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RESEARCH ARTICLE

Coalitional extreme desirability in finitely additive exchange economies

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Abstract We define a new notion of extreme desirability for economies in coalitional form. Through this, we obtain a finitely additive core-Walras equivalence theorem for an exchange economy with a measure space of agents and an infinite dimensional commodity space, whose positive cone has possibly empty interior.

Keywords Coalitional economies · Coalitional preferences · Core-Walras equivalence · Finitely additive measures

JEL Classification D5 · D51

1 Introduction

When looking for the existence of a Walrasian equilibrium or for core-Walras equivalence results, one of the major problems in dealing with models with an infinite dimensional commodity space is the possible emptiness of the order cone. Unfortunately, this prevents the use of classical separation arguments for proving the existence of prices which support Walrasian allocations in many of the spaces that are of interest for economic and financial models such as, for example, the L_p spaces, $p \in [1, \infty)$.

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Therefore, conditions which can amend this difficulty have been extensively studied, the most popular ones used in models with individual agents and preferences being the cone condition of Chichilnisky and Kalman (1980), the properness of Mas-Colell (1986) and the extremely desirable commodity of Rustichini and Yannelis (1991) (these three conditions prove to be equivalent for complete preorders, see Chichilnisky 1993). We refer the reader also to Aliprantis et al. (2000) for a complete overview on the topic.

Starting from the work of Aumann (1964), the idea that when an economy has sufficiently many traders everyone acts like a price-taker and only coalitions matter, has been formalized through the assumption of a space of negligible agents, each endowed with individual preferences on bundles of commodities. More precisely, both in the case of a finite and an infinite dimensional commodity space, the primitives in these models are a measure space of agents, a σ-algebra on the space (where each element represents a coalition), and a nonatomic measure. Clearly, if the nonatomic measure is non-trivial, this implies that the space is uncountable. In other words, in a countably additive setting the space of agents has to be uncountable. To overcome this lack of realism and of economic meaningfulness, many authors turned to the use of finitely additive measures. Also, the fact that only the bargaining power of the coalitions can influence the final outcomes, has suggested to work directly with coalitions themselves (Vind 1964). After the one of Armstrong and Richter (1984), many works have thus faced the problem of core-Walras equivalence in a coalitional finitely additive setting (Basile 1993; Donnini and Graziano 2009). However, in all these models, when the commodity space is of infinite dimension the non-emptiness of the positive cone is directly assumed.

The aim of our work is twofold: on the one hand, we introduce in a coalitional framework a condition which plays the same role of the *extremely desirable commodity assumption* combined with the *additivity condition* in individualistic models (Rustichini and Yannelis 1991), thus allowing to work with commodity spaces whose cone has possibly empty interior; on the other one, we obtain a core-Walras equivalence theorem working in a finitely additive context. We emphasize that, in the literature, there are other countably additive coalitional models that make use of properness-like conditions in order to obtain core-Walras equivalence theorems. We recall the works of Zame (1986), and the recent one of Greinecker and Podczeck (2013): the last one includes the case of all Banach lattices, at the cost of strengthening some measure theoretic hypotheses. The fact that properness-like assumptions are crucial in order to account for spaces whose positive cone has empty interior is also well emphasized by the recent work of Bhowmik and Graziano (2015), who made use precisely of the above-mentioned conditions for individuals to extend the classical Theorem of Vind (1972) to the case of an ordered Banach space whose positive cone may have empty interior with the presence of atoms in the agents space.

We define the notion of *coalitional extreme desirability,* and we also replace the *additivity condition* of Rustichini and Yannelis (1991) (see also Angeloni and Martellotti 2007; Bhowmik and Graziano 2015) by a weaker condition that, contrary to the additivity condition, is in fact satisfied for instance by balls in L_p spaces, $p \geq 1$. Then, after introducing a set of natural hypotheses on coalitional preferences, we prove a finitely additive core-Walras equivalence theorem for an exchange economy with a Banach lattice of commodities with the Radon–Nikodym property (RNP), without any assumption on the interior of the positive orthant. We also show that, when the commodity space is separable and has the RNP (Rustichini and Yannelis 1991), individualistic Theorem 6.1 can be deduced from our coalitional result.

We point out that there is a huge variety of spaces of interest in economic models and that satisfy the RNP (see Diestel and Uhl 1977), among which all reflexive spaces. Furthermore, our new condition is satisfied when one moves from a classical individualistic model to the derived coalitional one, as in Armstrong and Richter (1984). The present paper also represents a first preliminary step towards extensions to richer models.

In Sect. 2, we describe the model, recall some definitions and introduce the notion of coalitional extreme desirability. In Sect. 3, we provide our finitely additive core-Walras equivalence result under coalitional extreme desirability. Section 4 is devoted to the comparisons with Rustichini and Yannelis' (1991) result and with other coalitional models.

2 The model

Let (Ω, Σ) be a measurable space, where Σ is an algebra on Ω, and let P be a strongly nonatomic finitely additive (f.a.) probability on it, that is, for every $A \in \Sigma$ and $\varepsilon \in (0, 1)$, there exists $B \subset A$, $B \in \Sigma$, such that $P(B) = \varepsilon P(A)$ (see also Bhaskara Rao and Bhaskara Rao 1983). For example, if $\Omega = [0, 1)$ and $\Sigma = \{\bigcup_{i=1}^{n} [a_i, b_i) : [a_i, b_i) \cap [a_j, b_j) = \emptyset, i \neq j, 0 \leq a_i < b_i < 1\}$, the Lebesgue measure on Σ is strongly nonatomic.[1] (Ω, Σ, P) is the space of agents, and elements in Σ are referred to as *coalitions*. Σ^+ denotes the class of non-negligible coalitions.

Let X be a Banach lattice with the Radon–Nikodym property (RNP), with positive cone X^+ representing the *commodity space*. By $\geqslant$ we shall denote the vector order in X, that is, $x \geqslant y$ if $x - y \in X^+$. The symbol Δ_n denotes the set $\{(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} \theta_i = 1\}$.

A *coalitional exchange economy* $\mathcal{E} = ((\Omega, \Sigma, P), (\succ_F)_{F \in \Sigma}, e)$ for (Ω, Σ, P) and X, is described as follows. An *allocation* is any f.a. measure $m : \Sigma \to X^+$ and, when $m \ll P$ [in the $\varepsilon - \delta$ sense of Bhaskara Rao and Bhaskara Rao (1983)], we shall refer to m as a *consumption allocation.*

$\mathcal{M}$ denotes the set of consumption allocations in $ba(\Sigma, X^+)$. A consumption allocation specifies the way commodities $x \in X^+$ are assigned to coalitions.

The *initial endowment* is an allocation $e : \Sigma \to X^+$: $e \equiv P$ that is, $e \ll P$ and $P \ll e$.

For any $m \in \mathcal{M}$, and for any coalition $F \in \Sigma$, we shall denote by $m_{|F}$ the vector measure defined on Σ by $m_{|F}(E) = m(E \cap F)$, for every $E \in \Sigma$.

A *simple allocation* is any allocation s of the form $s = \sum_{i=1}^{q} y_i P_{|H_i}$, where $\{H_i\}_i$ is a decomposition of Ω.

[1] Other two simple examples can be constructed in the following way. Let P be a f.a. and semiconvex measure on a σ-algebra, and let $\mathcal{A}$ be its Stone algebra. Then, the measure $\widetilde{P}$ corresponding to P, is strongly nonatomic on $\mathcal{A}$. Or, if P is f.a. and semiconvex on a σ-algebra, then P is strongly nonatomic on the algebra generated by a filtering family $(\Omega_t)_t$.

In particular, for each $x \in X^+$, we shall indicate by xP the *uniform allocations* $\hat{x}$ of Cheng (1991).

The positive cone $(X^*)^+$ of the norm dual of X represents the *price space,* and its elements x^* are termed as *prices.* The value of a commodity x at price x^* is given by the evaluation $x^*(x)$.

For each coalition $F \in \Sigma^+$, a *preference relation* $\succ_F$ on $\mathcal{M} \times \mathcal{M}$ is assigned.

The following assumptions on preferences are natural for coalitional form economies.

(T) $\succ_F$ is transitive and irreflexive.

(I) **(I.i)** If $m_1, m_2 \in \mathcal{M}$, and $m_2 \succ_F m_1$, then, for each subcoalition $G \in \Sigma^+$, $G \subseteq F$, $m_2 \succ_G m_1$; **(I.ii)** if $m_2 \succ_F m_1, m_2 \succ_G m_1$, then $m_2 \succ_{F \cup G} m_1$, for each $F, G \in \Sigma^+$.

(WM) (*weak monotonicity*) For any $m \in \mathcal{M}$ and $x \in X^+ \setminus \{0\}$, $m + xP \succ_\Omega m$.

(S) (*selfishness*) If $m_1, m_2, m_3 \in \mathcal{M}$, $F \in \Sigma^+$ are such that $m_{1|_F} = m_{2|_F}$ then $[m_3 \succ_F m_1 \iff m_3 \succ_F m_2]$ and $[m_1 \succ_F m_3 \iff m_2 \succ_F m_3]$.

(A) (*availability*) $e(\Omega) \gg 0$ (where the notation $x \gg 0$ means that $x^*(x) > 0$ for every nonzero price $x^* \in (X^*)^+$).

Assumptions **(I), (S)** and **(A)**, and variants of **(WM)**, have natural economic interpretations and are common to most of coalitional finitely additive models (see Armstrong and Richter 1984; Basile 1993; Basile and Graziano 2001; Donnini and Graziano 2009). Instead, some coalitional models (e.g., Donnini and Graziano 2009) do not assume transitivity of preferences. However, we point out that transitivity is a standard assumption in countably additive coalitional models (see for example Zame 1986 and the recent work of Greinecker and Podczeck 2013) as well as for the coalitional models derived from individualistic ones (e.g., Armstrong and Richter 1984), where transitivity of coalitional preferences derives from that of individual preferences. Therefore, this assumption does not appear too demanding.

It is easily seen that **(WM)** is implied by the usual monotonicity assumed in Armstrong and Richter (1984).

Note that **(WM)**, jointly with condition **(T)**, implies the following weak form of transitivity **(WT)**, which should be compared with condition (V.3) in Cheng (1991):

(WT) if $f \succ_\Omega g$ and $x \in X^+ \setminus \{0\}$ then $f + xP \succ_\Omega g$.

We also assume the following form of continuity:

(C) For every $F \in \Sigma^+$, any $\alpha, \beta \in \mathcal{M}$ with $\beta \succ_F \alpha$, and any $\tau > 0$, there exist $F_0 \in \Sigma^+$, $F_0 \subseteq F$, and $\rho(\tau) > 0$ such that $P(F \setminus F_0) < \tau$ and, for every simple allocation s with $\|s - \beta\| < \rho$, $s \succ_{F_0} \alpha$.

Definition 2.1 $m \in \mathcal{M}$ is *feasible* provided $m(\Omega) = e(\Omega)$.

Denote by $\mathcal{F}$ the set of feasible allocations.

Given $m_1, m_2 \in \mathcal{M}$ and $F \in \Sigma^+$, we say that m_1 *blocks* m_2 via F if $m_1 \succ_F m_2$ and $m_1(F) = e(F)$.

$\alpha \in \mathcal{F}$ is called a *core allocation* if no allocation $m \in \mathcal{M}$ blocks α via any $F \in \Sigma^+$.

$\alpha \in \mathcal{F}$ is called a *Walrasian allocation,* if there is a nonzero price $x^* \in (X^*)^+$ such that, for each coalition $F \in \Sigma^+$, $x^*(\alpha(F)) \leq x^*(e(F))$, and $x^*(\beta(F)) > x^*(e(F))$

whenever β is a consumption allocation with $\beta \succ_F \alpha$. The pair (x^*, α) is called a *Walrasian equilibrium.*

We now extend the additivity condition as well as the notion of extreme desirability of Rustichini and Yannelis (1991) to coalitional preferences, in the following way.

Definition 2.2 Coalitional preferences $\succ_F$ are called *proper* if

(P) there are $u \in X^+$, and an open, convex, solid neighborhood U of 0 in X, fulfilling the following two conditions:
- **(P.i)** there exists $\vartheta > 0$ such that, for every $n \in \mathbb{N}$, $y_1, \ldots, y_n \in X^+ \cap U^c$, and every $(t_1, \ldots, t_n) \in \Delta_n$, the vector $\sum_{i=1}^n t_i y_i \notin \vartheta U$;
- **(P.ii)** setting $H = \bigcup_{t>0} t(u + U)$, if $y \in X^+$, $t \in [0, 1]$, $v \in X^+$ are such that $z = (y + tu - v) \in \overline{(y + H)} \cap X^+$, then $zP \succ_\Omega yP$.

u is called an *extremely desirable commodity* with respect to U.

Let us briefly comment on these two properties.

Observe first that condition **(P.ii)** can be equivalently formulated in the more familiar form: setting $H = \bigcup_{t>0} t(u + U)$, if $y \in X^+$, $t \in [0, 1]$, $v \in X^+$ are such that $z = (y + tu - v) \in (y + H) \cap X^+$, then $zP \succ_\Omega yP$ (use for example Aliprantis and Border 2006, Lemma 5.28 page 182).

Under this form, one immediately deduces that **(P)** is satisfied by the coalitional model derived from an individualistic one, where individual preferences satisfy assumptions $(A.10)$ and $(A.11)$ in Rustichini and Yannelis (1991), or the properness of Angeloni and Martellotti (2007).

Condition **(P.i)** replaces the so-called *additivity condition* in Rustichini and Yannelis (1991); in Angeloni and Martellotti (2007) it had been noted that the additivity condition can be equivalently reformulated in the form : $U^c \cap X^+$ is *convex*, and another equivalent formulation, expressed in terms of linear functionals and half-spaces thus determined, appears in Martellotti (2008).

The main advantage of the weaker form of the additivity condition **(P.i)** above, is that it is immediately satisfied by the balls of some important spaces.

For instance, **(P.i)** holds if $U = \rho X_1$ (X_1 denotes the unitary open ball) and $X = L^p(T, \mathcal{A}, \mu)$ for some measure space $(T, \mathcal{A}, \mu)$ and $p \geq 1$. In fact, if q is the conjugate exponent of p, then **(P.i)** holds with $\vartheta = \frac{\rho}{\sqrt[q]{q}}$. By Schwartz's inequality, for every $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n_+$, one immediately has:

$$\sum_{i=1}^n \theta_i \leq \sqrt[q]{q} \sqrt[p]{\sum_{i=1}^n \theta_i^p}.$$

Indeed, write $\sum_{i=1}^n \theta_i = \theta \cdot \mathbf{1}$ as a scalar product between two vectors in $\mathbb{R}^n$. Hence:

$$\sum_{i=1}^n \theta_i = \theta \cdot \mathbf{1} \leq \|\mathbf{1}\|_q \cdot \|\theta\|_p = \sqrt[q]{q} \sqrt[p]{\sum_{i=1}^n \theta_i^p}.$$

Thus, if $\theta \in \Delta_n$, then $\sum_{i=1}^n \theta_i^p \geq \frac{1}{q^{\frac{p}{q}}}$.

Then, as $\|y_i\|_p \geq \rho$ one finds:

$$\left\|\sum_{i=1}^n t_i y_i\right\|_p^p = \int_A \left(\sum_{i=1}^n t_i y_i\right)^p d\mu \geq \int_A \left(\sum_{i=1}^n t_i^p y_i^p\right) d\mu$$
$$= \sum_{i=1}^n t_i^p \|y_i\|_p^p \geq \rho^p \sum_{i=1}^n t_i^p \geq \frac{\rho^p}{q^{\frac{p}{q}}}$$

whence $\left\|\sum_{i=1}^n t_i y_i\right\|_p \geq \frac{\rho}{\sqrt[q]{q}} = \vartheta$.

Similarly, if X is a real Hilbert space, and again $U = \rho X_1$, then **(P.i)** is satisfied with $\vartheta = \frac{1}{\sqrt{2}}$.

In fact, as in the above computation, one finds:

$$\left\|\sum_{i=1}^n t_i y_i\right\|^2 = \left\langle \sum t_i y_i, \sum t_i y_i \right\rangle \geq \sum t_i^2 \|y_i\|^2 \geq \rho^2 \sum t_i^2 \geq \frac{\rho^2}{2}.$$

It should be underlined that the space $L^1(T, \mathcal{A}, \mu)$ does not enjoy however the RNP.

Finally, if $(X^+)^\circ \neq \emptyset$ and coalitional preferences satisfy **(WM)**, then **(P.ii)** is automatically satisfied.

3 Main result

Coalitional extreme desirability allows us to prove core-Walras equivalence.

Before proving it, we recall a lemma from Martellotti (2007).

Lemma 3.1 *Let P be a strongly nonatomic f.a. probability on an algebra Σ. Then, for every $\varepsilon > 0$, and every finite decomposition of E, $\{E_1, \ldots E_\ell\} \subseteq \Sigma$, there exists a decomposition of E, say $\{F, F_1, \ldots, F_q\}$, such that $P(F) = P(E \setminus \bigcup F_q) < \varepsilon$, $P(F_1) = \cdots = P(F_q)$, and $\{F_1, \ldots, F_q\}$ is finer than the corresponding decomposition of $E \setminus F$ by $\{E_1, \ldots E_\ell\}$.*

Let now α be an allocation in the core. Define the set:

$$\mathcal{K} = \bigcup_{F \in \Sigma^+} \left[\{\gamma(F) : \gamma \in \mathcal{M}, \gamma \succ_F \alpha\} - e(F)\right].$$

By **(WM)**, $\mathcal{K}$ is nonempty. Moreover, consider the sets $W = \vartheta U$, with ϑ determined by **(P.i)**, and $C = \bigcup_{t>0} t(u + W)$.

Lemma 3.2 $\overline{\mathcal{K}} \cap (-C) = \emptyset$

Proof Since $-C$ is open, one can prove equivalently that $\mathcal{K} \cap (-C) = \emptyset$.

Let by contradiction $z = \gamma(F) - e(F) \in \mathcal{K} \cap (-C)$ be fixed, and take $\varepsilon > 0$ such that $z + \varepsilon X_1 \subset (-C)$.

Let $\delta = \delta(\frac{\varepsilon}{7})$ be determined by the absolute continuity of γ and e w.r.t. P. By **(C)**, we can choose $\tau = \delta$ and determine $F_0 \subseteq F$, $F_0 \in \Sigma^+$, with $P(F \backslash F_0) < \tau$, and the corresponding $\rho(\tau) = \rho$.

As X has the RNP, we can choose a simple allocation $s = \sum_{i=1}^{q} y_i P_{|F_i}$, where $\{F_i\}_i$ is a decomposition of F_0, such that $\|\gamma - s\| < \min\{\frac{\varepsilon}{7}, \rho\}$ (see Uhl 1967); thus $s \succ_{F_0} \alpha$.

Now, setting $z_0 = s(F_0) - e(F_0)$, we obtain:

$$\begin{aligned}\|z_0 - z\| &= \|s(F_0) - \gamma(F) + e(F) - e(F_0)\| \\ &\leq \|s(F_0) - \gamma(F_0)\| + \|\gamma(F \backslash F_0)\| + \|e(F \backslash F_0)\| < \frac{3}{7}\varepsilon.\end{aligned}$$

Hence, $s(F_0) - e(F_0) \in (-C)$, with $s \succ_{F_0} \alpha$.

W.l.o.g. we can assume that $P(F_1) = \cdots = P(F_n) = \xi$. In fact, otherwise, applying Lemma 3.1 we can reduce to a subset $\widetilde{F} \subseteq F_0$ with $P(F_0 \backslash \widetilde{F}) < \tau$, and to a decomposition $\{E_1, \ldots, E_n\}$ of $\widetilde{F}$ with $P(E_1) = \cdots = P(E_n) = \xi$. Hence, consider $\widetilde{z} = s(\widetilde{F}) - e(\widetilde{F})$. We get:

$$\begin{aligned}&\|z - \widetilde{z}\| \leq \|z - z_0\| + \|z_0 - \widetilde{z}\| \\ &\leq \frac{3}{7}\varepsilon + \|s(F_0) - \gamma(F_0)\| + \|\gamma(F_0 \backslash \widetilde{F})\| + \|\gamma(\widetilde{F}) - s(\widetilde{F})\| + \|e(F_0 \backslash \widetilde{F})\| < \varepsilon\end{aligned}$$

and so, $s(\widetilde{F}) - e(\widetilde{F}) \in (-C)$, with $s \succ_{\widetilde{F}} \alpha$.

Therefore, $s(F_0) - e(F_0) \in -t(u + W)$ for some $t > 0$, that means:

$$\sum_{i=1}^{q} y_i \xi - e(F_0) = -t(u + v_0) \tag{1}$$

for some $v_0 \in W$, or else:

$$\sum_{i=1}^{q} y_i + w - v = \frac{e(F_0)}{\xi} \in X^+$$

where $w = \frac{t}{\xi}u$, $v = -\frac{t}{\xi}v_0 \in \frac{t}{\xi}W$.

As in Rustichini and Yannelis (1991), we can choose w.l.o.g. that $v \in X^+$.

Again, similarly to the proof in Rustichini and Yannelis (1991), for every choice of $(t_1, \ldots, t_q) \in \Delta_q$, one finds with the Riesz Decomposition Property of X, $v_1, \ldots v_q \in X^+$ with $v_1 + \ldots + v_q = v$ and $y_i + t_i w \geqslant v_i$, $i = 1, \ldots, q$.

Set now $H = \bigcup_{t>0} t(u + U)$, and remember that (u, U) is a properness pair.

Define $d_i : [0, 1] \to \mathbb{R}$ as $d_i(t) = \text{dist}\left[y_i + tw - v_i, (y_i + H) \cap X^+\right]$, and $f : \Delta_q \to \Delta_q$ as:

$$f(t_1, \ldots, t_q) = \left(\frac{t_i + d_i(t_i)}{1 + \sum_{i=1}^q d_i(t_i)} \right)_{i=1,\ldots,q}$$

and let $(\overline{t_1}, \ldots, \overline{t_q})$ be a fixed point:

$$\overline{t_i} \sum_{i=1}^q d_i(\overline{t_i}) = d_i(\overline{t_i}).$$

We claim that $\sum_{i=1}^q d_i(\overline{t_i}) > 0$.

Suppose by contradiction that $d_i(\overline{t_i}) = 0$ for each $i = 1, \ldots, q$, that is:

$$(y_i + \overline{t_i}w - v_i) \in \overline{(y_i + H) \cap X^+} \subseteq \overline{(y_i + H)} \cap X^+;$$

then, by **(P.ii)** and **(I.i)**, $(y_i + \overline{t_i}w - v_i)P \succ_{F_i} y_i P$ and, by **(I.ii)** and **(T)**, we have:

$$\sum_{i=1}^q (y_i + \overline{t_i}w - v_i)P_{|F_i} \succ_{F_0} s \succ_{F_0} \alpha.$$

Set now $s_0 = \sum_{i=1}^q (y_i + \overline{t_i}w - v_i)P_{|F_i} + \alpha|_{\Omega \setminus F_0}$. Then, $s_0 \succ_{F_0} \alpha$. But, from (1), also $s_0(F_0) = e(F_0)$; therefore s_0 blocks α via F_0, which contradicts the assumption that α is in the core.

Then, we turn to the case $\sum_{i=1}^q d_i(\overline{t_i}) > 0$.

Although pretty similar to that in Rustichini and Yannelis (1991), we include some details here, to show how the weaker formulation of the additivity condition (this is the unique point where it has been used in Rustichini and Yannelis 1991) still leads to a contradiction. In fact, for those indices i for which $d_i(\overline{t_i}) = 0$, one has necessarily that $\overline{t_i} = 0$ too and, conversely, if $\overline{t_i} = 0$, then $d_i(\overline{t_i}) = 0$. Hence, we can split $\{1, \ldots, q\}$ into $I = \{i : d_i(\overline{t_i}) = 0\} = \{i : \overline{t_i} = 0\}$ and $J = \{1, \ldots, q\} \setminus I$.

For $i \in I$, necessarily $v_i = 0$; in fact, if $v_i \neq 0$ one would find that:

$$y_i \geqslant y_i - v_i = y_i + \overline{t_i}w - v_i$$

and $y_i - v_i \neq y_i$, whence, by **(WM)**, $y_i P \succ_\Omega (y_i - v_i)P$.

But this leads to the conclusion that $(y_i - v_i) \notin \overline{(y_i + H)} \cap X^+$ for otherwise, because of **(P)**, the converse $(y_i - v_i)P \succ_\Omega y_i P$ should hold.

Therefore, $d_i(\overline{t_i}) > 0$ for $i \in I$, a contradiction to the very definition of I.

Hence, $v = \sum_{i=1}^q v_i = \sum_{i \in J} v_i$.

For $i \in J$, as $d_i(\overline{t_i}) > 0$, necessarily $(y_i + \overline{t_i}w - v_i) \notin \overline{X^+ \cap (y_i + H)}$ and, since $(y_i + \overline{t_i}w - v_i) \in X^+$ (for $y_i \in X^+$ and we have chosen $v_i \leqslant y_i + \overline{t_i}w$), *a fortiori* $\overline{t_i}w - v_i \notin H$, i.e., $\overline{t_i}\frac{t}{\xi}u - v_i \notin H$.

Thus, $v_i \notin \overline{t_i} \cdot \frac{t}{\xi}U$ for $i \in J$, and $\sum_{i \in J} \overline{t_i} = 1$.

Hence, $\frac{\xi}{t} \cdot \frac{v_i}{\overline{t_i}} \notin U$; by virtue of **(P.i)**:

$$\sum_{i \in J} \overline{t_i} \cdot \frac{\xi}{t} \cdot \frac{v_i}{\overline{t_i}} = \frac{\xi}{t} \sum_{i \in J} v_i = \frac{\xi}{t} v \notin \vartheta U$$

which precisely contradicts the original assumption that $v = \frac{t}{\xi} v_0 \in \frac{t}{\xi} W = \frac{t}{\xi} \vartheta U$.

Hence $\overline{\mathcal{K}} \cap (-C) = \emptyset$. $\square$

Lemma 3.3 *$\overline{\mathcal{K}}$ is convex.*

Proof Fix $z_1, z_2 \in \overline{\mathcal{K}}$, $\lambda \in (0, 1)$; to prove that $z_\lambda = \lambda z_1 + (1 - \lambda) z_2 \in \overline{\mathcal{K}}$ we have to prove that, for each $\varepsilon > 0$, there exists $\xi \in \mathcal{K}$ with $\|z_\lambda - \xi\| < \frac{\varepsilon}{2}$.

Since $z_i \in \overline{\mathcal{K}}$, there are allocations γ_i, $i = 1, 2$ and coalitions F_i, $i = 1, 2$: $\gamma_i \succ_{F_i} \alpha$ and:

$$\|\gamma_i(F_i) - e(F_i) - z_i\| < \varepsilon.$$

By **(C)**, after determining δ through the absolute continuity of γ_i, $i = 1, 2$ and e w.r.t. P, choose $\tau = \delta(\frac{\varepsilon}{12})$, and determine $\rho_i = \rho_i(\tau)$, and $E_i \subseteq F_i$, $E_i \in \Sigma^+$ with $P(F_i \backslash E_i) \leq \tau$, $i = 1, 2$. Set $\rho = \min\{\rho_1, \rho_2\}$. As previously, we can replace γ_1, γ_2, e by means of simple allocations s_1, s_2, η: $\|s_i - \gamma_i\| < \min\left\{\frac{\varepsilon}{12}, \rho\right\}$, $\|e - \eta\| < \frac{\varepsilon}{12}$ and $s_i \succ_{E_i} \alpha$.

Also, since s_1, s_2, η are simple, we can rearrange them on a common decomposition D and, by means of suitable refinements, D can be represented as $D = D_1 \cup D_2 \cup D_3$, where $D_1 = \{G_1, \ldots, G_n\}$ is a decomposition of $E_1 \backslash E_2$, $D_2 = \{H_1, \ldots, H_p\}$ of $E_1 \cap E_2$ and $D_3 = \{K_1, \ldots, K_q\}$ of $E_2 \backslash E_1$.

By the nonatomicity of P, we can find in each G_i a subset $G_{i,\lambda}$ such that $P(G_{i,\lambda}) = \lambda P(G_i)$ and, analogously, a subset $K_{j,(1-\lambda)} \subset K_j$ with $P(K_{j,(1-\lambda)}) = (1-\lambda) P(K_j)$; finally each H_ℓ splits into $H_{\ell,\lambda}$, $H_\ell \backslash H_{\ell,\lambda}$ with $P(H_{\ell,\lambda}) = \lambda P(H_\ell)$, $P(H_\ell \backslash H_{\ell,\lambda}) = (1 - \lambda) P(H_\ell)$.

Consider now:

$$S_1 = \bigcup_{i=1}^{n} G_{i,\lambda} \subset E_1 \backslash E_2, \quad S_2 = \bigcup_{\ell=1}^{p} H_{\ell,\lambda} \subset E_1 \cap E_2,$$

$$S_3 = \bigcup_{\ell=1}^{p} H_\ell \backslash H_{\ell,\lambda} \subset E_1 \cap E_2, \quad S_4 = \bigcup_{j=1}^{q} K_{j,(1-\lambda)} \subset E_2 \backslash E_1.$$

Note that $S_2 \cap S_3 = \emptyset$.

Also, $(s_1 - \eta)(S_1 \cup S_2) = \lambda(s_1 - \eta)(E_1)$ and $(s_2 - \eta)(S_3 \cup S_4) = (1-\lambda)(s_2 - \eta)(E_2)$.

Moreover, $s_1 \mathbf{1}_{S_1 \cup S_2} \succ_{S_1 \cup S_2} \alpha$ and $s_2 \mathbf{1}_{S_3 \cup S_4} \succ_{S_3 \cup S_4} \alpha$, whence:

$$s_1 \mathbf{1}_{S_1 \cup S_2} + s_2 \mathbf{1}_{S_3 \cup S_4} \succ_{S_1 \cup \ldots \cup S_4} \alpha.$$

Hence:

$$s_1(S_1 \cup S_2) + s_2(S_3 \cup S_4) - e(S_1 \cup \ldots \cup S_4) \in \mathcal{K}.$$

But:

$$\begin{aligned}
&\|(s_1 - \eta)(S_1 \cup S_2) - \lambda z_1\| \\
&\quad \leq \lambda(\|(s_1 - \eta)(E_1) - (\gamma_1 - e)(E_1)\| + \|(\gamma_1(F_1 \backslash E_1) - e(F_1 \backslash E_1))\| \\
&\qquad + \|z_1 - (\gamma_1 - e)(E_1)\|) \leq \frac{5}{6}\lambda\varepsilon
\end{aligned}$$

and, similarly:

$$\|(s_2 - \eta)(S_3 \cup S_4) - (1 - \lambda)z_2\| \leq \frac{5}{6}(1 - \lambda)\varepsilon.$$

Furthermore,

$$\|(s_1 - e)(S_1 \cup S_2) - (s_1 - \eta)(S_1 \cup S_2)\| < \frac{\varepsilon}{12}, \quad \|(s_2 - e)(S_3 \cup S_4) - (s_2 - \eta)(S_3 \cup S_4)\| < \frac{\varepsilon}{12}$$

so:

$$\|s_1(S_1 \cup S_2) + s_2(S_3 \cup S_4) - e(S_1 \cup \ldots \cup S_4) - z_\lambda\| \leq \frac{\varepsilon}{6} + \frac{5}{6}\lambda\varepsilon + \frac{5}{6}(1 - \lambda)\varepsilon = \varepsilon.$$

□

Theorem 3.1 *Under assumptions* **(T), (I), (WM), (S), (A), (C), (P)**, *α is in the core if and only if α is a Walrasian allocation.*

Proof As usual, it is straightforward to prove that every Walrasian allocation is in the core.

To prove the converse inclusion, note that, from Lemmas 3.2 and 3.3, since $-C$ is open we can strictly separate $\overline{\mathcal{K}}$ and $-C$ by means of a nonzero linear functional $x^* \in X^*$, namely we can find a nonzero $x^* \in X^*$, $t \in \mathbb{R}$:

$$x^*(x) \geq t > x^*(y), \quad x \in \overline{\mathcal{K}}, y \in -C.$$

Since $0 \in \overline{\mathcal{K}} \cap \overline{(-C)}$, necessarily $t = 0$; hence $x^*(x) \geq 0$ on $\mathcal{K}$. x^* is therefore positive, since $X^+ \backslash \{0\} \subset \mathcal{K}$.

It remains to prove that (x^*, α) is a Walrasian equilibrium.

Consider the allocation $\alpha + \varepsilon x P$, with $\varepsilon > 0$, $x \in X^+ \backslash \{0\}$. Using **(WM)** and **(I.i)**, we deduce $x^*(\alpha(F) + \varepsilon x P(F) - e(F)) \geq 0$, for each $F \in \Sigma^+$; hence, if $x^*(x) = 0$ we have $x^*(\alpha(F)) \geq x^*(e(F))$, otherwise, if $x^*(x) > 0$, letting $\varepsilon \downarrow 0$ we get $x^*(\alpha(F)) \geq x^*(e(F))$, for each $F \in \Sigma^+$. Furthermore, if $x^*(\alpha(F)) > x^*(e(F))$, for some $F \in \Sigma^+$, then $x^*(\alpha(\Omega)) = x^*(\alpha(F)) + x^*(\alpha(\Omega \backslash F)) > x^*(e(F)) + x^*(e(\Omega \backslash F)) =$

$x^*(e(\Omega))$, contradicting the feasibility of α. Hence, $x^*(\alpha(F)) = x^*(e(F))$, for every $F \in \Sigma^+$.

Now let $F \in \Sigma^+$ and $\beta \in \mathcal{M}$: $\beta \succ_F \alpha$. It is then impossible that $x^*(\beta(F)) < x^*(e(F))$. Suppose $x^*(\beta(F)) = x^*(e(F))$.

Now two cases can occur:

(I) $x^*(\beta(F)) > 0$. Then, by $\beta \ll P$ and the continuity of x^*, there exists $\tau > 0$ such that each $F_0 \in \Sigma^+$, $F_0 \subseteq F$ with $P(F \setminus F_0) < \tau$ has $x^*(\beta(F_0)) > 0$.

By **(I.i)**, on each such F_0 we have $\beta \succ_{F_0} \alpha$.

Applying **(C)**, corresponding to the above τ, we determine $\rho(\tau) > 0$ and $F_0 \subseteq F$, $F_0 \in \Sigma^+$ (with $P(F \setminus F_0) < \tau$), such that $\beta \succ_{F_0} \alpha$. Let now s be a simple allocation, $s = \sum x_i P|_{F_i}$ such that $\|s - \beta\| < \frac{\rho}{3}$, where $\{F_i\}_i$ is a decomposition of F_0.

Then, $s \succ_{F_0} \alpha$ and at least one of the F_is has strictly positive $x^*\beta$—measure. For the sake of simplicity, let us assume that it is F_1. We can as well assume that $0 < P(F_1) < 1$ (otherwise, by the nonatomicity of P, we can split F_1 into F_1^1, F_1^2, with $P(F_1^1) = P(F_1^2) = \frac{1}{2} P(F_1)$, and substitute $x_1 \mathbf{1}_{F_1}$ with $x_1 \mathbf{1}_{F_1^1} + x_1 \mathbf{1}_{F_1^2}$). Now, we can choose $G_1 \in \Sigma^+$, $G_1 \subseteq F_1$ such that:

$$P(G_1)\|x_1\| \leq \frac{\rho}{3};$$

$$\frac{x^*\beta(G_1)}{P(G_1)} \geq \frac{x^*\beta(F_1)}{P(F_1)}.$$

Notice that such a set exists, since the range of the 2-valued measure $(P, x^*\beta)$ has convex closure, i.e., its closure is a zonoid (see the Appendix for a detailed explanation).

Consider now $\sigma = \beta(F_1) P_{|G_1} + s_{|\Omega \setminus G_1}$. We have:

$$\|s - \sigma\| = \|\beta(F_1) - x_1\| P(G_1) \leq \|\beta(F_1) - x_1 P(F_1)\| + \|x_1\|(1 - P(F_1))P(G_1)$$
$$< \frac{\rho}{3} + \|x_1\|(1 - P(F_1))P(G_1) < \frac{2}{3}\rho.$$

Hence, $\|\sigma - \beta\| < \rho$, and so $\sigma \succ_{F_0} \alpha$. For $\gamma = \sigma_{|G_1} + \beta_{|\Omega \setminus G_1}$, it holds $\gamma \succ_F \alpha$, therefore $\gamma(F) - e(F) \in \mathcal{K}$. So,

$$\begin{aligned} x^*(e(F)) &\leq x^*(\gamma(F)) = x^*(\sigma(G_1)) + x^*(\beta(F \setminus G_1)) \\ &= x^*(\beta(F_1))P(G_1) + x^*(\beta(F)) - x^*(\beta(G_1)) = x^*(\beta(F)) \\ &\quad + x^*[\beta(F_1)P(G_1) - \beta(G_1)]. \end{aligned}$$

As $P(F_1) < 1$, we have $\frac{x^*\beta(G_1)}{P(G_1)} \geq \frac{x^*\beta(F_1)}{P(F_1)} > x^*\beta(F_1)$, and so the previous inequalities yield $x^*(e(F)) \leq x^*(\gamma(F)) < x^*(\beta(F)) = x^*(e(F))$, thus we have reached a contradiction.

(II) $x^*(\beta(F)) = 0$

In this case, $x^*(e(F)) = 0$. Take the allocation $\gamma = \alpha + e(F)P$. Since, by the equivalence of e and P, we have that $e(F) \in X^+ \setminus \{0\}$, by **(WM)** it holds $\gamma \succ_\Omega \alpha$ and $x^*(\gamma(\Omega)) = x^*(\alpha(\Omega)) + x^*(e(F)) = x^*(e(\Omega))$.

So, by **(A)**, we are again in the conditions of case **(I)** above, namely, the allocation γ and the grand coalition Ω can play the role of β and F of the previous case to lead us to a contradiction. □

Remark 3.1 Our core-Walras equivalence theorem also holds when P is nonatomic and Σ is an algebra with the Seever property (that is, if $\{A_n\}$ is an increasing sequence and $\{B_n\}$ a decreasing sequence of sets in Σ, with $A_n \subset B_n$, for every n, there is a $C \in \Sigma$ such that $A_n \subset C \subset B_n$ for all n) as Lemma 3.1 can be proved under these hypotheses as well.

Remark 3.2 In coalitional models, it is standard to work with cone $\mathcal{M}$ of allocations (see, for example, Armstrong and Richter 1984; Cheng 1991). According to one referee's suggestion, we want to point out that, in our case, the proof line can also be applied to the case of any Banach lattice X, provided the class $\mathcal{M}$ of allocations is bounded to integral allocations (namely those allocations admitting a density in $L^1_X(P)$). This would hence allow also L^1 as commodity space.

Remark 3.3 Notice that Theorem 3.1 can also be proved, *ceteris paribus*, assuming the following weaker form of continuity in place of **(C)**:

(C*) Let $F \in \Sigma^+$, and $\alpha, \beta \in \mathcal{M}$ with $\beta \succ_F \alpha$. Then, for every $\tau > 0$, there exists $\rho(\tau) > 0$ such that, for every simple allocation s with $\|s - \beta\| < \rho$, there exists $F_0 = F_0(s) \in \Sigma^+$, $F_0 \subseteq F$, with $P(F \setminus F_0) < \tau$ and $s \succ_{F_0} \alpha$.

Together with the standard continuity assumption (see, for example, Greinecker and Podczeck 2013):

(C)** or every $F \in \Sigma^+$, any $\alpha, \beta \in \mathcal{M}$ with $\beta \succ_F \alpha$, and any $\tau > 0$, there exists $\varepsilon \in (0, 1)$ and $F_0 \in \Sigma^+$, $F_0 \subseteq F$, such that $P(F \setminus F_0) < \tau$ and $\varepsilon\beta \succ_{F_0} \alpha$.

Indeed, it is immediate to get convinced that assumption **(C*)** is enough to prove Lemmas 3.2 and 3.3, while assumption **(C**)** can be used to do the final step of Theorem 3.1 in order to show that (x^*, α) is a Walrasian equilibrium, namely, to prove that if $F \in \Sigma^+$, and $\beta \succ_F \alpha$, then $x^*(\beta(F)) > x^*(e(F))$.

In fact, take $F \in \Sigma^+$ and $\beta \in \mathcal{M} : \beta \succ_F \alpha$. It is impossible that $x^*(\beta(F)) < x^*(e(F))$.

Suppose then that $x^*(\beta(F)) = x^*(e(F))$. Suppose first that $x^*(\beta(F)) > 0$. By (C^{**}), for some $\varepsilon \in (0, 1)$ and $F_0 \subseteq F$, $F_0 \in \Sigma^+$, we have $\varepsilon\beta \succ_{F_0} \alpha$. Moreover, by $\beta \ll P$ and the continuity of x^*, F_0 can be taken so that $x^*(\beta(F_0)) > 0$. So, setting $\gamma = \varepsilon\beta_{|F_0} + \beta_{|F \setminus F_0}$, by **(I)** and **(S)**, we get $\gamma \succ_F \alpha$, and hence $\gamma(F) - e(F) \in \mathcal{K}$. So $x^*(e(F)) \le x^*(\gamma(F)) = x^*(\beta(F)) + (\varepsilon - 1)x^*(\beta(F_0)) < x^*(\beta(F)) = x^*(e(F))$, a contradiction. Using **(A)**, the case $x^*(\beta(F)) = 0$ can now be treated exactly as case **(II)** in the last part of Theorem 3.1.

Remark 3.4 If one takes $\mathcal{M}$ to be the cone of simple allocations, Theorem 3.1 can be proved by replacing hypothesis **(C)** with **(C**)**.

In fact, in this case, the continuity assumption needs to be used just in the last part of the proof, namely, to prove that (x^*, α) is a Walrasian equilibrium, and this follows from exactly the same line of the previous remark.

Remark 3.5 As for the comparison with other coalitional finitely additive models, in most of the coalitional literature in such a setting, the separation argument is deduced from the assumption that the interior of X_+ is nonempty (Vind 1964; Armstrong and Richter 1984; Basile and Graziano 2001). To our knowledge, the only papers attempting to avoid this assumption are Cheng (1991) and Donnini and Graziano (2009). In both of these papers the commodity spaces are not Banach lattices. Moreover, in the first one, the author makes use of a sort of properness (conditions (V.1)–(V.3)) that we cannot compare to ours, because the consumption set in Cheng (1991) is the whole space X. Also, in Donnini and Graziano (2009), the authors suggest a condition surrogating the non-emptiness of the interior of the cone, the so defined (K–V) condition. However, in this case, one cannot dispense from involving the production set and, therefore, as the authors themselves mentioned, the result is not applicable to pure exchange economies.

4 Comprehensiveness and countably additive case

We now show that, when X is separable and has the RNP, and countable additivity is assumed, our result, under the hypotheses discussed in **Remark** 3.3, covers the individualistic core-Walras equivalence theorem of Rustichini and Yannelis (1991). The translation of the individualistic model into the coalitional one is the standard one of Armstrong and Richter (1984).

Proposition 4.1 *Assumptions* (*A.5.*) *and* (*A.7.*) *of Theorem 6.1 of Rustichini and Yannelis (1991) imply assumptions* ($\mathbf{C}^*$) *and* ($\mathbf{C}^{**}$).

Proof Take $\alpha \in \mathcal{M}$ where $\alpha = \int a \mathrm{d}P$, with $a : \Omega \longrightarrow X^+$. Then, by (*A.7.*) of Rustichini and Yannelis (1991), the set $\Gamma(\omega) = \left\{x \in X^+ : x \succ_\omega a(\omega)\right\}$ is measurable, for every $\omega \in \Omega$. Take $E \in \Sigma^+$, and $\beta = \int b \mathrm{d}P \in \mathcal{M}$, $\beta \succ_E \alpha$, that is, $b(\omega) \succ_\omega a(\omega)$, P-a.e. in E. From (*A.5.*) of Rustichini and Yannelis (1991), there exists $\rho(\omega) > 0$ such that $b(\omega) + \rho(\omega)X_1 \subset \Gamma(\omega)$, P-a.e. in E. We claim that $\omega \mapsto \rho(\omega)$ can be taken to be measurable. Indeed, as X is separable, the multifunction defined by $G(\omega) = X^+ \setminus \Gamma(\omega)$ is measurable (Riečan and Neubrunn 1997, page 261), and so it is also weakly measurable. As $\Gamma(\omega)$ is open and $b(\omega) \in \Gamma(\omega)$, P-a.e. in E, it holds $d(b(\omega), G(\omega)) > 0$, P-a.e. in E. Notice that $G(\omega)$ is complete; hence, by Theorem III.7 page 66 of Castaing and Valadier (1977), there exists a sequence $\{g_n\}_n$ of measurable selections of G such that $G(\omega) = \overline{\{g_n(\omega), n \in \mathbb{N}\}}$. Hence, $d(b(\omega), G(\omega)) = \inf_n \|b(\omega) - g_n(\omega)\|$, so $\omega \mapsto d(b(\omega), G(\omega))$ is measurable and the same holds for $\omega \mapsto \frac{1}{2}d(b(\omega), G(\omega))$. Hence, define $\rho(\omega) = \frac{1}{2}d(b(\omega), G(\omega))$ and notice that $\rho(\omega) > 0$ and $b(\omega) + \rho(\omega)X_1 \subset \Gamma(\omega)$, as $\rho(\omega) < d(b(\omega), G(\omega))$ and, if $y \in X^+$ is such that $\|b(\omega) - y\| < \rho(\omega)$, then $y \notin G(\omega)$ and so $y \in \Gamma(\omega)$.

Consider now $E_n = \left\{\omega \in E : \rho(\omega) > \frac{1}{n}\right\}$. We have $P(E_n) \to P(E)$.

Fix $0 < \tau < 2P(E)$, and choose $\overline{n}$ such that $\frac{1}{\overline{n}} < \frac{\tau}{2}$ and $P(E \setminus E_{\overline{n}}) < \frac{\tau}{2}$.

Set now $\rho(\tau) = \frac{1}{\overline{n}^2}$, and let s be a simple allocation such that $\|s - \beta\| < \frac{1}{\overline{n}^2}$, that is,

$$\int_\Omega \left\| \frac{\mathrm{d}s}{\mathrm{d}P} - b \right\| \mathrm{d}P < \frac{1}{\overline{n}^2}.$$

Then, for $H = \left\{\omega \in E : \left\| \frac{\mathrm{d}s}{\mathrm{d}P}(\omega) - b(\omega) \right\| > \frac{1}{\overline{n}}\right\}$, it holds $P(H) \leq \frac{1}{\overline{n}}$. Set $F = E_{\overline{n}} \backslash H$: we have $E \backslash F = (E \backslash E_{\overline{n}}) \cup H$, and hence $P(E \backslash F) \leq \frac{\tau}{2} + \frac{1}{\overline{n}} < \tau$. Notice that $\overline{n}$ can be chosen in such a way that $P(E_{\overline{n}}) > \frac{\tau}{2}$. Hence, as $P(H) \leq \frac{1}{\overline{n}}$, then $P(F) = P(E_{\overline{n}}) - P(E_{\overline{n}} \cap H) > \frac{\tau}{2} - \frac{1}{\overline{n}} > 0$. So, $F \neq \emptyset$.

As $F \subset E_{\overline{n}}$, then, for $\omega \in F$, $\rho(\omega) > \frac{1}{\overline{n}}$ and, as $F \cap H = \emptyset$, then $\|\frac{\mathrm{d}s}{\mathrm{d}P}(\omega) - b(\omega)\| \leq \frac{1}{\overline{n}}$. Hence, $\|\frac{\mathrm{d}s}{\mathrm{d}P}(\omega) - b(\omega)\| < \rho(\omega)$ and so, $\frac{\mathrm{d}s}{\mathrm{d}P}(\omega) \in \Gamma(\omega)$, P-a.e. in F, that is, $s \succ_F \alpha$. Hence, $(\mathbf{C}^*)$ holds.

Fix now $\tau > 0$, and choose an $\overline{n}$ such that:

$$P(\{\omega \in E : \|b(\omega)\| > \overline{n}\}) < \frac{\tau}{2}$$

and

$$P\left(\left\{\omega \in E : \rho(\omega) \leq \frac{1}{\overline{n}}\right\}\right) < \frac{\tau}{2}.$$

Take $F = E \backslash (A_1 \cup A_2)$, where $A_1 = \{\omega \in E : \|b(\omega)\| > \overline{n}\}$, and $A_2 = \left\{\omega \in E : \rho(\omega) \leq \frac{1}{\overline{n}}\right\}$. Then, $P(E \backslash F) < \tau$ and, for $\omega \in F$, $b(\omega) + \frac{1}{\overline{n}} X_1 \subset \Gamma(\omega)$. Take $\varepsilon \in \left(1 - \frac{1}{\overline{n}^2}, 1\right)$, and consider $\varepsilon\beta$. Then, pointwise in F, one has $\|b(\omega) - \varepsilon b(\omega)\| = (1 - \varepsilon)\|b(\omega)\|$. Then, $\|b(\omega) - \varepsilon b(\omega)\| < \frac{1}{\overline{n}^2}\|b(\omega)\| \leq \frac{1}{\overline{n}}$, so $\varepsilon b(\omega) \in b(\omega) + \frac{1}{\overline{n}} X_1 \subset \Gamma(\omega)$, whence $\varepsilon b(\omega) \succ_\omega a(\omega)$ in F, therefore $\varepsilon\beta \succ_F \alpha$, and so $(\mathbf{C}^{**})$ is proved. $\square$

It is now routine to show that the assumptions of Rustichini and Yannelis (1991) imply all our other coalitional assumptions, provided the initial endowment in their model has P-a.e. nonzero values. Hence, for separable commodity spaces with the RNP, our theorem covers their Theorem 6.1.

We now shortly compare the result in Sect. 3 with other countably additive coalitional core-Walras equivalence theorems existing in the literature. A deeper investigation is postponed to a future work.

(a) The validity of Proposition 4.1 above is deeply dependent on the assumption of the separability of the space X: this precludes the comparison with results such as Corollary 4 in Evren and Hüsseinov (2008).
(b) Zame (1986) proved a coalitional core-Walras equivalence result in the countably additive setting (Theorem 2). Although his result cannot be deduced directly from Theorem 3.1 when X enjoys the RNP, it is worthwhile to mention that a

similar result can be proved along the same lines of our proof, actually without the requirement that e has relatively compact range (if we assume a countably additive setting, relative compactness of the range follows directly from the nonatomicity of P, the absolute continuity of e w.r.t. P, and Theorem 10, page 266 in Diestel and Uhl, 1977). In fact, **(C-1)–(C-5)** of Zame either coincide or imply **(T), (I), (S)** and **(WM)**, and **(P)** can replace **(C-9)**. Indeed, according to the statement of the author, the individualistic analogous of assumption **(C9)** is "quite a bit stronger" than Mas-Colell (1986) properness (hence than Rustichini and Yannelis', 1991, condition). Moreover, conditions **(C-6)** and **(C-8)** can be used to prove the initial part of Lemma 3.2 as well as Lemma 3.3.

(c) Zame's framework has been recently reconsidered by Greinecker and Podczeck (2013); as in our work, their aim is to significantly extend the class of Banach lattices on which a coalitional core-Walras equivalence result holds. However, the point of view of the two approaches is substantially different. While we offer an approach adapted to a new class of possible Banach lattices, their effort is based on the idea of completely abandoning any requirement on the commodity space, and to focus on the measure theoretic properties of the space of agents and on a strengthening of the nonatomicity notion of the probability P. The models also differ in some of the assumptions on preferences: indeed, their assumption **(P.7)** is implied by our **(C)**, when $\mathcal{M}$ is the cone of simple allocations, while we have both a weaker monotonicity and a "quite weaker"properness-like assumption (they assume **(C9)** of Zame 1986). We think that, one of the appealing features of our approach, is the easiness in detecting whether a Banach lattice enjoys the RNP (see Diestel and Uhl 1977, pages 217–219).

(d) In the countably additive case, the proof of the convexity of the set $\overline{\mathcal{K}}$ can be shortened and given analogously to Armstrong and Richter (1984), Lemma 4, in the following way.

Lemma 4.1 *$\overline{\mathcal{K}}$ is convex.*

Proof Fix $z_1, z_2 \in \overline{\mathcal{K}}$, $t_1, t_2 \in [0, 1]$ such that $t_1 + t_2 = 1$; to prove that $z = t_1 z_1 + t_2 z_2 \in \overline{\mathcal{K}}$, we have to prove that, for each $\varepsilon > 0$, there exists $\xi \in \mathcal{K}$ with $\|z - \xi\| < \varepsilon$.

Since $z_i \in \overline{\mathcal{K}}$, there are allocations γ_i, $i = 1, 2$ and coalitions F_i,$i = 1, 2$: $\gamma_i \succ_{F_i} \alpha$ and $\|\gamma_i(F_i) - e(F_i) - z_i\| < \frac{\varepsilon}{14}$.

Let $\delta > 0$ be determined by $\gamma_i, e \ll P$, $i = 1, 2$ and, by (C), choose $\tau = \delta\left(\frac{\varepsilon}{14}\right)$, and determine $\rho(\tau) = \min\{\rho_1(\tau), \rho_2(\tau)\}$. As in Lemma 3.3, we can replace γ_1, γ_2 by means of simple allocations s_1, s_2: $\|s_i - \gamma_i\| < \frac{\varepsilon}{14}$ and $s_i \succ_{G_i} \alpha$, where $G_i \in \Sigma^+$, $G_i \subseteq F_i$ and $P(F_i \setminus G_i) < \tau$. Now, from $s_i \ll P$ and $e \ll P$, it follows that each s_i and e are nonatomic and so, as X has the RNP, the range of s_i and e have convex closure (see Uhl 1969). Hence, as in Armstrong and Richter (1984), Lemma 4, we can choose two disjoint measurable sets $E_1 \subseteq G_1$ and $E_2 \subseteq G_2$, such that $\|s_i(E_i) - t_i s_i(G_i)\| < \frac{\varepsilon}{14}$ and $\|e(E_i) - t_i e(G_i)\| < \dfrac{\varepsilon}{14}$, $i = 1, 2$. Define the allocation s as the allocation which equals s_i on E_i, $i = 1, 2$, and e outside $E_1 \cup E_2$. Let $E = E_1 \cup E_2$. Hence $s \succ_E \alpha$, therefore $\xi = s(E) - e(E) \in \mathcal{K}$. Moreover, an easy computation shows that $\|s(E) - e(E) - (t_1 z_1 + t_2 z_2)\| < \varepsilon$. □

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5 Appendix

In this Appendix we will give an explanation of the fact that, in Theorem 3.1 it is possible to choose $G_1 \in \Sigma^+$, $G_1 \subseteq F_1$ such that:

$$P(G_1)\|x_1\| \leq \frac{\rho}{3};$$

$$\frac{x^*\beta(G_1)}{P(G_1)} \geq \frac{x^*\beta(F_1)}{P(F_1)}.$$

Since we are assuming that P is strongly nonatomic, P satisfies the Darboux Property, that is for every $\tau > 0$ and every $E \in \Sigma$ one can decompose E into finitely many disjoint Σ-measurable sets, each with probability P less than τ.

All allocations are assumed to be absolutely continuous with respect to P in the $\varepsilon - \delta$-sense; hence $x^*\beta$ will also fulfill the Darboux Property. Since x^* is a positive functional, we are reasoning on a $\mathbb{R}^2_+$-valued finitely additive measure on an algebra Σ.

By means of a Stone argument, Σ is transformed into a pure algebra (i.e., containing no countable unions), the *Stone algebra*, where therefore P and $x^*\beta$ transfer to countably additive measures which we shall denote by $\widetilde{P}$ and $\widetilde{x^*\beta}$. By a standard argument in measure theory, one can extend each of these two set functions to a nonnegative measure on the generated σ-algebra, and this measure will automatically inherit the Darboux Property; let us denote by $\widetilde{\widetilde{P}}$ and $\widetilde{\widetilde{x^*\beta}}$ these two further extensions (see Martellotti 2001, Sect. 2, for a complete treatment of this construction).

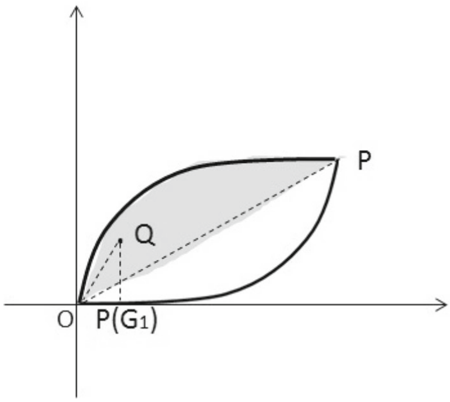
Now we are in a countably additive setting, where all forms of nonatomicity are equivalent.

The Liapounoff's Theorem then implies that the range of the pair $\left(\widetilde{\widetilde{P}}, \widetilde{\widetilde{x^*\beta}}\right)$ is a compact convex subset of the positive orthant which contains the origin and is symmetric w.r.t the middle point of the line joining the origin with $\left(\widetilde{\widetilde{P}}(\Omega), \widetilde{\widetilde{(x^*\beta)}}(\Omega)\right)$. Classically, the range of a finite dimensional, nonatomic, countably additive measure is called a *zonoid* (see Bolker 1969).

Furthermore, we know that the image under $(\widetilde{P}, \widetilde{x^*\beta})$ of the Stone algebra is dense in this zonoid, as well as we know that the image of the Stone algebra under $(\widetilde{P}, \widetilde{x^*\beta})$ precisely coincides with the range of $(P, x^*\beta)$.

In conclusion, the image of Σ under the pair $(P, x^*\beta)$ is dense in a zonoid of $\mathbb{R}^2$.

Due to its symmetry properties, a two-dimensional zonoid (which, in the case of nonnegative measures, is a subset of the positive orthant) will look like the "*leaf*" in the picture below.



The above argument can be analogously applied to the coalition F_1 instead of the whole grand coalition Ω.

Hence the image of the trace algebra Σ_{F_1} under the pair $(P, x^*\beta)$ is dense in a form of the type in the following picture, where the endpoint P has coordinates $(P(F_1), x^*\beta(F_1))$.

Now it is enough to note that the ratios involved in the inequality:

$$\frac{x^*\beta(G_1)}{P(G_1)} \geq \frac{x^*\beta(F_1)}{P(F_1)}$$

represent the slope of the segments joining O with Q, if $Q = (P(G_1), x^*\beta(G_1))$, and O with P.

Hence, the two requirements simply reduce to finding a set on the upper part of the *leaf*, so that the slope of the joining segment is greater than that of the diagonal, and with first coordinate $P(G_1)$ small enough.

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