CHAPTER 18

Asian Options: Payoffs and Pricing Models

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In this chapter, we describe and compare alternative procedures for pricing Asian options. Asian options are derivatives contracts written on an average price. More precisely, prices of an underlying security (or index) are recorded on a set of dates during the lifetime of the contract. At the option's maturity, a payoff is computed as a deterministic function of an average of these prices. As reported by Falloon and Turner (1999), the first contract linked to an average price was traded in 1987 by Bankers & Trust in Tokyo, hence the attribute 'Asian'.

Asian options are quite popular among commodity derivative traders and risk managers. This is due to several reasons.

Primarily, Asian options smooth possible market manipulations occurring near the expiry date. In general, the longer the averaging period, the smoother the path. This is shown in Figures 18.1 and 18.2. The first figure shows three simulated paths of the underlying and of its average: the strong oscillations in the underlying path disappear as we consider the time average. The second figure presents simulated paths of the two quantities and the simulated distributions one year in the future: the one that refers to the arithmetic average appears much less dispersed than the one referring to the underlying.

Secondly, Asian options provide a suitable hedge for firms facing a stream of cash flows. This is the case, for instance, with commodity end-users that are financially exposed to average prices. Asian-style options, and other options written on alternative definitions of average prices, are effective hedging devices in commodity markets. Eydeland and Wolyniec (2003) provide an example of how these derivatives play an important role in price risk management performed by local delivery companies in the gas market. Moreover, oil markets often use these securities to stabilize cash flows that stem from meeting obligations to clients.

A few examples of Asian options traded on organized markets are:

- The New York Mercantile Exchange (NYMEX) and Intercontinental Continental Exchange (ICE) offer several average price products which are linked to energy

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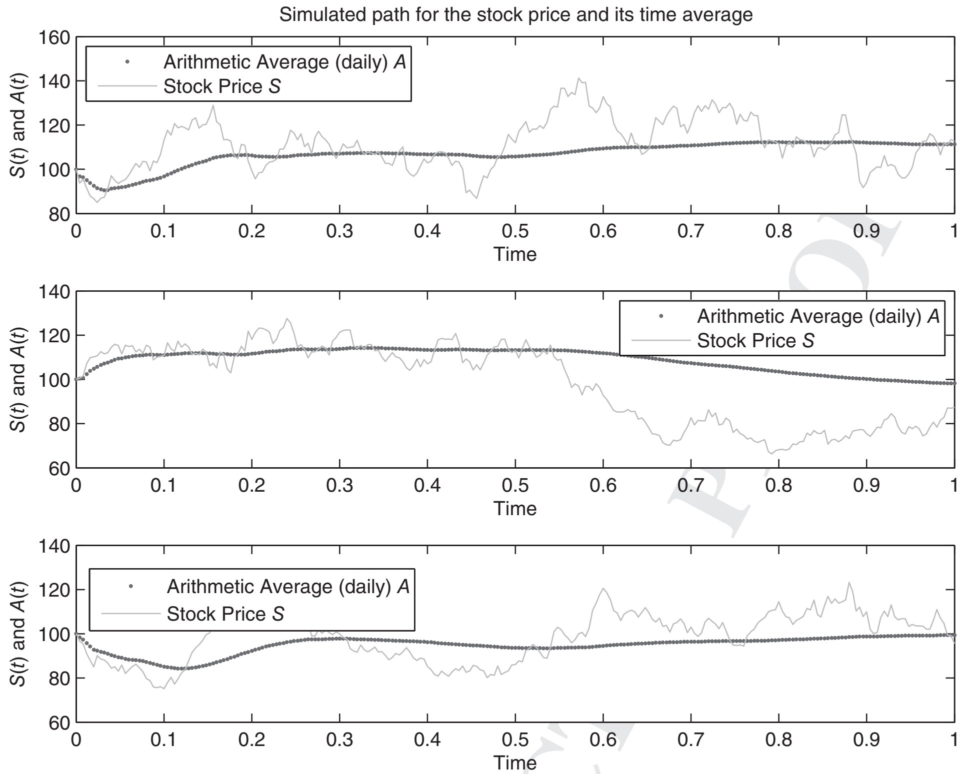

FIGURE 18.1 Simulated paths of spot price and its time (arithmetic) average

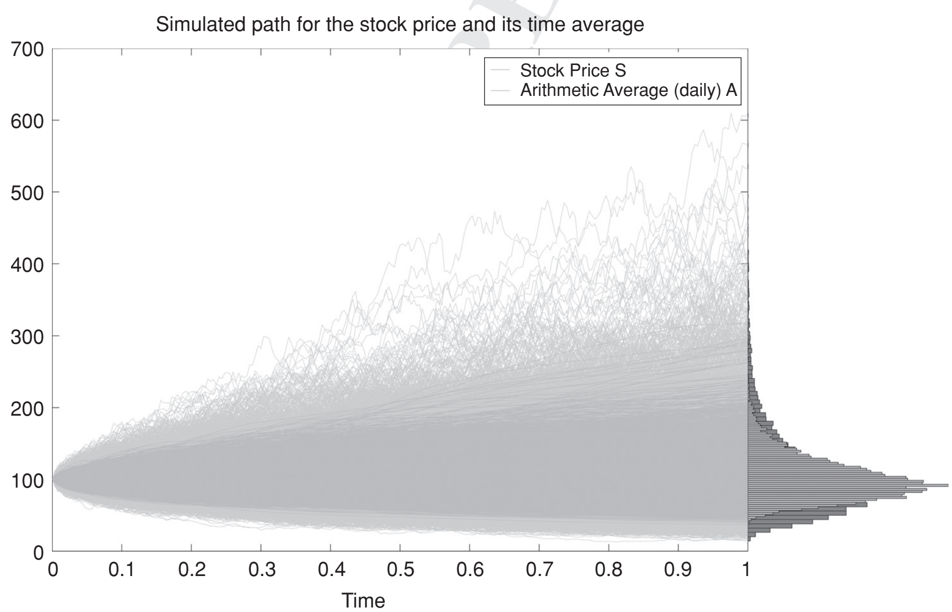

FIGURE 18.2 Simulated paths and terminal distribution of spot price and arithmetic average

TABLE 18.1 Average price options traded at NYMEX

WTI Average Price Option
Underlying Futures
Light Sweet Crude Oil Futures (CL)
Contract Unit
On expiration of a call option, the value will be the difference between the average daily settlement price during the calendar month of the first nearby underlying Light Sweet Crude Oil Futures and the strike price multiplied by 1000 barrels, or zero, whichever is greater. On expiration of a put option, the value will be the difference between the strike price and the average daily settlement price during the calendar month of the first nearby underlying Light Sweet Crude Oil Futures multiplied by 1000 barrels, or zero, whichever is greater.
Price Quotation
US dollars and cents per barrel
Option Style
Average Price non-early exercisable option
Minimum Fluctuation
$0.01 per barrel
Expiration of Trading
Trading ends the last business day of the calendar month
Listed Contracts CME ClearPort and Open Outcry: 72 consecutive months CME Globex: 1 consecutive month
Strike Prices
Twenty strike prices in increments of $0.50 (50) per barrel above and below the at-the-money strike price, and the next 10 strike prices in increments of $2.50 above the highest and below the lowest existing strike prices for a total of at least 61 strike prices. The at-the-money strike price is nearest to the previous day's close of the underlying futures contract. Strike price boundaries are adjusted according to the futures price movements. In addition, options trading can be conducted in strike price increments of $0.01.
Settlement Type
Financial
Exchange Rule
These contracts are listed with, and subject to, the rules and regulations of NYMEX

Source: http://www.cmegroup.com/trading/energy/crude-oil/light-sweet-crude_contractSpecs_options.html#prodType=AVP.

products, e.g. Brent Average Price Options and WTI (West Texas Intermediate) Average Price Options. Details are provided in Tables 18.1 and 18.2.

- The London Metal Exchange (LME) offers Traded Average Price Options (TAPOs) based on the LME Monthly Average Settlement Price (MASP) for several metals, such as for copper grade A, high-grade primary aluminium, standard lead, primary nickel, special high-grade zinc, aluminium alloy and tin. Because many users in the industry price their physical material on the basis of the LME MASP, brokers developed off-exchange average price option products, known as Asians, which quickly became popular, particularly with large producers. To meet this growing demand, the LME developed the TAPO contracts. TAPO contracts complement existing LME futures and traded options contracts. Details for a copper TAPO contract are provided in Table 18.3.
- The Chicago Mercantile Exchange (CME) launched trading for three new cash-settled petroleum crack spread average price options contracts in July, 2009. These new average

TABLE 18.2 Average options on Brent traded at ICE

ICE BRENT AVERAGE PRICE OPTION

Description

The Brent Average Price Option is based on the underlying ICE Brent 1st Line Future (I) and will automatically exercise into the settlement price of the 1st Line Future on the day of expiry of the options contract.

Contract Symbol I

Hedge Instrument: The delta hedge for the Brent Average Price Option is the ICE Brent 1st Line Swap Future (I)

Contract Size

1000 barrels

Unit of Trading

Any multiple of 1000 barrels

Currency

US dollars and cents

Trading Price Quotation

One cent ($0.01) per barrel

Settlement

Price Quotation: One tenth of one cent ($0.001) per barrel

Minimum Price Fluctuation

One tenth of one cent ($0.001) per barrel

Last trading day

Last trading day of the contract month

Option Type

Options are Asian-style and will be automatically exercised on the expiry day if they are in-the-money. The swap future resulting from exercise immediately goes to cash settlement, relieving market participants of the need to concern themselves with liquidation or exercise issues. If an option is out-of-the-money it will expire automatically. It is not permitted to exercise the option on any other day or in any other circumstances than the last trading day. No manual exercise is permitted.

Expiry

19:30 London Time (14:30 EST). Automatic exercise settings are pre-set to exercise contracts which are one minimum price fluctuation or more 'in-the-money' with reference to the relevant reference price. Members cannot override automatic exercise settings or manually enter exercise instructions for this contract. The reference price will be a price in USD and cents per barrel equal to the average of the settlement prices as made public by ICE for the Brent 1st Line Swap Future for the contract month. When exercised against, the Clearing House, at its discretion, selects sellers against which to exercise on a pro rata basis.

Option Premium / Daily Margin

The premium on the Brent Average Priced Option is paid/received on the business day following the day of trade. Net Liquidating Value (NLV) will be re-calculated each business day based on the relevant daily settlement prices. For buyers of options, the NLV credit will be used to off set their Original Margin (OM) requirement; for sellers of options, the NLV debit must be covered by cash or collateral in the same manner as the OM requirement. OM for all options contracts is based on the options delta.

Strike Price Intervals

Minimum $0.50 increment strike prices: $1.00 strikes from $20 to $240; $0.50 strikes, 20 strikes above and below ATM. The at-the-money strike price is the closed interval nearest to the previous business day's settlement price of the underlying contract.

Contract Series

Up to 72 consecutive months

Final Payment Date

Two Clearing House business days following the last trading day. Business Days: Publication days for ICE.

Source: https://www.theice.com/productguide/ProductSpec.shtml;jsessionid=2A767FBF878E8F31ADADC5ED9B70B639?specId=11523783.

TABLE 18.3 LME traded average price options (TAPO's) on copper specifications

TAPO Contract Specifications

Contract Metals
Copper grade A, high-grade primary aluminium, standard lead, primary nickel, special high-grade zinc, aluminium alloy and tin.

Contract Date
The business day on which the contract is traded

Contract Period
Calendar months up to 27 months forward for copper grade A, high-grade primary aluminium, primary nickel, special high-grade zinc, and 15 months forward for standard lead, aluminium alloy and tin. Contracts can be traded daily up to and including the penultimate business day of the current month.

Option Type
Calls and puts based on the monthly average settlement price (MASP). No early exercise. Fixed period: The period between the first business day of the current month and the last business day of the month (inclusive).

Strike Price
$1 gradations

Currency
US dollars

Minimum Tick Size
0.01 USD (one cent)

Premium Payment
Next business day after the contract is traded.

Exercise
The exercise process is automatic once the LME monthly average settlement price is made official. A TAPO contract that is 'in-the-money' generates two futures trades per member which are equal and opposite in tonnage. One trade corresponds to the MASP and the other to the original strike price of the option.

Settlement Date
Settlement is two business days after exercise. The futures trades settle as per LME rules and regulations.

Margining
Like all existing LME contracts, TAPOs are margined using the SPAN methodology.

MASP
The arithmetic average of all settlement prices determined during the fixing period. This becomes an official LME price on the last day of the current month at 3.00 pm.

Source: http://www.lme.com/en-gb/trading/contract-types/tapos/.

options are the gasoil–Brent crude oil crack spread options, the heating oil–crude oil crack spread options, and the RBOB–crude oil crack spread options.

- The freight options currently traded are contracts to settle the difference between the average spot freight rate over a prespecified period of time and an agreed strike price. Freight options in the dry bulk market are traded on the Baltic Capesize Index (BCI), Baltic Panamax Index (BPI) and Baltic Supramax Index (BSI). The Baltic indices are calculated on a daily basis by the Baltic Exchange based on data supplied by a panel of independent international shipbrokers, and are reported in the market at 13:00 h London time. Freight

TABLE 18.4 Payoff structures of European and Asian options. $S(T)$ is the underlying price at option maturity T, whilst $A(T)$ stands for some form of averaging of the underlying asset price – see equations (18.1) and (18.2)

Type	Call	Put
European options	$(S(T) - K)^+$	$(K - S(T))^+$
Fixed-strike Asian options	$(A(T) - K)^+$	$(K - A(T))^+$
Floating strike Asian options	$(S(T) - A(T))^+$	$(A(T) - S(T))^+$

options on a Baltic index settle the difference between the arithmetic average of the spot Baltic assessments over the trading days of the settlement month and an agreed strike price. The options are executed between two counterparties through a broker primarily as an OTC contract, though the majority of the trades are subsequently cleared through a clearing house and quoted in terms of implied volatility.[1]

- Other examples include commodity-linked bonds on average bond prices and Asian-style catastrophe (CAT) insurance options with payoffs depending on the accumulated catastrophic losses, see Chang *et al.* (2010).

18.1 PAYOFF STRUCTURES

The payoff structure of plain vanilla European options, fixed and floating strike Asian options, is illustrated in Table 18.4. The quantity A in this table represents the (possibly weighted) arithmetic average of spot prices over a given time frame up to the option expiry. If we let $S(t)$ be the underlying spot price at time t, then A is given by

$$A(T) := A(0, T) = \sum_{i=0}^{N} w(t_i)S(t_i), \tag{18.1}$$

where $T = t_N$ refers to the option expiry and $t_i, i = 0, \ldots, N$, with $t_0 = 0$, refer to the so-called monitoring dates, that is, the dates at which the underlying price is taken for entering in the

[1]Implied volatilities of Baltic options assessments, i.e. implied volatility for an at-the-money option in the Dry Bulk Option market submitted by brokers at 17.30 (London), are published by the Baltic Exchange (see the website www.balticexchange.com). Options are at-the-money, i.e. strikes are set equal to the prevailing forward freight agreement rate. Market prices can be recovered, in line with market practice, by inserting the quoted volatility in the Asian option price formula of Turnbull and Wakeman and Levy, to be discussed later in this chapter. The market quotes are for forward start freight call options on the BCI, BPI and BSI for the next four quarters (+1Q, +2Q, +3Q, +4Q) and the next two calendar years (+1CAL and +2CAL). Each quarter contract consists of three options that expire at the end of each month in the quarter of interest, whereas a calendar contract is a strip of 12 monthly options. If on 4 January, 2008 an investor holds the BCI+1Q, this contract comprises three freight options which settle at the end of April 2008, May and June 2008. The settlement prices of each of these options are given by the average of the BCI spot rates over the trading days of the respective settlement month. The main characteristics of freight market indexes are illustrated in Chapter 8. Additional information can be found in Nomikos *et al.* (2013).

computation of the average, w_i is the weight attributed to each observation (with the constraint that the sum of weights equals 1). The most common weighting scheme is equally, that is $w(t_i) = 1/(N+1)$. In general, the first monitoring date is the trade date.

It is common practice to price Asian options assuming that the average is recorded continuously over the option lifetime rather than at discrete dates. Therefore, the sum in formula (18.1) is replaced by an integral as follows:

$$A(T) = \int_0^T w(u)S(u)du. \tag{18.2}$$

Few variants to the above expression are possible, for example a partial average option for which the time interval taken into account for the average calculation is a subset of the full life of the option. In a forward starting option, the calculation of the average starts at a later instant with respect to the trade date, so that the time-to-maturity period is always larger than the time averaging period. In the partial averaging case, (18.1) becomes

$$A(t_k, t_m) = \sum_{i=k}^{m} w(t_i)S(t_i)$$

where $0 < k < m < N$ and in the forward starting case, (18.1) becomes

$$A(t_k, t_N) = \sum_{i=k}^{N} w(t_i)S(t_i).$$

In the continuous monitoring case, the partial averaging and the forward starting options are respectively defined as $\int_{t_k}^{t_m} w(u)S(u)du$ and $\int_{t_k}^{T} w(u)S(u)du$.

Market practice assumes that the underlying asset price evolves according to Black–Scholes lognormal dynamics. Unfortunately, given this model setup, the average depends on a sum of correlated lognormal variates and the probability distribution of the average does not admit a simple analytical expression. Consequently, numerical approximations need to be developed for the purpose of pricing arithmetic Asian options. We review these in the next section.

18.2 PRICING ASIAN OPTIONS IN THE LOGNORMAL SETTING

This section illustrates the most common procedures for pricing Asian options in the Black–Scholes lognormal setting, that is assuming that the risk-neutral process for the underlying asset is a geometric Brownian motion satisfying

$$dS(t) = (r - q)\, S(t)\, dt + \sigma S(t)\, dW(t), S_0 = s_0, \tag{18.3}$$

where $W(t)$ is a standard Brownian motion, r is the continuously compounding rate of interest, q the continuous dividend yield and σ is the instantaneous percentage price volatility. It is convenient also to notice that (see Chapter 12)

$$S(t) = s_0 e^{\left(r-q-\frac{\sigma^2}{2}\right)t+\sigma W(t)}. \tag{18.4}$$

Given that the most common weighting is equally, we also have

$$A(T) = \frac{s_0}{T}\int_0^T e^{\left(r-q-\frac{\sigma^2}{2}\right)u+\sigma W(u)}du \quad \text{(continuous monitoring)},$$

or

$$A(T) = \frac{s_0}{N+1}\sum_{i=0}^{N} e^{\left(r-q-\frac{\sigma^2}{2}\right)i\Delta+\sigma W(i\Delta)} \quad \text{(discrete monitoring)}.$$

In both cases, continuous and discrete monitoring, the fixed strike Asian option fair price is given by

$$e^{-rT}\widetilde{E}_0\left(A(T)-K\right)^+, \tag{18.5}$$

where $\widetilde{E}_0$ denotes expectation under the risk-neutral probability measure. The pricing problem consists of finding the distribution function of $A(T)$.

If the option is into the averaging period, the above expectation can be computed by adjusting the strike price to take into account the average observed so far. In practice, if we let T_1 be the length of the averaging period so far, and $T - T_1$ the length of the remaining averaging period, the option price is given by

$$e^{-r(T-T_1)}\widetilde{E}_{T_1}\left(A(0,T)-K\right)^+. \tag{18.6}$$

The average price can be decomposed as

$$A(0,T) = \frac{T_1}{T}A(0,T_1) + \frac{T-T_1}{T}A(T_1,T).$$

Substituting this expression in (18.6), we observe that the payoff now becomes

$$e^{-r(T-T_1)}\frac{T}{T-T_1}(\widetilde{E}_0(A(T_1,T)) - \hat{X}),$$

where the modified strike price $\hat{X}$ is

$$\hat{X} = \frac{T}{T-T_1}\left(X - \frac{T_1}{T}A(0,T_1)\right)$$

and $A(0, T_1)$ is the average realized so far. In particular, if $\hat{X}$ is negative, that is

$$\frac{T_1}{T} A(T_1) > X,$$

it means that the call option at maturity will be exercised for sure: the average so far is so high that the remaining averaging period cannot make the option become out-of-the-money at maturity (vice versa, the put option will not be exercised for sure). Therefore, the call option value will be

$$e^{-r(T-T_1)} \frac{T}{T-T_1} \left(\widetilde{E}_{T_1}(A(T_1, T)) - \hat{X}\right),$$

whilst the put option will be worthless. The computation of the expectation of the average is discussed in Boxes 18.1 and 18.2, depending on the monitoring convention. The extension to Asian options on futures prices is considered in Box 18.3.

Albeit there exist very accurate procedures to compute the expectation in (18.5) under a geometric Brownian motion, we restrict attention here to those that combine accuracy and implementation simplicity. To do this we briefly illustrate:

1. Approximation of the average distribution by fitting integer moments (Ju, 2002; Levy, 1992; Milevsky and Posner, 1998; Turnbull and Wakeman, 1991).
2. Computation of lower bound for the price (Rogers and Shi, 1992; Thompson, 1998).
3. Monte Carlo simulation (see, e.g., the discussion in Fu *et al.*, 1998.

The first method derives a probability distribution sharing a number of moments with the distribution of the price average. The second approach aims to calculate tight lower bounds for the exact option price. The third prices an Asian option resorting to simulation.

Other procedures, which are very accurate but whose implementation is not straightforward, such as the eigenfunction method in Lewis (1998), the numerical solution of the pricing partial differential equation (PDE) (Rogers and Shi, 1992; Vecer, 2001), the upper bound provided by Thompson (1998) and Rogers and Shi (1992), numerical inversion of a single Laplace transform (Geman and Yor, 1993; Lewis, 2002; Shaw, 1998) or of a double transform (Cai and Kou, 2012; Fusai, 2004) are not illustrated here. A detailed comparison among these procedures can be found in Fusai and Roncoroni (2008), chapter 15.

18.2.1 Moment Matching

Moment matching is the most popular approach for pricing Asian options. The average price is assigned an arbitrary probability density function constrained to match a number of moments of $A(T)$. Unfortunately, this method does not provide any assessment of the approximation error. This procedure consists of two steps:

1. Derive a closed-form expression for the moments of $A(T)$

$$\mu_n = \widetilde{E}_0 \left[A_T^n\right].$$

Expressions for their computation are provided in Boxes 18.1 and 18.2, depending on the monitoring convention we adopt.

2. Choose and fit an arbitrary density function to a number of selected moments. Specifically, we consider lognormal and Edgeworth series approximations.

BOX 18.1 MOMENTS OF $A(T)$ IN THE CONTINUOUSLY MONITORED CASE

If we consider the continuously monitored case, we have that (see Geman and Yor, 1993)

$$\mu_n := \frac{s_0^n}{T^n}\frac{n!}{\lambda^{2n}}\left\{\sum_{j=0}^{n} d_j^{(\gamma/\lambda)} \exp\left[\left(\frac{\lambda^2 j^2}{2} + \lambda j \gamma\right)T\right]\right\}, \tag{18.7}$$

where

$$d_j^{(\beta)} = 2^n \prod_{\substack{0\le i\le n\\ i\ne j}} [(\beta+j)^2 - (\beta+i)^2]^{-1},$$

$$\lambda = \sigma, \qquad \gamma = \frac{r-q-\sigma^2/2}{\sigma}. \tag{18.8}$$

Care has to be taken in computing moments when $r = q$ (a practical approach is to use the above expressions, setting $r = q + 0.000001$). In particular, notice that if $r = q$, then $\mu_1 = s_0$ and

$$\mu_2 = \frac{2e^{\sigma^2 T} - 2(1+\sigma^2 T)}{\sigma^4 T^2}.$$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%COMPUTING MOMENTS OF THE ARITH AVERAGE%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function dj=djbeta(n, beta, j)
term = 1;
for i = 0:n
  if abs(i - j)>0
      term = term * (1 / ((beta + j) ^ 2 - (beta + i) ^ 2));
  end
end

  dj = term * (2 ^ n);

function mn=moment_n(n, v, lambda, t)
term = 0;
```

```
fac = 1;
for j = 0:n %Step -1
  term = term + ...
          djbeta(n, v / lambda, j) * ...
    exp((lambda * lambda * j * j / 2 + lambda * j * v) * t);
  if (j > 0)
      fac = fac * j;
  end
end
  mn = fac * term / (lambda ^ (2 * n));
```

By way of illustration, let us suppose that $r = 0.05$, $q = 0$, $\sigma = 0.2$, $\Delta = 1/12$ (i.e. 1 month) and $T = 1$, so that $m = 0.0025$. We have

n	1	2	3	4
μ_n	1.0254	2.1034	3.2367	4.4281

BOX 18.2 MOMENTS OF $A(T)$ IN THE DISCRETELY MONITORED CASE

In the following, we exploit a recursive formulation to compute the moments of $A(T)$. Given (18.4), let us define the log-price increment over a time step of length Δ

$$Z_k^\Delta \equiv m\Delta + \sigma X_k^\Delta, \quad k = 1, \dots, N,$$

where $m = r - q - \frac{\sigma^2}{2}$, $\Delta = T/N$ and X_k^Δ is the increment of the Brownian motion, so that $X_k^\Delta \sim \mathcal{N}(0, \Delta)$. We are interested in the moments of

$$\begin{aligned}\sum_{k=0}^{N} S_{\Delta k} &= s_0 + s_0 e^{Z_1} + s_0 e^{Z_1+Z_2} + \cdots + s_0 e^{Z_1+\cdots+Z_N} \\ &= s_0\left(1 + e^{Z_1^\Delta}\left(1 + e^{Z_2^\Delta}\left(\cdots\left(1 + e^{Z_N^\Delta}\right)\right)\right)\right).\end{aligned}$$

Starting from $L_T^\Delta \equiv e^{Z_1^\Delta}$ and introducing recursively the quantities

$$L_k^\Delta \equiv e^{Z_k^\Delta}\left(1 + L_{k+1}^\Delta\right), \quad k = N-1, \dots, 1, \tag{18.9}$$

we have $A(T) \equiv S_0 \left(1 + L_1^\Delta\right) / (N+1)$. Recursion (18.9) translates into a formula for the moments of the arithmetic average. Indeed, from the independence of Z_k^Δ and L_{k+1}^Δ as well as from the definition of Z_k^Δ, we obtain

$$\widetilde{E}\{(L_k^\Delta)^n\} = \widetilde{E}\{(e^{Z_k^\Delta}(1 + L_{k+1}^\Delta))^n\} \tag{18.10}$$

$$= \widetilde{E}\{e^{nZ_k^\Delta}\}\widetilde{E}\{(1 + L_{k+1}^\Delta)^n\} \tag{18.11}$$

$$= \widetilde{E}\{e^{nZ_k^\Delta}\}\widetilde{E}\left\{\sum_{q=0}^{n} \frac{n}{q} (L_{k+1}^\Delta)^q\right\} \tag{18.12}$$

$$= \phi_\Delta(n) \sum_{q=0}^{n} \frac{n}{q} \widetilde{E}\{(L_{k+1}^\Delta)^q\}, \tag{18.13}$$

where

$$\phi_\Delta(n) = \tilde{E}\left(e^{nZ_k^\Delta}\right) = e^{(r-q-\frac{\sigma^2}{2})\Delta n + \frac{1}{2}\sigma^2\Delta n^2}.$$

The recursion starts with

$$\widetilde{E}\{(L_N^\Delta)^n\} \equiv \widetilde{E}\{e^{nZ_1^\Delta}\} = \phi_\Delta(n). \tag{18.14}$$

The moments of the arithmetic average can be computed as follows:

$$\widetilde{E}\left((A(T))^n\right) = \widetilde{E}\left(\frac{S_0^n\left(1 + L_1^\Delta\right)^n}{(N+1)^n}\right) = \frac{S_0^n}{(N+1)^n} \sum_{j=0}^{n} \binom{n}{j} \widetilde{E}\left\{\left(L_1^\Delta\right)^j\right\}. \tag{18.15}$$

By way of illustration, let us suppose that $r = 0.05$, $q = 0$, $\sigma = 0.2$, $\Delta = 1/12$ (i.e. 1 month) and $T = 1$, so that $m = 0.0025$. In addition, we have

$$\phi(1) = 1.0042,\ \phi(2) = 1.0117,\ \phi(3) = 1.0228, \quad \phi(4) = 1.0373.$$

We can create the following table, which in each row provides the first four moments of $L_k^\Delta, k = 12, \ldots, 1$ at each time step.

Month	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
0	1	1	1	1	1
1	1	1.004175	1.011735	1.022755	1.037347
2	1	2.012544	4.067261	8.25413	16.82108
3	1	3.025122	9.199047	28.11916	86.40204
4	1	4.041928	16.43998	67.28876	277.1516
5	1	5.062980	25.82335	132.6866	686.8423
6	1	6.088295	37.38291	231.4961	1445.835
7	1	7.117891	51.15282	371.1677	2719.374
8	1	8.151787	67.16767	559.4264	4710.006
9	1	9.189999	85.46251	804.279	7660.109
10	1	10.23255	106.0728	1114.022	11854.55
11	1	11.27945	129.0346	1497.25	17623.5
12	1	12.33072	154.3842	1962.864	25345.33

For example, in the last row we can read the moments of $(L_1)^n$, $n = 0, \ldots, 4$. They are computed from the moments in the previous row via

$$\begin{aligned}
\widetilde{E}\{(L_1)^1\} &= 1 \times 1 + 1 \times 11.2794 = 12.3307,\\
\widetilde{E}\{(L_1)^2\} &= 1 \times 1 + 2 \times 11.2794 + 1 \times 129.0346 = 154.3842,\\
\widetilde{E}\{(L_1)^3\} &= 1 \times 1 + 3 \times 11.2794 + 3 \times 129.0346 + 1 \times 1497.2503 = 1962.8637,\\
\widetilde{E}\{(L_1)^4\} &= 1 \times 1 + 4 \times 11.2794 + 6 \times 129.0346 + 4 \times 1497.2503 + 1\\
&\quad \times 17623.5018 = 25345.3255.
\end{aligned}$$

The first four moments of $A(1)$ can then be computed as follows (let us assume that $s_0 = 1$):

$$\begin{aligned}
\widetilde{E}\{(A(1))^1\} &= \frac{1 + 12.3307}{13} = \frac{13.3307}{13},\\
\widetilde{E}\{(A(1))^2\} &= \frac{11 \times 1 + 2 \times 12.3307 + 1 \times 154.3842}{13^2} = \frac{180.0456}{13^2},\\
\widetilde{E}\{(A(1))^3\} &= \frac{11 \times 1 + 3 \times 12.3307 + 3 \times 154.3842 + 1 \times 1962.8637}{13^3} = \frac{2464.0083}{13^3},\\
\widetilde{E}\{(A(1))^4\} &= \frac{11 \times 1 + 4 \times 12.3307 + 6 \times 154.3842 + 4 \times 1962.8637 + 1 \times 25345.3255}{13^4}\\
&= \frac{34173.4080}{13^4}.
\end{aligned}$$

18.2.1.1 Lognormal Approximation (Turnbull–Wakeman–Levy formula) In this approximation, see Levy (1992) and Turnbull and Wakeman (1991), we assume that the average $A(T)$ is lognormally distributed with mean m and variance v^2. The parameters m and v^2 are chosen to match exactly the mean and variance of the arithmetic average, given in

Boxes 18.1 and 18.2. Owing to its simplicity, this approximation has gained large popularity. The approximated Asian call option price turns out to be given by the modified Black–Scholes formula

$$c_{\log} = s_0 e^{m+v^2/2-rT}\mathcal{N}(d_1) - e^{-rT}\mathcal{N}(d_2), \tag{18.16}$$

where

$$\begin{aligned} m &= 2\log\mu_1 - \frac{1}{2}\log\mu_2, \qquad & v^2 &= \log\mu_2 - 2\log\mu_1, \\ d_1 &= \frac{\ln(s_0/K) + m + v^2}{v}, \qquad & d_2 &= d_1 - v. \end{aligned} \tag{18.17}$$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%PRICING ASIAN OPTIONS VIA LOGNORMAL DISTRIBUTION%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function res= AsianCalllog(Spot, strike, rf, sigma, t)

%Compute the first two moments of the Average
  m1 = moment_n(1, (rf - sigma * sigma / 2) / sigma, sigma, t) / t;
  m2 = moment_n(2, (rf - sigma * sigma / 2) / sigma, sigma, t) / t ^ 2;

%Fit the parameters of the lognormal density
  m = 2 * log(m1) - log(m2) / 2;
  v = sqrt(log(m2) - 2 * log(m1));

%Compute the Levy approximation
  d1 = (log(Spot / strike) + m + v * v) / v;
  d2 = d1 - v;
  esp = m + v * v / 2 - rf * t;
  nd1 = normcdf((log(Spot / strike) + m + v * v) / v,0,1);
  nd2 = normcdf((log(Spot / strike) + m + v * v) / v - v,0,1);

%The result
  res = Spot * exp(esp) * nd1 - exp(-rf * t) * strike * nd2;
```

18.2.1.2 Edgeworth Series Approximation The lognormal approximation only captures the mean and variance of the average. In order to fit the third and fourth moment as well, that is, skewness and kurtosis of the average, Turnbull and Wakeman (1991) proposed to adopt a fourth-order Edgeworth series expansion of the true (but unknown) distribution of $A(T)$ around the lognormal. This approximation works as follows.

Let k_n be the difference in the nth cumulant[2] between the exact distribution f and the approximate lognormal distribution l, namely $k_n = \chi_n(f) - \chi_n(l)$. We have:

$$\begin{aligned}
\chi_1(f) &= \mu_1, \\
\chi_2(f) &= \mu_2 - \mu_1^2, \\
\chi_3(f) &= \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3, \\
\chi_4(f) &= \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4.
\end{aligned}$$

Parameters m and v^2 are set according to expression (18.17), so that $k_1 = k_2 = 0$, while the cumulants of the approximating lognormal distribution can be computed as

$$\chi_n(l) = \exp\left(nm + \frac{1}{2}n^2v^2\right), \qquad n = 1, 2, 3, 4.$$

The approximate Asian option price is given by

$$c_{\text{edg}} = c_{\log} + e^{-rT}\frac{s_0}{T}\left[-\frac{k_3}{6}\frac{\partial f_{\log}(y; m, v^2)}{\partial y} + \frac{k_4}{24}\frac{\partial^2 f_{\log}(y; m, v^2)}{\partial y^2}\right]_{y=TK/s_0}, \tag{18.18}$$

where $c_{\log}$ is defined in formula (18.16) and $f_{\log}(y; m, v^2)$ is the lognormal density with parameters m and v^2:

$$f_{\log}\left(y; m, v^2\right) = \frac{1}{\sqrt{2\pi v^2}y}\exp\left(-\frac{(\ln y - m)^2}{2v^2}\right), \quad y > 0.$$

The main problem of the Edgeworth series is that increasing the number of matched moments does not guarantee an improvement in the resulting approximation. Since the distribution of $A(T)$ is not univocally determined by its moments, the approximation (18.18) may even lead to a negative-valued density.[3]

To overcome this problem, Ju (2002) considers the Edgeworth series for approximating the distribution of $\ln A(T)$ with a normal distribution, and he obtains the following approximation:

$$c_{\text{Ju}} = c_{\log} + e^{-rT}K\left[z_1 n(y) + z_2\frac{\partial n(y)}{\partial y} + z_3\frac{\partial^2 n(y)}{\partial y^2}\right]_{y=\ln(K/s_0)}, \tag{18.19}$$

[2]Cumulants of a random variable are defined as coefficients in the Taylor expansion of the logarithm of the moment-generating function about the origin, and are related to moments.

[3]A discussion in the context of Asian options under which the Edgeworth expansion is positive and unimodal can be found in Ju (2002).

where $c_{\log}$ is given in (18.16), $n(y) = n(y; m, v^2)$ is the Gaussian density with mean m and variance v^2 given in (18.17):

$$n(y; m, v^2) = \frac{1}{\sqrt{2\pi v^2}} \exp\left(-\frac{(y-m)^2}{2v^2}\right),$$

and derivatives are computed as

$$\frac{\partial n(y; m, v^2)}{\partial y} = -\frac{(y-m)}{v^2} n(y; m, v^2),$$
$$\frac{\partial^2 n(y; m, v^2)}{\partial y^2} = \frac{(m^2 - v^2 - 2my + y^2)}{v^4} n\left(y; m, v^2\right).$$

The remaining coefficients are as follows:

$$\begin{aligned}
z_1 &= -\sigma^4 T^2 \left(\frac{1}{45} + \frac{x}{180} - \frac{11x^2}{15120} - \frac{x^3}{2520} + \frac{x^4}{113400}\right) \\
&\quad - \sigma^6 T^3 \left(\frac{1}{11340} - \frac{13x}{30240} - \frac{17x^2}{226800} + \frac{23x^3}{453600} + \frac{59x^4}{5987520}\right), \\
z_2 &= -\sigma^4 T^2 \left(\frac{1}{90} + \frac{x}{360} - \frac{11x^2}{30240} - \frac{x^3}{5040} + \frac{x^4}{226800}\right) \\
&\quad + \sigma^6 T^3 \left(\frac{31}{22680} + \frac{11x}{60480} - \frac{37x^2}{151200} - \frac{19x^3}{302400} + \frac{953x^4}{59875200}\right), \\
z_3 &= \sigma^6 T^3 \left(\frac{2}{2835} - \frac{x}{60480} - \frac{2x^2}{14175} - \frac{17x^3}{907200} + \frac{13x^4}{124700}\right), \\
x &= rT.
\end{aligned}$$

Other approximations based on the moments are given for example in Milevsky and Posner (1998), but they do not appear to be very accurate and therefore we do not consider them.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%PRICING ASIAN OPTIONS USING THE JU APPROXIMATION%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function res=AsianCallJu(Spot, strike, rf, sigma, t)
%parameters
rt = rf * t;
sgt = sigma * sqrt(t);
k = t* strike/ Spot;
y = log(strike / Spot);
%Compute the first two moments of the average
m1 = moment_n(1, (rt - sgt * sgt / 2) / sgt, sgt, 1);
m2 = moment_n(2, (rt - sgt * sgt / 2) / sgt, sgt, 1);
```

```
%compute m and v^2 (here just v)
meanlog = 2 * log(m1) - log(m2) / 2;
v = log(m2) - 2 * log(m1);
%additional terms
z1 = -(t ^ 2) * (sigma ^ 4) * (1 / 45 + rt / 180 - ...
   11 * rt * rt / 15120 - (rt ^ 3) / 2520 + (rt ^ 4) / 113400) - ...
    (t ^ 3) * (sigma ^ 6) * (1 / 11340 - 13 * rt / 30240 - ...
   17 * rt * rt / 226800 + 23 * (rt ^ 3) / 453600 + ...
   59 * (rt ^ 4) / 5987520);
z2 = -t * t * (sigma^ 4) * (1 / 90 + rt / 360 - ...
   11 * rt * rt / 30240 - (rt ^ 3) / 5040 + (rt ^ 4) / 226800) + ...
   (t ^ 3) * (sigma ^ 6) * (31 / 22680 + 11 * rt / 60480 - ...
   37 * rt * rt / 151200 - 19 * (rt ^ 3) / 302400 + ...
   953 * (rt ^ 4) / 59875200);

z3 = (t ^ 3) * (sigma ^ 6) * (2 / 2835 - rt / 60480 - ...
   2 * rt * rt / 14175 - 17 * (rt ^ 3) / 907200 + ...
   13 * (rt ^ 4) / 124700);

n = exp(-(y - meanlog) * (y - meanlog) / (2 * v)) / sqrt(2 * pi * v);
dn = -n * (y - meanlog) / v;
d2n = n * (meanlog ^ 2 - v - 2 * v * y + y ^ 2) / v ^ 2;
correction = z1 * n + z2 * dn + z3 * d2n;

%compute the price according to the lognormal approximation
calllog = AsianCalllog(Spot, strike, rf, sigma, t);

%Ju approximation
res= calllog + exp(-rt) * strike * correction;
```

BOX 18.3 ASIAN OPTIONS ON FUTURES PRICES

As illustrated in the introduction, in organized commodity and energy markets traded average options are based on futures or forward prices rather than on spot. This is equivalent to assuming that the cost-of-carry on the underlying asset is zero (i.e., the dynamics of the lognormal futures price is now $dF = \sigma F dW(t)$). If we use a moment-based formula for pricing the option, we observe that the futures price being a martingale, the expected value of the average is equal to the current futures price and therefore we have to compute only the variance of the average. For example, Haug (2006) shows that the Turnbull and Wakeman formula becomes

$$e^{-rT}(FN(d_1) - KN(d_2)), \tag{18.20}$$

where

$$d_1 = \frac{\ln(F/X) + T\sigma_A^2/2}{\sqrt{T\sigma_A^2}},\ d_2 = d_1 - \sqrt{T\sigma_A^2},$$

and

$$T\sigma_A^2 = \ln\left(\frac{2e^{\sigma^2 T} - 2(1+\sigma^2 T)}{\sigma^4 T^2}\right).$$

18.2.2 Lower Price Bound

Rogers and Shi (1992) and Thompson (1998) obtain lower and upper bounds for the Asian option price. For a lower bound, the idea is simple and powerful. Consider the random variable

$$X = \frac{s_0}{T}\int_0^T e^{(r-\sigma^2/2)s+\sigma W_s}\,\mathrm{d}s - K.$$

The Asian option price is given by $\widetilde{E}_0(X^+)$. Using the iterated rule for conditional expectations, the fact that $X^+ \geq X$ and the positiveness of X^+, we have

$$\widetilde{E}_0(X^+) = \widetilde{E}_0[\widetilde{E}_0(X^+|Z)] \geq \widetilde{E}_0[\widetilde{E}_0(X|Z)^+] := c_{\mathrm{low}},$$

for any conditioning variable Z. Rogers and Shi (1992) propose using $Z = \int_0^T W_s\,\mathrm{d}s$, and provide an analytical expression for the lower bound c_{low}.[4] Thompson (1998) obtained the same lower bound via a simpler expression

$$c \geq c_{\mathrm{low}} = e^{-r}\left(\int_0^1 s_0 e^{\alpha t+\sigma^2 t/2}\mathcal{N}\left(\frac{-\gamma^* + \sigma t(1-t/2)}{1/\sqrt{3}}\right)\mathrm{d}t - K\mathcal{N}\left(\frac{-\gamma^*}{1/\sqrt{3}}\right)\right), \quad (18.21)$$

where $\alpha = r - \sigma^2/2$ and the option maturity T has been standardized to 1.[5] Here $\mathcal{N}(x)$ denotes the standard normal cumulative function and γ^* is the unique solution to the equation

$$\int_0^1 s_0 \exp\left(3\gamma^*\sigma t(1-t/2) + \alpha t + \frac{1}{2}\sigma^2(t - 3t^2(1-t/2)^2)\right)\mathrm{d}t = K. \quad (18.22)$$

[4]They also discuss how to measure the accuracy of the lower bound.

[5]For a general T, r and σ must be replaced by rT and $\sigma\sqrt{T}$, respectively.

Computation of γ^* can be done using standard root finder routines (e.g., the bisection method). A Matlab script providing the implementation of the above formula is given here.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%ASIAN PRICE LOWER BOUND%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [asian_premium_lower]=...
get_AsianCall_Lower(spot, strike, t, sigma, rf)

%Scale parameters
rft=rf*t; sigmat=sigma*sqrt(t);
A = (rf - sigma * sigma / 2);

%find optimal value of gamma
gstar = get_gamma(spot,strike, rf, sigma);

arg2 = (-gstar) / (1 / (3 ^ 0.5));

%discount factor
df=exp(-rft);

%lower bound
asian_premium_lower = ...
    (quadgk(@(s) spot * exp(A * s + sigmat * sig-
mat * s / 2) .* ...
    normcdf((-gstar + sigmat * s .* (1 - s / 2)) ...
    / (1 / (3 ^ 0.5)),0,1),0,1) ...
    -strike*normcdf(arg2))*df;

%%Auxiliary function: Find optimal value of gamma
function res =get_gamma(spot, strike, rf , sigma)

sg2half=0.5*sigma*sigma;
A = (rf - sg2half);

res =fsolve(@(trialGamma) ...
 spot*quadgk(@(t) ...
 (exp(A * t + sg2half*t .*(1-3*t.*(1-t/2).*(1-t/2)) ...
 + 3*sigma*trialGamma*t.*(1-t/2))),0,1)-strike,0.0);
```

18.2.3 Monte Carlo Simulation

Monte Carlo simulation is a popular pricing technique due to its flexibility in dealing with complex payoffs and sophisticated dynamics. However, it is much slower with respect to alternative methods in achieving an acceptable precision. Indeed the accuracy can be ameliorated by increasing the number of simulations, but this also increases the computational cost. For this reason, it is often implemented by adopting reduction of variance techniques. This issue is indeed highly related to the pricing of Asian options and all Monte Carlo simulations proposed in the literature have been applying one of the variance reduction techniques. The basic crude Monte Carlo scheme is as follows:

1. Fix the number N of monitoring dates and the time step $\Delta = T/N$, so that the monitoring dates are $t_i = i \times \Delta$.
2. Starting from $S(0) = s_0$, simulate the spot prices at times $i\Delta$ along the jth path discretizing the solution (18.4):

$$S^{(j)}(i\Delta) = S^{(j)}((i-1)\Delta) \times e^{\left(r-\frac{\sigma^2}{2}\right)\Delta+\sigma\left(W^{(j)}(i\Delta)-W^{(j)}((i-1)\Delta)\right)}, i = 1, \dots, N, \tag{18.23}$$

   with $j = 1, \dots, m$ where m is the number of simulations. The increment $W^{(j)}(i\Delta) - W^{(j)}((i-1)\Delta)$ is simulated according a $\mathcal{N}(0, \Delta)$. For example, we can set

$$W^{(j)}(i\Delta) - W^{(j)}((i-1)\Delta) = \sqrt{\Delta} \times \Phi^{-1}\left(u_i^{(j)}\right), \tag{18.24}$$

   where u is a uniform(0,1) random variable and Φ^{-1} is the inverse cumulative distribution function of the standard normal distribution.
3. Update the average according to

$$A^{(j)}(i\Delta) = \frac{i-1}{i} \times A^{(j)}((i-1)\Delta) + \frac{S^{(j)}(i\Delta)}{i}, A^{(j)}(0) = s_0. \tag{18.25}$$

4. Compute the discounted Asian option payoff in the jth path:

$$\pi^{(j)} = e^{-r \times N \times \Delta}(A^{(j)}(n\Delta) - K)^+. \tag{18.26}$$

5. The option price is estimated by repeating steps 2 to 4 m times and discounting the payoff along each path and then averaging across simulations:

$$\hat{c} = \frac{1}{m}\sum_{j=1}^{m} \pi^{(j)}. \tag{18.27}$$

6. We can evaluate the accuracy of the estimate by computing the standard error:

$$se = \sqrt{\frac{\hat{\sigma}^2}{m}}, \tag{18.28}$$

   where

$$\hat{\sigma}^2 = \frac{1}{m}\sum_{j=1}^{m}(\pi^{(j)} - \hat{c})^2.$$

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7. The confidence interval at a given confidence level α, say $\alpha = 95\%$, is given by

$$\hat{c} \pm z_{1-\frac{\alpha}{2}} \times se,$$

where z_α is the quantile at the level α of the standard normal distribution.

A spreadsheet implementation of the above scheme is given in Figure 18.3. Cells B4:F13 refer to simulated standard normal random variables.[6] Cells G3:K13 refer to the simulation of $m = 5$ stock price paths over the next 10 days, according to equations (18.23) and (18.24). Cells L3:P13 refer to the simulation of the price average along time, see equation (18.25). In cells L18:P18 we compute the Asian option payoff for each simulated path, see equation (18.26). Finally, in cells L20 and L21 we average the payoffs across simulations, see equation (18.27), and then we discount it. Finally, we also compute the standard error of the estimate, see equation (18.28).

In Box 18.4, we illustrate how to modify the above Monte Carlo procedure in order to simulate the spot price consistently with the observed market forward curve.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%Asian Price by MC Simulation%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [asian_mc, asian_mc_a, se, se_a]=...
get_AsianCall_MC(spot, strike, t, sigma, rate_cc, div_cc, nsimul,
                 ndates)

rf=rate_cc;
df=exp(-rf*t);

%Assigning Time step
dt=t/ndates;
timestep=[0:dt:t]';

%Simulate increments dW
dW=randn(ndates,nsimul)*dt^0.5;

%Simulate increments dlog-Price
dlogS=(rf-div_cc-sigma*sigma/2)*dt+sigma*dW;

%Simulate log-prices
logS=log(spot)+[zeros(1,nsimul);
        cumsum(dlogS)];

%Get spot price paths
prices=exp(logS);
```

[6]The generation of the Gaussian random variable can be done in Excel using the cell formula `=NORM.S.INV(RAND())`.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1		Gaussian Random Number Simulations					Stock Price Simulations					Average Price Simulation				
2	Days	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
3	0	0	0	0	0	0	100	100	100	100	100	100	100	100	100	100
4	1	1.8221	-0.5164	0.6529	-0.2364	-1.7514	102.3214	99.3391	100.8193	99.6915	97.7992	101.1607	99.6695	100.4097	99.8458	98.8996
5	2	0.2093	0.0201	0.1253	2.5657	1.0475	102.5824	99.3544	100.9692	102.9697	99.0938	101.6346	99.5645	100.5962	100.8871	98.9644
6	3	0.8114	0.7554	0.6232	-1.6116	-0.1169	103.6304	100.2984	101.7582	100.8820	98.9376	102.1336	99.7480	100.8867	100.8858	98.9577
7	4	0.1104	1.0138	-1.1960	-0.3965	-0.9186	103.7649	101.5827	100.2205	100.3674	97.7849	102.4598	100.1149	100.7535	100.7821	98.7231
8	5	-0.2008	-1.5653	-0.6048	0.8771	-0.0415	103.4913	99.5813	99.4469	101.4770	97.7238	102.6317	100.0260	100.5357	100.8979	98.5566
9	6	-0.0189	1.3680	1.1475	-1.1853	1.5090	103.4563	101.3094	100.8909	99.9569	99.5972	102.7495	100.2093	100.5864	100.7635	98.7052
10	7	-1.5770	1.2193	-0.9852	-0.9897	-1.3226	101.4030	102.8738	99.6316	98.7036	97.9351	102.5812	100.5424	100.4671	100.5060	98.6089
11	8	1.5317	0.1025	0.6385	2.2352	-0.2277	103.3766	102.9971	100.4296	101.5239	97.6437	102.6696	100.8151	100.4629	100.6191	98.5017
12	9	0.0422	-0.2706	0.6627	-2.1106	0.9301	103.4216	102.6349	101.2650	98.8396	98.7895	102.7448	100.9971	100.5431	100.4412	98.5305
13	10	-0.2276	-1.0012	0.5876	-0.4821	0.7257	103.1140	101.3333	102.0103	98.2290	99.6905	102.7784	101.0277	100.6765	100.2401	98.6359
14																
15			=NORM.S.INV(RAND())					=H12*EXP((0.02-0.3*0.3/2)*(1/252)+0.2*C13*(1/250)^0.5)							=AVERAGE(J3:J13)	
16																
17											Simulation	1	2	3	4	5
18											Payoff	2.7784	1.0277	0.6765	0.2401	0.0000
19																
20											Price Estimate	0.94376	=AVERAGE(L17:P17)*EXP(-0.02*10/250)			
21											Std. Error	0.49143	=STDEV(L18:P18)/5^0.5			

FIGURE 18.3 Spreadsheet implementation of the Monte Carlo simulation for pricing an Asian option. Parameters: $r = 2\%$, $\sigma = 20\%$, $\Delta = 1/250$, 10 averaging points, 5 simulations, $s_0 = 100$, $K = 100$, $T - t = 10/250$

```
%Get time average
avg=sum(prices,1)./(ndates+1);

%Payoff
payoff =max(avg-strike,0)*df;

%Asian price
asian_mc=mean(payoff);

%Standard error of MC estimate
se=std(payoff)/sqrt(nsimul);
```

TABLE 18.5 Monte Carlo price estimates and standard errors for an arithmetic average option varying the number m of simulations. Parameters: $S(0) = 8.2$, $K = 8.5$, $r = 3\%$, $\sigma = 50\%$, $T = 0.5$, $N = 8$

m	10,000	20,000	40,000	80,000	160,000
MC	0.7776	0.7652	0.7553	0.7662	0.7661
s.e.	0.0117	0.0081	0.0057	0.0041	0.0029
ratio of s.e.		1.44	1.42	1.41	1.41

In Table 18.5 we report for different number m of simulations the MC estimate and the corresponding standard error. Notice that as we double the number of simulations the standard error reduces by a factor approximately equal to $\sqrt{2}$, as expected. Indeed, the standard error decreases as $1/\sqrt{m}$. So in order to reduce by a factor of 10 the standard error, we need to increase by a factor of 10^2 the number of simulations. This can be quite computationally intensive.

BOX 18.4 HOW TO INCLUDE FORWARD CURVES IN THE MONTE CARLO SIMULATION?

To simulate the spot price taking into account the information in the futures curve is particularly relevant in commodity markets where the futures term structure can show peculiar shapes, due to seasonality effects. An example is shown in Figure 18.4, where the evolution of the natural gas term structure in March 2007 is illustrated. For this reason, it would be important, if we decide to price Asian options via Monte Carlo, to take into account the futures term structure shape. This can be done according to the following procedure:

- Let $F(0, T)$ be the forward price quoted at time 0 for maturity T.
- It is well known that under the risk-neutral measure

$$\widetilde{E}_0(S(T)) = F(0, T).$$

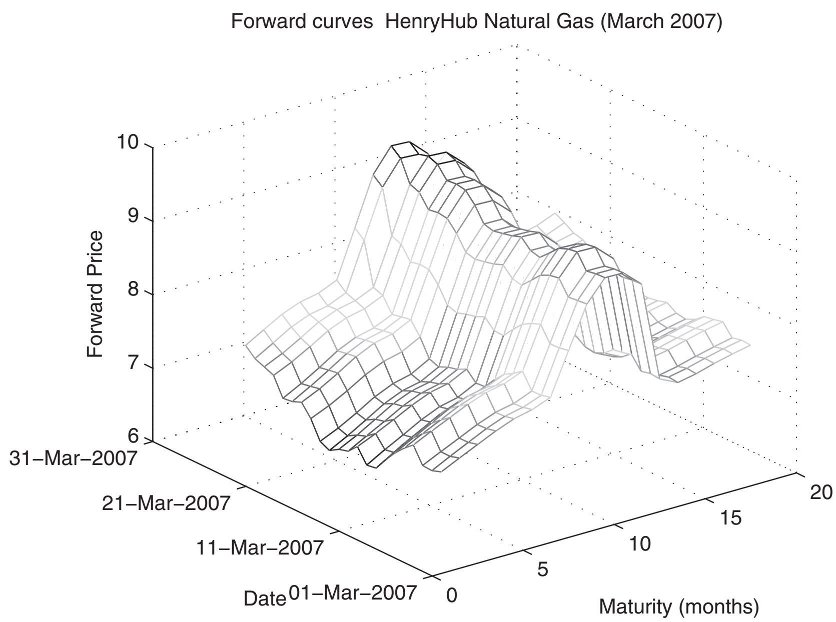

FIGURE 18.4 Natural gas forward curve evolution between 1 March and 31 March 2007

- If we assume a GBM process for the underlying with constant volatility, we have

$$S(T) = F(0,T) \times e^{-\frac{\sigma^2}{2}T+\sigma W(T)}.$$

- Given that we have forward quotations $F(0, T_i)$ for different maturities T_i, we can write

$$S(T_i) = F(0,T_i) \times e^{-\frac{\sigma^2}{2}T_i+\sigma W(T_i)},$$
$$S(T_{i+1}) = F(0,T_{i+1}) \times e^{-\frac{\sigma^2}{2}T_{i+1}+\sigma W(T_{i+1})},$$

and therefore

$$\frac{S(T_{i+1})}{S(T_i)} = \frac{F(0,T_{i+1}) \times e^{-\frac{\sigma^2}{2}T_{i+1}+\sigma W(T_{i+1})}}{F(0,T_i) \times e^{-\frac{\sigma^2}{2}T_i+\sigma W(T_i)}}.$$

In the presence of a forward curve $F(0,T)$, the simulation of the GBM process can be performed according to

$$S(T_{i+1}) = S(T_i) \times \frac{F(0,T_{i+1})}{F(0,T_i)} \times e^{-\frac{\sigma^2}{2}(T_{i+1}-T_i)+\sigma(W(T_{i+1})-W(T_i))},$$

starting from s_0 and using the fact that

$$W(T_{i+1}) - W(T_i) \sim \mathcal{N}(0, T_{i+1} - T_i).$$

The procedure:

1. Assign σ, and the forward curve.
2. Assign the monitoring dates and interpolate the forward curve at these date; compute the ratios

$$f(i+1) = \frac{F(0, T_{i+1})}{F(0, T_i)}.$$

3. Simulate the increments

$$\epsilon(i+1) = -\frac{\sigma^2}{2}(T_{i+1} - T_i) + \sigma(W(T_{i+1}) - W(T_i)).$$

4. Starting from the initial date ($T_0 = 0$), $S\left(T_0\right) = F(0, 0)$, compute the simulated prices

$$S(T_{i+1}) = S(T_i) \times f(i+1) \times e^{\epsilon(i+1)}.$$

5. Update the computation of the average and at maturity compute the Asian option payoff.
6. Repeat the previous steps a large number of times and then average the discounted payoff.

18.2.3.1 Improving the Accuracy of Monte Carlo Simulation

The accuracy of the above basic Monte Carlo simulation scheme can be improved by resorting to variance reduction techniques. The most common ones in the context of Asian option pricing are

- antithetic variate, and
- control variate.

`Antithetic variate` attempts to reduce the variance of the simulation error by introducing negative dependence between pairs of replications. In fact, this method works well if the covariance between the payoffs in the standard path and in the antithetic one is negative. Unfortunately, the preservation of negative correlation in the payoffs is not always guaranteed, so sometimes this procedure can be ineffective. `Control variate` exploits information about the errors in estimates of known quantities to reduce the error in an estimate of an unknown quantity. Both techniques are illustrated briefly in Boxes 18.5 and 18.6.

BOX 18.5 IMPROVING MONTE CARLO VIA VARIANCE REDUCTION: ANTITHETIC VARIATE

This procedure is based on the fact that if $u_i^{(j)} \sim U(0,1)$ (i.e., we do a random extraction from a standard uniform random variable), then

$$W^{(j)}(i) - W^{(j)}(i-1) = \sqrt{\Delta} \times \Phi^{-1}\left(u_i^{(j)}\right) \sim \mathcal{N}(0,\Delta).$$

It is also true that

$$-(W^{(j)}(i) - W^{(j)}(i-1)) = -\sqrt{\Delta} \times \Phi^{-1}\left(u_i^{(j)}\right) \sim \mathcal{N}(0,\Delta),$$

that is, dW and its opposite $-dW$ have the same Gaussian distribution (the mean is assumed to be zero). This observation can be exploited to better sample from the Gaussian distribution with respect to using two independent draws.

The idea of antithetic simulation consists of using both random numbers to get the so-called **antithetic path**

$$S^{(j)}(i\Delta) = S^{(j)}((i-1)\Delta) \times e^{\left(r-q-\frac{\sigma^2}{2}\right)\Delta + \sigma\left(W^{(j)}(i)-W^{(j)}(i-1)\right)}, i = 1, \ldots, N,$$
$$S_A^{(j)}(i\Delta) = S_A^{(j)}((i-1)\Delta) \times e^{\left(r-q-\frac{\sigma^2}{2}\right)\Delta + \sigma\left(W_A^{(j)}(i)-W_A^{(j)}(i-1)\right)}, i = 1, \ldots, N,$$

where

$$W_A^{(j)}(i) - W_A^{(j)}(i-1) = -(W^{(j)}(i) - W^{(j)}(i-1)).$$

The Asian option price is then computed using both paths

$$\frac{e^{-r\times N\times\Delta}}{m}\sum_{j=1}^{m}\frac{(A^{(j)}(n)-K)^+ + \left(A_A^{(j)}(n)-K\right)^+}{2},$$

where $A^{(j)}$ and $A_A^{(j)}$ are the time averages computed according to the standard path and to the antithetic one.

To grasp the benefit of this method, let us consider the following parameter set: $S(0) = 8.2$, $K = 8.5$, $r = 3\%$, $\sigma = 50\%$, $T = 0.5$, $N = 8$, with 100,000 and 200,000 simulations respectively. We have the results in the following table.

	100,000 MC runs			**200,000 MC runs**		
Method	**Price**	**s.e.**	**Seconds**	**Price**	**s.e.**	**Seconds**
Crude MC	0.76663	0.003671	0.7	0.76182	0.00257	3.6
Antithetic	0.76615	0.00194	1.3	0.762289	0.00136	6.5

We complete here the previous Matlab script including antithetic simulation.

```
%Antithetic Simulation
dlogS_a=(rf-q-sigma*sigma/2)*dt-dW;
logS_a=log(spot)+[zeros(1,nsimul);

cumsum(dlogS_a)];
prices_a=exp(logS_a);
avg_a=sum(prices_a,1)./(ndates+1);
payoff_a =max(avg_a-strike,0)*df;

asian_mc_a=(mean(payoff_a)+mean(payoff))/2;
se_a=std((payoff_a+payoff)/2)/sqrt(nsimul);
```

A simulated path and its antithetic path for the underlying and the average are illustrated in Figure 18.5.

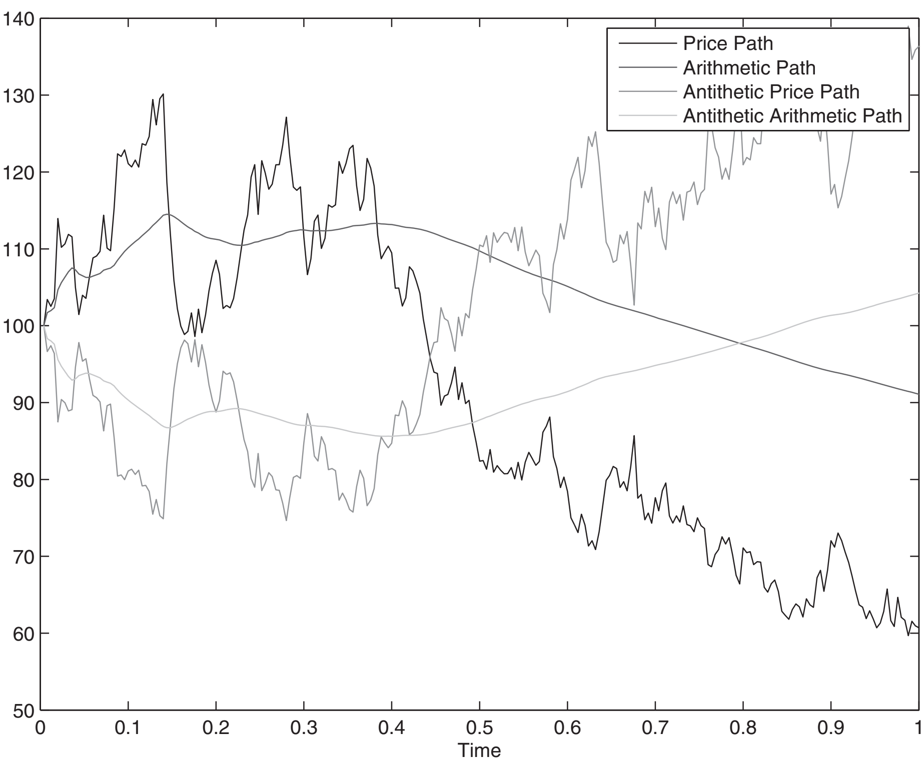

FIGURE 18.5 Simulated and antithetic path for the underlying and its time average

BOX 18.6 IMPROVING MONTE CARLO VIA VARIANCE REDUCTION: CONTROL VARIATES

Let us suppose that we need to estimate $\widetilde{E}(Y)$, where Y represents some payoff function. Using MC simulation we generate m i.i.d. replicas of Y, $Y^{(j)}$ say, and we estimate the above expected value by

$$\bar{Y} = \frac{1}{m}\sum_{j=1}^{m} Y^{(j)}.$$

The accuracy of this estimate can be measured by the variance of the estimator, that is σ^2/m (the square root of this quantity is called the standard error of the MC estimate). Let us now suppose that on each replication we can calculate another output, say $Z^{(j)}$ along $Y^{(j)}$. Moreover, let us suppose that $\widetilde{E}(Z)$ is known.

Then for any fixed number b we can calculate the additional quantity

$$Y^{(j)}(b) = Y^{(j)} - b(Z^{(j)} - \widetilde{E}(Z)),$$

which still provides an unbiased estimate of $\widetilde{E}(Y)$. Each $Y^{(j)}(b)$ has variance $\sigma_Y^2(b)$:

$$\begin{aligned}\sigma_Y^2(b) \equiv \widetilde{V}ar(Y^{(j)}(b)) &= \widetilde{V}ar(Y^{(j)} - b(Z^{(j)} - \widetilde{E}(Z))) \\ &= \sigma_Y^2 + b^2\sigma_Z^2 - 2b\sigma_{YZ}.\end{aligned}$$

The control variate estimator is given by

$$\bar{Y}(b) = \frac{1}{m}\sum_{j=1}^{m} Y^{(j)}(b),$$

and has variance

$$\frac{\sigma_Y^2(b)}{m}.$$

The optimal value of b that minimizes the variance of the control variate estimator is

$$b^* = \frac{\sigma_{YZ}}{\sigma_Z^2},$$

where σ_{YZ} is the covariance of the (simulated) values of Y and Z and σ_Z^2 is the variance of the control variate. The variance of $\bar{Y}(b^*)$ relative to the variance of $\bar{Y}$ is

$$\frac{\sigma_Y^2(b^*)}{\sigma_Y^2} = 1 - \rho_{ZY}^2.$$

We can make the following remarks:

- The higher the correlation ρ_{ZY} between Z and Y, the higher will be the reduction in the variance. Notice that very high negative correlations can be of some help.
- If the computational effort per replication is roughly the same with and without a control variate, then $1 - \rho_{ZY}^2$ measures the computational speed-up resulting from the use of a control; in other words, the number of simulations of $Y^{(j)}$ required to achieve the same variance as m replications of the control variate estimator is $N/(1 - \rho_{ZY}^2)$.
- Given that the ratio $1/(1 - \rho_{ZY}^2)$ approaches 1 very fast as $|\rho_{ZY}|$ decreases away from 1, in order to be effective the control variate must have a very high degree of correlation with Y.
- In practice, b^* must be estimated. We can run some preliminary simulation and estimate it using the sample counterparts.

The main issue in the implementation of the control variate technique is to find a convenient control variate. For arithmetic Asian options, it turns out that a good control variate is given by the geometric average G_T, defined as

$$G_T = e^{\frac{1}{T}\int_0^T \ln S(u)du} \tag{18.29}$$

in the continuous monitoring case, and

$$G_T = \left(\prod_{k=0}^{N} S_{\Delta k}\right)^{\frac{1}{N+1}}, \tag{18.30}$$

in the discrete one. The payoff of a geometric fixed strike call option is then $(G_T - K)^+$. For typical values of the volatility parameter, the correlation between geometric and arithmetic average is very high, as we can verify via Monte Carlo simulation, from the following table (here σ refers to the volatility of the underlying asset and ρ to the correlation between arithmetic and geometric average):

σ	$\rho\ (A(T), G(T))$
1%	1
10%	0.9998
50%	0.9960
100%	0.9820

In order to make the CV procedure effective, we also need a closed-form formula for $\widetilde{E}(G_T - K)^+$. If the asset evolves according to a GBM process, geometric Asian options can be priced in closed form according to the formula

$$c_{geo} = s_0 e^{(m-r)T} \mathcal{N}(d_1) - Ke^{-rT} \mathcal{N}(d_2), \tag{18.31}$$

with

$$d_1 = \frac{\ln\left(\frac{s_0}{K}\right) + \left(m + \frac{v^2}{2}\right) T}{v\sqrt{T}}, d_2 = d_1 - v\sqrt{T}.$$

Here, m depends on the monitoring frequency:

- discrete time monitoring

$$m = \frac{1}{2}\left(r - \frac{(N+2)}{6(N+1)}\sigma^2\right),\ v = \sqrt{\frac{2N+1}{6(N+1)}}\sigma; \tag{18.32}$$

- continuous time monitoring

$$m = \frac{1}{2}\left(r - \frac{\sigma^2}{6}\right),\ v = \frac{\sigma}{\sqrt{3}}. \tag{18.33}$$

Notice that in the above formula, the quantity v^2 refers to the variance of the log-geometric average and that the discrete monitoring version tends to the continuous one as we let the number of monitoring dates go to infinity, that is $N \to \infty$.

We therefore have the following algorithm:

1. Simulate the arithmetic and the geometric average.
2. Compute the Monte Carlo estimate of the two Asian options, MC^{Arit} and MC^{Geo} say.
3. Assuming $b = 1$, the control variate estimate is given by

$$CV^{Asia} = MC^{Arit} - MC^{Geo} + c_{geo}.$$

18.3 A COMPARISON

We discuss here the effectiveness of the alternative procedures described so far. A more detailed discussion and numerical examples can be found in Fusai and Roncoroni (2008). The experiment conducted here is under different sets of input parameters, as reported in Table 18.6. Numerical results are given in Table 18.7. In this table, we also add an upper bound, computed according to the procedure described in Fusai and Roncoroni (2008). The

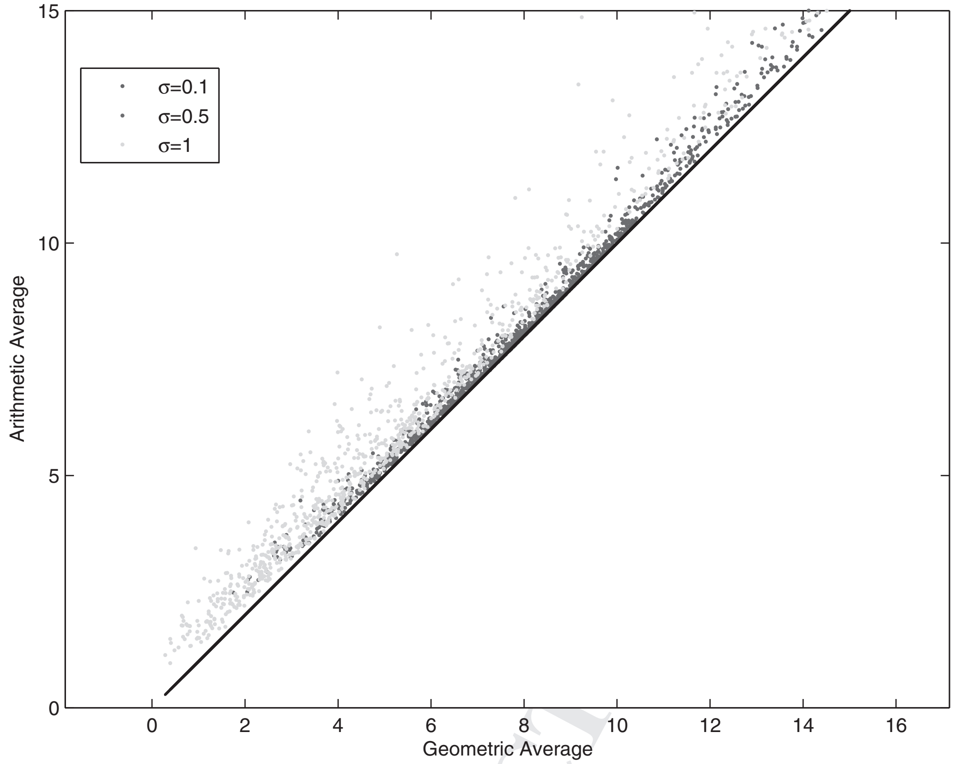

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FIGURE 18.6 Simulated values of the geometric and arithmetic average for different volatility levels. The black line is the 45° line; notice that the arithmetic average is always higher than the geometric one, so this also provides a lower bound to the exact price

quantity $\sigma\sqrt{T}$ determines the accuracy of the method: in general, the lower its value, the more difficult it is to find an accurate numerical method (in some sense, this is curious because one is supposed to believe the contrary). However, the accuracy of the lower bound and of Monte Carlo simulation improves at smaller values of the parameter σ (or lower volatility or smaller time to maturity). Interestingly, the lower bound provides an exact approximation up to the third digit of Ju's method (and in Case 4 the two methods provide the same result up to the

TABLE 18.6 Parameter set

Example	s_0	K	r	σ	T	$\sigma\sqrt{T}$
1	1.9	2	0.05	0.5	1	0.5
2	2	2	0.05	0.5	1	0.5
3	2.1	2	0.05	0.5	1	0.5
4	2	2	0.02	0.1	1	0.1
5	2	2	0.18	0.3	1	0.3
6	2	2	0.0125	0.25	2	0.3535
7	2	2	0.05	0.5	2	0.7071

TABLE 18.7 Approximate prices for an Asian option under alternative numerical methods

Example	Lower	Levy	Edge	Ju	MC crude	SE crude	MC CV	SE CV	Upper
1	0.1931	0.1954	0.1948	0.1929	0.186	0.0037	0.1881	0.0022	0.1938
2	0.2463	0.2498	0.2451	0.2462	0.2491	0.0043	0.2445	0.0024	0.2470
3	0.3061	0.3106	0.3014	0.3061	[illegible]	0.0048	[illegible]	0.0026	0.3069
4	0.0560	0.0561	0.0560	0.0560	0.0554	0.0008	0.0561	0.0004	0.0560
5	0.2184	0.2198	0.2175	0.2184	0.2138	0.0026	0.2182	0.0011	0.2185
6	0.1722	0.1735	0.1735	0.1722	0.1693	0.0028	0.1718	0.0016	0.1724
7	0.3498	0.3592	0.3639	0.3497	0.3445	0.0065	0.3446	0.0038	0.3526

MC = Monte Carlo, Lower = R–S–T lower bound, Levy = moment matching (lognormal approximation), edge = moment matching (Edgeworth series expansion), Ju = moment matching using normal series expansion, MC crude = plain Monte Carlo, SE crude = standard error plain Monte Carlo, MC CV = control variate Monte Carlo, SE CV = standard error control variate Monte Carlo, Upper = upper bound.

fourth digit). Edgeworth approximation sometimes returns a price estimate below the lower bound, showing that raising the number of fitted moments does not necessarily provide a better approximation. The Monte Carlo estimate, if the number of simulations is not large enough, can fall outside the lower–upper range.

18.4 THE FLEXIBLE SQUARE-ROOT MODEL

Commodity-linked derivatives should be priced consistently with all market price information available at the valuation time. In particular, traders need models which produce prices taking into account three sets of information:

1. The quoted forward/futures prices of the commodity, provided they are available.[7]
2. A time-varying volatility coefficient, a feature allowing our model to fit either the term structure of implied volatilities or a time-dependent, that is, seasonal, spot price historical volatility.
3. Spot price dynamics exhibiting mean reversion in their trend, a quality shown by some important classes of commodity prices, among which we cite agriculturals and energy-related products such as electricity and gas.

These features usually reflect properties related to the physical use of the commodity for industrial or consumption processes.

The specialized literature has examined these issues in great detail. Routledge *et al.* (2000) underline the impact of periodical components on the price dynamics of most commodities. Eydeland and Wolyniec (2003) show that the predictable component of electricity price dynamics is bound by weather and consumption-related features. Todorova (2004) notes

[7]For the purpose of our analysis, we assume interest rates are deterministic. This amounts to treating forward and futures prices as equivalent.

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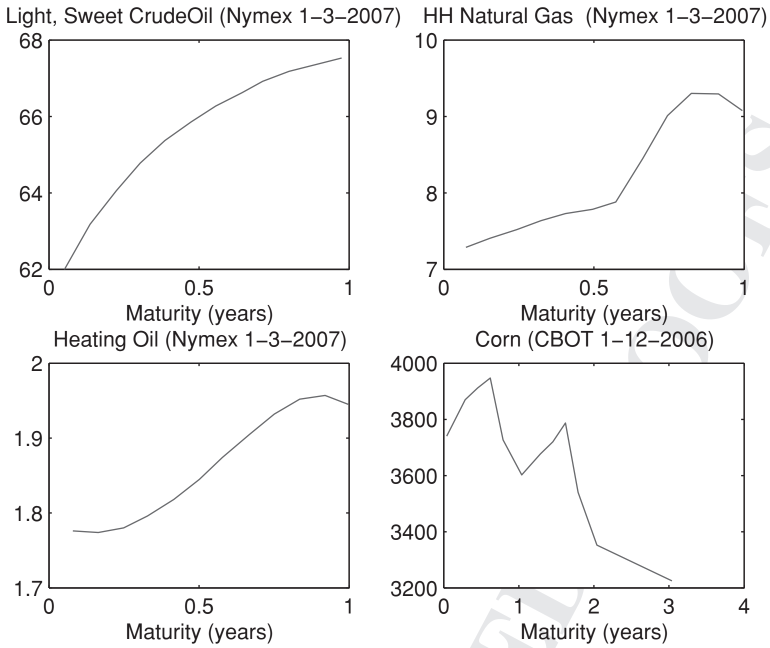

FIGURE 18.7 Futures curves for a sample of energy and agricultural commodities

that oil and gas markets show seasonal components affecting expected future spot prices, while Richter and Sorensen (2000) and Lien and Koekebakker (2004) find strong evidence of seasonality effects upon agricultural commodity prices. For most commodities, mean reversion is a stylized fact empirically accepted by several studies. In energy markets, the relevance of this property may vary across products and over time within the same commodity. For instance, Bessembinder *et al.* (1995) find clear evidence of mean reversion across 11 commodity markets, pointing out strong patterns for agriculturals and crude oil (see also Pindyck, 2001), and weak patterns for metals. Schwartz (1997) and Casassus and Collin-Dufresne (2005), among others, confirm the existence of a mean-reversion property in crude oil, copper, gold and silver. The case of electricity markets is rather peculiar. Geman and Roncoroni (2006) discover the existence of two competing mean-reversion effects in most US power markets: one is the traditional smooth reversion to average prices; the other stems from the spiky behaviour of electricity spot prices during periods of capacity congestion.

Figure 18.7 displays futures curves for light, sweet crude oil, natural gas and heating oil as quoted at NYMEX on 1 March, 2007, and corn as reported by CBOT on 1 December, 2006. The time-dependent component is plainly visible in the reported graphs. In particular, corn exhibits a clear seasonal pattern, which should be considered while pricing options on averages. Table 18.8 gives the implied volatility for different maturities for three different commodities (crude oil, heating oil and natural gas).

We now present a simple, yet effective method to make the spot dynamics include all price information implied by the quoted forward/futures curve, if any. This task can be achieved by letting the risk-neutral drift of spot price dynamics be time dependent. Moreover, the spot

TABLE 18.8 Term structures of implied volatilities for different commodities and maturities. Data from 4 October, 2013

Month	1st	2nd	3rd	6th	12nd	18th	24th	30th	60th
Crude	18.94	19.74	19.63	19.55	18.6	16.87	16.94	19.41	19.66
Heating oil	17.51	18.01	18.15	17.36	15.55	15.55	15.55	17.62	18.08
Natural gas	28.63	29.14	29.48	27.35	25.6	21.43	21.12	28.74	29.21

price volatility is allowed to reproduce any time pattern assigned by the user. We remark that the importance of assuming a time-varying drift goes beyond the ability to fit a quoted forward/future curve. For instance, Cartea and Williams (2007) point out that gas price dynamics exhibit a time-varying historical trend and market price of risk. Therefore, estimating these quantities may represent a viable alternative to directly fitting the risk-neutral price drift to forward quotes. This option can be useful whenever forward/futures quotes are not available or their reliability is limited by, say, liquidity constraints.

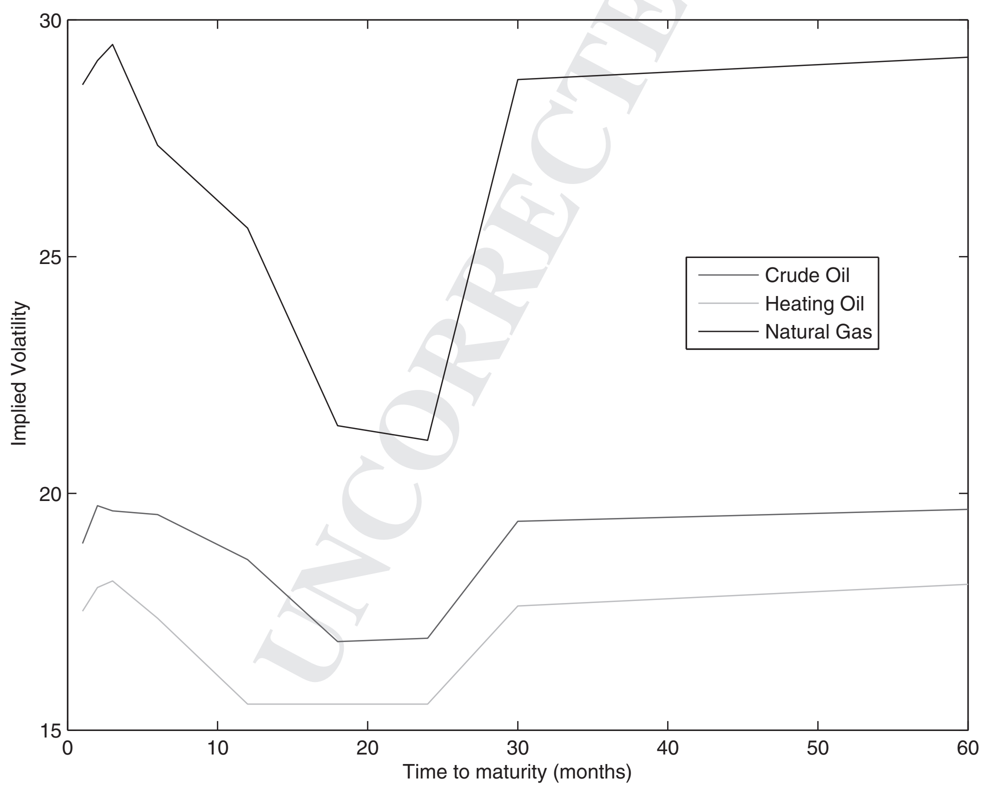

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FIGURE 18.8 Periodical component affecting the volatility of three energy commodities. Data from 4 October 2013

18.4.1 General Setup

We adopt the same setup as before, so the time horizon $[0, T]$ is split into a number $N + 1$ of Δ-spaced monitoring dates $0, \Delta, \ldots, N\Delta = T$ (the equally spaced assumption is made only for notational convenience, but it can easily be released). Our goal is to compute analytical formulae for fixed maturity options whose payoff structure depends on $\sum_{j=0}^{N} w_j S_{j\Delta}$ ($\sum_j w_j = 1$).

The option can be priced following a three-step algorithm devised by Fusai *et al.* (2008), which we now sketch here for the reader's convenience.

To this end, we start by assuming that the moment generating function (mgf) corresponding to the joint probability density of the pair consisting of the spot price $S_{N\Delta}$ and the cumulated spot price $\sum_{j=0}^{N} w_j S_{j\Delta}$ under the selected monitoring rule is known. This function is defined as

$$(\gamma, \mu) \rightarrow v_{0,x}(N, \Delta; \gamma, \mu) := \widetilde{E}_0\left(e^{-\gamma S_{n\Delta} - \mu \sum_{j=0}^{N} \alpha_j S_{j\Delta}}\right).$$

Table 18.9 illustrates instances of this function which correspond to traded options in the energy markets.

The fixed strike Asian-style option price can then be written as

$$C_{0,x}^{T}(k) = e^{-rT}\left(\frac{1}{2\pi\sqrt{-1}}\int_{a_l-\sqrt{-1}\infty}^{a_l+\sqrt{-1}\infty} e^{\mu k}\frac{v_{0,x}(\Delta, N; 0, \mu)}{\mu^2}d\mu + \sum_{j=0}^{N}\alpha_j F_{0,j\Delta} - K\right) \quad (18.34)$$

where x is the starting spot price ($x = s_0$) and a_l is a positive free parameter. If the analytical computation of the above integral in the complex plane is not possible, numerical evaluation is required. The use of the Fourier–Euler algorithm proposed by Abate and Whitt (1992) leads

TABLE 18.9 Moment generating functions relevant for a sample set of popular plain vanilla and Asian-style energy derivatives

Option	γ	μ	α_j	mgf $v_{0,x}(N,\Delta;\gamma,\mu)$
Standard European	any	0	—	$\widetilde{E}_0[e^{-\gamma S_{N\Delta}}]$
Fixed strike std. Asian	0	any	$\frac{1}{N+1}$	$\widetilde{E}_0\left[e^{-\frac{\mu}{N+1}\sum_{j=0}^{N} S_{j\Delta}}\right]$
Fixed strike vol. weighted	0	any	$\frac{V_j}{\sum_i V_i}$	$\widetilde{E}_0\left[e^{-\frac{\mu}{\sum_i V_i}\sum_{j=0}^{N} V_j S_{j\Delta}}\right]$
Floating strike std. Asian	any	$-\gamma$	$\frac{1}{N+1}$ $\frac{1}{N+1}$	$\widetilde{E}_0\left[e^{-\gamma\left(S_{n\Delta} - \frac{1}{N+1}\sum_{j=0}^{N} S_{j\Delta}\right)}\right]$

to the following numerical inversion formula:

$$\mathcal{L}^{-1}\left[\frac{v_{0,s_0}^{N,\Delta}(0,\mu)}{\mu^2}\right](k) \approx \sum_{m=0}^{M}\binom{M}{m}2^{-m}d_{P+m}(k),$$

with

$$d_P(k) = \frac{e^{a_l/2}}{2k}Re\left(\frac{v_{0,s_0}^{N,\Delta}\left(0,\frac{a_l}{2k}\right)}{\mu^2}\right) + \frac{e^{a_l/2}}{k}\sum_{j=1}^{P}(-1)^j\, Re\left(\frac{v_{0,s_0}^{N,\Delta}\left(0,\frac{a_l+2j\pi i}{2k}\right)}{\mu^2}\right),$$

where $Re(x)$ is the real part of x, and P and M are suitable constants. We suggest adopting the following parametric setting: $a_l = 18.4$, $M = 25$, $N = 15$ (see Fusai and Roncoroni, 2008 for details).

The floating strike Asian-style option price can be priced by a similar formula, see Fusai *et al.* (2008). Option Greeks, such as Delta and Gamma, can be obtained by differentiating the Fourier transform of the option price with respect to the standing spot price x. Finally, we refer to the above-mentioned paper for the discussion of the continuous monitoring version.

The price dynamics that makes it possible to exploit the above procedure is the square-root process (see Chapter 12). We consider here the diffusion case, whilst the extension to allow for the inclusion of jumps is given in Marena *et al.* (2013).

In particular, the following two spot price dynamics make possible the explicit computation of the mgf.

- **Specification 1.** Square-root process with time-varying drift and volatility:

$$\begin{aligned} dS_t &= \theta_t S_t dt + \sigma_t \sqrt{S_t} dW_t, \\ S_0 &= x, \end{aligned} \tag{18.35}$$

where

- the time-varying drift is chosen to fit the market observed term structure of forward prices $F(0, t)$ observed at time 0 for maturities t up to time $T = N\Delta$ by setting

$$\theta_T = \partial_T \ln \frac{F(0,T)}{x} = \partial_T \ln F(0,T); \tag{18.36}$$

- $\left(\sigma_t\right)_{t\geq 0}$ is a deterministic time-varying spot price volatility. σ_t^2 represents the time t variance of instantaneous price variations per unit of price value S_t and is expressed in $1/time$ units;
- $\left(W_t\right)_{t\geq 0}$ is a standard Brownian motion;
- x is the spot price, which can be estimated using $F(0, 0)$.

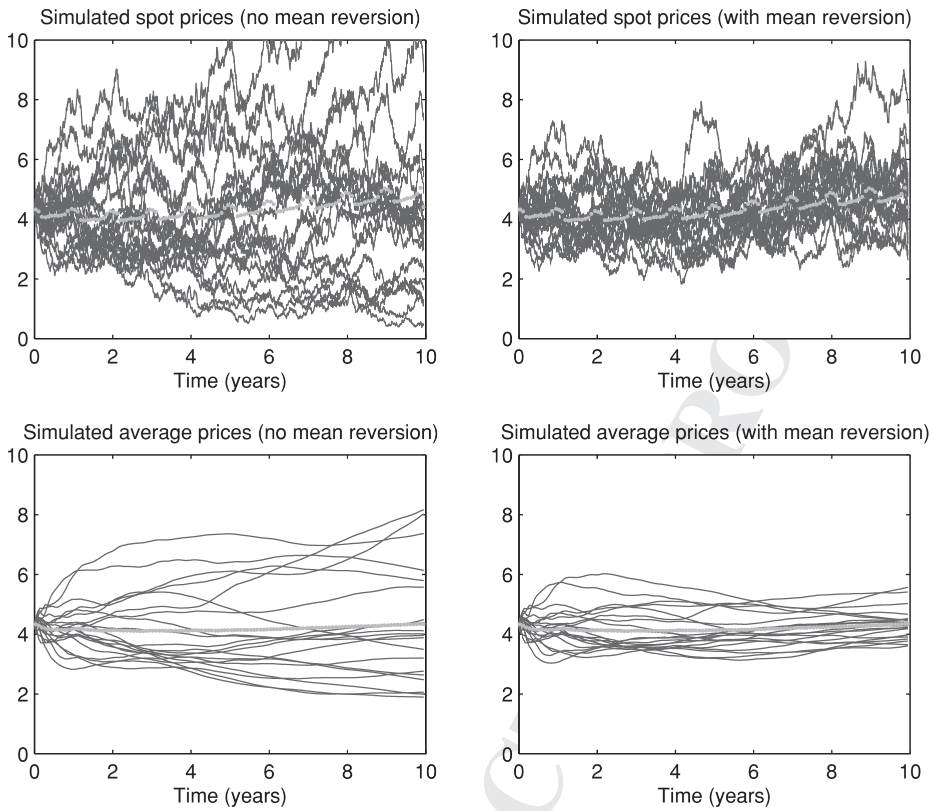

FIGURE 18.9 Simulated dynamics of the spot prices (top panels) and their time average (bottom panels). On the left, we simulate the dynamics according to model (18.35), whilst on the right dynamics (18.37) are simulated. The solid red curves refer to the market forward curve

- **Specification 2.** Square-root mean-reverting process with a time-varying trend:

$$dS_t = \beta(\eta_t - S_t)dt + \sigma_t \sqrt{S_t} dW_t, \tag{18.37}$$

where the additional parameters are
- β, the mean-reversion constant frequency expressed in $1/time$ units;
- $(\eta_t)_{t\geq 0}$, a deterministic time-varying price trend spot quotes revert to, that is selected such that the model fits the forward/futures price curve quoted in the market, by imposing

$$\eta(T) = F(0, T) + \frac{1}{\beta}\partial_T F(0, T). \tag{18.38}$$

Figure 18.9 shows simulated paths of the natural gas dynamics (18.35) and (18.37) and of their time-averaged prices. Both dynamics are fully consistent with the observed term structure of natural gas futures prices up to 5 years in the future (the futures curve refers to market quotes on 18 January, 2014 at CME). However, the mean-reverting model does

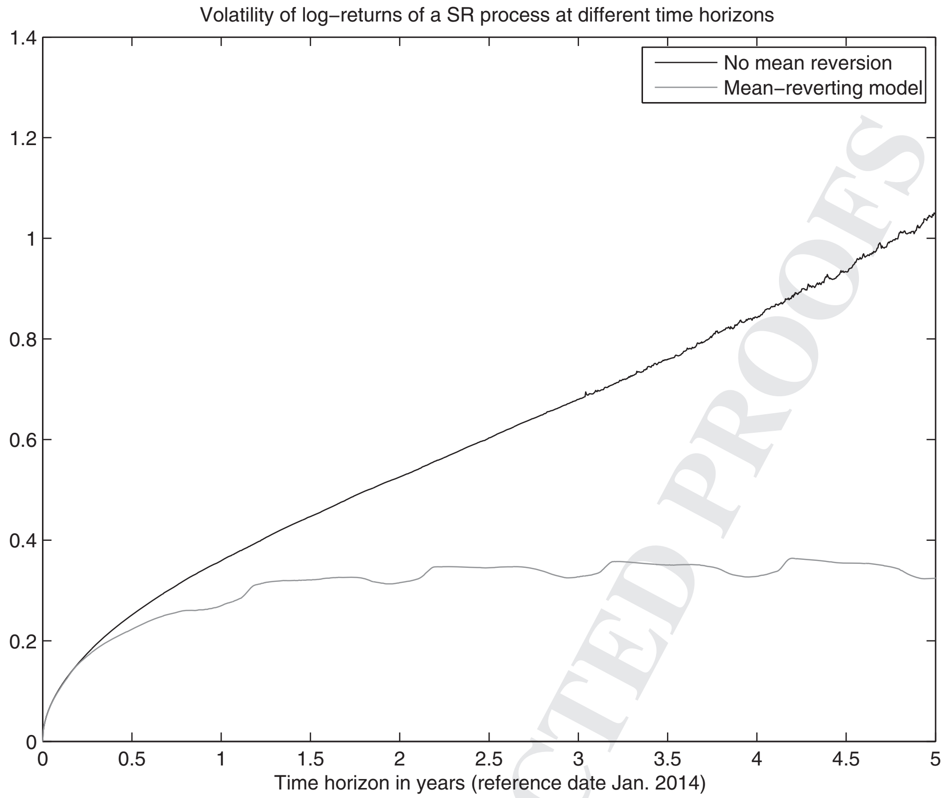

FIGURE 18.10 Volatility of log-returns in the SR models (18.35) and (18.37)

limit the probability of having future spot prices that deviate largely from the forward curve. Large deviations are instead possible in the model without mean reversion. In other words, the mean-reversion property generates a volatility of the log-return prices that tend to stabilize to a constant value as we consider longer and longer horizons, whilst without mean reversion the volatility increases with the time horizon. This is illustrated in Figure 18.10. Notice that in the mean-reverting model the seasonal shape of the forward curve affects the volatility of log-returns.

In both specifications, the drift terms are selected so that the dynamics is consistent with the market observed (or eventually the user's specified) forward price curve $(F_{0,T}, T \geq 0)$ quoted in the market, that is the restriction

$$\widetilde{E}_t(S_T) = F(0,T) \tag{18.39}$$

is satisfied. In addition, in both specifications, the drift restrictions that allow us to fit the market forward curve, that is the expressions (18.36) and (18.38), only apparently require a (numerical) differentiation with respect to the maturity. Indeed, see expressions (18.45) and (18.46) below, where this is not the case because the computation of the moment generating function turns out to require the values of the forward curve itself and not of its derivatives as well. Finally, we remark that the dynamics in (18.35) with constant parameters has been introduced by Dassios and Nagaradjasarma (2006), whilst (18.37) was introduced first by Cox *et al.* (1985) in order to model interest rate dynamics.

Proposition 18.4.1[8] provides the expression for the mgf in the case of time-dependent spot price drift and volatility.

Proposition 18.4.1 *Under commodity spot price dynamics* (18.35)*, the moment generating function of the pair* $(S_{n\Delta}, \sum_{j=0}^{N} \alpha_j S_{j\Delta})$ *given the information available at time* 0 *is:*

$$v_{0,x}^{\theta}(N, \Delta; \gamma, \mu) = e^{-\Lambda_0^{\theta}(\Delta;\gamma,\mu)x}, \tag{18.40}$$

where the function $\Lambda_j^{\theta}(\Delta; \gamma, \mu)$ *satisfies the recursive equation*

$$\Lambda_j^{\theta}(\Delta; \gamma, \mu) = A_{j\Delta}^{\theta}\left(\Delta; \Lambda_{j+1}^{\theta}(\Delta; \gamma, \mu)\right) + \mu\alpha_j,$$

for $j = N-1, N-2, \ldots, 1, 0$*, with starting value*

$$\Lambda_N^{\theta}(\Delta, \gamma, \mu) = \gamma + \mu\alpha_N.$$

Here A^{θ} *is defined as*

$$A_t^{\theta}(\Delta; \gamma) = \frac{\gamma \frac{F_{0,t+\Delta}}{F_{0,t}}}{1 + \frac{\gamma}{2} F_{0,t+\Delta} \int_t^{t+\Delta} \frac{\sigma_s^2}{F_{0,s}} ds}, \tag{18.41}$$

and $y = S(t)$*. In addition, as byproduct, we obtain the moment generating function of the spot price*

$$\widetilde{E}_t(e^{-\gamma S_{t+\Delta}}) = v_{0,x}^{\theta}(1, \Delta; \gamma, 0) \tag{18.42}$$

and of the arithmetic average

$$\widetilde{E}_t\left(e^{-\mu \sum_{j=0}^{N} \alpha_j S_{j\Delta}}\right) = v_{0,x}^{\theta}(N, \Delta; 0, \mu). \tag{18.43}$$

If we replace in formula (18.34), the expression of the mgf in (18.42), we can price plain vanilla options according to the square-root process. If instead we use (18.43), we can price fixed strike Asian options. The expression of the joint mgf is relevant for pricing floating strike options, not considered here.

A practical illustration of the procedure is given in Box 18.7, where the pricing of an Asian option consistent with the market forward curve and the market term structure of implied volatility is considered. Notice that implied volatility usually refers to percentage

[8] The proof can be found in Fusai *et al.* (2008).

volatility of log-returns. In order to obtain the volatility to be used in the square-root model, we can use the following transformation:

$$\sigma_{GBM} S = \sigma_{SR} \sqrt{S},$$

so that

$$\sigma_{SR} = \sigma_{GBM} \sqrt{S}.$$

This transformation is implemented in the numerical example considered in Box 18.7.

BOX 18.7 PRICING ASIAN OPTIONS IN A FLEXIBLE FRAMEWORK

We provide here the Matlab code for the implementation of the pricing procedure. A numerical example is also presented.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%Implement the Asian Price SQUARE Root Model%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function res=...
Asian_SR(strike, FwdCurve, VolCurve, r, t, n, mAW, nAW, AAW )

%Computing the expected value of the arithmetic average
dt=t/n;
momasia=sum(int_FwdCurve(dt*[0:n], FwdCurve))/(n+1);

%Inversion of Laplace transform
eul=AW_LTinversion(@(mu)...
     SR_LT_Avg_dt(mu, FwdCurve, VolCurve, t, n), strike, ...
            mAW, nAW, AAW);

%Pricing Formula
res=exp(-r*t)*(eul + momasia - strike);

%Laplace transform wrt strike of the asian option in the
%discrete monitoring case CIR model
function res=SR_LT_Avg_dt(mu, FwdCurve, VolCurve, t, n)
    res=SR_DoubleLT_dt(0, mu, FwdCurve, VolCurve, t, n)/mu^2;

%%%Moment Generating Function of the SR Process
function res=SR_DoubleLT_dt(gam, mu, FwdCurve, VolCurve, t, n)
    res=exp(-int_FwdCurve(0, FwdCurve)*...
        Lambdafunction(t/n, gam, mu, FwdCurve, VolCurve,  n));
```

```
%%%Function Lambda in Proposition 1
function aDiscr=Lambdafunction(dt, gam, mu, FwdCurve,
                               VolCurve, n)
aDiscr = gam + mu/(n + 1);
for j = n - 1:-1:0
    aDiscr =Afunction(j*dt,(j+1)*dt, aDiscr, FwdCurve,
                      VolCurve) ...
                            + mu/(n + 1);
end

%%%Function A in Proposition 1
function  res=Afunction(t, T, gam, FwdCurve, VolCurve)
    den= 1 + 0.5*gam*int_FwdCurve(T,FwdCurve)*...
        quadgk(@(x) (int_VolCurve(x, VolCurve).^2)./...
              int_FwdCurve(x,FwdCurve), t, T);
    num=gam*int_FwdCurve(T,FwdCurve)/int_FwdCurve(t,FwdCurve);

    res=num./den;

%%%Interpolating the forward curve
function iFwdCurve = int_FwdCurve(t, FwdCurve)
pp=interp1(FwdCurve(:,1),FwdCurve(:,2),'pchip','pp');
f = @(x) ppval(pp,x);
iFwdCurve=f(t);

%%%Interpolating the volatility curve
function iVolCurve = int_VolCurve(t, VolCurve)
pp=interp1(VolCurve(:,1),VolCurve(:,2),'pchip','pp');
f = @(x) ppval(pp,x);
iVolCurve=f(t);

%%%Inversion of the Laplace transform
function euler1=AW_LTinversion(LTf_s, tvar, m, n, A)
%downloaded from
%http://www.columbia.edu/~ww2040/6711F12/tools.html
%and modified by Gianluca Fusai
a=zeros(m+n+1,1); S=zeros(n+m+1,1);

anot=exp(A/2)*arrayfun(LTf_s,A/(2*tvar))/(2*tvar);
anot=anot+exp(A/2)*real(arrayfun(LTf_s,(A/2+1i*pi)/tvar)...
                    *exp(1i*pi))/tvar;

k=[1:(m+n+1)];
a=exp(A/2)*real(arrayfun(LTf_s, (A/2+(1+k)*1i*pi)/tvar)...
```

```
                    *exp(1i*pi))/tvar;
S(1)=anot-a(1);

for k=2:n+m+1
   S(k)=S(k-1)+((-1)^k)*a(k);
end

euler1=S(n);
for k=1:m
   euler1=euler1+nchoosek(m,k)*S(n+k);
end
euler1=(2^(-m))*euler1;
```

Let us consider the following example:

```
%A Numerical Example
%Parameters: risk free rate and time to maturity
r=0.0525; ttm=1;
%Parameters LT inversion
mAW=12; nAW=20; AAW=18.4;
%Market Forward Curve
ng_fwd=[0 7.1409
0.074   7.288
0.1534  7.405
0.2438  7.52
0.3233  7.635
0.4055  7.73
0.4959  7.785
0.5726  7.88
0.663   8.45
0.7452  9.01
0.8247  9.3
0.9151  9.295
0.9945  9.075;


%Term Structure of Percentage Volatility
ng_vol=[0.  13.8
0.0833333   16.8
0.166667    14.1
0.25   16
0.333333    16.8
0.416667    20.4
0.5 22.7
0.583333    26.7
```

```
0.666667    22.5
0.75   18.7
0.833333    18.4
0.916667    16.1
1 13.8];

%Pricing for different monitoring dates and strike at 1
strike=mean(ng_fwd(:,2)); %fix strike approximately at-the-money

%Change GBM vol into SR vol
ng_vol(:,2)=(ng_vol(:,2)/100)*sqrt(max(ng_fwd(:,2)))

res(1)=Asian_SR(strike, ng_fwd, ng_vol, r, ttm, 12, mAW, nAW, AAW)
res(2)=Asian_SR(strike, ng_fwd, ng_vol, r, ttm, 24, mAW, nAW, AAW)
res(3)=Asian_SR(strike, ng_fwd, ng_vol, r, ttm, 36, mAW, nAW, AAW)
res(4)=Asian_SR(strike, ng_fwd, ng_vol, r, ttm, 48, mAW, nAW, AAW)
```

We have the results in the following table:

N	12	24	36	48
Premium	0.3602	0.3648	0.3661	0.3667
MC	0.3652	0.3628	0.3593	0.3564

An analytical expression for the mgf of the underlying spot price $S_{t+\Delta}$ is available in the mean-reverting case as well, and is given in the following proposition 18.4.2:[9]

Proposition 18.4.2 *Under spot price dynamics* (18.37), *the moment generating function of the pair* $(S_{N\Delta}, \sum_{j=0}^{N} \alpha_j S_{j\Delta})$ *given the information available at time* 0 *is*

$$v_{0,x}^{\beta}(n, \Delta; \gamma, \mu) = e^{-\Lambda_0^{\beta}(\Delta;\gamma,\mu)x - \sum_{j=0}^{N-1} B_{j\Delta}^{\beta}\left(\Delta;\Lambda_{j+1}^{\beta}\left(\Delta;\gamma,\mu\right)\right)}, \tag{18.44}$$

where the function $\Lambda_j^{\beta}(\Delta; \gamma, \mu)$ *satisfies the recursive equation*

$$\Lambda_j^{\beta}(\Delta; \gamma, \mu) = A_{j\Delta}^{\beta}\left(\Delta; \Lambda_{j+1}^{\beta}(\Delta; \gamma, \mu)\right) + \mu\alpha_j,$$

[9] The proof can be found in Fusai *et al.* (2008).

for $j = N - 1, n - 2, \ldots, 0$*, with starting value*

$$\Lambda_N^{\beta}(\Delta, \gamma, \mu) = \gamma + \mu \alpha_N.$$

Here $A_{j\Delta}^{\beta}$ *and* B_t^{β} *are respectively given by*

$$A_t^{\beta}(\Delta; \gamma) = \frac{\gamma e^{-\beta\Delta}}{1 + \frac{\gamma}{2}\int_t^{t+\Delta} \sigma_s^2 e^{-\beta(t+\Delta-s)} ds}, \tag{18.45}$$

$$B_t^{\beta}(\Delta; \gamma) = \gamma F_{0,T} - F_{0,t} A_t^{\beta}(\Delta; \gamma) - \frac{1}{2}\int_t^{t+\Delta} F_{0,s}\sigma_s^2 A_s^{\beta}(\Delta; \gamma)^2 \, ds, \tag{18.46}$$

and $y = S(t)$*. In addition, the moment generating function of* $S_{t+\Delta}$ *is* $v_{t,y}^{\beta}(1, \Delta; \gamma, 0)$*, whilst the moment generating function of the arithmetic average is* $v_{0,x}^{\beta}(N, \Delta; 0, \mu)$*.*

18.4.2 Numerical Results

In the original paper by Fusai *et al.* (2008) a few numerical tests have been conducted in order to examine:

- The discrepancy of prices stemming from the alternative assumptions of a discrete vs. continuous monitoring rule. For barrier options, Fusai *et al.* (2006) showed that price differences can be very large in spite of a relatively high monitoring frequency. For Asian options, the story is a bit different. As expected, price differences between discrete and continuous monitoring rules decrease as long as the number of monitoring dates increases. In addition, the convergence of the discretely monitored option price to the continuously monitored one is almost linear in the monitoring frequency, much faster than is known to occur for barrier options; that is, approximately like $1/\sqrt{N}$.
- The impact of including market information about the forward prices in the spot price dynamics for the purpose of pricing Asian-style options. This analysis is conducted using quotes taken from the Natural Gas Market at NYMEX. It turns out that a non-flat forward curve produces highly significant option price deviations from figures obtained in the case where such information is not accounted for by the underlying spot price model.
- The price differences between square-root and lognormal model. Option prices using the square-root model specification accurately approximate quotes stemming from the model assuming lognormal dynamics, provided the volatility coefficient is adequately chosen to reproduce prices of plain vanilla at-the-money options. This fact constitutes a major result since the new flexible method allows us to price Asian-style option prices in real time, with great accuracy and allowing for time-varying volatility, fitting a forward curve. Vice versa, numerical approximation for the geometric Brownian motion case requires intensive calculations and much greater computational time. This result is quite robust across the examined spectrum of parameters, the only case where significant discrepancies are observed related to deeply out-of-the-money options.

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- Finally, the impact of including information about the time structure of historical volatility in the pricing device. A test on corn price data quoted at CBOT is performed. It turns out that using this information may result in significant price discrepancies compared with the quotes obtained using the market model represented by the geometric Brownian motion. These results suggest that when pricing Asian-style options in market contexts where a seasonal component strongly affects the evolution of spot price volatility, one should include this information as precisely as possible. This remark is particularly important for several commodity markets, such as energy and agriculturals, where the time variation of volatility is significantly pronounced.

18.4.3 A Case Study

In this final section we consider as a case study the computation of the fair value of Asian-style options taking into account market information. We consider the forward curve on Brent as quoted on 24 December, 2013 at ICE. Values across all delivery months are reported in Table 18.10, where we indicate the exact day of trading termination and the time to maturity of the contract as expressed in year units.

Our final goal is to assess Asian-style option prices under a realistic market setting in the square-root model with mean reversion.

We begin by defining values for each of the input quantities indicated in step 0 of the pricing algorithm stated earlier. Our base case assumes that:

- Current time is 24 December, 2013.
- Options expire on 14 March, 2014 (the average is computed over 59 working days), 13 June, 2014 (124 working days) and 13 November, 2014 (233 working days).
- Averages are computed based on daily monitoring, that is, $\Delta = 1/250$ years.
- Strike index K is assumed to match the at-the-money level, defined as

$$\overline{Avg}_{0,N} := \frac{1}{N+1}\sum_{j=0}^{N} F(0, j\Delta),$$

  where $N = T/\Delta$.
- For each maturity, interest rate r is linearly interpolated from LIBOR quotes on value date. Quotes are given in Table 18.11. Interpolated values at option maturities are respectively 0.2096%, 0.2797% and 0.4071%. These values are converted to continuous compounding using the conversion formula $\ln(1 + \text{LIBOR})$.
- Mean-reversion frequency is set equal to $\beta = 0.1$ p.a.
- Current spot price is set equal to the shortest maturity futures price, i.e. 111.99 USD.
- Spot price volatility of log-returns under the GBM assumption is assumed equal to 0.20, a typical value for crude oil (see e.g., the implied volatility quotes in Table 18.8. This figure is transformed into spot price volatility in the square-root model according to

$$S \times 0.2 = \sqrt{S} \times \sigma_{SR},$$

TABLE 18.10 Crude Oil Futures prices quoted on 24 December, 2013 at ICE

Delivery	Maturity (mm/dd/yy)	Days to expiry	Settlement	Volume
Feb14	1/16/14	23	111.99	26179
Mar14	2/13/14	51	111.65	7590
Apr14	3/14/14	80	111.33	2495
May14	4/15/14	112	110.95	936
Jun14	5/15/14	142	110.54	2538
Jul14	6/13/14	171	110.08	221
Aug14	7/16/14	204	109.55	125
Sep14	8/14/14	233	109.05	245
Oct14	9/15/14	265	108.55	47
Dec14	11/13/14	324	107.59	1862

Source: https://www.theice.com/productguide/ProductSpec.shtml?specId=219#data.

so that

$$\sigma_{SR} = \sqrt{S} \times 0.2 = \sqrt{111.99} \times 0.2$$

and we set $\sigma_{SR} = 2$.

We price the Asian option using the analytical method described above. To benchmark our results we consider Monte Carlo simulation. In particular, given that with respect to the GBM dynamics no exact solution of the considered square-root stochastic equations is possible, we discretize the mean-reverting dynamics according to the Euler scheme using a time step Δ:

$$S(t+\Delta) = S(t) + \left(\int_t^{t+\Delta} \eta(s)ds - \beta S(t)\Delta\right) + F(0, t+\Delta) - F(0, t) + \sigma\sqrt{S(t)}\epsilon(t)\sqrt{\Delta},$$

where $\epsilon(t)$ is a sequence of i.i.d. standard Gaussian random variables. In particular, notice that the term $\int_t^{t+\Delta} \eta(s)ds$ can be approximated as follows:

$$\begin{aligned}\int_t^{t+\Delta} \eta(s)ds = & \int_t^{t+\Delta} \left(F(0,s) + \partial_s F(0,s)\right) ds \\ = & \tfrac{(F(0,t+\Delta)+F(0,t))\Delta}{2} + F(0, t+\Delta) - F(0, t).\end{aligned}$$

TABLE 18.11 US$ LIBOR rates across varying times-to-maturity. Quotes as of 24 December, 2013

3m (93 days)	6m (185 days)	12m (370 days)
0.24585	0.34940	0.58360

TABLE 18.12 Asian-type option prices (exact and Monte Carlo estimate) for the mean-reverting model, varying the strike and the time horizon

Days	Strike	Exact	MC estimate	Confidence interval (3 std. errors)	
59	111.8683	1.9389	1.9403	1.9129	1.9676
124	111.5131	2.7947	2.8104	2.7706	2.8501
233	110.7821	3.7801	3.8022	3.748	3.8564

TABLE 18.13 Values of the forward curve at different horizons when it is set at a constant value. These values are used for pricing Asian options in Table 18.14

Days	59	124	233
min	111.56	110.793	109.05
avg	111.868	111.513	110.782
max	111.99	111.99	111.99

In conclusion, the Euler discretization allows us to simulate the spot price path according to

$$S(t+\Delta) = S(t) + \left(\int_t^{t+\Delta} F(0,s)ds - \beta S(t)\Delta \right) + F(0,t+\Delta) - F(0,t) + \sigma\epsilon(t)\sqrt{S(t)\Delta}.$$

Pricing results are given in Table 18.12. Exact and Monte Carlo estimates (100,000 simulations, time step $\Delta = 1/365$) agree quite well: exact prices always fall inside the three standard errors confidence interval.

The relevance of incorporating the market forward curve is examined in Tables 18.13 and 18.14 where different assumptions on the forward curve are examined. In particular, for each maturity, we price the Asian option using the market observed forward curve and we compare the price obtained assuming that the forward curve is flat at three different levels: the minimum forward price up to the option expiry, the average forward price up to the option expiry and the maximum forward price up to the option expiry. These values are reported in Table 18.13, whilst the corresponding Asian option prices are given in Table 18.14.

Finally, Table 18.15 shows that option prices decrease with an increase in the speed at which prices tend to revert back to their long-term trend. In fact, higher mean reversion reduces underlying price dispersions, so reducing the likelihood of ending up in-the-money.

TABLE 18.14 Asian option prices in the SR mean-reverting model at different horizons under different shapes of the forward curve: flat at the minimum (min)/average (avg)/maximum (max) level and the market observed one (market). Values are given in Table 18.13

Days	59	124	233
min	1.7868	2.4422	2.9551
avg	1.8506	2.6848	3.6281
market	1.9389	2.7947	3.7801
max	1.9998	3.0377	4.4083

TABLE 18.15 Asian option price vs. speed of mean reversion

β	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
Price	3.7801	3.6919	3.6068	3.5246	3.4452	3.3686	3.2945	3.2229	3.1536	3.0867

Further extensions might encompass pricing of Asian-style options written on a basket of prices and empirical examination of implied calibration of the model on plain vanilla quotes and comparison between model and market assessment of Asian options. Interesting results for basket options with possible applications to Asian options have been proposed in Caldana *et al.* (2014).

18.5 CONCLUSIONS

It is commonly held that pricing Asian-style options under the market model represented by a geometric Brownian motion is a difficult task. First, no closed-form expression exists for the fair option value. Second, information embedded within the standing market forward curve is neglected during the valuation process. Finally, prices are derived irrespective of the seasonal path exhibited by the spot price volatility or mean-reversion properties.

In this chapter, we have discussed standard models for pricing Asian options and more recent modelling achievements, particularly useful for pricing options in commodity markets where traders must quickly produce quotes compatible with the market view expressed in terms of forward prices, typically showing a seasonal behaviour and with a mean-reversion feature.

For the sake of completeness, we would like to mention more recent work by Cerny and Kyriakou (2011), Fusai and Meucci (2007), Fusai and Kyriakou (2014) and Marena *et al.* (2013). They consider the pricing of discrete monitoring Asian options in a more general framework allowing jumps and stochastic volatility. Finally, Ballota *et al*. (2014) consider the hedging problem of Asian options in a non-Gaussian setting.

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