end{aligned}
\tag{5.6}
$$

and rescale the anti-holomorphic background as

$$
\bar{\mathcal{F}}_{\mu\nu} \to z\,\bar{\mathcal{F}}_{\mu\nu}\ . \tag{5.7}
$$

The partition function Z_k does not depend on the arbitrary parameters x, y and z, because x and y appear only through a change of integration variables which leaves invariant the measure in (5.4), while z appears through a change of the anti-holomorphic background which only appears inside the gauge fermion Ξ' as shown in (4.12). Thus, we can choose these parameters to simplify as much as possible the structure of Z_k. In particular, if we take the limit

$$
x \to \infty \quad , \quad y \to 0 \quad , \quad z \to \infty \tag{5.8}
$$

with

$$
x^2 y \to \infty \quad , \quad \frac{z}{x^2} \to \infty\,, \tag{5.9}
$$

the moduli action (5.6) reduces to

$$
\begin{aligned}
\mathcal{S}'(\mathcal{F},\bar{\mathcal{F}}) = \mathrm{tr}\Bigg\{ &\frac{g}{2}\, D_m D^m - \frac{g}{2}\,\lambda_m \left(\mathrm{i}\,\sqrt{2}\,[\chi,\lambda^m] + f^{mn}\,\lambda_n\right) \\
&+\frac{t}{4}\, a'_\mu\, \bar{\mathcal{F}}^{\mu\nu}\left(\mathrm{i}\,\sqrt{2}\,[\chi,a'_\nu] - \frac{1}{2}\mathcal{F}_{\nu\rho}\, a'^{\rho}\right) + \frac{t}{4}\, M'_\mu\, \bar{\mathcal{F}}^{\mu\nu} M'_\nu \\
&+s\, h^m \left(\lambda_m\,\eta + \frac{\mathrm{i}}{\sqrt{2}}\, D_m\bar{\chi}\right)\Bigg\} + \dots\ .
\end{aligned}
\tag{5.10}
$$

Here we have introduced the coupling constants

$$
g = \frac{x^4}{g_0^2}\,, \qquad t = \frac{z}{x^2}\,, \qquad s = \frac{x^2 y}{g_0^2}\,, \tag{5.11}
$$

JHEP07(2009)092

which all tend to ∞ because of eq. (5.9), and have denoted with ... the terms of the first three lines of eq. (5.6) which are subleading in this limit. The integrals over a'^{μ}, M'^{μ}, D_m, λ_m, $\bar{\chi}$ and η can now be easily performed since they are all Gaussian.

To evaluate these integrals we choose the deformation parameters f_{mn} and h_m as in ref. [17], namely to take the matrix f along the Cartan directions $H^a_{\mathrm{SO}(7)}$ of SO(7), i.e.

$$f = \vec{f}\cdot\vec{H}_{\mathrm{SO}(7)} = \sum_{a=1}^{3} f_a\, H^a_{\mathrm{SO}(7)} = \begin{pmatrix} \mathrm{i}f_1\sigma_2 & 0 & 0 & 0 \\ 0 & \mathrm{i}f_2\sigma_2 & 0 & 0 \\ 0 & 0 & \mathrm{i}f_3\sigma_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{with } \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{5.12}$$

and the vector h with only h_7 non-vanishing.[13] When these parameters are inserted in (5.10), the fermion λ_7 lacks an explicit "mass term" from the $(f\lambda\lambda)$ coupling but it becomes effectively "massive" thanks to the $(\lambda_7\eta)$ term proportional to h_7 and thus can be integrated without problems. Actually, it is easy to integrate out the entire quartet formed by D_7, λ_7, $\bar{\chi}$ and η and realize that it yields just a numerical constant independent of g, s, h_7 and χ. Indeed, even if these quantities do appear in the interactions among the quartet components, they can be scaled away by a change of integration variables that leaves the integration measure invariant.

Once the quartet has been integrated, we can safely set $h_7 = 0$. Thus, the deformation matrix (4.10) becomes

$$\mathcal{F} = -2 \begin{pmatrix} \mathrm{i}E_1\sigma_2 & 0 & 0 & 0 \\ 0 & \mathrm{i}E_2\sigma_2 & 0 & 0 \\ 0 & 0 & \mathrm{i}E_3\sigma_2 & 0 \\ 0 & 0 & 0 & \mathrm{i}E_4\sigma_2 \end{pmatrix} \tag{5.13}$$

with

$$\begin{aligned} E_1 &= \frac{1}{2}(f_1 - f_2 - f_3) \quad , \quad E_2 = \frac{1}{2}(f_2 - f_3 - f_1)\,, \\ E_3 &= \frac{1}{2}(f_3 - f_1 - f_2) \quad , \quad E_4 = \frac{1}{2}(f_1 + f_2 + f_3)\,, \end{aligned} \tag{5.14}$$

such that

$$E_1 + E_2 + E_3 + E_4 = 0\ . \tag{5.15}$$

At this point, we are left with the integral over a'^{μ}, M'^{μ}, the six "massive" fermions $\lambda_1, \ldots, \lambda_6$ (which we will label with an index $\hat{m} = 1, \ldots, 6$) and the corresponding six auxiliary bosons $D_{\hat{m}}$, plus of course the integral over χ. From eq. (5.10), we see that the relevant action for these fields is extremely simple and given by

$$\mathrm{tr}\left\{\frac{g}{2}\, D_{\hat{m}} D^{\hat{m}} - \frac{g}{2}\,\lambda_{\hat{m}}\left(Q'^2\lambda\right)^{\hat{m}} + \frac{t}{4}\, a'_{\mu}\,\bar{\mathcal{F}}^{\mu\nu}\left(Q'^2 a'\right)_{\nu} + \frac{t}{4}\, M'_{\mu}\,\bar{\mathcal{F}}^{\mu\nu} M'_{\nu}\right\} \tag{5.16}$$

[13] Even if this is not the most general configuration, it is the most convenient one for the following computations.

with the deformed BRST charge acting as in eq. (4.7). The integral we have to compute is then

$$I = \int \left\{ da'^{\mu} dM'^{\mu} dD_{\hat{m}} d\lambda_{\hat{m}} \right\} \mathrm{e}^{-\mathrm{tr}\left\{ \frac{g}{2} D_{\hat{m}} D^{\hat{m}} - \frac{g}{2} \lambda_{\hat{m}} (Q'^2 \lambda)^{\hat{m}} + \frac{t}{4} a'_{\mu} \bar{\mathcal{F}}^{\mu\nu} (Q'^2 a')_{\nu} + \frac{t}{4} M'_{\mu} \bar{\mathcal{F}}^{\mu\nu} M'_{\nu} \right\}}$$
$$\sim \frac{\mathrm{Pf}_{(\mathrm{adj},\mathbf{1},\mathbf{6}\subset\mathbf{7})}(g\, Q'^2)\ \ \mathrm{Pf}_{(\mathrm{symm},\mathbf{1},\mathbf{8}_s)}(\tfrac{t}{2}\, \bar{\mathcal{F}})}{\det^{1/2}_{(\mathrm{adj},\mathbf{1},\mathbf{6}\subset\mathbf{7})}(g)\ \det^{1/2}_{(\mathrm{symm},\mathbf{1},\mathbf{8}_s)}(\tfrac{t}{2}\, \bar{\mathcal{F}} Q'^2)} \tag{5.17}$$

up to numerical coefficients. The origin of the various terms in the above expression is clear: $\mathrm{Pf}_{(\mathrm{adj},\mathbf{1},\mathbf{6}\subset\mathbf{7})}(g\, Q'^2)$ comes from the integration of the six fermions $\lambda_{\hat{m}}$ which transform in the adjoint representation of SO(k), are singlets of SO(8) and form a 6-vector inside the **7** of SO(7), as indicated by the labels on the Pfaffian symbol. Similarly, $\mathrm{Pf}_{(\mathrm{symm},\mathbf{1},\mathbf{8}_s)}(\frac{t}{2}\, \bar{\mathcal{F}})$ comes from the integration of the fermions M'^{μ}; $\det^{1/2}_{(\mathrm{adj},\mathbf{1},\mathbf{6}\subset\mathbf{7})}(g)$ comes from the integration of the six bosons $D_{\hat{m}}$ and finally $\det^{1/2}_{(\mathrm{symm},\mathbf{1},\mathbf{8}_s)}(\frac{t}{2}\, \bar{\mathcal{F}} Q'^2)$ comes from the integration of the bosons a'^{μ}. Exploiting the properties of the Pfaffians, we can simplify eq. (5.17) and get

$$I \sim \frac{\mathrm{Pf}_{(\mathrm{adj},\mathbf{1},\mathbf{6}\subset\mathbf{7})}(Q'^2)}{\det^{1/2}_{(\mathrm{symm},\mathbf{1},\mathbf{8}_s)}(Q'^2)}\ . \tag{5.18}$$

As expected, all dependence on g, t and the anti-holomorphic background $\bar{\mathcal{F}}$ has dropped out from the final result, which instead depends on the holomorphic background $\mathcal{F}$ given in (5.13) and on χ (the last instanton moduli to be integrated) through the action of the deformed BRST charge.

Combining everything and absorbing all numerical factors in the overall normalization coefficient, we finally obtain

$$Z_k = \mathcal{N}_k \int \{d\chi\}\ \frac{\mathrm{Pf}_{(\mathrm{adj},\mathbf{1},\mathbf{6}\subset\mathbf{7})}(Q'^2)\ \mathrm{Pf}_{(\mathbf{k},\mathbf{8}_v,\mathbf{1})}(Q'^2)}{\det^{1/2}_{(\mathrm{symm},\mathbf{1},\mathbf{8}_s)}(Q'^2)}\ . \tag{5.19}$$

As suggested by eq. (4.7), it is convenient to redefine $\mathrm{i}\sqrt{2}\chi \to \chi$ and $\mathrm{i}\sqrt{2}\phi \to \phi$, so that the new χ variable becomes real and

$$Q'^2 \bullet = T_{\mathrm{SO}(k)}(\chi) \bullet - T_{\mathrm{SO}(8)}(\phi) \bullet + T_{\mathrm{SO}(7)}(\mathcal{F}) \bullet\ . \tag{5.20}$$

Furthermore, for ease of notation we set

$$\mathcal{P}(\chi) \equiv \mathrm{Pf}_{(\mathrm{adj},\mathbf{1},\mathbf{6}\subset\mathbf{7})}(Q'^2)\,,\ \ \mathcal{R}(\chi) \equiv \mathrm{Pf}_{(\mathbf{k},\mathbf{8}_v,\mathbf{1})}(Q'^2)\,,\ \ \mathcal{Q}(\chi) \equiv \det{}^{1/2}_{(\mathrm{symm},\mathbf{1},\mathbf{8}_s)}(Q'^2) \tag{5.21}$$

and, after a suitable redefinition of the overall normalization, we rewrite the partition function as follows

$$Z_k = \mathcal{N}_k \int \left\{ \frac{d\chi}{2\pi\mathrm{i}} \right\} \frac{\mathcal{P}(\chi)\, \mathcal{R}(\chi)}{\mathcal{Q}(\chi)}\ . \tag{5.22}$$

Since the integrand is singular when the denominator $\mathcal{Q}(\chi)$ vanishes and tends to one when $\chi \to \infty$, the integral (5.22) is naively divergent and must be suitably defined to make

JHEP07(2009)092

sense. Here we follow the same prescription of ref. [17], and cure the singularities along the integration path by giving the zeroes of $\mathcal{Q}(\chi)$ a small positive imaginary part moving them in the upper-half complex plane, and regulate the divergence at infinity by interpreting the χ-integral as a contour integral. Even if this prescription as it stands does not seem to be fully justified and lacks a rigorous derivation from first principles, there is clear evidence of its validity in results of ref. [17] and their numerous generalizations discussed for example in refs. [45][49], as well as in the agreement with numerical analysis based on Monte-Carlo methods [50].

Using this prescription, the instanton partition function (5.22) will then be expressed as a finite sum of residues evaluated at the poles of the integrand, showing that the integral over the instanton moduli effectively localizes on the zeroes of $\mathcal{Q}(\chi)$ and thus receives contributions only from those configurations for which the bosonic "kinetic" terms vanish. This is completely similar to the localization of the integrals over the instanton moduli space in $\mathcal{N}=2$ super Yang-Mills theories in four dimensions discussed in refs. [18] and [31][35].

6 Explicit expressions and results for low k

6.1 $k=1$

The 1-instanton partition function Z_1 is particularly simple: in fact, for $k=1$ there are no λ_m's and no χ's, so that the factor $\mathcal{P}(\chi)$ is not generated and no contour integral has to be evaluated. Furthermore, for $k=1$ the factor $\mathcal{R}(\chi)$ reduces just to Pf (ϕ), as already observed after eq. (5.3), while from eq. (5.17) we see that the integration over a'^{μ} and M'^{μ} reduces to

$$\int \left\{da'^{\mu} dM'^{\mu}\right\} \mathrm{e}^{-\left\{\frac{1}{8}a'_{\mu}\bar{\mathcal{F}}^{\mu\nu}\mathcal{F}_{\rho\nu}a'^{\rho}+\frac{1}{4}M'_{\mu}\bar{\mathcal{F}}^{\mu\nu}M'_{\nu}\right\}} \;\sim\; \frac{1}{\det^{1/2}(\mathcal{F})} \;\sim\; \frac{1}{\mathcal{E}} \tag{6.1}$$

where have defined

$$\mathcal{E} \;\equiv\; E_1 E_2 E_3 E_4 \;. \tag{6.2}$$

Thus, for $k=1$ we simply have

$$Z_1 = \mathcal{N}_1 \,\frac{\mathrm{Pf}\,\phi}{\mathcal{E}} \;. \tag{6.3}$$

Notice that the factor $1/\mathcal{E}$ in the above result can be interpreted as the regulated volume of the eight-dimensional $\mathcal{N}=1$ superspace. In fact, for $k=1$ the moduli a' and M' are identified with the superspace coordinates (see eq. (3.11)), so that from (6.1) we can obtain the effective identification[14]

$$\int d^8x\, d^8\theta \;\longleftrightarrow\; \frac{(2\pi)^4}{\mathcal{E}} \;. \tag{6.4}$$

This is the eight-dimensional analogue of the effective rule that appears in the instanton calculus in four dimensions using localization and ϵ-deformation methods [18, 36].

[14] The factors of π's are introduced for later convenience, but it is easy to trace their origin in the Gaussian integration over the eight bosonic moduli a'^{μ}.

JHEP07(2009)092

6.2 $k > 1$

Let us now consider the cases with $k > 1$. To perform the integration over the χ's we can exploit the SO(k) invariance of the integrand in (5.22) and, at the price of introducing a Vandermonde determinant[15] $\Delta(\chi)$, bring the χ's to the Cartan subalgebra, whose generators we denote as $H^i_{\mathrm{SO}(k)}$, i.e.

$$\chi \;\to\; \vec{\chi}\cdot\vec{H}_{\mathrm{SO}(k)} \;=\; \sum_{i=1}^{\mathrm{rank\,SO}(k)} \chi_i\, H^i_{\mathrm{SO}(k)}\ . \tag{6.5}$$

Then the partition function becomes

$$Z_k = \mathcal{N}_k \int \prod_i \left(\frac{d\chi_i}{2\pi \mathrm{i}}\right) \Delta(\vec{\chi})\, \frac{\mathcal{P}(\vec{\chi})\,\mathcal{R}(\vec{\chi})}{\mathcal{Q}(\vec{\chi})}\ . \tag{6.6}$$

Again, we have absorbed all numerical factors produced by the "diagonalization" of χ into a redefinition of the normalization coefficient $\mathcal{N}_k$.

Without any loss of generality we can assume that also the vacuum expectation values of the scalar ϕ belong to the Cartan directions $H^u_{\mathrm{SO}(8)}$ of SO(8) and thus have the following block-diagonal form

$$\phi = \vec{\phi}\cdot\vec{H}_{\mathrm{SO}(8)} = \sum_{u=1}^{4} \phi_u\, H^u_{\mathrm{SO}(8)} = \begin{pmatrix} \mathrm{i}\phi_1\sigma_2 & 0 & 0 & 0 \\ 0 & \mathrm{i}\phi_2\sigma_2 & 0 & 0 \\ 0 & 0 & \mathrm{i}\phi_3\sigma_2 & 0 \\ 0 & 0 & 0 & \mathrm{i}\phi_4\sigma_2 \end{pmatrix}\ . \tag{6.7}$$

With these choices, Q'^2 corresponds to infinitesimal Cartan actions which can be diagonalized in any representation by going to the basis provided by the weights.

Let consider, for instance, the charged moduli μ' which we relabel as

$$\mu'^{I}{}_{U} \;\to\; \mu'^{\vec{\pi}}{}_{\vec{\gamma}} \;\sim |\vec{\pi},\vec{\gamma}\rangle\,, \tag{6.8}$$

where $\vec{\pi}$ belongs to the set of weights of the vector representation $\mathbf{k}$ of SO(k), while $\vec{\gamma}$ is a weight of the vector representation $\mathbf{8}_v$ of SO(8). Then, from (5.20) we have

$$Q'^2\,|\vec{\pi},\vec{\gamma}\rangle = \Big(T_{\mathrm{SO}(k)}(\vec{\chi}) - T_{\mathrm{SO}(8)}(\vec{\phi})\Big)\,|\vec{\pi},\vec{\gamma}\rangle = \big(\vec{\chi}\cdot\vec{\pi} - \vec{\phi}\cdot\vec{\gamma}\big)|\vec{\pi},\vec{\gamma}\rangle\ . \tag{6.9}$$

Notice that the variables $\mu'^{\vec{\pi}}{}_{\vec{\gamma}}$ are in general complex, and their conjugate moduli are[16] $\mu'^{-\vec{\pi}}{}_{-\vec{\gamma}}$; the couples of conjugate moduli are therefore labeled by half of the possible pairs of weights $(\vec{\pi},\vec{\gamma})$. Hence, the complex fermionic integration over the $\mu'^{\vec{\pi}}{}_{\vec{\gamma}}$'s yields

$$\mathcal{R}(\vec{\chi}) = \prod_{\vec{\pi}\in\mathbf{k}} \prod_{\vec{\gamma}\in\mathbf{8}_v}^{(+)} \Big(\vec{\chi}\cdot\vec{\pi} - \vec{\phi}\cdot\vec{\gamma}\Big)\ . \tag{6.10}$$

[15] Notice that this operation is formally acceptable only when χ is real, which is what we have argued at the end of the previous section.

[16] All representations appearing in our expressions are real, namely correspond to weight sets that are closed under parity.

JHEP07(2009)092

Here the product over $\vec{\gamma}$ is limited to half of the weights, that we refer to as the "positive" ones; this is the meaning of the superscript $(+)$ appearing above. The weights of the vector representation $\mathbf{8}_v$ of SO(8) are expressed in terms of the versors $\vec{e}_u$ $(u = 1, \dots 4)$ spanning the weight space as $\pm\vec{e}_u$. Taking $\vec{e}_u$ as the positive ones, we obtain

$$\mathcal{R}(\vec{\chi}) = \prod_{u=1}^{4} \prod_{\vec{\pi}\in\mathbf{k}} \left(\vec{\chi}\cdot\vec{\pi} - \phi_u\right) . \tag{6.11}$$

We can proceed in a similar way for the six moduli $\lambda_{\hat{m}}$, finding

$$\mathcal{P}(\vec{\chi}) = \prod_{\vec{\rho}\in\mathrm{adj}} \prod_{\vec{\alpha}\in\mathbf{6}\subset\mathbf{7}}^{(+)} \left(\vec{\chi}\cdot\vec{\rho} - \vec{f}\cdot\vec{\alpha}\right) = \prod_{a=1}^{3} \prod_{\vec{\rho}\in\mathrm{adj}} \left(\vec{\chi}\cdot\vec{\rho} - f_a\right) . \tag{6.12}$$

Indeed, the positive weights of the $\mathbf{6} \subset \mathbf{7}$ representation of SO(7) correspond simply to the versors $\vec{e}_a$ $(a = 1, 2, 3)$ of the weight space. Finally, considering the moduli a'^{μ}, we get

$$\mathcal{Q}(\vec{\chi}) = \prod_{\vec{\sigma}\in\mathrm{symm}} \prod_{\vec{\beta}\in\mathbf{8}_s}^{(+)} \left(\vec{\chi}\cdot\vec{\sigma} - \vec{f}\cdot\vec{\beta}\right) = \prod_{A=1}^{4} \prod_{\vec{\sigma}\in\mathrm{symm}} \left(\vec{\chi}\cdot\vec{\sigma} - E_A\right) . \tag{6.13}$$

Here E_A $(A = 1, \dots 4)$ denote the scalar products of the background $\vec{f}$ given in (5.12) with the four positive weights of the spinor representation of SO(7), and correspond precisely to the parameters introduced in (5.14). Also the Vandermonde determinant can be expressed in terms of the non-zero weights of the adjoint representation of SO(k):

$$\Delta(\vec{\chi}) = \prod_{\vec{\rho}\in\mathrm{adj}\neq\vec{0}} \vec{\chi}\cdot\vec{\rho} \, . \tag{6.14}$$

All the above expressions become explicit using the weight sets of the various representations provided in appendix C. As an illustration, let us discuss $\mathcal{R}(\vec{\chi})$ given in (6.11). When $k = 2n$, the rank of SO(k) is n. Denoting the versors of the $\mathbb{R}^n$ weight space as $\vec{e}_i$, the weights $\vec{\pi}$ of the vector representation $\mathbf{2n}$ are simply $\vec{\pi} = \pm\vec{e}_i$, so that from (6.11) we get

$$\mathcal{R}(\vec{\chi}) = \prod_{u=1}^{4} \prod_{i=1}^{n} (\chi_i - \phi_u)(-\chi_i - \phi_u) = \prod_{u=1}^{4} \prod_{i=1}^{n} (\phi_u^2 - \chi_i^2) \, . \tag{6.15}$$

For $k = 2n+1$, the rank of SO(k) is again n but now the vector representation contains in addition to the weights $\pm\vec{e}_i$ also a null weight $\vec{0}$. As a consequence, we find an extra factor of Pf ϕ; indeed

$$\mathcal{R}(\vec{\chi}) = \prod_{u=1}^{4} (-\phi_u) \prod_{i=1}^{n} (\chi_i - \phi_u)(-\chi_i - \phi_u) = \mathrm{Pf}\,\phi \prod_{u=1}^{4} \prod_{i=1}^{n} (\phi_u^2 - \chi_i^2) \, . \tag{6.16}$$

Let us notice that also the adjoint and symmetric representations of SO(k) contain null weights, which lead to terms independent of $\vec{\chi}$ in the products (6.12) and (6.13). In

JHEP07(2009)092

particular, the adjoint representation has n null weights both for $k = 2n$ and $k = 2n+1$, leading to

$$\mathcal{P}(\vec{\chi}) = \big(\mathrm{Pf}\, f\big)^n \prod_{a=1}^{3} \prod_{\vec{\rho}\in \mathrm{adj}\neq\vec{0}} (\vec{\chi}\cdot\vec{\rho} - f_a)\ , \tag{6.17}$$

where

$$\mathrm{Pf}\, f = f_1 f_2 f_3\ . \tag{6.18}$$

The symmetric representation, instead, has n null weights when $k = 2n$, and $n+1$ when $k = 2n+1$, so that

$$\mathcal{Q}(\vec{\chi}) = \mathcal{E}^n \prod_{A=1}^{4} \prod_{\vec{\sigma}\in \mathrm{symm}\neq\vec{0}} (\vec{\chi}\cdot\vec{\sigma} - E_A) \quad \text{for } k = 2n \tag{6.19}$$

and

$$\mathcal{Q}(\vec{\chi}) = \mathcal{E}^{n+1} \prod_{A=1}^{4} \prod_{\vec{\sigma}\in \mathrm{symm}\neq\vec{0}} (\vec{\chi}\cdot\vec{\sigma} - E_A) \quad \text{for } k = 2n+1\,, \tag{6.20}$$

where $\mathcal{E}$ is the quantity defined in (6.2).

Using these explicit expressions we can perform the final integrations over the χ's and obtain the instanton partition functions Z_k given in (6.6). As discussed at the end of section 5, the χ-integrals are understood as contour integrals in the upper-half complex plane and the singularities at the zeroes of the polynomial $\mathcal{Q}(\vec{\chi})$ are avoided by giving the deformation parameters E_A a small positive imaginary part, according to the prescriptions of ref. [17]. In particular, we choose

$$\mathrm{Im}\, E_1 > \mathrm{Im}\, E_2 > \mathrm{Im}\, E_3 > \mathrm{Im}\, E_4 > \mathrm{Im}\, \frac{E_1}{2} > \ldots > \mathrm{Im}\, \frac{E_4}{2} > 0\ . \tag{6.21}$$

Let us apply this to the simplest non-trivial case, namely $k = 2$, where we have

$$Z_2 = \mathcal{N}_2\, \frac{\mathrm{Pf}\, f}{\mathcal{E}} \int \frac{d\chi}{2\pi \mathrm{i}}\, \frac{\prod_{u=1}^{4}(\phi_u^2 - \chi^2)}{\prod_{A=1}^{4}(2\chi - E_A)(-2\chi - E_A)}\ . \tag{6.22}$$

The integration prescription described above leads to express Z_2 as a sum over the residues of the integrand at $\chi = E_A/2$:

$$Z_2 = \mathcal{N}_2\, \frac{\mathrm{Pf}\, f}{2\,\mathcal{E}} \sum_{A=1}^{4} \frac{\prod_{u=1}^{4}(\phi_u^2 - (E_A/2)^2)}{\prod_{B\neq A}(E_A - E_B)\prod_B(-E_A - E_B)}\ . \tag{6.23}$$

If we perform the algebra, and use the relations (5.14) between the quantities E_A and the three independent parameters f_a, in the end we get

$$\begin{aligned} Z_2 = \mathcal{N}_2 \Bigg\{ \frac{\big(\mathrm{Pf}\, \phi\big)^2}{4\,\mathcal{E}^2} + \frac{1}{\mathcal{E}} \Bigg[& \frac{\mathrm{Tr}\, \phi^4 - \frac{1}{2}\big(\mathrm{Tr}\, \phi^2\big)^2}{256} \\ & + \frac{\mathrm{Tr}\, f^2\, \mathrm{Tr}\, \phi^2}{2048} + \frac{\mathrm{Tr}\, f^4 - \frac{5}{4}\big(\mathrm{Tr}\, f^2\big)^2}{16384} \Bigg] \Bigg\}\ . \end{aligned} \tag{6.24}$$

JHEP07(2009)092

Here we have rewritten the resulting polynomials in the eigenvalues f_a and ϕ_u in terms of invariants constructed with the matrices ϕ and f in order to get expressions that, although derived choosing ϕ and f in the Cartan directions, are valid generically. For instance, the terms of order ϕ^4 in (6.24) arise in the form

$$\sum_{u>v} \phi_u^2 \phi_v^2 = -\frac{1}{4}\left(\operatorname{Tr}\phi^4 - \frac{1}{2}\big(\operatorname{Tr}\phi^2\big)^2\right) , \tag{6.25}$$

since, according to eq. (6.7), in the block-diagonal case we have

$$\operatorname{Tr}\phi^2 = -2\sum_u \phi_u^2 , \quad \operatorname{Tr}\phi^4 = 2\sum_u \phi_u^4 . \tag{6.26}$$

For $k = 3$, the integral to be computed reads

$$Z_3 = \mathcal{N}_3 \frac{\operatorname{Pf}\phi \operatorname{Pf} f}{\mathcal{E}^2} \int \frac{d\chi}{2\pi \mathrm{i}} \frac{\chi^2 \prod_{u=1}^4 (\phi_u^2 - \chi^2) \prod_{a=1}^3 (f_a^2 - \chi^2)}{\prod_{A=1}^4 (2\chi - E_A)(-2\chi - E_A)(\chi - E_A)(-\chi - E_A)} . \tag{6.27}$$

The integration prescription leads now to the sum over two classes of residues, those in $\chi = E_A/2$ and those in $\chi = E_A$. After the algebra has been carried out, this sum reduces to

$$\begin{aligned} Z_3 = \mathcal{N}_3 \operatorname{Pf}\phi \Bigg\{ \frac{\big(\operatorname{Pf}\phi\big)^2}{12\,\mathcal{E}^3} + \frac{1}{\mathcal{E}^2}\Bigg[\frac{\operatorname{Tr}\phi^4 - \frac{1}{2}\big(\operatorname{Tr}\phi^2\big)^2}{256} + \frac{\operatorname{Tr} f^2 \operatorname{Tr}\phi^2}{2048} \\ + \frac{\operatorname{Tr} f^4 - \frac{5}{4}\big(\operatorname{Tr} f^2\big)^2}{16384} \Bigg] + \frac{1}{96\mathcal{E}} \Bigg\} . \end{aligned} \tag{6.28}$$

In the cases $k = 4$ and $k = 5$ the rank of $\mathrm{SO}(k)$ equals 2 and we have therefore to perform a double contour integral over χ_1 and χ_2. In appendix C we give some details about the classes of residues that contribute to these integrations. The complete resulting expressions for Z_4 and Z_5 are too cumbersome to report them explicitly; however, we report the terms with the highest power of $\mathcal{E}$ in the denominator, namely

$$Z_4 = \mathcal{N}_4 \frac{\big(\operatorname{Pf}\phi\big)^4}{48\,\mathcal{E}^4} + \cdots , \tag{6.29a}$$

$$Z_5 = \mathcal{N}_5 \frac{\big(\operatorname{Pf}\phi\big)^5}{240\,\mathcal{E}^5} + \cdots , \tag{6.29b}$$

which will be useful for the calculations described in the next section.

7 The prepotential and its gravitational corrections

From the instanton partition functions Z_k computed in the previous section, we define the "grand-canonical" partition function

$$\mathcal{Z} = \sum_{k=0}^{\infty} Z_k \, \mathrm{e}^{2\pi \mathrm{i} \tau k} = \sum_{k=0}^{\infty} Z_k \, q^k \tag{7.1}$$

JHEP07(2009)092

where we have conventionally set $Z_0 = 1$, and, as in (2.12), defined $q \equiv \exp(2\pi i\tau)$. This allows us to obtain the non-perturbative contributions to the effective action of the D7-branes. However, to do so one has first to take into account the fact that the k-th order in the q-expansion receives contributions not only from genuine k-instanton configurations but also from "disconnected" ones, corresponding to copies of instantons of lower numbers k_i such that $\sum k_i = k$ [18]. Thus, to isolate the connected components we have to take the logarithm of $\mathcal{Z}$. Moreover, as we have explicitly shown in the previous sections, the partition functions Z_k have been obtained by integrating over *all* moduli, including the "center of mass" coordinates x^μ and their superpartners θ^α defined in (3.11). In absence of deformations these zero-modes do not appear in the moduli action and the integration over them would diverge, producing the (infinite) "supervolume" of the eight-dimensional base manifold. In presence of SO(7) deformations, instead, as we remarked around eq. (6.4), the integration over the superspace coordinates yields a factor of $(2\pi)^4/\mathcal{E}$. Therefore, to obtain the integral over the centered moduli only, it is sufficient to remove this factor. Having done so, we can promote the vacuum expectation value ϕ appearing in $\mathcal{Z}$ to the full fledged dynamical superfield $\Phi(x,\theta)$ and, after removing the RR deformation, obtain the non-perturbative contributions to the effective action of the D7-branes, namely

$$S^{(\text{n.p.})} = \frac{1}{(2\pi)^4}\int d^8x\, d^8\theta\, F^{(\text{n.p.})}\big(\Phi(x,\theta)\big) \tag{7.2}$$

with the "prepotential" $F^{(\text{n.p.})}(\Phi)$ given by

$$F^{(\text{n.p.})}(\Phi) = \mathcal{E}\,\log\mathcal{Z}\Big|_{\phi\to\Phi, f\to 0}\,. \tag{7.3}$$

Expanding in instanton contributions we can write

$$F^{(\text{n.p.})}(\Phi) = \sum_{k=1}^{\infty} F_k\, q^k\,\Big|_{\phi\to\Phi, f\to 0} \tag{7.4}$$

and, using (7.1), express recursively each F_k in terms of the partition functions Z_k and of the coefficients F_j with $j<k$, according to

$$\begin{aligned}
F_1 &= \mathcal{E}Z_1\,,\\
F_2 &= \mathcal{E}Z_2 - \frac{F_1^2}{2\mathcal{E}}\,,\\
F_3 &= \mathcal{E}Z_3 - \frac{F_2F_1}{\mathcal{E}} - \frac{F_1^3}{6\mathcal{E}^2}\,,\\
F_4 &= \mathcal{E}Z_4 - \frac{F_3F_1}{\mathcal{E}} - \frac{F_2^2}{2\mathcal{E}} - \frac{F_2F_1^2}{2\mathcal{E}^2} - \frac{F_1^4}{24\mathcal{E}^3}\,,\\
F_5 &= \mathcal{E}Z_5 - \frac{F_4F_1}{\mathcal{E}} - \frac{F_3F_2}{\mathcal{E}} - \frac{F_3F_1^2}{2\mathcal{E}^2} - \frac{F_2^2F_1}{2\mathcal{E}^2} - \frac{F_2F_1^3}{6\mathcal{E}^3} - \frac{F_1^5}{120\mathcal{E}^4}\,,\\
&\ldots.
\end{aligned} \tag{7.5}$$

The prepotential $F^{(\text{n.p.})}(\Phi)$ must be well-defined when the closed string deformation is turned off, and hence all coefficients F_k must be finite in the limit $f\to 0$. On the other

JHEP07(2009)092

hand, as is clear from the explicit expressions obtained in the previous section, the partition functions Z_k exhibit singularities of different orders, ranging from $O(f^{-4})$ (corresponding to $1/\mathcal{E}$) up to $O(f^{-4k})$ (corresponding to $1/\mathcal{E}^k$). Thus, for consistency of the whole procedure, in computing F_k all such divergences must disappear. Imposing the cancellation of the most divergent term fixes the overall normalization coefficients $\mathcal{N}_k$ but, once this choice is made, all the remaining cancellations of divergences must take place.

For $k = 1$, from eq. (6.3) we have directly

$$F_1 = \mathcal{N}_1 \operatorname{Pf} \phi \,. \tag{7.6}$$

For $k = 2$, we must insert the above result into eq. (7.5) and use the expression (6.24) for the partition function Z_2. The resulting contribution is

$$F_2 = \left(\frac{\mathcal{N}_2}{4} - \frac{\mathcal{N}_1^2}{2}\right) \frac{(\operatorname{Pf} \phi)^2}{\mathcal{E}} + \dots \,. \tag{7.7}$$

We fix the normalization $\mathcal{N}_2$ as

$$\mathcal{N}_2 = 2\mathcal{N}_1^2 \tag{7.8}$$

in order to cancel the most divergent term, and having done so, we find that all other divergences disappear, leaving

$$F_2 = 2\mathcal{N}_1^2 \left(\frac{\operatorname{Tr} \phi^4 - \frac{1}{2}(\operatorname{Tr} \phi^2)^2}{256} + \frac{\operatorname{Tr} f^2 \operatorname{Tr} \phi^2}{2048} + \frac{\operatorname{Tr} f^4 - \frac{5}{4}(\operatorname{Tr} f^2)^2}{16384} \right) \,. \tag{7.9}$$

We proceed in the same way at the next order, $k = 3$. Using eq. (6.28) and the above expressions for F_1 and F_2, one can see from eq. (7.5) that the most divergent term of F_3 reads

$$F_3 = \frac{(\operatorname{Pf} \phi)^3}{\mathcal{E}^2} \left(\frac{\mathcal{N}_3}{12} - \frac{\mathcal{N}_2 \mathcal{N}_1}{4} + \frac{\mathcal{N}_1^3}{3} \right) + \dots = \frac{(\operatorname{Pf} \phi)^3}{\mathcal{E}^2} \left(\frac{\mathcal{N}_3}{12} - \frac{\mathcal{N}_1^3}{6} \right) + \dots \,, \tag{7.10}$$

so that we have to choose

$$\mathcal{N}_3 = 2\mathcal{N}_1^3 \,. \tag{7.11}$$

Once this is done, all other divergences cancel and we are simply left with

$$F_3 = \frac{\mathcal{N}_1^3}{48} \operatorname{Pf} \phi \,. \tag{7.12}$$

It is interesting to note that the contributions from odd instanton numbers $k = 2n + 1$ have to contain the factor $\operatorname{Pf} \phi$ which, being quartic, saturates already the dimensionality of the prepotential. Thus, in these cases, there is no room for f-dependent terms.

So far, the only ambiguity left is the overall normalization factor $\mathcal{N}_1$. Considering the ratio $F_3/F_1 = \mathcal{N}_1^2/48$, we see that by setting

$$\mathcal{N}_1 = 8 \,, \tag{7.13}$$

it takes the value $d_3/d_1 = 4/3$ as in the heterotic theory (see eq. (2.7c)). With this choice all possible ambiguities are fixed, and no further adjustments are possible. For $k = 4$,

JHEP07(2009)092

the partition function Z_4 can be computed as indicated in appendix C. The cancellation of most divergent term in the expression of F_4 following from eq. (7.5), requires that $\mathcal{N}_4 = 2\mathcal{N}_1^4 = 8192$. Using this, we then find

$$F_4 = \frac{1}{4}\mathrm{Tr}\,\phi^4 - \frac{1}{4}(\mathrm{Tr}\,\phi^2)^2 + \frac{3}{32}\mathrm{Tr}\,\phi^2\,\mathrm{Tr}\,f^2 + \frac{3}{256}\left(\mathrm{Tr}\,f^4 - \frac{5}{4}(\mathrm{Tr}\,f^2)^2\right) \,. \tag{7.14}$$

In the case $k = 5$, having computed Z_5 along the lines described in appendix C, the cancellation of the highest divergence in F_5 requires that $\mathcal{N}_5 = 2\mathcal{N}_1^5 = 65536$, after which we get

$$F_5 = \frac{48}{5}\,\mathrm{Pf}\,\phi \,. \tag{7.15}$$

Making the replacement $\phi \to \Phi(x,\theta)$ and taking the limit $f \to 0$ in the above results, we obtain the non-perturbative contributions to the prepotential according to eq. (7.4). Up to instanton number $k = 5$, our findings are summarized in

$$\begin{aligned} F^{(\mathrm{n.p.})}(\Phi) = \mathrm{Tr}\,\Phi^4 \left(\frac{1}{2}\,q^2 + \frac{1}{4}\,q^4 + \dots\right) - (\mathrm{Tr}\,\Phi^2)^2 \left(\frac{1}{4}\,q^2 + \frac{1}{4}\,q^4 + \dots\right) \\ + 8\,\mathrm{Pf}\,\Phi \left(q + \frac{4}{3}\,q^3 + \frac{6}{5}\,q^5 + \dots\right) \,, \end{aligned} \tag{7.16}$$

which perfectly match the expectations from the heterotic string, as one can see by comparing eq.s (7.16) and (2.7).[17]

Actually, if we refrain from taking the limit $f \to 0$, our method allows to obtain also the instanton-induced gravitational corrections to the prepotential. Indeed, once the factor $1/\mathcal{E}$ is removed from $\log \mathcal{Z}$ as indicated in (7.3), we are allowed not only to replace ϕ with the full dynamical gauge superfield $\Phi(x,\theta)$ as we have done so far, but also to replace the constant RR background f with a full-fledged dynamical gravitational superfield, in complete analogy with what happens in the $\mathcal{N} = 2$ SYM theories in four dimensions [36]. The reason is that the (7×7) matrix f_{mn} defines, through eq. (4.1), an (8×8) anti-symmetric tensor $\mathcal{F}_{\mu\nu}$, which can be interpreted as the graviphoton field-strength. In turn, $\mathcal{F}_{\mu\nu}$ can be considered as the lowest component of a eight-dimensional bulk chiral superfield $\mathcal{W}_{\mu\nu}(x,\theta)$ defined as

$$\mathcal{W}_{\mu\nu}(x,\theta) = \mathcal{F}_{\mu\nu}(x) + \theta\,\chi_{\mu\nu} + \frac{1}{2}\,\theta\gamma^{\rho\sigma}\theta\,\mathcal{R}_{\rho\sigma\mu\nu}(x) + \cdots \tag{7.17}$$

where $\chi_{\mu\nu}$ is the gravitino field-strength and $\mathcal{R}_{\rho\sigma\mu\nu}$ is the Riemann curvature tensor. Notice that since the matrix f_{mn} parameterizes the 21 components of $\mathcal{F}_{\mu\nu}$ that are related to the rotation in a 7-dimensional subspace as indicated in eq. (4.1), the graviphoton field-strength is subject to the constraint

$$(P_1^+)^{\mu\nu}_{\rho\sigma}\,\mathcal{F}_{\mu\nu} = 0 \,, \tag{7.18}$$

where P_1^+ is the octonionic projector described in appendix A.1. This constraint can be viewed as the eight-dimensional analogue of the self-duality constraint that is imposed on the graviphoton background in four dimensions [18, 36].

[17] The t_8 structure is produced, according to eq.s (A.35) and (A.36), by integrating the prepotential over $d^8\theta$ to obtain the effective Lagrangian.

JHEP07(2009)092

In this way we can obtain the non-perturbative prepotential, including gravitational corrections, which is therefore given by

$$F^{(\text{n.p.})}(\Phi,\mathcal{W}) = \mathcal{E}\,\log\mathcal{Z}\Big|_{\phi\to\Phi,f\to\mathcal{W}}\,. \tag{7.19}$$

The first few contributions at low instanton numbers can be read from eq.s (7.6), (7.9) and (7.14). However, to express the result in a covariant form, it is convenient to first take advantage of the following trace identities

$$\begin{aligned}\operatorname{Tr} f^2 &= \frac{1}{4}\operatorname{Tr}\mathcal{F}^2\,,\\ \operatorname{Tr} f^4 - \frac{5}{4}(\operatorname{Tr} f^2)^2 &= -\frac{1}{8}\Big(\operatorname{Tr}\mathcal{F}^4 + \frac{1}{4}(\operatorname{Tr}\mathcal{F}^2)^2\Big)\,,\end{aligned} \tag{7.20}$$

so that we obtain

$$\begin{aligned}F^{(\text{n.p.})}(\Phi,\mathcal{W}) = F^{(\text{n.p.})}(\Phi) &+ \frac{1}{2^6}\operatorname{Tr}\mathcal{W}^2\operatorname{Tr}\Phi^2\left(q^2+\frac{3}{2}\,q^4+\cdots\right)\\ &- \frac{1}{2^{10}}\Big(\operatorname{Tr}\mathcal{W}^4+\frac{1}{4}(\operatorname{Tr}\mathcal{W}^2)^2\Big)\left(q^2+\frac{3}{2}\,q^4+\cdots\right).\end{aligned} \tag{7.21}$$

Once we perform the integration over the fermionic superspace coordinates, this expression shows that instantons with even topological charge induce in the D7-brane effective action non-perturbative purely gravitational terms proportional to $\operatorname{Tr}\mathcal{R}^4+\frac{1}{4}(\operatorname{Tr}\mathcal{R}^2)^2$, and mixed gauge/gravitational terms proportional to $\operatorname{Tr}\mathcal{R}^2\operatorname{Tr}F^2$. The relative coefficients of the instanton corrections for the various structures are again in perfect agreement with the expectations from the heterotic string calculations, as indicated in eq. (2.5).

8 Conclusions

In this paper we have analyzed in detail the integral over the D-instanton moduli in the type I′ theory. Such matrix integrals are different from the D(–1) matrix integrals in type IIB since they possess mixed moduli from the D7/D(–1) sectors. They also differ from "ordinary" instantonic brane systems, such as the D3/D(–1) system, because the mixed moduli are only fermionic; they are instead similar to so-called "exotic" instantons. We have shown that localization techniques similar to the ones that were successful for type IIB matrix integrals and for the $\mathcal{N}=2$ instanton calculus in four dimensions allow to perform the integration also in the D7/D(-1) system for generic values of the instanton number k. The outcome of the computation is the quartic prepotential for the SO(8) gauge multiplet $\Phi(x,\theta)$ on a stack of D7-branes. Up to $k=5$ and taking into account also the tree-level and one-loop contributions discussed in section 3.1, the explicit result we find is

$$\begin{aligned}F(\Phi) = \operatorname{Tr}\Phi^4&\left[\frac{\mathrm{i}\pi\tau}{12}+\frac{1}{2}q^2+\frac{1}{4}q^4+\ldots\right]\\ &+\left(\operatorname{Tr}\Phi^2\right)^2\left[\frac{1}{32}\log\left(\operatorname{Im}\tau\operatorname{Im}U\,|\eta(U)|^4\right)-\frac{1}{4}q^2-\frac{1}{4}q^4+\ldots\right]\\ &+8\operatorname{Pf}\Phi\left[q+\frac{4}{3}q^3+\frac{6}{5}q^5+\ldots\right].\end{aligned} \tag{8.1}$$

JHEP07(2009)092

This expression matches perfectly the results of the heterotic $\big[\mathrm{SO}(8)\big]^4$ theory as given in eq. (2.13) upon use of the duality relations (2.11). Our computation represents thus an explicit quantitative check of the heterotic/type I′ duality.

We expect that the techniques we utilized may be useful in dealing with the moduli space integrals for other instances of "exotic" instanton systems, also the four-dimensional ones of potential phenomenological relevance.

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A Conventions and notations

A.1 SO(7) and SO(8) gamma-matrices

SO(7) can be embedded into SO(8) in such a way that the vector representation **8** of SO(8) is identified with the spinor representation $\mathbf{8}_s$ of SO(7). This embedding is best described by using an explicit realization of the Clifford algebras in $d = 7$ and $d = 8$ based on the octonionic structure constants.

The Clifford algebra in 7 dimensions,

$$\big\{\tau^i, \tau^j\big\}_{\alpha\beta} = -2\delta^{ij}\,\delta_{\alpha\beta}\,, \tag{A.1}$$

can be realized by the matrices (8×8)-matrices τ^i $(i = 1, \ldots, 7)$ with elements

$$(\tau^i)_{\alpha\beta} = \delta^{i8}_{\alpha\beta} + C^{-\,i8}_{\alpha\beta} \qquad (\alpha, \beta = 1, \ldots, 8)\,, \tag{A.2}$$

where we made use of the totally antisymmetric, (anti)-selfdual four-index tensors in $d = 8$ $C^{\pm}_{\mu\nu\rho\sigma}$. In turn, these tensors are expressed as

$$C^{\pm}_{ijk8} = c_{ijk} \quad , \quad C^{\pm}_{ijk\ell} = \pm\frac{1}{3!}\,\epsilon_{ijk\ell mnp}\,c_{mnp} \tag{A.3}$$

in terms of the octonionic structure constants c_{mnp} $(m, n, \ldots = 1, \ldots, 7)$, with c a totally antisymmetric tensor whose only non-zero elements can be taken to be

$$c_{127} = c_{163} = c_{154} = c_{253} = c_{246} = c_{347} = c_{567} = 1\ . \tag{A.4}$$

The tensor c_{mnp} enjoys various properties, such as

$$c_{mpr}c_{nrq} = -\delta_{mn}\delta_{pq} + \delta_{mq}\delta_{np} + \frac{1}{6}\epsilon_{mnpqstu}c_{\mathrm{stu}} \equiv -\delta_{mp,nq} + \frac{1}{6}\epsilon_{mnpqstu}c_{\mathrm{stu}} \tag{A.5}$$

and

$$c_{mpq}c_{npq} = 6\,\delta_{mn}\,, \quad \epsilon_{mnpqstu}c_{\mathrm{stu}}c_{rpq} = 24\,c_{mnr}\ . \tag{A.6}$$

These properties imply the existence of useful identities for the tensors $C^{\pm}_{\mu\nu\rho\sigma}$, such as

$$C^{\pm\mu\nu\rho\sigma} C^{\pm}_{\rho\sigma\tau\omega} = 6\,\delta^{\mu\nu}_{\tau\omega} \pm 4\, C^{\mu\nu}_{\tau\omega} \ . \tag{A.7}$$

The SO(7) generators T^{ij} in the spinorial representation $\mathbf{8}_s$, satisfying the so(7) algebra

$$\Big[T^{mn}, T^{pq}\Big] = \delta^{np}\, T^{mq} - \delta^{mp}\, T^{nq} + \delta^{mq}\, T^{np} - \delta^{nq}\, T^{mp} \,, \tag{A.8}$$

are defined as

$$T^{mn} = -\frac{1}{4}\,[\tau^m, \tau^n] \equiv -\frac{1}{2}\,\tau^{mn} \ . \tag{A.9}$$

Using the definition (A.2) and eq. (A.7), one can show that

$$\big(T^{mn}\big)_{AB} = \frac{1}{2}\Big(\delta^{mn}_{AB} + C^{-}{}^{mn}_{AB}\Big) \ . \tag{A.10}$$

A generic SO(7) group element in the spinor representation can then be parametrized as

$$R(f) = \mathrm{e}^{\frac{1}{2} f_{mn}\, T^{mn}} \tag{A.11}$$

and the infinitesimal SO(7) variation of any field X^A transforming in the spinor representation is

$$\delta X^A = \frac{1}{2}\, f_{mn} \big(T^{mn}\big)^{AB}\, X_B = -\frac{1}{2}\, \mathcal{F}^{AB}_{\mathbf{21}}\, X_B \tag{A.12}$$

where we have introduced (the subscript **21** will become clear later)

$$\mathcal{F}^{AB}_{\mathbf{21}} = \frac{1}{2}\, f_{mn} \big(\tau^{mn}\big)^{AB} \ . \tag{A.13}$$

The SO(7) generators t^{mn} in the vector representation $\mathbf{7}$, satisfying the so(7) algebra with the *same* normalization as in eq. (A.8) are given by

$$\big(t^{mn}\big)_{pq} = \delta^{mn}_{pq} = \delta^m_p\, \delta^n_q - \delta^m_q\, \delta^n_p \ . \tag{A.14}$$

Thus, in the vector representation, the SO(7) group element with parameters f_{mn} is represented by

$$r(f) = \mathrm{e}^{\frac{1}{2} f_{mn}\, t^{mn}} \tag{A.15}$$

and the infinitesimal variation of any field ϕ^m transforming in the vector representation is

$$\delta\phi^p = \frac{1}{2}\, f_{mn} \big(t^{mn}\big)^{pq} \phi_q = \frac{1}{2}\, f_{mn}\, \delta^{mn,pq}\, \phi_q = f^{pq}\, \phi_q \ . \tag{A.16}$$

The SO(8) Clifford algebra can be realized by taking the eight gamma matrices γ^μ to be (we use now $\mu = 1, \ldots 8$, while $m = 1, \ldots 7$)

$$\gamma^m = \mathrm{i}\tau^m \otimes \sigma^1 \,, \quad \gamma^8 = \mathbf{1}_8 \otimes \sigma^2 \ . \tag{A.17}$$

These matrices satisfy indeed

$$[\gamma^\mu, \gamma^\nu] = 2\delta^{\mu\nu} \ . \tag{A.18}$$

Note that this is a Weyl basis, since the chirality matrix γ is represented by

$$\gamma \equiv -\gamma^1\gamma^2\dots\gamma^8 = -\tau^1\tau^2\dots\tau^7\otimes\sigma^3 = \mathbf{1}\otimes\sigma^3 \ . \tag{A.19}$$

Note also that in their realization given in eq. (A.17) all the gamma matrices are antisymmetric; the charge conjugation matrix can thus be taken to be simply the identity matrix.

The two-index gamma-matrices

$$\gamma^{\mu\nu} \equiv \frac{1}{2}\left[\gamma^\mu,\gamma^\nu\right] \ , \tag{A.20}$$

which are again anti-symmetric, are given, according to eq. (A.17), by

$$\gamma^{mn} = -\tau^{mn}\otimes\mathbf{1}_2 \ , \quad \gamma^{m8} = -\tau^m\otimes\sigma^3 \ . \tag{A.21}$$

For the anti-chiral block we find explicitly

$$\bar\gamma^{\mu\nu}_{\dot\alpha\dot\beta} = C^{-\mu\nu}_{\dot\alpha\dot\beta} + \delta^{\mu\nu}_{\dot\alpha\dot\beta} \tag{A.22}$$

while for the chiral block we can write

$$(\gamma^{mn})_{\alpha\beta} = \delta^{mn}_{\alpha\beta} + C^{-mn}_{\alpha\beta} \ , \quad (\gamma^{m8})_{\alpha\beta} = -\delta^{m8}_{\alpha\beta} - C^{-m8}_{\alpha\beta} \tag{A.23}$$

or, splitting the spinor index α into $(a,8)$ with $a=1,\dots 7$,

$$(\gamma^{\mu\nu})_{ab} = \delta^{\mu\nu}_{ab} - C^{+\mu\nu}_{ab} \ , \quad (\gamma^{\mu\nu})_{a8} = -\delta^{\mu\nu}_{a8} + C^{+\mu\nu}_{a8} \ . \tag{A.24}$$

The 28-dimensional space of anti-symmetric 8×8 matrices, namely the adjoint space of SO(8), admits an orthogonal decomposition $\mathbf{28}\to\mathbf{21}+\mathbf{7}$ enforced by the following projectors:

$$\begin{aligned}(P_1^+)^{\mu\nu}_{\rho\sigma} &= \frac{1}{8}\left(\delta^{\mu\nu}_{\rho\sigma} + C^{+\mu\nu}{}_{\rho\sigma}\right) \ , \\ (P_2^+)^{\mu\nu}_{\rho\sigma} &= \frac{3}{8}\left(\delta^{\mu\nu}_{\rho\sigma} - \frac{1}{3}C^{+\mu\nu}{}_{\rho\sigma}\right) \ .\end{aligned} \tag{A.25}$$

Indeed, it is straightforward to check that

$$(P_1^+)^2 = P_1^+ \ , \quad (P_2^+)^2 = P_2^+ \ , \quad P_1^+P_2^+ = P_2^+P_1^+ = 0 \ , \quad P_1^+ + P_2^+ = \mathbf{1} \tag{A.26}$$

using the properties of the tensor $C^+_{\mu\nu\rho\sigma}$, see eq. (A.7). Since the tensor $C^+_{\mu\nu\rho\sigma}$ is traceless, the dimensionality of the two eigenspaces are easily obtained by taking the trace of the projectors:

$$\dim\,\mathrm{Ker}(P_1^+) = 21 \ , \quad \dim\,\mathrm{Ker}(P_2^+) = 7 \ . \tag{A.27}$$

The $\mathrm{Ker}(P_1^+)$ subspace is spanned by the 21 matrices $(\tau^{mn})_{\mu\nu}$ corresponding to (twice) the SO(7) spinorial generators in which we identify the indices A in the $\mathbf{8}_s$ of SO(7) with the indices μ in the vector of SO(8). Indeed one can verify that

$$(P_1^+)^{\mu\nu}_{\rho\sigma}(\tau^{mn})_{\mu\nu} \propto \left(\delta^{\mu\nu}_{\rho\sigma} + C^{+\mu\nu}{}_{\rho\sigma}\right)(\tau^{mn})_{\mu\nu} = 0 \ . \tag{A.28}$$

JHEP07(2009)092

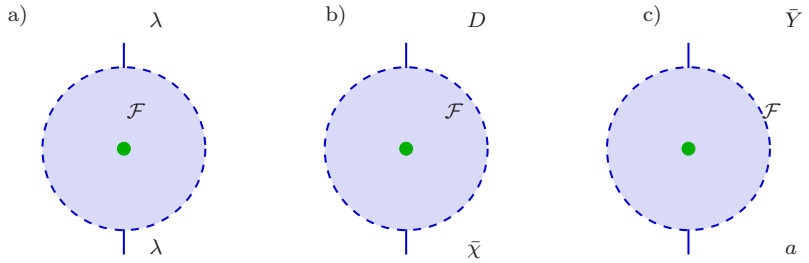

Figure 2. Disk diagrams describing the interactions of a holomorphic RR field-strength vertex (in the interior of the disk) with moduli vertices. The boundary of the disk is on the D(–1)'s.

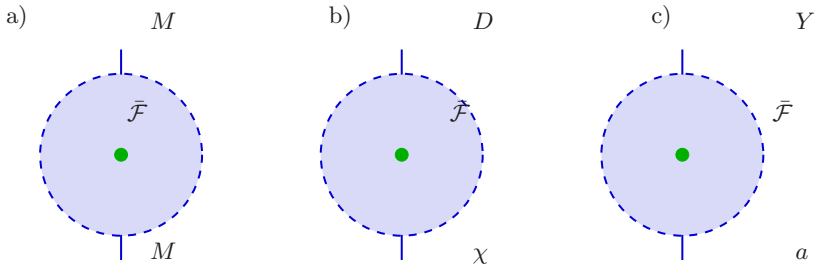

Figure 3. Disk diagrams describing the interactions of an anti-holomorphic RR field-strength vertex (in the interior of the disk) with moduli vertices. The boundary of the disk is on the D(-1)'s.

The $\mathrm{Ker}(P_2^+)$ subspace is instead spanned by the 7 matrices $(\tau^m)_{\mu\nu}$, namely the SO(7) matrices with the above identification of spinorial indices of SO(7) and vector indices of SO(8):

$$(P_2^+)^{\mu\nu}_{\rho\sigma}(\tau^m)_{\mu\nu} \propto \left(\delta^{\mu\nu}_{\rho\sigma} - \frac{1}{3}C^{+\mu\nu}{}_{\rho\sigma}\right)(\tau^m)_{\mu\nu} = 0\ . \tag{A.29}$$

Thus, there is a non-standard[18] embedding of SO(7) into SO(8) in which the adjoint representation of the latter, whose elements are antisymmetric 8×8 matrices $\mathcal{F}_{\mu\nu}$, decomposes into $\mathbf{21}\oplus\mathbf{7}$ as follows:

$$\mathcal{F}_{\mu\nu} = \mathcal{F}^{\mathbf{21}}_{\mu\nu} + \mathcal{F}^{\mathbf{7}}_{\mu\nu} = \frac{1}{2}f_{mn}(\tau^{mn})_{\mu\nu} + h_m(\tau^m)_{\mu\nu}\ . \tag{A.30}$$

Eq.s (A.28) and (A.29) imply the following relations, useful in the computation of the diagram in figure 3 a):

$$(\tau^{mn})_{\mu\nu}\left(\delta^{\mu\nu}_{\rho\sigma} - C^{+\mu\nu}{}_{\rho\sigma}\right) = +4\,(\tau^{mn})_{\rho\sigma}\,, \tag{A.31}$$

$$(\tau^{m})_{\mu\nu}\left(\delta^{\mu\nu}_{\rho\sigma} - C^{+\mu\nu}{}_{\rho\sigma}\right) = -4\,(\tau^{m})_{\rho\sigma}\ . \tag{A.32}$$

[18] In a standard embedding, the adjoint representation **21** of SO(7) corresponds simply to the restriction of $\mathcal{F}_{\mu\nu}$ to its elements $\mathcal{F}_{mn}$, while the **7** corresponds to $\mathcal{F}_{m8}$.

JHEP07(2009)092

The following identities are instead useful for the computation of the diagram in figure 2 a):

$$(\tau^{mn})_{\mu\nu}(\bar\gamma^{\mu\nu})_{pq} = -8\delta^{mn}_{pq}\,, \qquad (\tau^{mn})_{\mu\nu}(\bar\gamma^{\mu\nu})_{p8} = 0\,, \tag{A.33}$$

$$(\tau^{m})_{\mu\nu}(\bar\gamma^{\mu\nu})_{p8} = +8\delta^{m}_{p}\,, \qquad (\tau^{m})_{\mu\nu}(\bar\gamma^{\mu\nu})_{pq} = 0\,. \tag{A.34}$$

A.2 The t_8 tensor

The explicit expression of the totally anti-symmetric 8-index tensor t_8 can be read from eq. (2.2). Several of its properties are given, for instance, in appendix B of [37]. Here, let us just recall how it appears from the integration over the superspace coordinates $d^8\theta$ (or $d^8\bar\theta$) of chiral (or anti-chiral) superfields such as those in eq. (3.2) or eq. (7.17), see for example appendix 9.A of ref. [38]. For bi-linear operators of the form

$$R^{\mu\nu} = \frac{1}{4}(\gamma^{\mu\nu})_{\alpha\beta}\theta^\alpha\theta^\beta\,, \quad \bar R^{\mu\nu} = \frac{1}{4}(\bar\gamma^{\mu\nu})_{\dot\alpha\dot\beta}\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}\,, \tag{A.35}$$

one finds

$$\int d^8\theta\big(R^{\mu_1\mu_2}\cdots R^{\mu_7\mu_8}\big) = t_+^{\mu_1\mu_2\ldots\mu_7\mu_8}\,, \quad \int d^8\bar\theta\big(\bar R^{\mu_1\mu_2}\cdots \bar R^{\mu_7\mu_8}\big) = t_-^{\mu_1\mu_2\ldots\mu_7\mu_8} \tag{A.36}$$

with the antisymmetric tensors $t_\pm$ being related to t_8 and to the Levi-Civita tensor ϵ_8 by

$$t_\pm = t_8 \pm \frac{1}{2}\epsilon_8\,. \tag{A.37}$$

B Vertex operators and disk amplitudes

In this appendix we give some details on the evaluation of the string diagrams that describe the interaction between the instanton moduli and the constant RR background.

As explained in refs. [23, 36, 44], to simplify the procedure it is convenient to first rewrite the quartic interactions among a_μ, χ and $\bar\chi$ appearing in $\mathcal{S}'_{\rm quartic}$ of eq. (3.13b) in a cubic form. This can be done by introducing two new auxiliary fields Y_μ and $\bar Y^\mu$, so that we can replace $\mathcal{S}'_{\rm quartic}$ by

$$\begin{aligned}\mathcal{S}'_{\rm aux} =& \frac{1}{g_0^2}\,{\rm tr}\,\Big\{\frac{1}{2}D_mD^m + \frac{1}{2}D_m(\tau^m)_{\mu\nu}\,[a^\mu,a^\nu] \\ &+ \bar Y^\mu Y_\mu + [a^\mu,\bar\chi]\,Y_\mu + \bar Y^\mu\,[a_\mu,\chi] + \frac{1}{2}\,[\bar\chi,\chi]^2\Big\}\,.\end{aligned} \tag{B.1}$$

It is easy to see that $\mathcal{S}'_{\rm aux}$ reduces to $\mathcal{S}'_{\rm quartic}$ when the auxiliary fields Y_μ and $\bar Y^\mu$ acquire their on-shell values:

$$Y_\mu = -\,[a_\mu,\chi]\,, \quad \bar Y^\mu = -\,[a^\mu,\bar\chi]\,. \tag{B.2}$$

The entire moduli action $\mathcal{S}'_{\rm cubic}+\mathcal{S}'_{\rm mixed}+\mathcal{S}'_{\rm aux}$ (with the first two terms given in eq.s (3.19) and (3.20)) can be obtained by computing "scattering" amplitudes among the vertex operators representing the various instanton moduli, including the auxiliary ones. In standard CFT notations (see for example refs. [23, 37] for details), these vertex operators are

$$V_a = (2\pi\alpha')^{1/2}\,a_\mu\,\psi^\mu\,{\rm e}^{-\varphi}\,, \quad V_\chi = (2\pi\alpha')^{1/2}\,\chi\,\bar\Psi\,{\rm e}^{-\varphi}\,, \quad V_{\bar\chi} = (2\pi\alpha')^{1/2}\,\bar\chi\,\Psi\,{\rm e}^{-\varphi}\,, \tag{B.3}$$

JHEP07(2009)092

for the neutral moduli of the Neveu-Schwarz sector, and

$$V_M = (2\pi\alpha')^{3/4}\, M_\alpha\, S^\alpha\, S^-\, \mathrm{e}^{-\varphi/2}\,, \quad V_\lambda = (2\pi\alpha')^{3/4}\, \lambda_{\dot\alpha}\, S^{\dot\alpha}\, S^+\, \mathrm{e}^{-\varphi/2}\,, \tag{B.4}$$

for those of the Ramond sector. For the fermionic charged moduli, corresponding to open strings with eight mixed ND directions, we have instead

$$V_\mu = (2\pi\alpha')^{3/4}\, \mu\, \Delta\, S^+\, \mathrm{e}^{-\varphi/2}\ . \tag{B.5}$$

Finally, the vertex operators for the auxiliary moduli are

$$V_D = (2\pi\alpha')\, D_m (\tau^m)_{\mu\nu}\ :\psi^\mu\psi^\nu:\,, \quad V_Y = (2\pi\alpha')\, Y_\mu\ :\bar\Psi\psi^\mu:\,, \quad V_{\bar Y} = (2\pi\alpha')\, \bar Y_\mu\ :\Psi\psi^\mu: \tag{B.6}$$

in the neutral sector, and

$$V_w = (2\pi\alpha')\, w\, \Delta\, S^{\dot\alpha=\dot 8} \tag{B.7}$$

in the charged sector. In writing these vertex operators, we have neglected all numerical factors in the normalizations and only inserted the appropriate powers of $(2\pi\alpha')$ that are needed to give the moduli the canonical dimensions (not the ADHM ones). Indeed, as we have shown in the main text, the result of the integration over the moduli space is insensitive to the numerical coefficients of the various structures.

Notice that in eq. (B.7) we have selected the $\dot\alpha = \dot 8$ component of the spin field $S^{\dot\alpha}$, since the BRST charge used in section 3 is precisely the $\dot\alpha = \dot 8$ component of the supersymmetry charge $Q^{\dot\alpha}$, which is preserved by both the D7- and the D(–1)-branes, and given by

$$Q^{\dot\alpha} = \oint \frac{dz}{2\pi\mathrm{i}}\, S^{\dot\alpha}(z)\, S^+(z) \mathrm{e}^{-\varphi(z)/2}\ . \tag{B.8}$$

Using this information, and applying the techniques discussed in refs. [22, 23], one can check the BRST transformation properties reported in eq. (3.22), as well as

$$QM^\mu = \mathrm{i}\sqrt{2}\, Y^\mu\,, \quad QY^\mu = -[M^\mu, \chi]\ . \tag{B.9}$$

This stringy approach to the instanton calculus allows to easily compute also the interactions between moduli and bulk gravitational fields. In particular, we are interested in the interactions with RR field-strengths $\mathcal{F}$ and $\bar{\mathcal{F}}$, which correspond to the disk diagrams represented in figures 2 and 3. These can be computed using standard CFT techniques by inserting in the disk interior the following RR vertex operators

$$\begin{aligned} V_{\mathcal{F}} &= (2\pi\alpha')^{1/2}\, \mathcal{F}_{\mu\nu}\, (\gamma^{\mu\nu}\gamma)_{\alpha\beta}\, S^\alpha\, S^-\, \mathrm{e}^{-\varphi/2}\ \widetilde{S}^\beta\, \widetilde{S}^-\, \mathrm{e}^{-\widetilde{\varphi}/2}\,, \\ V_{\bar{\mathcal{F}}} &= (2\pi\alpha')^{1/2}\, \bar{\mathcal{F}}_{\mu\nu}\, (\gamma^{\mu\nu}\gamma)_{\dot\alpha\dot\beta}\, S^{\dot\alpha}\, S^+\, \mathrm{e}^{-\varphi/2}\ \widetilde{S}^{\dot\beta}\, \widetilde{S}^+\, \mathrm{e}^{-\widetilde{\varphi}/2}\,, \end{aligned} \tag{B.10}$$

where the matrices $\gamma^{\mu\nu}$ and γ have been defined in eq.s (A.20) and (A.19).

Let us now give some details on the computation of the disk diagram represented in figure 2a, which corresponds to the following amplitude

$$\Big\langle\!\Big\langle V_\lambda V_\lambda V_{\mathcal{F}} \Big\rangle\!\Big\rangle \ \equiv\ C_0 \int \frac{dx_1\, dx_2\, dz d\bar z}{dV_{\mathrm{CKG}}}\ \times \big\langle V_\lambda(x_1)\, V_\lambda(x_2)\, V_{\mathcal{F}}(z,\bar z)\big\rangle \tag{B.11}$$

JHEP07(2009)092

where C_0 is the normalization of D(–1) disk amplitudes [23]

$$C_0 = \frac{2}{(2\pi\alpha')^2}\frac{1}{g_0^2} \tag{B.12}$$

and $dV_{\rm CKG}$ is the volume of the conformal Killing group. As usual, the open string punctures x_i are integrated along the real axis with $x_1 \geq x_2$ while the closed string puncture z is integrated on the upper half complex plane. More explicitly, after reflecting the right movers on the disk boundary, we have

$$\begin{aligned}\left\langle\!\!\left\langle V_\lambda V_\lambda V_{\mathcal{F}} \right\rangle\!\!\right\rangle = \frac{2}{g_0^2}\,\mathrm{tr}\left(\lambda_{\dot\alpha}\lambda_{\dot\beta}\mathcal{F}_{\mu\nu}(\gamma^{\mu\nu}\gamma)_{\alpha\beta}\right)\int\frac{dx_1\,dx_2\,dzd\bar z}{dV_{\rm CKG}}\;\times \\ \times\left\langle\,(S^{\dot\alpha}S^+\mathrm{e}^{-\varphi/2})(x_1)(S^{\dot\beta}S^+\mathrm{e}^{-\varphi/2})(x_2)(S^\alpha S^-\mathrm{e}^{-\varphi/2})(z)S^\beta S^-\mathrm{e}^{-\varphi/2}(\bar z)\right\rangle\,.\end{aligned} \tag{B.13}$$

The correlator appearing in the second line above can be obtained by decomposing the ten-dimensional four-point function of spin fields in 8+2 dimensions. Due to the anti-symmetry in $(\alpha\beta)$ of the polarization factor, the only relevant structure in this correlator is

$$\frac{\frac{1}{2}\,(\gamma^\rho)^{\dot\alpha\alpha}(\gamma_\rho)^{\dot\beta\beta}}{\left[(x_1-z)(x_1-\bar z)(x_2-z)(x_2-\bar z)\right]}\,. \tag{B.14}$$

Then, inserting this into (B.13) and exploiting the $\mathrm{Sl}(2,\mathbb{R})$ invariance to fix $x_1 \to \infty$ and $z \to \mathrm{i}$, we are left with the following elementary integral

$$2\mathrm{i}\int_{-\infty}^{\infty} dx_2\,\frac{1}{1+x_2^2} = 2\pi\mathrm{i}\,, \tag{B.15}$$

so that, after some algebra, we find

$$\left\langle\!\!\left\langle V_\lambda V_\lambda V_{\mathcal{F}} \right\rangle\!\!\right\rangle = -\frac{1}{16g_0^2}\,\mathrm{tr}\left\{\lambda_{\dot\alpha}(\gamma^{\mu\nu})^{\dot\alpha\dot\beta}\lambda_{\dot\beta}\,\mathcal{F}_{\mu\nu}\right\} \tag{B.16}$$

where we have clumped the remaining numerical factors in the normalization of the background field $\mathcal{F}$. With similar calculations, one can compute all other diagrams in figure 2 obtaining

$$\begin{aligned}\left\langle\!\!\left\langle V_D V_{\bar\chi} V_{\mathcal{F}} \right\rangle\!\!\right\rangle &= \frac{\mathrm{i}}{g_0^2}\,\frac{1}{8\sqrt{2}}\,\mathrm{tr}\left\{D_m(\tau^m)_{\mu\nu}\,\bar\chi\,\mathcal{F}^{\mu\nu}\right\}\,,\\ \left\langle\!\!\left\langle V_{\bar Y} V_a V_{\mathcal{F}} \right\rangle\!\!\right\rangle &= \frac{\mathrm{i}}{g_0^2}\,\frac{1}{2\sqrt{2}}\,\mathrm{tr}\left\{\bar Y_\mu\,a_\nu\,\mathcal{F}^{\mu\nu}\right\}\,.\end{aligned} \tag{B.17}$$

Likewise, for the diagrams with the anti-holomorphic background represented in figure 3 we find

$$\begin{aligned}\left\langle\!\!\left\langle V_M V_M V_{\mathcal{F}} \right\rangle\!\!\right\rangle &= -\frac{1}{16g_0^2}\,\mathrm{tr}\left\{M_\alpha(\gamma^{\mu\nu})^{\alpha\beta}M_\beta\,\bar{\mathcal{F}}_{\mu\nu}\right\}\,,\\ \left\langle\!\!\left\langle V_D V_\chi V_{\bar{\mathcal{F}}} \right\rangle\!\!\right\rangle &= \frac{\mathrm{i}}{g_0^2}\,\frac{1}{8\sqrt{2}}\,\mathrm{tr}\left\{D_m(\tau^m)_{\mu\nu}\,\chi\,\bar{\mathcal{F}}^{\mu\nu}\right\}\,,\\ \left\langle\!\!\left\langle V_Y V_a V_{\bar{\mathcal{F}}} \right\rangle\!\!\right\rangle &= \frac{\mathrm{i}}{g_0^2}\,\frac{1}{2\sqrt{2}}\,\mathrm{tr}\left\{Y_\mu\,a_\nu\,\bar{\mathcal{F}}^{\mu\nu}\right\}\,.\end{aligned} \tag{B.18}$$

JHEP07(2009)092

From the last lines of eq.s (B.17) and (B.18), we see that the presence of a RR background induces two extra terms in $\mathcal{S}'_{\rm aux}$, so that the latter must be replaced according to

$$\mathcal{S}'_{\rm aux} \to \mathcal{S}'_{\rm aux} - \frac{\mathrm{i}}{g_0^2}\frac{1}{2\sqrt{2}}\,\mathrm{tr}\,\left\{\bar{Y}_\mu\, a_\nu\, \mathcal{F}^{\mu\nu} + Y_\mu\, a_\nu\, \bar{\mathcal{F}}^{\mu\nu}\right\}\ . \tag{B.19}$$

As a consequence, the equations of motion of the auxiliary fields change and eq. (B.2) must be replaced by

$$Y_\mu = -\,[a_\mu,\chi] + \frac{\mathrm{i}}{2\sqrt{2}}\,\mathcal{F}_{\mu\nu}a^\nu\,,\quad \bar{Y}^\mu = -\,[a^\mu,\bar{\chi}] + \frac{\mathrm{i}}{2\sqrt{2}}\,\bar{\mathcal{F}}^{\mu\nu}a_\nu\ . \tag{B.20}$$

Thus, eliminating Y_μ and $\bar{Y}^\mu$ we recover the new $\mathcal{F}$-dependent quartic action

$$\begin{aligned}\mathcal{S}'_{\rm quartic}(\mathcal{F},\bar{\mathcal{F}}) =& \mathcal{S}'_{\rm quartic} + \frac{1}{g_0^2}\,\mathrm{tr}\left\{\frac{\mathrm{i}}{2\sqrt{2}}\,[a_\mu,\bar{\chi}]\,\mathcal{F}^{\mu\nu}a_\nu\right.\\ &\left. + \frac{\mathrm{i}}{2\sqrt{2}}\,[a_\mu,\chi]\,\bar{\mathcal{F}}^{\mu\nu}a_\nu + \frac{1}{8}\,\bar{\mathcal{F}}^{\mu\nu}a_\nu\mathcal{F}_{\mu\rho}a^\rho\right\},\end{aligned} \tag{B.21}$$

which reproduces the expression given in eq.s (4.3) and (4.9) of the main text.

Furthermore, from eq.s (B.16)–(B.18) we obtain the following background-dependent cubic terms:

$$\begin{aligned}\mathcal{S}_{\rm cubic}(\mathcal{F},\bar{\mathcal{F}}) =& \mathcal{S}_{\rm cubic} + \frac{1}{g_0^2}\,\mathrm{tr}\left\{\frac{1}{16}\,\lambda_{\dot\alpha}(\gamma^{\mu\nu})^{\dot\alpha\dot\beta}\lambda_{\dot\beta}\,\mathcal{F}_{\mu\nu} + \frac{1}{16}\,M_\alpha(\gamma^{\mu\nu})^{\alpha\beta}M_\beta\,\bar{\mathcal{F}}_{\mu\nu}\right.\\ &\left. + \frac{\mathrm{i}}{8\sqrt{2}}D_m(\tau^m)_{\mu\nu}\,\bar{\chi}\,\mathcal{F}^{\mu\nu} + \frac{\mathrm{i}}{8\sqrt{2}}D_m(\tau^m)_{\mu\nu}\,\chi\,\bar{\mathcal{F}}^{\mu\nu}\right\}.\end{aligned} \tag{B.22}$$

To compare this expression with that used in section 4, we have first to decompose the background fluxes as in eq. (A.30) and use the relabelled fermion moduli defined in eq. (3.17). Then, performing the traces on the τ-matrices, one can show that the couplings involving D_m receive contributions only from the $\mathcal{F}^{\mathbf{7}}$ and $\bar{\mathcal{F}}^{\mathbf{7}}$ components, given by

$$\frac{\mathrm{i}}{8\sqrt{2}}\mathcal{F}^{\mu\nu}\,D_m(\tau^m)_{\mu\nu}\,\bar{\chi} = -\frac{\mathrm{i}}{\sqrt{2}}h^m D_m\,\bar{\chi}\,,\quad \frac{\mathrm{i}}{8\sqrt{2}}\bar{\mathcal{F}}^{\mu\nu}\,D_m(\tau^m)_{\mu\nu}\,\chi = -\frac{\mathrm{i}}{\sqrt{2}}\bar{h}^m D_m\,\chi\ . \tag{B.23}$$

On the other hand, using eq.s (A.32) and (A.33) we can rewrite the fermionic bilinears which appear in the first line of eq. (B.22) as follows

$$\begin{aligned}\frac{1}{16}\,\mathcal{F}_{\mu\nu}\,\lambda_{\dot\alpha}(\gamma^{\mu\nu})^{\dot\alpha\dot\beta}\lambda_{\dot\beta} &= \frac{1}{16}\,\mathcal{F}_{\mu\nu}\,\lambda_m(\gamma^{\mu\nu})^{mn}\lambda_n + \frac{1}{8}\mathcal{F}_{\mu\nu}\,\lambda_m(\gamma^{\mu\nu})^{m8}\eta\\ &= -\frac{1}{2}\,f^{mn}\lambda_m\lambda_n + h^m\lambda_m\eta\\ \frac{1}{16}\,\bar{\mathcal{F}}_{\mu\nu}\,M_\alpha(\gamma^{\mu\nu})^{\alpha\beta}M_\beta &= \frac{1}{16}\,\bar{\mathcal{F}}_{\mu\nu}\,M^\rho\left(\delta^{\mu\nu}_{\rho\sigma} - C^{+\mu\nu}{}_{\rho\sigma}\right)M^\sigma\\ &= \frac{1}{8}\,\bar{f}^{mn}(\tau_{mn})_{\rho\sigma}M^\rho M^\sigma - \frac{1}{4}\,\bar{h}^m(\tau_m)_{\rho\sigma}M^\rho M^\sigma\ .\end{aligned} \tag{B.24}$$

From these expressions we retrieve the terms of eq.s (4.3), (4.9) and (4.11) of the main text for $\bar{h}_m = 0$.

JHEP07(2009)092

C Details on the SO(k) integrals

Weight sets of SO($2n+1$). This group has rank n. If we denote by $\vec{e}_i$ the versors in the $\mathbb{R}^n$ weight space,

- the set of the $2n+1$ weights $\vec{\pi}$ of the vector representation is given by

$$\pm\,\vec{e}_i\,, \qquad \vec{0} \text{ with multiplicity } 1; \tag{C.1}$$

- the set of $n(2n+1)$ weights $\vec{\rho}$ of the adjoint representation (corresponding to the two-index antisymmetric tensor) is the following:

$$\pm\,\vec{e}_i \pm \vec{e}_j\ (i<j)\,, \qquad \pm\,\vec{e}_i\,, \qquad \vec{0} \text{ with multiplicity } n; \tag{C.2}$$

- the $(n+1)(2n+1)$ weights of the two-index symmetric tensor[19] are

$$\pm\,\vec{e}_i \pm \vec{e}_j\ (i<j)\,, \qquad \pm\,\vec{e}_i\,, \qquad \pm\,2\vec{e}_i\,, \qquad \vec{0} \text{ with multiplicity } n+1\,. \tag{C.3}$$

Weight sets of SO($2n$). This group has rank n. If we denote by $\vec{e}_i$ the versors in the $\mathbb{R}^n$ weight space,

- the set of the $2n$ weights $\vec{\pi}$ of the vector representation is given by

$$\pm\,\vec{e}_i; \tag{C.4}$$

- the set of $n(2n-1)$ weights $\vec{\rho}$ of the adjoint representation (corresponding to the two-index antisymmetric tensor) is the following:

$$\pm\,\vec{e}_i \pm \vec{e}_j\ (i<j)\,, \qquad \vec{0} \text{ with multiplicity } n; \tag{C.5}$$

- the $n(2n+1)$ weights of the two-index symmetric tensor[20] are

$$\pm\,\vec{e}_i \pm \vec{e}_j\ (i<j)\,, \qquad \pm\,2\vec{e}_i\,, \qquad \vec{0} \text{ with multiplicity } n\,. \tag{C.6}$$

SO(7) and its spinorial weights. The SO(7) rotation group parametrized by the RR fluxes f_{mn} defined in eq. (4.1) act on the moduli a^μ, M^μ in its spinorial representation $\mathbf{8}_s$. The set of weights of this representation is

$$\vec{\beta} = \frac{1}{2}(\pm\vec{e}_1 \pm \vec{e}_2 \pm \vec{e}_3) \tag{C.7}$$

and we define as "positive" weights those for which the product of the three signs is -1:

$$\begin{aligned} &\vec{\beta}_1 = \frac{1}{2}(-\vec{e}_1+\vec{e}_2+\vec{e}_3)\,, \quad \vec{\beta}_2 = \frac{1}{2}(\vec{e}_1-\vec{e}_2+\vec{e}_3)\,, \quad \vec{\beta}_3 = \frac{1}{2}(\vec{e}_1+\vec{e}_2-\vec{e}_3)\,, \\ &\vec{\beta}_4 = \frac{1}{2}(-\vec{e}_1-\vec{e}_2-\vec{e}_3)\,, \end{aligned} \tag{C.8}$$

so that the combinations

$$E_A = \vec{f}\cdot\vec{\beta}_A \tag{C.9}$$

are exactly the combinations introduced in eq. (5.14) in the text.

[19] In fact, this is not an irreducible representation: it decomposes into the $(n+1)(2n+1)-1$ traceless symmetric tensor plus a singlet. One of the $\vec{0}$ weights corresponds to the singlet.

[20] Again, this is not an irreducible representation, since it contains a singlet.

JHEP07(2009)092

Integration in the cases $k = 4, 5$. The group SO(4) has rank 2, and the poles of the integrand of eq. (6.6) are determined by the polynomial $\mathcal{Q}(\chi_1, \chi_2)$. According to eq. (6.19) and to the set of weights in eq. (C.6), the χ-dependent part of $\mathcal{Q}$ (i.e., the one determined from the non-zero weights) is

$$\begin{aligned}\prod_{A=1}^{4}(2\chi_1 - E_A)(-2\chi_1 - E_A)(2\chi_2 - E_A)(-2\chi_2 - E_A) \\ (\chi_1 - \chi_2 - E_A)(-\chi_1 + \chi_2 - E_A)(\chi_1 + \chi_2 - E_A)(-\chi_1 - \chi_2 - E_A)\ .\end{aligned} \tag{C.10}$$

Let's label the various types of monomials from 1 to 8 in the order appearing above. With the prescriptions given in eq. (6.21), it is straightforward to see that all poles in the integrand of eq. (6.6) are simple (in certain cases, apparent double poles are compensated by zeroes of the Vandermonde determinant). We have to sum the residues over different possible classes of poles. For instance, we could, from the χ_1 integral, pick up the residue from a simple pole determined by the 5^{th} factor in eq. (C.10):

$$\chi_1 = \chi_2 + E_A\ . \tag{C.11}$$

After substituting this value in the remaining terms of the integrand, we integrate over χ_2 and we can again pick up contributions from various possible poles. For instance, suppose that we choose the one coming from the third factor:

$$\chi_2 = \frac{E_B}{2} \tag{C.12}$$

and make this replacement in all remaining factors of the integrand to compute the residue. The choices eq. (C.11) and eq. (C.12) are possible for all A, B, so we have to sum the residues over A, B independently; let us write this particular contribution to the integral as

$$\sum_{A,B}(5,3)\ . \tag{C.13}$$

With this condensed notation, it is straightforward to check that the contributions to the integral are the following:

$$\sum_{A\neq B}(1,3) + \sum_{A,B}(1,6) + \sum_{A,B}(5,3) + \sum_{A,B}(7,2) + \sum_{A\neq B}(7,3) + \sum_{A,B}(7,6)\ . \tag{C.14}$$

In fact, there are also other contributions that, however, cancel in pairs:

$$0 = \left(\sum_{A\neq B}(1,7) + \sum_{A\neq B}(7,1)\right) + \left(\sum_{A>B}(5,7) + \sum_{A<B}(7,5)\right)\ . \tag{C.15}$$

Evaluating explicitly the sums in eq. (C.14) one obtains Z_4; inserting it in eq. (7.5) one determines F_4, as described in the main text.

Let us now move to SO(5), which again has rank 2. According to eq. (6.20) and to the set of weights in eq. (C.3), the χ-dependent part of $\mathcal{Q}$ (i.e., the one determined from the

non-zero weights) is

$$\begin{aligned}\prod_{A=1}^{4}&(2\chi_1-E_A)(-2\chi_1-E_A)(2\chi_2-E_A)(-2\chi_2-E_A)\\&(\chi_1-\chi_2-E_A)(-\chi_1+\chi_2-E_A)(\chi_1+\chi_2-E_A)(-\chi_1-\chi_2-E_A)\\&(\chi_1-E_A)(-\chi_1-E_A)(\chi_2-E_A)(-\chi_2-E_A)\ .\end{aligned} \tag{C.16}$$

Let us label the various types of monomials from 1 to 12 in the order appearing above. One can check that only simple poles appear and, using the condensed notation introduced above, the classes of residues that contribute to Z_5 are the following:

$$\begin{aligned}&\sum_{A\neq B}(1,3)+\sum_{A,B}(1,6)+\sum_{A,B}(1,11)+\sum_{A,B}(5,3)+\sum_{A,B}(5,11)+\sum_{A,B}(7,2)+\sum_{A\neq B}(7,3)\\&\quad+\sum_{A,B}(7,6)+\sum_{A,B}(7,10)+\sum_{A\neq B}(7,11)+\sum_{A,B}(9,3)+\sum_{A,B}(9,6)+\sum_{A\neq B}(9,11)\,,\end{aligned} \tag{C.17}$$

having already taken into account the pairwise cancellation of other classes of contributions:

$$\begin{aligned}0=&\left(\sum_{A\neq B}(1,7)+\sum_{A\neq B}(7,1)\right)+\left(\sum_{A>B}(5,7)+\sum_{A<B}(7,5)\right)\\&+\left(\sum_{A>B}(5,9)+\sum_{A<B}(9,5)\right)+\left(\sum_{A<B}(7,9)+\sum_{A>B}(9,7)\right)\ .\end{aligned} \tag{C.18}$$

Explicitly evaluating the sums in eq. (C.17) one obtains Z_5; inserting it in eq. (7.5) one determines F_5, as described in the main text.

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JHEP07(2009)092