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Interfaces with Other Disciplines

General lattice methods for arithmetic Asian options

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ABSTRACT

In this research, we develop a new discrete-time model approach with flexibly changeable driving dynamics for pricing Asian options, with possible early exercise, and a fixed or floating strike price. These options are ubiquitous in financial markets but can also be recast in the framework of real options. Moreover, we derive an accurate lower bound to the price of the European Asian options under stochastic volatility. We also survey theoretical aspects; more specifically, we prove that our tree method for the European Asian option in the binomial model is unconditionally convergent to the continuous-time equivalent. Numerical experiments confirm smooth, monotonic convergence, highly precise performance, and robustness with respect to changing driving dynamics and contract features.

1. Introduction

Tree approaches are classic all-purpose tools in fields of finance and operations research. For example, Muroi and Suda (2013) have combined with discrete Malliavin calculus to compute price sensitivities. In addition, the modelling of operational problems sharing salient features with the modelling of options with early exercise opportunities has been highlighted via several researches; (e.g., see Nadarajah, Margot, & Secomandi, 2017 and references therein). More specifically, Zmeškal (2010) has fused with real American options, as trees are a standard practical method for appraising options with possible early exercise (see also Chockalingam & Muthuraman, 2015 for likely alternatives) and handling management decisions, and a fuzzy methodology in order to allow for vagueness of the input parameters. De Reyck, Degraeve, and Vandenborre (2008) used decision trees to model uncertainty in projects. In general, real options are usually evaluated on trees as they tend to be more understandable and transparent (see, for example, Guthrie, 2009). Other earlier contributions include, for example, Ekvall (1996) who developed a lattice approach for valuing multivariate contingent claims that could handle American-type exercise.

Asian options are among the most popular path-dependent options actively traded in financial markets, such as exchange rates, interest rates and commodities, due to their appealing payoffs dependent on the averages of underlying asset prices during a pre-specified time window. They can be used to hedge thinly traded assets over a certain period of time. Also, due to averaging, their payoff is less susceptible to market manipulations at maturity compared to plain vanilla options. More importantly, the application of Asian options to investment and management problems (real options) receives increasing attention in the literature (see, for example, Driouchi, Bennett, & Simpson, 2010) which motivates this paper.

More specifically, we revisit the long-standing problem of valuing non-linear derivatives contingent on the arithmetic average and provide several advances. We can distinguish between arithmetic average options, with a fixed or floating strike price, and with European or American-type exercise. The distribution of the underlying arithmetic average asset price is not known. For this reason, exact closed-form solutions for pricing arithmetic Asian options are inexistent, however numerical methods can be employed to solve the pricing problem.

Papers in the literature on pricing Asian options adopt, for example, transform techniques, analytical approximations based on moment matching, Monte Carlo simulation and partial differential equations (PDEs); it is beyond our scope to provide a repetitorium, rather we refer to Fusai and Kyriakou (2016) for more details. Despite the overwhelming volume of the literature on Asian options, the state of affairs for them is still quite not complete. Our aim is to contribute to their already well-publicized success by developing reliable approaches suitable for new practices, such as model calibration and real option applications.

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Pricing American options with Asian features is hard, especially under general driving dynamics. To this end, we resort to a discrete-time model approach. Commonly, trees offer a convenient way of visualizing simplified models of stochastic dynamics for the underlying asset price, which makes them attractive for pedagogical purposes and computation of derivative prices. They are easy to explain and implement and are described virtually in every textbook on derivatives.

Evaluating arithmetic average options in a discrete-time model can be quite cumbersome as the number of alternative average realizations grows fast with the number of time steps. Hull and White (1993) circumvent this by pricing the option for only certain designated values of the average at each level in the lattice, using linear interpolation to estimate the option price at the other average values. Facing the same challenge, Chalasani, Jha, and Varikooty (1998) adapt instead the lower bound of Rogers and Shi (1995) in the binomial model and combine with interpolation. Chalasani, Jha, Egriboyun, and Varikooty (1999) additionally allow for early-exercise provision. Other contributions with early-exercise feature are limited, for example, to floating strikes (see Hansen & Jørgensen, 2000) or, to lognormal price dynamics (e.g., see Zvan, Forsyth, & Vetzal, 1998), or incur notable speed-accuracy imbalances (e.g., see Ritchken, Sankarasubramanian, & Vijh, 1993). Lo, Wang, and Hsu (2008) extend the model of Chalasani et al. (1998) by considering higher moments of the underlying asset return distribution and apply an Edgeworth binomial lattice. On the other hand, Reynaerts, Vanmaele, Dhaene, and Deelstra (2006) adhere to an alternative bound-based approach by putting in less information than Chalasani et al. (1998), implying some loss of accuracy but improvement of computational ease. Neave and Ye (2003) derive bounds by combining paths and exploiting the structural information in the binomial trees to simplify computations. Succinctly, amidst others, the aforementioned contributions rely on path grouping and approximation techniques and bounds, which represent their main sources of error and drawback, in addition to model restriction for the underlying state variable. Instead, we do pricing on a *magic* tree in the sense that we do not require explicit access to paths, but rather only their distributional properties which allows us to obviate any kind of approximation and computational challenge. Also, as access to paths is not imminent, the computational burden is not affected despite the fact that the tree does not recombine; furthermore, the computational effort reduces perceptibly by exploiting state space reduction.

The work in this paper is concerned with several overarching themes. We propose a discrete-time model approach for Asian options of European or American exercise, with a fixed or floating strike price, in a one or two-dimensional setting depending on the contract type and the asset price process as we summarize in Table 1 and we explain in the paper. Our technique is precise, simple and easily adaptable to a general class of discrete-time models that are able to reproduce stylized properties of the asset prices in the various markets, such as random jumps and/or stochastic volatility, while maintaining excellent convergence properties. As a case in point, we prove the convergence of the European Asian option price in the binomial model of Cox, Ross, and Rubinstein (1979) to the continuous-time equivalent, while we demonstrate smooth, monotonic convergence by means of several numerical examples under alternative driving dynamics.

The remainder of the paper is organized as follows. In Section 2, we present our discrete-time model framework and exemplify some specifications, with extended details given in the appendix. In Section 3, we propose our novel tree approach for pricing arithmetic Asian options with different payoff structures and possible early exercise. In Sections 4 and 5, we focus on the specific treatment of models with stochastic volatility. Section 6 presents the proof of the consistency of our method with a PDE approach and a study of the convergence of the proposed methods to the continuous-time model. In Section 7, we provide various numerics that demonstrate the accuracy and scale of applicability of our methods. Section 8 concludes the paper. Extended option payoff structures and supporting theoretical results are collected in additional appendices.

2. The discrete-time model

In a N-period discrete-time model, the time period $[0, T]$ is partitioned into N equal time steps of length $\Delta := T/N$. The price of the underlying asset under the risk neutral probability $\mathbb{P}$ at arbitrary time $n\Delta \leq T$ is given by

$$S_n = S_0 e^{\sum_{j=0}^{n} \xi_j}, \tag{1}$$

where $\xi_0 := 0$ and $\{\xi_j\}_{j=1}^N$ is a sequence of discrete random variables with probability distribution

$$\xi_j := \begin{cases} \ln x_1 & p_1(Y_{j-1}) \\ \ln x_2 & p_2(Y_{j-1}) \\ \vdots & \vdots \\ \ln x_d & p_d(Y_{j-1}) \end{cases}, \tag{2}$$

$$\sum_{i=1}^{d} p_i(Y_{j-1}) = 1,$$

where $x_1 > x_2 > \cdots > x_d$ and $\{Y_j\}_{j=1}^N$ is a (possibly multi-dimensional) Markov process. In particular, we consider three classes of models:

Model 1. $\ln S$ is an independent increment process (e.g., a discrete-time analogue of a Lévy process), in which case the distribution of ξ_j does not depend on Y_j, i.e., $p_i(Y_{j-1}) = p_i$ for each $j = 1, \ldots, N$ and $i = 1, \ldots, d$. This is, for example, the case of the binomial model of Cox et al. (1979) and the bivariate tree model of Hilliard and Schwartz (2005), although various alternative lattice specifications are encompassed, such as Jarrow and Rudd (1983), Boyle (1988), Omberg (1988), Amin (1991) and Tian (1993).

Model 2. S is a one-dimensional diffusion with $Y_j := S_j$ for each $j = 1, \ldots, N$. This class of one-dimensional diffusion models nests a variety of popular asset pricing models, such as exponential Ornstein–Uhlenbeck, Brennan–Schwartz, Cox–Ingersoll–Ross and the constant elasticity of volatility (CEV) models (see Cai, Li, & Shi, 2014). In this paper, we consider in more details the binomial lattice model approach of Hilliard (2014) applied to the CEV model.

Model 3. S has stochastic variance V and $Y_j := (S_j, V_j)$. Here, we study the two-dimensional binomial lattice of Akyıldırım, Dolinsky, and Soner (2014) applied to the Heston model.

In Appendix A, we put under the microscope each of the models above separately, present class-specific constructions and narrow down to model-specific cases to facilitate the exposition.

In the above models, absence of arbitrage follows by imposing that the risk neutral process $e^{-rn\Delta}S_n$ is a martingale, where r is the continuously compounded risk-free rate of interest. The parameters of the discrete distribution are chosen so that the required moments either match exactly those of the continuous distribution, or in the limit as $\Delta \to 0$, so that the discrete-time Markov chain converges weakly to the continuous-time stochastic model. We can generalize further by taking into account deterministic time-inhomogeneities: the parameters describing the local behaviour will now be time-dependent but non-random. Thereby, the construction (1)–(2) represents a flexible parametric and tractable family of models, depending on the choice of $\{\xi_j\}$, that is able to reproduce the whole range of option prices across strikes and maturities.

Table 1
Summary of our methods and models.

Model	Type	
	Fixed strike	Floating strike
	European	
Independent increments	Section 3.1 (Eqs. (8) and (9)) Variable Z (Eq. (3))	Section 3.3 (Eqs. (17) and (18)) Variable Z (Eq. (15))
One-dimensional diffusions	Section 3.1 (Eqs. (8) and (9)) Variables S,Z (Eqs. (1) and (3))	Section 3.3 (Eqs. (17) and (18)) Variables S,Z (Eqs. (1) and (15))
Stochastic volatility	Section 5 Lower bound (Eq. (26))	Section 4 (Eq. (23)) Variables X,ν (Eqs. (20) and (21))
	American	
Independent increments	Section 3.2 (Eq. (16)) Variables S,Z (Eqs. (1) and (15))	Section 3.3 (Eqs. (17) and (18)) Variable Z (Eq. (15))
One-dimensional diffusions	Section 3.2 (Eq. (16)) Variables S,Z (Eqs. (1) and (15))	Section 3.3 (Eqs. (17) and (18)) Variable Z (Eqs. (1) and (15))
Stochastic volatility	–	Section 4 (Eq. (23)) Variables X,ν (Eqs. (20) and (21))

3. Tree approach for arithmetic Asian options

In this section, we present a tree method for pricing Asian options of European or American-style exercise, with fixed or floating strike price. Our result covers all possible variations of this contract in terms of payoff specification, option exercise and monitoring frequency of the underlying, in a general and practically useful model framework. We also expand to the cases of a forward start option and an Asian option with a fixed finite monitoring frequency (details are deferred to Appendix B and Appendix C). In what follows, we consider different cases separately.

3.1. European fixed strike option

The payoff at maturity $T = N\Delta$ of an Asian call option with fixed strike has form

$$\left(\frac{\sum_{n=0}^{N} S_n}{N+1} - K\right)^+ = \left(\frac{\sum_{n=0}^{N} S_n \Delta}{T+\Delta} - K\right)^+,$$

where $(\cdot)^+$ denotes the positive part function, K is the strike price and S_n is given by (1). Define the process Z as

$$Z_j = \frac{\sum_{n=0}^{j} S_n \Delta - K(T+\Delta)}{S_j} = \frac{Z_{j-1}}{e^{\xi_j}} + \Delta, \quad 0 < j \le N, \tag{3}$$

where the second equality follows from (1). By recursive substitution, we get that

$$Z_N = Z_j \prod_{k=j+1}^{N} e^{-\xi_k} + \Delta \sum_{i=j+1}^{N-1} \prod_{k=i+1}^{N} e^{-\xi_k} + \Delta. \tag{4}$$

The (forward) price of the option is then given by

$$\frac{\mathbb{E}\left(S_N Z_N^+\right)}{T+\Delta} = \frac{S_0 e^{rT}}{T+\Delta} \bar{\mathbb{E}}\left(Z_N^+\right), \tag{5}$$

where $\bar{\mathbb{E}}(\cdot)$ is the expected value under the new measure $\bar{\mathbb{P}}$ with the numéraire given by the underlying asset price S.[1] The Radon–Nikodym derivative is given by

$$\left.\frac{d\bar{\mathbb{P}}}{d\mathbb{P}}\right|_n = \frac{S_n}{S_0\, e^{r\,n\,\Delta}},$$

and the probability distribution of ξ_j under $\bar{\mathbb{P}}$ is given by

$$\bar{p}_n(Y_{n-1}) = p_n(Y_{n-1}) \frac{S_n}{S_0\, e^{r\,n\,\Delta}}.$$

[1] For a general treatment of change of numéraire techniques, readers are referred to Geman, Karoui, and Rochet (1995).

Results (3) and (4) imply that (5) can be evaluated recursively backwards from maturity. To this end, we build truncated ranges for z at each time step j, $[z_{L,j}, z_{U,j}]$, $z_{L,j} \le 0 \le z_{U,j}$. From (5), we set $z_{L,N} = 0$ and by reversing recursion (3) we get for the lower cut-off point

$$z_{L,j-1} = \left(z_{L,j} - \Delta\right)x_1, \quad 0 < j \le N, \tag{6}$$

noting that $z_{L,j} < 0$ for $j < N$, where x_1 is the largest one-period return value (see Eq. (2)). If $Z_0 \ge 0$, then from the inverse relation (3), we have that $Z_j > 0$ for any $0 \le j \le N$ and the option is eventually exercised for sure. Hence, we set $z_{U,0} = 0$, and from (3) we get by forward propagation in time

$$z_{U,j} = \frac{z_{U,j-1}}{x_d} + \Delta, \quad 0 < j \le N, \tag{7}$$

where x_d is the smallest one-period return value yielding the upper cut-off range point. If $Z_0 < z_{L,0}$, then $Z_N < 0$ and the option expires out-of-the-money surely. Following the previous analysis, we define

$$c(y, z, N) = z^+, \tag{8}$$

$$c(y,z,j) = \begin{cases} z\bar{\mu}_j + \Delta & \text{for } z > z_{U,j} \\ \sum_{i=1}^{d} \bar{p}_i(y)\, c\left(Y_{j+1}, \frac{z}{x_i} + \Delta, j+1\right) & \text{for } z_{L,j} < z < z_{U,j} \\ 0 & \text{for } z < z_{L,j} \end{cases} \tag{9}$$

for $0 \le j < N$, where $\bar{\mu}_j = \bar{E}_{j-1}(e^{-\xi_j})$, $\{\bar{p}_i(y)\}_{i=1}^{d}$ is the probability distribution of ξ_j under the measure $\bar{\mathbb{P}}$ and $Y_{j+1} = Y_{j+1}(y, x_i)$. In Model 2 for example, where $Y = S$, we have that $Y_{j+1}(y, x_i) = y\, x_i$. The forward price of the option is then given by

$$\frac{S_0 e^{rT}}{T+\Delta} c\left(Y_0, \Delta - \frac{K}{S_0}(T+\Delta), 0\right).$$

Note that, in Model 1, recursion (9) does not depend on y, hence it reduces to a one-dimensional non-recombinant tree. In Model 2, we have a two-dimensional tree, which is recombinant in Y (but not in Z). In Model 3, we end up with a three-dimensional tree, which is computationally demanding; for this, we propose a two-dimensional alternative lattice method and a lower bound for stochastic volatility models, whose discussion is postponed to Sections 4 and 5.

Back to our discussion of recursion (9), we implement, for computational convenience equally spaced grids for z at each time step j: $\mathbf{z}_j := \left\{z_{m,j}\right\}_{m=0}^{n_z-1}$, where $z_{m,j} := z_{L,j} + m\delta z$. Note that, in general, if $z_{m,j}$ is a grid point, it is not guaranteed that $z_{m,j}/x_i + \Delta$ (see Eq. (9)) will also be, as the tree model for Z does not recombine. More specifically, if

$$\frac{z_{m,j}}{x} + \Delta \in \left(z_{L,j+1} + m^x_{j+1}\delta z, \quad z_{L,j+1} + \left(m^x_{j+1} + 1\right)\delta z\right),$$
$$x = x_1, x_2, \ldots, x_d,$$

where

$$m^x_{j+1} := \left\lfloor \frac{z_{m,j}/x + \Delta - z_{L,j+1}}{\delta z} \right\rfloor,$$

$\lfloor\cdot\rfloor$ denoting the floor function, then, in practice, $c\left(y, z_{m,j}/x + \Delta, j+1\right)$ can be obtained by interpolation; we opt for linear interpolation. For a twice differentiable function $c(y, z, j+1)$ in z, for each value of y and j, and bounded second derivative,

$$\begin{aligned} c\left(y, \frac{z_{m,j}}{x} + \Delta, j+1\right) &= \alpha^x_{j+1} c\left(y, z_{L,j+1} + m^x_{j+1}\delta z, j+1\right) + \left(1 - \alpha^x_{j+1}\right) \\ &\quad \times c\left(y, z_{L,j+1} + \left(m^x_{j+1} + 1\right)\delta z, j+1\right) + \gamma^x_{j+1}, \end{aligned} \tag{10}$$

where

$$\alpha^x_{j+1} := \frac{z_{L,j+1} + \left(m^x_{j+1} + 1\right)\delta z - z_{m,j}/x - \Delta}{\delta z}$$

and the error is given by

$$\begin{aligned} \gamma^x_{j+1} :&= \frac{1}{2} \frac{\partial^2 c(y, z^*_{m^x_{j+1}}, j+1)}{\partial z^2} \left(\frac{z_{m,j}}{x} + \Delta - z_{L,j+1} - m^x_{j+1}\delta z\right) \\ &\quad \times \left(z_{L,j+1} + (m^x_{j+1} + 1)\delta z - \frac{z_{m,j}}{x} - \Delta\right) \end{aligned}$$

for some $z^*_{m^x_{j+1}} \in (z_{L,j+1} + m^x_{j+1}\delta z, \quad z_{L,j+1} + (m^x_{j+1} + 1)\delta z)$. From (9) and (10), we get that the accumulated interpolation error at time step j and node m is

$$\begin{aligned} \varepsilon_{m,j} :&= \sum_{i=1}^{d} \bar{p}_i(y) \left[\alpha^{x_i}_{j+1} \varepsilon_{m^{x_i}_{j+1}, j+1} + (1 - \alpha^{x_1}_{j+1}) \varepsilon_{m^{x_i}_{j+1}+1, j+1}\right] \\ &\quad + \sum_{i=1}^{d} \bar{p}_i(y)\, \gamma^{x_i}_{j+1}. \end{aligned} \tag{11}$$

The norm of the interpolation error is given by

$$\left\|\varepsilon_j\right\| := \max_y \max_m \left|\varepsilon_{m,j}\right|. \tag{12}$$

Also,

$$\left|\gamma^x_{j+1}\right| \le M(\delta z)^2, \tag{13}$$

where

$$M = \max_y \max_j \max_m \left|\frac{\partial^2 c(y, z^*_m, j+1)}{\partial z^2}\right|.$$

From (12) and (13), recursion (11) becomes

$$\left\|\varepsilon_j\right\| \le \left\|\varepsilon_{j+1}\right\| + M(\delta z)^2,$$

and, at time zero,

$$\|\varepsilon_0\| \le \|\varepsilon_N\| + NM(\delta z)^2 = \frac{TM(\delta z)^2}{\Delta}$$

as the terminal payoff is evaluated exactly. Hence, if $(\delta z)^2$ has a larger order than Δ, i.e.,

$$\delta z < \sqrt{\Delta}, \tag{14}$$

the norm of the accumulated interpolation error converges to zero as $\Delta \to 0$.

3.2. American fixed strike option

An American option can be exercised before maturity. For this, it is necessary to re-define the process Z based on weights $1/(j+1)$ for arbitrary j:

$$Z_j = \frac{\frac{1}{j+1}\sum_{n=0}^{j} S_n}{S_j} = \frac{j}{j+1}\frac{Z_{j-1}}{e^{\xi_j}} + \frac{1}{j+1}, \quad 0 \le j \le N. \tag{15}$$

The original recursion (8)–(9) is adapted for (15) and the early-exercise feature leading to the following recursion

$$\begin{aligned} \tilde{c}(y, z, N) &:= \left(z - \frac{K}{S}\right)^+, \\ c(y, z, j) &:= \sum_{i=1}^{d} \bar{p}_i(y)\, \tilde{c}\left(Y_{j+1}, \frac{j+1}{j+2}\frac{z}{x_i} + \frac{1}{j+2}, j+1\right), \; 0 \le j < N \\ \tilde{c}(y, z, j) &:= \max\left(c(y, z, j), z - \frac{K}{S}\right), \quad 0 \le j < N, \end{aligned} \tag{16}$$

where in (16) the holder chooses between the continuation value and the exercise payoff of the option. $\tilde{c}$ is the value of the option immediately before the exercise opportunity. At the end of the recursion, the forward price of the option is given by

$$S_0 e^{rT} c(Y_0, 1, 0).$$

By explicit dependence of the terminal payoff on $S = Y$, the pricing problem remains two-dimensional even under the simplest Model 1.

3.3. Floating strike option

For the case of an Asian option with a floating strike price, we adhere to the definition of Z in (15). Then, pricing the particular option of European put type with coefficient $\bar{K} \ge 0$ amounts to calculating

$$\mathbb{E}\left[S_N\left(Z_N - \bar{K}\right)^+\right] = S_0 e^{rT} \bar{E}\left[\left(Z_N - \bar{K}\right)^+\right] = S_0 e^{rT} c(1, 0)$$

recursively backwards based on

$$c(y, z, N) = \left(z - \bar{K}\right)^+, \tag{17}$$

$$c(y, z, j) = \sum_{i=1}^{d} \bar{p}_i(y)\, c\left(Y_{j+1}, \frac{j+1}{j+2}\frac{z}{x_i} + \frac{1}{j+2}, j+1\right), \quad 0 \le j < N. \tag{18}$$

In the case of the American-type option, the holder chooses between the continuation value and the early-exercise payoff of the option

$$\tilde{c}(y, z, j) = \max\left(c(y, z, j), z - \bar{K}\right), \quad 0 \le j < N,$$

where

$$c(y, z, j) = \sum_{i=1}^{d} \bar{p}_i(y)\, \tilde{c}\left(Y_{j+1}, \frac{j+1}{j+2}\frac{z}{x_i} + \frac{1}{j+2}, j+1\right), \quad 0 \le j < N,$$

initialized by

$$\tilde{c}(y, z, N) = \left(z - \bar{K}\right)^+.$$

It is worth noting that, by nature of the payoff of the floating strike Asian option, S is factorized out and, by change of measure, the pricing problem becomes one-dimensional for both European and American options under Model 1.

4. Tree method for arithmetic Asian options and stochastic volatility

The discrete-time stochastic volatility model proposed in Akyıldırım et al. (2014) combined with the method proposed in the previous section leads to a three-dimensional tree that is computationally quite unmanageable. Hence, we propose a two-dimensional recombinant tree method for European and American floating strike Asian options; the fixed strike case is treated separately in the next section. Consider the asset price process S with stochastic variance V defined by the stochastic differential equations (A.3). For the sake of exemplification, we focus here on the Heston model specification (A.4). We revisit the continuous-time variable $Z(t)$ defined in Rogers and Shi (1995),

$$Z(t) = \frac{1}{t}\frac{\int_0^t S(u)du}{S(t)}. \tag{19}$$

Additionally, we employ the change of variable

$$X(t) = \frac{\ln Z(t)}{\sqrt{V_0}} + \rho\frac{V(t)}{\eta}, \tag{20}$$

$$\nu(t) = \frac{2}{\eta}\sqrt{V(t)}, \tag{21}$$

that leads to a transformed system of stochastic differential equations driven by independent Brownian motions. The dynamics of X and ν under the measure $\bar{\mathbb{P}}$ are described by

$$dX(t) = \mu_X(X(t), \nu(t), t)\, dt + \sigma_X(\nu(t))\, d\bar{W}(t),$$
$$d\nu(t) = \mu_\nu(\nu(t))\, dt + d\bar{B}(t),$$

where $\bar{W}$ and $\bar{B}$ are independent Brownian motions and

$$\begin{aligned}\mu_X(x,\nu,t) &:= \frac{1}{t\sqrt{V_0}}\left(e^{-\sqrt{V_0}\,(x-\frac{\rho\eta}{4}\nu^2)} - 1\right) - \frac{1}{\sqrt{V_0}}\left(r + \frac{1}{2}\frac{\eta^2\nu^2}{4}\right)\\ &\quad + \frac{\rho}{\eta}\left(k\bar{v} + (\rho\eta - k)\frac{\eta^2\nu^2}{4}\right),\\ \sigma_X(\nu) &:= \frac{\eta\,\nu\sqrt{1-\rho^2}}{2\sqrt{V_0}},\\ \mu_\nu(\nu) &:= \frac{2k\bar{v}/\eta^2 - 1/2}{\nu} - \frac{1}{2}(k - \eta\rho)\,\nu.\end{aligned}$$

Then, we apply a two-stage tree approach to ν and X. More specifically, from Hilliard (2014) we get for $\nu(t)$

$$\nu_n = \nu_0 + \sum_{j=1}^{n}\zeta_j,$$

where

$$\zeta_j := \begin{cases}\sqrt{\Delta}, & p_j = 1/\left(1 + e^{-2\,\mu_\nu(\nu_{j-1})\,\sqrt{\Delta}}\right)\\ -\sqrt{\Delta}, & 1 - p_j\end{cases},$$

and from Akyıldırım et al. (2014) we get for $X(t)$

$$X_n = X_0 + \sum_{j=1}^{n}\varkappa_j + \alpha_n\varkappa_n, \tag{22}$$

where

$$\varkappa_j := \begin{cases}\sqrt{\Delta}, & q_j(X_{j-1}, \nu_{j-1})\\ -\sqrt{\Delta}, & 1 - q_j\end{cases}.$$

The coefficients $\{\alpha_j\}$ and the probabilities $\{q_j\}$ are chosen by matching the first two moments of the continuous-time distribution of the increment of X, i.e.,

$$\bar{E}_{j-1}(\varkappa_j) = \mu_X(X_{j-1}, \nu_{j-1}, j-1)\,\Delta + o(\Delta),$$

$$\bar{E}_{j-1}(\varkappa_j^2) = \sigma_X^2(\nu_{j-1})\Delta + o(\Delta).$$

The normalization of the variable $\ln Z(t)$ by $\sqrt{V_0}$ allows us to control the explosive behaviour of $\mu_X(x)$ for $x \to -\infty$ and, hence, to guarantee that the probabilities $\{q_j\}$ fall in the range [0,1].

Moreover, we note that ζ_j and $\varkappa_j$ are independent, hence, for example, the joint probability of an upward movement of both X and ν is given by the product $q_j p_j$. Therefore, floating strike Asian options are priced through the following recursion

$$\begin{aligned}c(x,\nu,N) &= \left(\exp\left(\sqrt{V_0}\,(x - \frac{\rho\eta}{4}\nu^2)\right) - \bar{K}\right)^+\\ c(x,\nu,j) &= q_{j+1}\,p_{j+1}\,c(x+\sqrt{\Delta}, \nu+\sqrt{\Delta}, j+1)\\ &\quad + q_{j+1}\,(1-p_{j+1})\,c(x+\sqrt{\Delta}, \nu-\sqrt{\Delta}, j+1)\\ &\quad + (1-q_{j+1})\,p_{j+1}\,c(x-\sqrt{\Delta}, \nu+\sqrt{\Delta}, j+1)\\ &\quad + (1-q_{j+1})\,(1-p_{j+1})\,c(x-\sqrt{\Delta}, \nu-\sqrt{\Delta}, j+1),\end{aligned} \tag{23}$$

for $0 \le j < N$. Note that in (23) interpolation is not required. Finally, the price at time zero is given by $c(X_0, \nu_0, 0)$. Recursion (23) can be adapted to the early exercise feature by replacing c by its continuation value.

The method presented in this section is not applicable to fixed strike Asian options.[2]

5. Lower bound for arithmetic Asian options with stochastic volatility model

In light of the limitation of the approach presented in the previous section under stochastic volatility (Model 3), in what follows we propose a lower bound for prices of European Asian options with a fixed or floating strike in the stochastic volatility model framework shown in A.3.

5.1. Fixed strike option

The idea for the derivation of a price bound stems from Fusai and Kyriakou (2016). More specifically,

$$\mathrm{LB}(\lambda) \le \mathbb{E}\left[(A_N - K)^+\right], \tag{24}$$

where

$$A_N := \frac{\sum_{n=0}^N S_n}{N+1}, \tag{25}$$

$$\mathrm{LB}(\lambda) := \mathbb{E}\left[(A_N - K)\mathbf{1}_{\{G_N > \lambda\}}\right], \tag{26}$$

and

$$G_N := \frac{\sum_{n=0}^N \ln S_n}{N+1}.$$

The replacing exercise-triggering event $\{G_N > \lambda\}$ and the actual $\{A_N > K\}$ relate closely aiming to minimize the distance between the lower bound and the true option price, while, at the same time, making the problem more analytically tractable compared to the original one.

We adopt here the asset price dynamics with variance factor process V by Akyıldırım et al. (2014), i.e.,

$$S_n = S_0 e^{\sum_{j=1}^n \xi_j + \alpha_n\xi_n}, \tag{27}$$

[2] The reason is that the relevant variable Z proposed by Rogers and Shi (1995) for this type of option, $Z(t) = \frac{\int_0^t S(u)du - KT}{S_0}$, is not positive, hence the transformation (20) cannot be used. The transformation is necessary, otherwise using directly the variables Z and V the recursion for fixed strike options performs very poorly.

where

$$\xi_j = \begin{cases} \sqrt{\Delta}, & p_j \\ -\sqrt{\Delta}, & 1 - p_j \end{cases}.$$

The coefficients $\{\alpha_j\}$ and the probabilities $\{p_j\}$ for $j = 1, \ldots, N$ are given by

$$\alpha_j = \frac{\sigma_S^2(V_{j-1}) - 1}{2},$$

$$p_j = \frac{\exp\left(r\Delta + \sqrt{\Delta}\alpha_{j-1}\xi_{j-1}\right) - \exp\left(-\sqrt{\Delta}(1+\alpha_j)\right)}{\exp\left(\sqrt{\Delta}(1+\alpha_j)\right) - \exp\left(-\sqrt{\Delta}(1+\alpha_j)\right)}.$$

where $\sigma_S(\cdot)$ is as in (A.4). From (27),

$$G_N = \ln S_0 + \sum_{j=0}^{N} \left(1 - \frac{j}{N+1}\right)\xi_j + \frac{1}{N+1}\sum_{j=1}^{N} \alpha_j \xi_j. \tag{28}$$

The lower bound (26) is given in terms of the Fourier inversion formula (see Goldberg, 1961, Theorem 5C)

$$\mathrm{LB}(\lambda) = \frac{e^{-\delta\lambda}}{2\pi}\int_{\mathbb{R}} e^{-iu\lambda}\Phi(u;\delta)du, \tag{29}$$

where the constant $\delta > 0$ ensures integrability and the Fourier transform is given by

$$\begin{aligned}
&\Phi(u;\delta) \\
&:= \int_{\mathbb{R}} e^{iu\lambda+\delta\lambda}\left\{\frac{1}{N+1}\sum\nolimits_{n=0}^{N} E\left[(S_n - K)\mathbf{1}_{\{G_N>\lambda\}}\right]\right\}d\lambda \\
&= \frac{S_0^{iu+\delta}}{iu+\delta}\left\{\frac{S_0}{N+1}\sum\nolimits_{n=0}^{N}\mathbb{E}\left[e^{\sum_{j=0}^{n}\xi_j+\alpha_n\xi_n+i(u-i\delta)\sum_{j=0}^{N}\left(1-\frac{j-\alpha_j}{N+1}\right)\xi_j}\right]\right. \\
&\quad \left. -K\mathbb{E}\left[e^{i(u-i\delta)\sum_{j=0}^{N}\left(1-\frac{j-\alpha_j}{N+1}\right)\xi_j}\right]\right\} \\
&= \frac{S_0^{iu+\delta}}{iu+\delta}\left\{\frac{S_0}{N+1}\sum\nolimits_{n=0}^{N}\Psi(-iw_{1,n},\ldots,-iw_{N,n})\right. \\
&\quad \left. -K\Psi(-iv_1,\ldots,-iv_N)\right\},
\end{aligned} \tag{30}$$

where the second equality follows from expressions (1) and (28), and

$$\Psi(u_1,\ldots,u_N) := \mathbb{E}\left(e^{i\sum_{j=1}^{N}u_j\xi_j}\right) \tag{31}$$

is the joint characteristic function of the random variables $\{\xi_j\}_{j=1}^{N}$. The characteristic function Ψ can be calculated by backward recursion. We define the conditional characteristic function

$$\varphi_n = \mathbb{E}_n\left(e^{i\sum_{j=n}^{N}u_j\xi_j}\right).$$

Starting from

$$\varphi_N = e^{iu_N\xi_N},$$

we calculate for each of $n = 1, \ldots, N$

$$\varphi_{n-1} = \mathbb{E}_{n-1}\left(e^{iu_{n-1}\xi_{n-1}}\varphi_n\right)$$

yielding eventually $\Psi = \varphi_0$. The coefficients $\{v_j\}$ and $\{w_{j,n}\}$ are given by

$$v_j := \quad i(u-i\delta)\left(1 - \tfrac{j-\alpha_j}{N+1}\right), \qquad 0 < j \le N \quad ,$$

$$w_{j,n} := \begin{cases} 1 + v_j, & 0 < j < n \\ 1 + \alpha_j + v_j, & j = n \\ v_j, & n < j \le N \end{cases}$$

Varying the free parameter λ in (26) leads to different lower bounds; we denote by λ^* the maximizer of (26):

$$\lambda^* := \arg\max_{\lambda} \mathrm{LB}(\lambda).$$

This satisfies the optimality condition

$$\mathbb{E}(A_N | Y_N = \lambda^*) = K. \tag{32}$$

5.2. Floating strike option

The case of the floating strike Asian option is dealt with similarly with an additional change of numéraire given by the underlying S. In particular,

$$\mathbb{E}\left[\left(A_N - \bar{K}S_N\right)^+\right] = S_0 e^{rT}\bar{\mathbb{E}}\left[\left(A_N S_N^{-1} - \bar{K}\right)^+\right],$$

and

$$\mathrm{LB}(\lambda) = \bar{\mathbb{E}}\left[\left(A_N S_N^{-1} - \bar{K}\right)\mathbf{1}_{\{G_N - \ln S_N > \lambda\}}\right] \le \bar{\mathbb{E}}\left[\left(A_N S_N^{-1} - \bar{K}\right)^+\right],$$

where G_N is given by (28). The lower bound is given from the Fourier transform representation (29) with

$$\begin{aligned}
\Phi(u;\delta) &= \frac{1}{iu+\delta}\left\{\frac{1}{N+1}\sum\nolimits_{n=0}^{N}\bar{\mathbb{E}}\left[e^{-\sum_{j=n+1}^{N}\xi_j-\alpha_n\xi_n-i(u-i\delta)\sum_{j=0}^{N}\frac{j-\alpha_j}{N+1}\xi_j}\right]\right. \\
&\quad \left. -\bar{K}\bar{\mathbb{E}}\left[e^{-i(u-i\delta)\sum_{j=0}^{N}\frac{j-\alpha_j}{N+1}\xi_j}\right]\right\} \\
&= \frac{1}{iu+\delta}\left\{\frac{1}{N+1}\sum\nolimits_{n=0}^{N}\bar{\Psi}(-iw_{1,n},\ldots,-iw_{N,n})\right. \\
&\quad \left. -\bar{K}\bar{\Psi}(-iv_1,\ldots,-iv_N)\right\},
\end{aligned}$$

where

$$\bar{\Psi}(u_1,\ldots,u_N) = \mathbb{E}\left(e^{-rN\Delta+i\sum_{j=1}^{N}(u_j-i)\xi_j}\right),$$

$$v_j = \quad -i(u-i\delta)\tfrac{j-\alpha_j}{N+1}, \qquad 0 < j \le N \quad ,$$

$$w_{j,n} = \begin{cases} v_j, & 0 < j < n \\ v_j - \alpha_j, & j = n \\ v_j - 1, & n < j \le N \end{cases}.$$

We note that the lower bounds for fixed and floating strike options can be adapted to Models 1 and 2. More details can be made available by the authors upon request, including an upper bound for the error of this lower bound price approximation.

6. Relationship with continuous-time diffusion model

In this section, we focus on the application of our tree method for pricing a European Asian option with a fixed strike price based on the construction in Section 3.1 in the binomial model setting of Cox et al. (1979) (see A.1). In what follows, we prove that our method is consistent with the PDE of Rogers and Shi (1995). This is important as we show that our tree method overcomes the well-known instability of Rogers and Shi (1995) PDE numerical schemes when the volatility is low (see Barraquand & Pudet, 1996 and Dubois & Leliévre, 2005). In fact, we prove the unconditional convergence of our method later in Section 6.2 in the diffusion model case and we demonstrate the convergence of the method via extensive numerical tests for the other models.

6.1. Discrete-time model and PDE consistency

Consider the PDE

$$\frac{\partial c}{\partial t} + \mathcal{G}c = 0, \tag{33}$$

where $c(z, t)$ is a sufficiently smooth function, $\mathcal{G}$ the differential operator

$$\mathcal{G} := (1 - rz)\frac{\partial}{\partial z} + \frac{1}{2}\sigma^2 z^2\frac{\partial^2}{\partial z^2},$$

and the relevant boundary condition is

$$c(z, T) = z^+.$$

The solution of (33) satisfies

$$c\left(\frac{-KT}{S_0}, 0\right) = \frac{e^{-rT}}{S_0}\mathbb{E}\left[\left(\int_0^T S(t)dt - KT\right)^+\right],$$

when $S(t)$ is represented by the geometric Brownian motion.

Proposition 1. *For any sufficiently smooth function $c(z, t)$,*

$$\lim_{\Delta\to 0}\frac{G_\Delta c - c}{\Delta} = \frac{\partial c}{\partial t} + \mathcal{G}c,$$

where the differential operator $\mathcal{G}$ is defined in (33) and we additionally define the operator

$$G_\Delta c(z,t) = \bar{p}c\left(\frac{z}{x_1}+\Delta, t+\Delta\right) + (1-\bar{p})c\left(\frac{z}{x_2}+\Delta, t+\Delta\right)$$

from (9).[3]

Proof. From the Taylor expansion

$$\begin{aligned} c\left(\frac{z}{x}+\Delta, t+\Delta\right) &= c(z,t) + \frac{\partial c(z,t)}{\partial t}\Delta + \frac{\partial c(z,t)}{\partial z}\left[z\left(\frac{1}{x}-1\right)+\Delta\right] \\ &\quad + \frac{1}{2}\frac{\partial^2 c(z,t)}{\partial z^2}\left[z^2\left(\frac{1}{x}-1\right)^2 + 2z\left(\frac{1}{x}-1\right)\Delta\right] + o(\Delta), \end{aligned}$$

we obtain

$$\begin{aligned} G_\Delta c(z,t) - c(z,t) &= \frac{\partial c(z,t)}{\partial t}\Delta + \frac{\partial c(z,t)}{\partial z}\left[z\bar{p}\left(\frac{1}{x_1}-\frac{1}{x_2}\right) + z\left(\frac{1}{x_2}-1\right)+\Delta\right] \\ &\quad + \frac{1}{2}\frac{\partial^2 c(z,t)}{\partial z^2}\left[z^2\bar{p}\left(\frac{1}{x_1}-1\right)^2 + z^2(1-\bar{p})\left(\frac{1}{x_2}-1\right)^2\right. \\ &\quad \left. + 2z\bar{p}\left(\frac{1}{x_1}-\frac{1}{x_2}\right)\Delta + 2z\left(\frac{1}{x_2}-1\right)\Delta\right] + o(\Delta). \end{aligned}$$

Using the expansions

$$\bar{p}\left(\frac{1}{x_1}-\frac{1}{x_2}\right) + \frac{1}{x_2} - 1 = -r\Delta + o(\Delta),$$

$$\bar{p}\left(\frac{1}{x_1}-1\right)^2 + (1-\bar{p})\left(\frac{1}{x_2}-1\right)^2 = \sigma^2\Delta + o(\Delta),$$

$$\bar{p}\left(\frac{1}{x_1}-\frac{1}{x_2}\right)\Delta + \left(\frac{1}{x_2}-1\right)\Delta = o(\Delta),$$

we further obtain

$$G_\Delta c(z,t) - c(z,t) = \left(\frac{\partial c(z,t)}{\partial t} + (1-rz)\frac{\partial c(z,t)}{\partial z} + \frac{1}{2}\sigma^2 z^2\frac{\partial^2 c(z,t)}{\partial z^2}\right)\Delta + o(\Delta),$$

hence the proposition is proved. □

From Proposition 1, we conclude that our discrete-time option price model approach is consistent with the PDE proposed by Rogers and Shi (1995). As highlighted, for example, in Dubois and Leliévre (2005), applying a standard finite difference scheme to this PDE results in instability for low volatility. On the contrary, we prove using probabilistic arguments in the next section that, for a fixed strike Asian option in the Cox et al. (1979) model, our method is unconditionally convergent. If we take into account also the interpolation error due to the non-recombinant tree, we recall from Section 3.1 the sufficient condition (14) for the error convergence to zero.

6.2. Convergence of the tree method

In this section, we prove the convergence of the actual European Asian option price with a fixed strike in the discrete-time model to the continuous-time equivalent when the underlying process is a diffusion. The challenge when proving the convergence resides in showing that the discrete average defined in (25) converges to the continuous average defined in Eq. (35). To prove this convergence, the assumptions of the Functional Limit Theorem (see Theorem 4 in Appendix D) must be satisfied.

Theorem 2. *Consider S_n defined in (1) with $E(\xi_j) = m\Delta + o(\Delta)$ and $\mathrm{Var}(\xi_j) = \sigma^2\Delta + o(\Delta)$, $j = 1, 2, \ldots, n$, and A_N defined in (25). Assume that*

$$\sup_N \mathbb{E}\left(A_N^2\right) < \infty. \tag{34}$$

Then,

$$\lim_{N\to\infty}\mathbb{E}\left[(A_N-K)^+\right] = \mathbb{E}\left[(A_T-K)^+\right],$$

where

$$A_T := \frac{1}{T}\int_0^T S(t)dt, \tag{35}$$

$$S(t) := S_0 e^{mt+\sigma W(t)}$$

and W is the standard Brownian motion.

Proof. Without loss of generality assume $T = 1$, hence $0 \le t \le 1$ and $\Delta = 1/N$. Define the random function

$$X_N(t) = \sum_{j=1}^{\lfloor Nt\rfloor}\xi_j - \frac{m}{N}\lfloor Nt\rfloor.$$

From the Functional Central Limit Theorem (see Theorem 4 in Appendix D),

$$X_N(t) \xrightarrow{d} \sigma W(t)$$

with respect to the Skorokhod topology on the space of càdlàg functions $D[0, 1]$. Hence,

$$\sum_{j=1}^{\lfloor Nt\rfloor}\xi_j \xrightarrow{d} mt + \sigma W(t).$$

S_n can be rewritten as

$$S_n = S_0 e^{\sum_{j=1}^n \xi_j} = S_0 e^{m\frac{n}{N} + X_n(1)}.$$

From the Integral Functional Convergence Theorem (see Theorem 5 in Appendix D), we have that[4]

$$\frac{1}{N}\sum_{n=1}^N e^{X_n(1)} \xrightarrow{d} \int_0^1 e^{\sigma W(t)}dt.$$

For $X_0(1) = 0$, we rewrite

$$\begin{aligned} A_N &= \frac{S_0}{N+1}\sum_{n=0}^N e^{m\frac{n}{N}+X_n(1)} \\ &= \frac{S_0 N}{N+1}\left(\frac{1}{N} + \frac{1}{N}\sum_{n=1}^N e^{m\frac{n}{N}+X_n(1)}\right) \xrightarrow{d} \int_0^1 S_0 e^{mt+\sigma W(t)}dt = A_1. \end{aligned}$$

We now prove that the pricing expectation of the discrete arithmetic Asian option converges to that of the continuous Asian option. Indeed,

$$\mathbb{E}\left[(A_N-K)^+\right] = \mathbb{E}\left(A_N \mathbf{1}_{\{A_N>K\}}\right) - K\mathbb{E}\left(\mathbf{1}_{\{A_N>K\}}\right),$$

where the second expected value can be written as

$$\mathbb{E}\left(\mathbf{1}_{\{A_N>K\}}\right) = 1 - P(A_N < K)$$

[3] Compared to (9), we have here accentuated, by slightly abusing the original notation, the dependence on time t.

[4] Note that for each $n \in \mathbb{N}$, $n \le N$, there exists $t \in [0, 1]$ such that $n = \lfloor Nt\rfloor$.

by convergence in distribution. Regarding the first expected value, define function $h : \mathbb{R} \to \mathbb{R}$: $h(a) = a\mathbf{1}_{\{a>K\}}$. The set of discontinuities of h is $D_h = \{K\}$ and $P(A_1 \in D_h) = 0$. Then, from the Continuous Mapping Theorem (see Theorem 6 in Appendix D),

$$h(A_N) \xrightarrow{d} h(A_1).$$

Next, we prove that $h(A_N)$ is uniformly integrable. A sufficient condition for the uniform integrability is that, for some $\epsilon > 0$,

$$\sup_N \mathbb{E}\left[|h(A_N)|^{1+\epsilon}\right] < \infty.$$

We fix $\epsilon = 1$ and obtain

$$\sup_N \mathbb{E}\left[h(A_N)^2\right] < \sup_N \mathbb{E}\left(A_N^2\right) < \infty,$$

by assumption (see Example 3). Finally, from convergence of mean (see Theorem 7 in Appendix D), we get

$$\lim_{N\to\infty} \mathbb{E}[h(A_N)] = \mathbb{E}[h(A_1)].$$

□

Example 3. The assumptions of Theorem 2 are satisfied in the Cox et al. (1979) model (see A.1). In fact, we have that

$$\mathbb{E}\left(\xi_j\right) = \left(r - \frac{\sigma^2}{2}\right)\Delta + o(\Delta), \quad \mathrm{Var}\left(\xi_j\right) = \sigma^2\Delta + o(\Delta),$$

and

$$\mathbb{E}\left(A_N^2\right) = \frac{S_0^2}{(N+1)^2}\left[\frac{1-e^{2r\frac{N+1}{N}}}{1-e^{2r\frac{1}{N}}} + 2\frac{(1-e^{2r\frac{N+1}{N}})(1-e^{r\frac{N-1}{N}})}{(1-e^{r\frac{1}{N}})(1-e^{2r\frac{1}{N}})}\right]. \quad (36)$$

Hence, for each fixed N, (36) is finite. Finally,

$$\lim_{N\to\infty} \mathbb{E}\left(A_N^2\right) = \frac{S_0^2}{2}\frac{(1-e^{2r})(1-e^r)}{r} < \infty.$$

We conclude that

$$\sup_N \mathbb{E}\left(A_N^2\right) < \infty.$$

7. Numerical results and analysis

For the purposes of our numerical experiments we consider the binomial model of Cox et al. (1979), the bivariate tree of Hilliard and Schwartz (2005) when the underlying process is a Merton (1976) jump diffusion, as a possible way of including sudden and extreme departures, the binomial tree of Hilliard (2014) to represent the CEV model and Akyıldırım et al. (2014) for the Heston stochastic volatility model. We consider options of European and American exercise type, fixed strike calls and floating strike puts.

In Figs. 1–3, we study the convergence of discrete-time model option prices to their continuous-time equivalents with increasing number of time steps. More specifically, in Fig. 1, in the case of European Asian options, we compute error patterns given by the distances of the computed option prices using our tree method (see Table 1 for the indicated cases) from the result of Fusai (2004) in the lognormal model (values are reported in the top panel of Table E.9). In the absence of an analogous universal benchmark in the American Asian option case (e.g., the methodology of Hansen & Jørgensen, 2000 does not generalize to fixed strike options; the PDE approach of Zvan et al., 1998 is limited to lognormal dynamics), we adhere to standard practice of computing differences of prices from our method (see Table 1) following successive time grid refinements and studying their pattern. Despite a widely documented in the literature convergence of tree models to Black–Scholes prices in a wavy fashion (e.g., see Broadie & Detemple, 1996 and Tian, 1999), our method is, in general, endowed with monotonic convergence, which is remarkably good for sufficiently large number of time steps. In particular, the error ratio converges to 2 quickly enough, implying that the error is almost exactly halved when we double the time steps, hence the convergence can be further accelerated by Richardson extrapolation. This powerful feature allows us to gauge the precision of the method and value options to the desired level of accuracy. Regardless of the driving dynamics, a similar behaviour is embedded in the jump diffusion and the CEV model as observed in Figs. 2 and 3. (More results can be made available by the authors upon request, for example, for one-dimensional diffusions.) In the absence of a true analytical benchmark in the European option case under continuous averaging in the continuous-time Merton model, we compute this using an accurate control variate Monte Carlo (CVMC) strategy as described in Fusai and Kyriakou (2016) (results are reported in the bottom panel of Table E.9). In the CEV model, for European options, the continuous-average continuous-time prices are calculated by Monte Carlo simulation. In both the jump diffusion and CEV models, for the American option, we adhere to a similar practice as in Fig. 1. The cases presented in Figs. 1 and 2 are accompanied by Tables 2 and 3 which contain converged prices of our method enhanced by extrapolation, corresponding to European and American Asian options with a fixed or floating strike, in the Black–Scholes or the Merton jump diffusion model. Our results in the former model choice match, for example, those from the implementation of the van Leer flux limiter of Zvan et al. (1998) and finite differences in Hansen and Jørgensen (2000). As

Table 2
The table reports prices of European and American continuous Asian options in the Black–Scholes model for different volatilities σ, fixed strikes K, and coefficients $\bar{K}$ for the floating-strike option by convergence of our tree method (refer to relevant Sections 3.1–3.3) based on the binomial model with increasing monitoring frequency. Parameters used are reported in the table; additional parameters: $S_0 = 100$, $r = 0.09$ per annum, and $T = 1$ year.

Asian fixed-strike call option					Asian floating-strike put option			
σ	K	European	σ	American	σ	$\bar{K}$	European	American
0.05	90	13.3782	0.05	13.4487	0.2	0.9	7.5982	12.7314
0.05	95	8.8088	0.05	8.8550	0.2	0.95	4.6567	8.5164
0.05	100	4.3082	0.05	4.3255	0.2	1	2.6210	5.0668
0.05	105	0.9583	0.05	0.9596	0.2	1.05	1.3582	2.6670
0.05	110	0.0521	0.05	0.0522	0.2	1.1	0.6513	1.2646
0.4	90	16.4999	0.2	15.518	0.4	0.9	11.4822	18.4072
0.4	95	13.5106	0.2	11.032	0.4	0.95	8.9622	14.7736
0.4	100	10.9237	0.2	7.297	0.4	1	6.8994	11.5957
0.4	105	8.7299	0.2	4.524	0.4	1.05	5.2476	8.9162
0.4	110	6.9034	0.2	2.637	0.4	1.1	3.9488	6.7346

Table 3
The table reports prices of European and American continuous Asian options in Merton (1976) model for different fixed strikes K and coefficients $\bar{K}$ for the floating-strike option by convergence of our tree method (refer to relevant Sections 3.1–3.3) based on the bivariate tree of Hilliard and Schwartz (2005) with increasing monitoring frequency. Model parameters are from Hilliard and Schwartz (2005) : $\sigma \in \{\sqrt{0.05}, 0.05\}$, $\lambda_J = 5$, $\sigma_J = \sqrt{0.05}$, and $\mu_J = -\sigma_J^2/2$; additional parameters: $S_0 = 40$, $T = 1$ year, $r = 0.08$ per annum.

Asian fixed-strike call option			Asian floating-strike put option		
σ	K	European	σ	$\bar{K}$	American
$\sqrt{0.05}$	30	11.48036	$\sqrt{0.05}$	0.75	13.9893
$\sqrt{0.05}$	35	8.0242	$\sqrt{0.05}$	0.875	10.0466
$\sqrt{0.05}$	40	5.3875	$\sqrt{0.05}$	1	6.9119
$\sqrt{0.05}$	45	3.5532	$\sqrt{0.05}$	1.125	4.6676
$\sqrt{0.05}$	50	2.3456	$\sqrt{0.05}$	1.25	3.1273
0.05	30	11.3290	0.05	0.75	13.6657
0.05	35	7.7147	0.05	0.875	9.6323
0.05	40	4.9377	0.05	1	6.3381
0.05	45	3.1121	0.05	1.125	4.2263
0.05	50	1.9748	0.05	1.25	2.8028

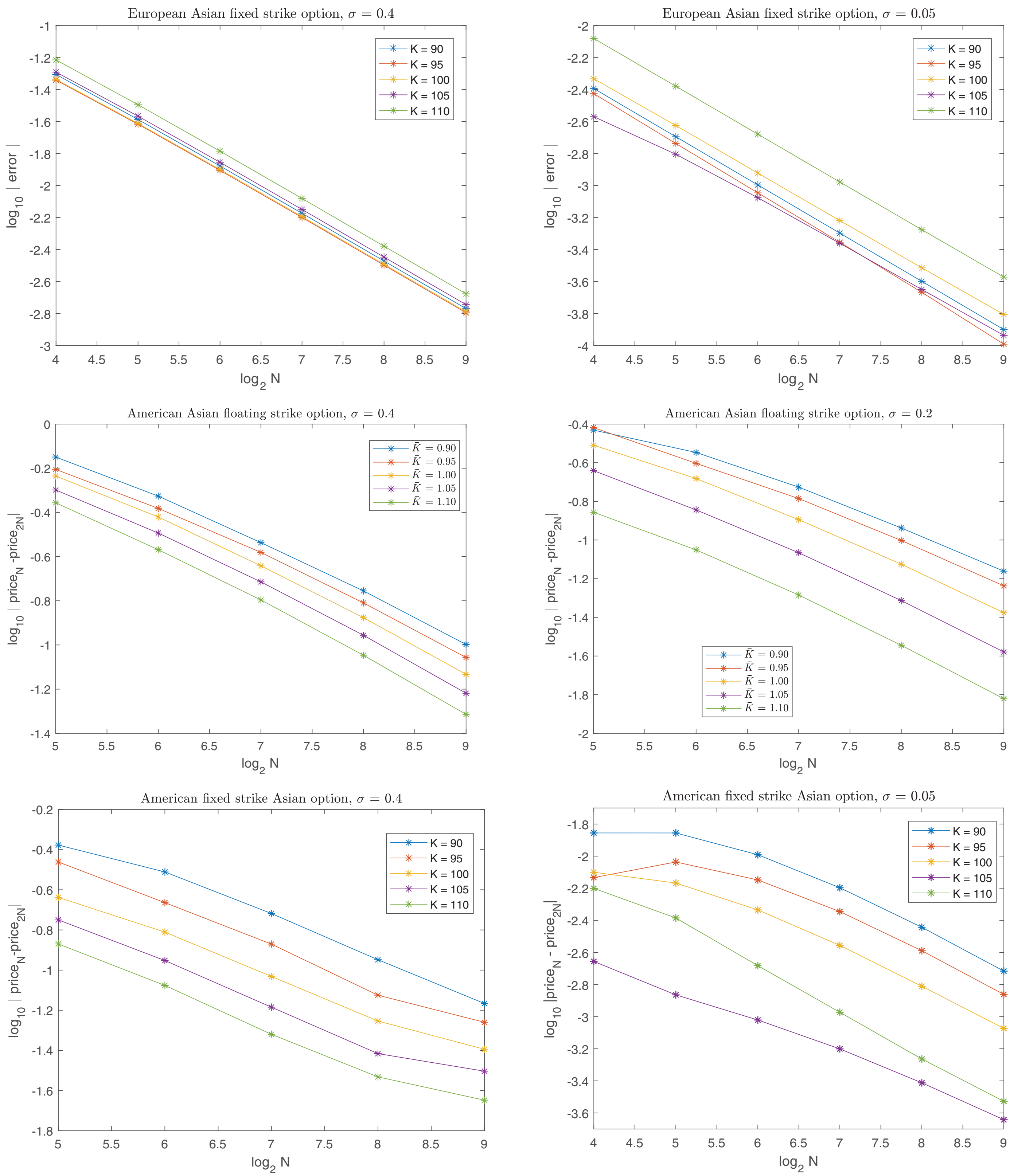

Fig. 1. Plots show convergence patterns with increasing number of time steps N of our method in the binomial model for European and American Asian options with different fixed strikes K, coefficients $\bar{K}$ for the floating-strike option (refer to relevant Sections 3.1–3.3), and volatilities σ. For more information about the computation of error, refer to Section 7. Parameters used are reported on the plots; additional parameters: $S_0 = 100$, $r = 0.09$ per annum, and $T = 1$ year.

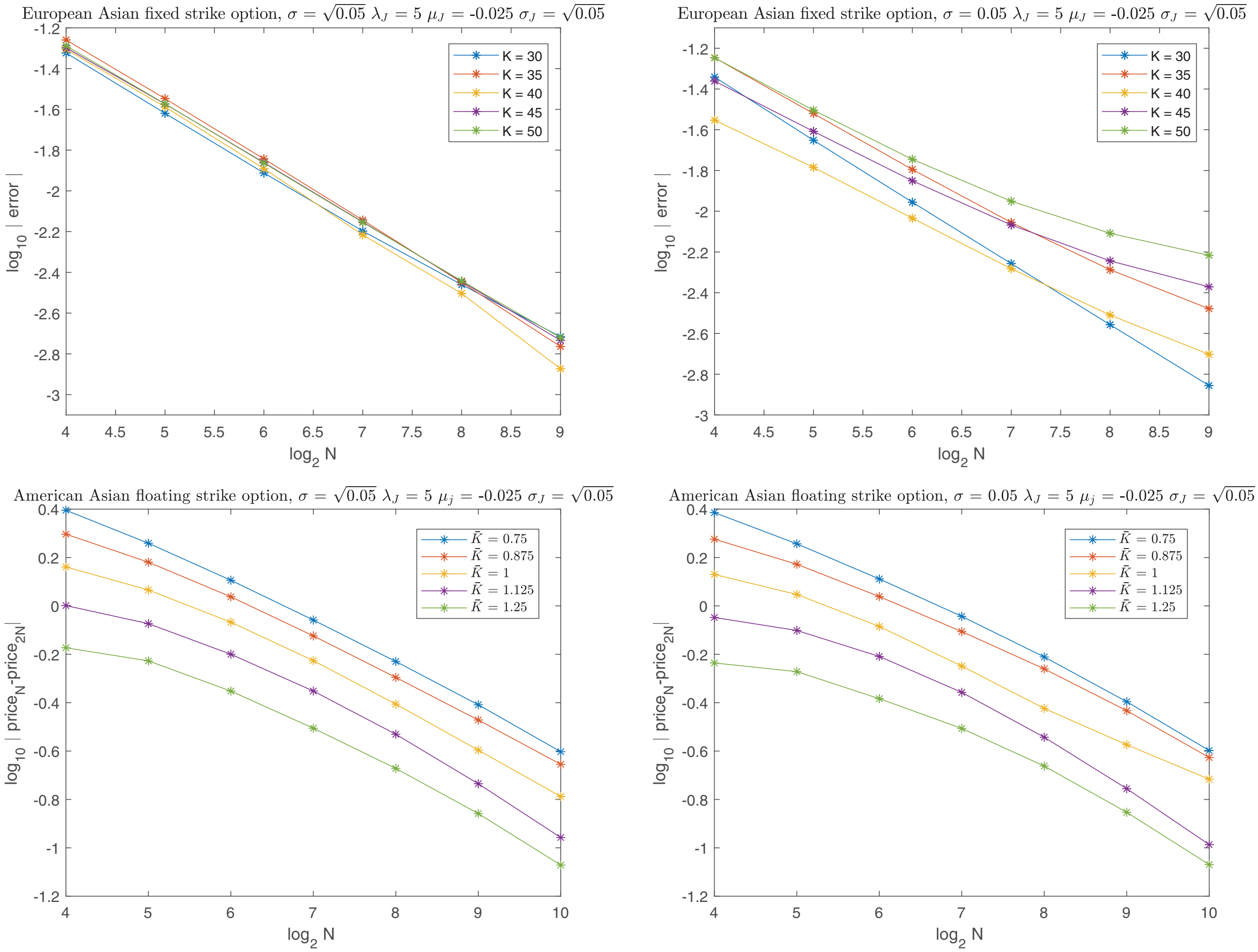

Fig. 2. Plots show convergence patterns with increasing number of time steps N of our method in the bivariate tree model of Hilliard and Schwartz (2005) for European and American Asian options (refer to relevant Sections 3.1 and 3.3). See also notes about the error computation in Section 7. Parameters used are reported on the plots; additional parameters: $S_0 = 40$, $T = 1$ year, and $r = 0.08$ per annum.

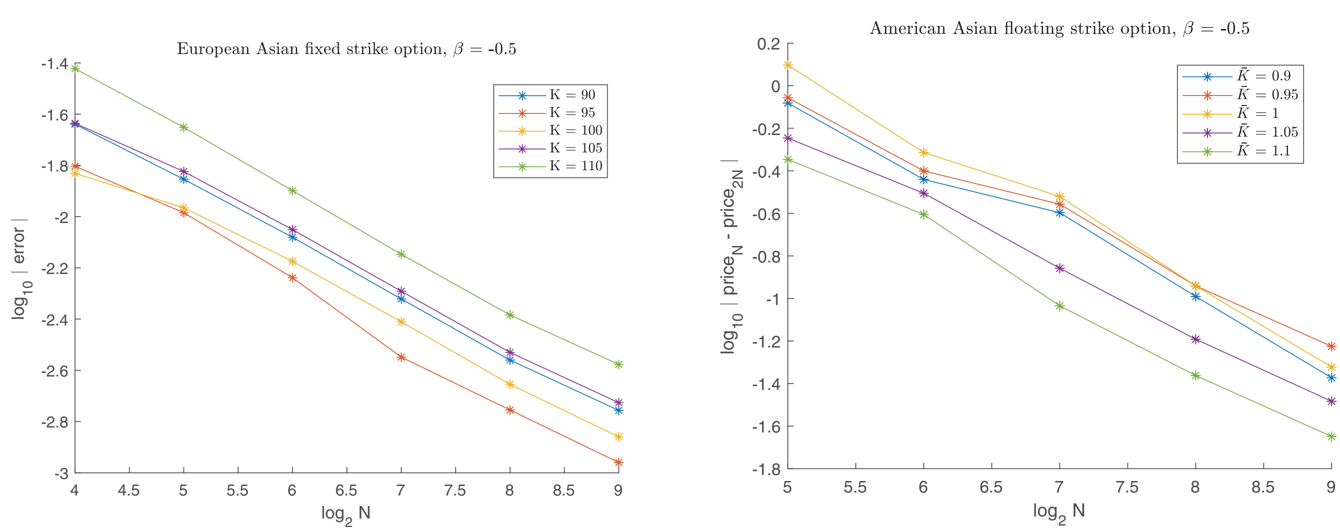

Fig. 3. Plots show convergence patterns with increasing number of time steps N of our method in the binomial tree of Hilliard (2014) for the CEV model and for European and American Asian options (refer to relevant Sections 3.1 and 3.3). See also notes about the error computation in Section 7. Parameters used are from Cai et al. (2014, Table 8): $\beta = -0.5$, $\delta S_0^{\beta} = 0.25$, $S_0 = 100$, $T = 1$ year, and $r = 0.05$ per annum.

Table 4
The table reports prices of European Asian call options with fixed strikes K in the binomial model. Comparisons are presented between our tree model (Tree) approach (9), the maximum lower bound of Fusai and Kyriakou (2016) (MLB), and the methods of Chalasani et al. (1998) and Lo et al. (2008) against control variate Monte Carlo (CVMC) price estimates (control variate given by the MLB) with standard errors (std. err.) also reported. Absolute % relative pricing errors with respect to CVMC price estimates are presented as well as the average error (AAPRE). Parameters used are reported in the table; additional parameters: $S_0 = 100$, $T = 1$ year and $N = 30$ time steps. CPU times are per option price.

K	σ	r	Tree	Abs. rel. err. (%)	MLB	Abs. rel. err. (%)	Chalasani et al.	Abs. rel. err. (%)	Lo et al.	Abs. rel. err. (%)	CVMC CV MLB	Std. err. $\times 10^{-5}$
95	0.05	0.05	7.17725	0.000	7.17725	0.000	7.178	0.010	7.177	0.004	7.17725	0.002
100	0.05	0.05	2.71194	0.000	2.71195	0.000	2.708	0.146	2.712	0.002	2.71195	0.003
105	0.05	0.05	0.33239	0.001	0.33236	0.011	0.309	7.038	0.332	0.118	0.33239	0.037
95	0.05	0.09	8.81080	0.000	8.81080	0.000	8.811	0.002	8.811	0.002	8.81080	0.000
100	0.05	0.09	4.30572	0.000	4.30572	0.000	4.301	0.110	4.306	0.006	4.30572	0.003
105	0.05	0.09	0.95674	0.002	0.95668	0.008	0.892	6.768	0.957	0.026	0.95675	0.056
90	0.1	0.05	11.94740	0.000	11.94738	0.000	11.949	0.013	11.947	0.003	11.94739	0.028
100	0.1	0.05	3.63480	0.001	3.63477	0.002	3.632	0.078	3.635	0.004	3.63485	0.045
110	0.1	0.05	0.31950	0.014	0.31938	0.054	0.306	4.240	0.319	0.172	0.31955	0.164
90	0.1	0.09	13.38507	0.000	13.38506	0.000	13.386	0.007	13.385	0.001	13.38507	0.014
95	0.1	0.09	8.90570	0.000	8.90569	0.000	8.91	0.048	8.91	0.048	8.90570	0.030
100	0.1	0.09	4.90875	0.001	4.90872	0.001	4.902	0.138	4.909	0.004	4.90879	0.044
105	0.1	0.09	2.06582	0.002	2.06567	0.009	2.03	1.736	2.07	0.200	2.06586	0.151
110	0.1	0.09	0.62077	0.004	0.62056	0.038	0.582	6.250	0.621	0.032	0.62080	0.148
90	0.3	0.05	13.92967	0.000	13.92834	0.009	13.929	0.004	13.928	0.012	13.92961	0.887
100	0.3	0.05	7.92504	0.001	7.92388	0.013	7.924	0.012	7.924	0.012	7.92493	0.770
110	0.3	0.05	4.04261	0.000	4.04103	0.039	4.040	0.065	4.041	0.040	4.04262	1.144
90	0.3	0.09	14.96211	0.000	14.96101	0.007	14.971	0.060	14.961	0.007	14.96207	0.709
100	0.3	0.09	8.81200	0.000	8.81087	0.013	8.807	0.057	8.811	0.012	8.81202	0.681
110	0.3	0.09	4.67376	0.003	4.67203	0.034	4.661	0.270	4.672	0.035	4.67364	1.066
90	0.5	0.09	18.15276	0.001	18.14721	0.031	18.15	0.016	18.14	0.071	18.15287	2.964
100	0.5	0.09	12.98962	0.000	12.98440	0.040	12.99	0.003	12.98	0.074	12.98957	2.830
110	0.5	0.09	9.07562	0.003	9.06972	0.063	9.08	0.051	9.07	0.059	9.07540	3.202
AAPRE				0.001		0.016		1.179		0.041		
time (seconds)			0.2		0.4		N.A.		N.A.			

Table 5
The table reports prices of European Asian call options with fixed strikes K in the binomial model. Comparisons are presented between our tree model approach (9), MLB and the methods of Neave and Ye (2003) and Hull and White (1993) against CVMC price estimates. See also notes in Table 4. Parameters used are reported in the table; additional parameters: $S_0 = 50$, $r = 0.1$ per annum, annual volatility $\sigma = 0.3$, and $N = 40$ time steps. CPU times are per option price. The CPU time of Neave and Ye (2003) method is from Nadarajah et al. (2017, p. 213). The CPU time of the Hull and White (1993) method corresponds to our implementation of the Matlab function *asianbycrr*.

K	T	Tree	Abs. rel. err. (%)	MLB	Abs. rel. err. (%)	Neave & Ye	Abs. rel. err. (%)	Hull & White	Abs. rel. err. (%)	CVMC CV MLB	Std. err. $\times 10^{-5}$
40	0.5	10.75389	0.000	10.75377	0.001	10.754	0.001	10.755	0.010	10.75389	0.146
45	0.5	6.35944	0.000	6.35921	0.004	6.360	0.009	6.363	0.056	6.35944	0.203
50	0.5	3.00817	0.000	3.00796	0.007	3.007	0.039	3.012	0.127	3.00817	0.187
55	0.5	1.10726	0.001	1.10690	0.033	1.104	0.295	1.108	0.066	1.10727	0.251
60	0.5	0.31801	0.004	0.31759	0.129	0.315	0.942	0.317	0.313	0.31800	0.328
40	1	11.54219	0.000	11.54173	0.004	11.544	0.016	11.545	0.024	11.54219	0.371
45	1	7.61226	0.000	7.61173	0.007	7.613	0.010	7.616	0.049	7.61227	0.399
50	1	4.52127	0.000	4.52071	0.013	4.519	0.051	4.522	0.016	4.52129	0.416
55	1	2.42240	0.001	2.42152	0.037	2.416	0.265	2.420	0.100	2.42242	0.530
60	1	1.18096	0.002	1.17971	0.104	1.173	0.672	1.176	0.418	1.18093	0.760
40	1.5	12.28160	0.000	12.28079	0.007	12.283	0.011	12.285	0.028	12.28159	0.564
45	1.5	8.66721	0.000	8.66635	0.010	8.667	0.002	8.670	0.033	8.66718	0.549
50	1.5	5.74387	0.001	5.74290	0.018	5.740	0.068	5.743	0.016	5.74393	0.573
55	1.5	3.59108	0.000	3.58961	0.041	3.582	0.252	3.585	0.169	3.59107	0.882
60	1.5	2.13287	0.001	2.13075	0.101	2.121	0.558	2.124	0.417	2.13290	1.221
40	2	12.95042	0.000	12.94929	0.009	12.952	0.012	12.953	0.020	12.95042	0.713
45	2	9.58046	0.000	9.57926	0.013	9.580	0.005	9.582	0.016	9.58047	0.688
50	2	6.79481	0.000	6.79339	0.021	6.789	0.085	6.792	0.041	6.79479	0.840
55	2	4.64116	0.001	4.63908	0.045	4.630	0.241	4.633	0.176	4.64118	1.192
60	2	3.06959	0.003	3.06659	0.101	3.053	0.543	3.057	0.413	3.06968	1.471
AAPRE			0.001		0.035		0.204		0.125		
time (seconds)		0.2		0.4		1.0		0.2			

already discussed, higher precision is possible if we extrapolate raw prices computed for a larger number of time steps.

To verify the accuracy of our proposed tree method, in Tables 4–7 we compare our results with the maximum lower bound (MLB) proposed in Fusai and Kyriakou (2016) and other important discrete-time option price model contributions. For different strikes, model and market parameters, monitoring frequencies and contract specifications, we report option prices as well as % relative pricing errors and the average error, obtained for each method against highly accurate reports from Monte Carlo simulation using the MLB as a control variate. More specifically, from Table 4, we see that, as expected, in all thirty-two parameter combinations our tree method is very close to the Monte Carlo price estimates (the latter are accurate to 4–6 decimals with 95% confidence): the average absolute % relative error of our tree prices against the Monte Carlo estimates is 0.001%. The raw MLB,

Table 6
The table reports prices of European Asian call options with fixed strikes K in the bivariate tree model of Hilliard and Schwartz (2005). Comparisons are presented between our tree model approach (9) and the MLB against CVMC price estimates. See also notes in Table 4. Model parameters are from Hilliard and Schwartz (2005): $\sigma \in \{\sqrt{0.05}, 0.1, 0.05\}$, $\lambda_J = 5$, $\sigma_J \in \{\sqrt{0.05}, 0.3\}$, and $\mu_J = -\sigma_J^2/2$; additional parameters: $S_0 = 40$, $T = 1$ year, $r = 0.08$ per annum, and $N = 200$ time steps. CPU times are per option price.

K	σ	σ_J	Tree	Abs. rel. err. (%)	MLB	Abs. rel. err. (%)	CVMC CV MLB	Std. err. $\times 10^{-4}$
30	$\sqrt{0.05}$	$\sqrt{0.05}$	11.47656	0.000	11.47188	0.044	11.4769	0.733
35	$\sqrt{0.05}$	$\sqrt{0.05}$	8.01953	0.003	8.01690	0.058	8.0216	0.712
40	$\sqrt{0.05}$	$\sqrt{0.05}$	5.38301	0.005	5.38170	0.077	5.3859	0.692
45	$\sqrt{0.05}$	$\sqrt{0.05}$	3.54873	0.008	3.54715	0.123	3.5515	0.778
50	$\sqrt{0.05}$	$\sqrt{0.05}$	2.34122	0.007	2.33810	0.207	2.3429	0.881
30	0.1	0.3	12.01148	0.002	12.01157	0.018	12.0137	1.273
35	0.1	0.3	8.81714	0.004	8.81728	0.042	8.8210	1.223
40	0.1	0.3	6.34105	0.008	6.34126	0.073	6.3459	1.370
45	0.1	0.3	4.58644	0.008	4.58672	0.075	4.5902	1.373
50	0.1	0.3	3.36379	0.013	3.36415	0.116	3.3681	1.463
30	0.05	$\sqrt{0.05}$	11.32546	0.000	11.32146	0.035	11.3254	0.639
35	0.05	$\sqrt{0.05}$	7.71002	0.003	7.70861	0.047	7.7123	0.638
40	0.05	$\sqrt{0.05}$	4.93491	0.008	4.93463	0.089	4.9390	0.646
45	0.05	$\sqrt{0.05}$	3.10849	0.011	3.10870	0.100	3.1118	0.638
50	0.05	$\sqrt{0.05}$	1.97045	0.011	1.96823	0.222	1.9726	0.791
AAPRE				0.006		0.088		
time (seconds)			0.8		1.0			

Table 7
The table reports prices of European Asian call options with fixed strikes K in the binomial model for the CEV diffusion. Comparisons are presented between our tree method (9) and Cai et al. (2014) against Monte Carlo (MC) price estimates. Parameters used are reported in the table; additional parameters: $S_0 = 100$, $\delta S_0^{\beta} = 0.25$, $T = 1$ year, $r = 0.05$ per annum and $N = 250$ time steps. CPU times are per option price. The CPU times in (Cai et al., 2014, Table 8) are reproduced here.

K	β	Tree	Abs. rel. err. (%)	Cai et al.	Abs. rel. err. (%)	MC	Std. err.
80	−0.5	21.7162	0.024	21.7143	0.033	21.7214	0.0084
90	−0.5	13.3322	0.015	13.3288	0.041	13.3342	0.0075
100	−0.5	6.8584	0.004	6.8537	0.065	6.8581	0.0058
110	−0.5	2.8640	0.061	2.8612	0.161	2.8658	0.0039
120	−0.5	0.9559	0.433	0.9554	0.484	0.9601	0.0022
80	−0.25	21.6731	0.032	21.6712	0.041	21.6800	0.0085
90	−0.25	13.2726	0.026	13.2690	0.053	13.2761	0.0076
100	−0.25	6.8536	0.006	6.8485	0.081	6.8541	0.0060
110	−0.25	2.9327	0.023	2.9296	0.128	2.9334	0.0041
120	−0.25	1.0413	0.261	1.0407	0.313	1.0440	0.0024
80	0.25	21.6025	0.005	21.6017	0.009	21.6037	0.0088
90	0.25	13.1594	0.011	13.1555	0.019	13.1580	0.0080
100	0.25	6.8462	0.043	6.8403	0.044	6.8433	0.0064
110	0.25	3.0755	0.046	3.0718	0.075	3.0741	0.0045
120	0.25	1.2285	0.123	1.2284	0.134	1.2301	0.0029
AAPRE			0.074		0.112		
time (seconds)		4.4		0.2			

i.e., not combined with Monte Carlo simulation, slightly underperforms, being a lower bound price approximation, still it is sufficiently accurate resulting in a lower average error of 0.014%. The prices from Chalasani et al. (1998) and the improved Lo et al. (2008) are less precise with average errors of 1.143% and 0.032%, respectively. In Table 5, we extend to comparisons with Neave and Ye (2003) and Hull and White (1993). As before, the tree method performs best with a 0.001% average error, or equivalently an observed overall accuracy of 4–5 decimals, whereas the MLB comes second with an average error of 0.035%. Our proposed method produces a price in a tenth of a second for 30–40 time steps, whereas Monte Carlo simulation is far more computationally intensive. Each reported time corresponds to one option price computed in Matlab R2018a based on an Intel Core i7 CPU at 2.50 GHz and 16.0 GB of RAM. The generally high accuracy of our tree method is transferred to the parametrization in Table 6 for Hilliard and Schwartz (2005) model with average errors of 0.006% and 0.088%, respectively, against the Monte Carlo benchmarks. We achieve a precision of 4 decimals in around one second with 200 time steps. In Table 7, we compare our tree method for European fixed strike options with the prices from Cai et al. (2014) for the CEV model and find an average error of 0.074% for the former (accuracy of 3 decimals) versus 0.112% for the latter. The computing time of our method is 4.4 seconds for 250 time steps. Cai et al. (2014) report a computing time of 0.2 seconds per option price calculated using their small-time expansion method (with the computing time of the coefficients of the expansion excluded). Finally, in Table 8, we produce option prices within the Heston stochastic volatility model framework, and compare our methods, i.e., the tree method for floating strike options and the MLB for fixed strike options (see Table 1), with the Monte Carlo results by Akyıldırım et al. (2014). Our discrete-time option prices return an average error of 0.42% and 0.21% for 300 time steps in 1.5 and 2.7 seconds, respectively, for floating and fixed strikes. The computing times of 7.6 seconds of the Monte Carlo implementations are from Akyıldırım et al. (2014).

Table 8
The left panel of the table reports prices of European Asian put options for different coefficients $\bar{K}$ for the floating-strike option in the stochastic volatility model. The tree method (23) is compared against Monte Carlo (MC) price estimates. The right panel reports prices of European Asian call options with fixed strike K in the stochastic volatility model. The maximum lower bound (26) is compared against MC price estimates. Parameters used are from Akyıldırım et al. (2014, Table 8): $S_0 = 50$, $V_0 = 0.01$, $k = 2.0$, $\theta = 0.01$, $\rho = 0.5$, $\eta = 0.1$, $r = 0.05$ per annum, $T = 1$ year and $N = 300$ time steps. CPU times are per option price. The times of the Monte Carlo implementations are from Akyıldırım et al. (2014, Table 10).

Asian floating-strike put option							Asian fixed-strike call option					
$\bar{K}$	$\bar{K}\,S_0$	Tree	Abs. err.	Abs. rel. err. (%)	MC	Std. err.	K	MLB	Abs. err.	Abs. rel. err. (%)	MC	Std. err.
0.88	44	4.82723	0.0175	0.36	4.84473	0.00088	44	6.9183	0.0048	0.07	6.9135	0.0113
0.9	45	3.89742	0.01736	0.44	3.91478	0.00086	45	5.973	0.0038	0.06	5.9692	0.0113
0.92	46	3.01976	0.01721	0.57	3.03697	0.00081	46	5.0298	0.0003	0.01	5.03	0.0112
0.94	47	2.22003	0.01335	0.6	2.23339	0.00074	47	4.1198	0.0068	0.17	4.113	0.011
0.96	48	1.5288	0.01366	0.89	1.54247	0.00064	48	3.2497	0.0013	0.04	3.2511	0.0105
0.98	49	0.97629	0.00798	0.81	0.98427	0.00053	49	2.4697	0.0053	0.22	2.4644	0.0098
1	50	0.57086	0.00013	0.02	0.57099	0.00041	50	1.797	0.0029	0.16	1.794	0.0089
1.02	51	0.30119	0.00103	0.34	0.30017	0.00029	51	1.251	0.0045	0.36	1.2555	0.0077
1.04	52	0.14436	0.00047	0.33	0.14389	0.0002	52	0.8437	0.0018	0.21	0.8454	0.0065
1.06	53	0.06448	0.00025	0.39	0.06423	0.00013	53	0.5511	0.0002	0.03	0.5509	0.0053
1.08	54	0.02585	0.00007	0.25	0.02578	0.00008	54	0.3487	0.0023	0.66	0.3464	0.0043
1.1	55	0.01005	0.00001	0.15	0.01003	0.00005	55	0.2153	0.0004	0.19	0.2157	0.0034
1.12	56	0.00367	0.00001	0.36	0.00366	0.00003	56	0.1301	0.0007	0.51	0.1308	0.0026
AAPRE				0.42						0.21		
time (seconds)		1.5			7.6			2.7			7.6	

8. Conclusions

In this paper, we propose a new discrete-time model approach to pricing options with American features and a payoff dependent on an arithmetic average price. The method lends itself to general driving dynamics.

A series of numerical tests demonstrate our fast solution mechanisms capable of generating monotonic and smooth convergence price patterns for European and American options under different model specifications, including tree constructions for asset price dynamics with independent log-increments, one-dimensional diffusions, and stochastic volatility models. Also, by exploiting the smooth convergence, we can easily accelerate this by extrapolation techniques.

Our research forms a fertile ground for further investigations. Due to the exceptional runtime-accuracy balances of our methods, we may efficiently build richer price evolution models that can better fit the market reality allowing, for example, time-inhomogeneity, and apply to computing the implied parameters. In addition, our choice of the particular product payoff structure was motivated in the first place by the scope in applications such as the capacity problem studied in Driouchi et al. (2010) with flexible expansion decision of American type aiming to better capture the favourable economic timing. Our method is endowed with robustness and flexibility to this end, and this is where our subsequent paper focuses the spotlight on.

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Appendix A. Model specifications

A1. Model 1: the case of independent log-increments

Under this model specification, the logarithm of the asset price $\ln S$ has independent increments. We consider, for illustration, the binomial model of Cox et al. (1979) and the bivariate tree model of Hilliard and Schwartz (2005). In the first one,

$$\xi_j := \begin{cases} \ln x_1 = \sigma\sqrt{\Delta}, & p_1 = \frac{e^{r\Delta} - x_2}{x_1 - x_2} \\ \ln x_2 = -\sigma\sqrt{\Delta}, & p_2 = 1 - p_1 \end{cases}.$$

In the Hilliard and Schwartz (2005) model,

$$\xi_j := \begin{cases} \ln x_{1,\omega}, & p_{1,\omega} = p\, q_{s+\omega+1} \\ \ln x_{2,\omega}, & p_{2,\omega} = (1-p)\, q_{s+\omega+1} \end{cases}, \tag{A.1}$$

where

$$x_{1,\omega} := e^{m\Delta + \sigma\sqrt{\Delta} + \omega h},$$
$$x_{2,\omega} := e^{m\Delta - \sigma\sqrt{\Delta} + \omega h},$$

and $\omega = 0, \pm 1, \pm 2, \ldots, \pm s$ is the number of (independent) jumps under Poisson compounding of size h, allowed up or down on each of the two nodes for the smooth (diffusion) factor. The probability of the up state of the smooth factor is

$$p := \frac{e^{r\Delta} - e^{m\Delta - \sigma\sqrt{\Delta}} \sum_{\omega=-s}^{s} e^{\omega h} q_{s+\omega+1}}{\left(e^{m\Delta + \sigma\sqrt{\Delta}} - e^{m\Delta - \sigma\sqrt{\Delta}}\right) \sum_{\omega=-s}^{s} e^{\omega h} q_{s+\omega+1}},$$

where $m := r - \sigma^2/2 - \lambda_J(e^{\mu_J + \sigma_J^2/2} - 1)$, μ_J and σ_J are, respectively, the jump-size mean and standard deviation, and λ_J the jump intensity. The jump probabilities q are calculated as shown in Hilliard and Schwartz (2005, Eq. (9)).

A2. Model 2: one-dimensional diffusion models

In this class of models, the asset price dynamics under the risk neutral measure $\mathbb{P}$ is generally given by

$$dS(t) = \mu(S(t), t)\, dt + \sigma(S(t), t)\, dW(t).$$

This set of models includes, for example, exponential Ornstein–Uhlenbeck, Brennan–Schwartz, Cox–Ingersoll–Ross and the CEV models. For illustration, we focus on the CEV model, with $\mu(s, t) := rs$ and $\sigma(s,t) := \delta s^{\beta+1}$, $\delta > 0$ and $\beta \in \mathbb{R}$, under the measure $\mathbb{P}$. As proposed by Hilliard (2014), in order to have a computationally simple lattice with recombining nodes, the instantaneous volatility must be constant. This can be achieved using the transformation

$$X(t) = \frac{1}{-\beta\,\delta\,S(t)^{\beta}}. \tag{A.2}$$

In the special case of $\beta = 0$, $X = \ln S$. Then, the binomial tree for X is given by

$$X_n = X_0 + \sum_{j=1}^{n} \varkappa_j,$$

where

$$\varkappa_j := \begin{cases} \sqrt{\Delta}, & p = 1/\left(1 + e^{-2\,\tilde{\mu}(X_{j-1})\,\sqrt{\Delta}}\right) \\ -\sqrt{\Delta}, & 1-p \end{cases},$$

and

$$\tilde{\mu}(x) = -r\beta x + \frac{1}{2}\frac{\beta+1}{\beta}\frac{1}{x}$$

is the drift of X. Finally, the distribution of the log-returns of S is

$$\xi_j = \begin{cases} \ln\left(\phi(X_{j-1} + \sqrt{\Delta}) - \phi(X_{j-1})\right), & p \\ \ln\left(\phi(X_{j-1} - \sqrt{\Delta}) - \phi(X_{j-1})\right), & 1-p \end{cases},$$

which follows by inversion of the transformation (A.2), resulting in

$$\phi(x) := \frac{1}{(-\beta\,\delta\,x)^{1/\beta}}.$$

For more details, we refer the interested readers to Hilliard (2014).

A3. Model 3: stochastic volatility models

We consider an asset price process S defined by the following stochastic differential equations under the measure $\mathbb{P}$

$$\begin{cases} dS(t) = S(t)\,r dt + S(t)\,\sigma_S(V(t))\left(\rho\,dB(t) + \sqrt{1-\rho^2}dW(t)\right) \\ dV(t) = \mu_V(V(t))dt + \sigma_V(V(t))\,dB(t) \end{cases} \tag{A.3}$$

for independent Brownian motions B and W, and general functions $\sigma_S(s, v)$, $\mu_V(v)$ and $\sigma_V(v)$. For example, in the Heston stochastic volatility model

$$\sigma_S(v) := \sqrt{v},\;\; \mu_V(v) := k(\bar{v} - v),\;\; \sigma_V(v) := \eta\,\sqrt{v}. \tag{A.4}$$

In Section 4, we present a two-dimensional tree construction of model (A.3) based on Akyıldırım et al. (2014).

Appendix B. Forward start option

Occasionally, the underlying asset is monitored only during part of the lifetime of the option, i.e., the averaging is based only on prices of the underlying during a deferred time interval $[a\Delta, N\Delta]$, $0 < a < N$ (e.g., see Reynaerts et al., 2006). It is common to call this a *forward start* Asian option.

Our proposed pricing approach can be flexibly adapted to the case of delayed averaging. First, the following modification of the process Z in (3) is relevant

$$Z_j := \frac{\frac{1}{N-a+1}\sum_{n=a}^{j} S_n - K}{S_j} = \frac{Z_{j-1}}{e^{\xi_j}} + \frac{1}{N-a+1}, \quad a < j \le N,$$

$$Z_a := \frac{1}{N-a+1} - \frac{K}{S_a}.$$

Hence, by the tower property of expectations, the price of the option is given by

$$\mathbb{E}\left(S_N Z_N^+\right) = S_0 e^{rT}\bar{\mathbb{E}}\left[c\left(Y_a, \frac{1}{N-a+1} - \frac{K}{S_a}, a\right)\right].$$

Then for $0 \le j < a$ the recursion (9) becomes

$$c\left(y, \frac{1}{N-a+1} - \frac{K}{s}, j\right) = \sum_{i=1}^{d} \bar{p}_i(y)\,c\left(Y_{j+1}, \frac{1}{N-a+1} - \frac{K}{s\,x_i}, j+1\right).$$

Appendix C. Discretely monitored Asian option

In the case of a discretely monitored Asian option, i.e., when the average is based on prices of the underlying monitored at certain discrete time points during part or the entire lifetime of the option, it is necessary to introduce another scale in the problem: $b := N/\tilde{N}$, $b \in \mathbb{Z}^+$, where $\tilde{N}$ is the number of averaging points and N the number of time steps. By analogy, in addition to the time step size $\Delta = T/N$, define the time interval $\tilde{\Delta} = b\Delta$ between successive equidistant averaging points.

Consider

$$S_{bn} = S_0 e^{\sum_{j=0}^{bn} \xi_j} = S_0 e^{\sum_{j=0}^{n} \Xi_{b,j}},$$

where $\Xi_{b,0} := 0$ and the random variables $\{\Xi_{b,j}\}_{j=1}^{\tilde{N}}$ are i.i.d. with

$$\Xi_{b,j} := \sum_{l=b(j-1)+1}^{bj} \xi_l.$$

Define

$$\tilde{Z}_j = \frac{\frac{1}{\tilde{N}+1}\sum_{n=0}^{j} S_{bn} - K}{S_{bj}} = \frac{\tilde{Z}_{j-1}}{e^{\Xi_{b,j}}} + \frac{1}{\tilde{N}+1}, \quad 0 < j \le \tilde{N}.$$

The option value function (9) is now given by

$$c(\tilde{z}, j) = \sum_{k=0}^{b} \frac{b!}{k!\,(b-k)!}\bar{p}^k(1-\bar{p})^{b-k} c\left(\frac{\tilde{z}}{x_1^k x_2^{b-k}} + \frac{1}{\tilde{N}+1}, j+1\right)$$
$$\text{for } \tilde{z} \in \left(\tilde{z}_{L,j}, \tilde{z}_{U,j}\right), \quad 0 < j \le \tilde{N}.$$

Appendix D. Convergence theorems

Theorem 4 (Functional Central Limit Theorem (Billingsley, 1968, Theorem 16.1))**.** *Suppose that random variables x_j are i.i.d. with mean 0 and finite variance σ^2. Define the random function X_n in the space $D[0,1]$ of càdlàg processes as*

$$X_n(t) = \frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor nt \rfloor} x_j,$$

where $t \in [0,1]$. Then,

$$X_n \xrightarrow{d} W,$$

where the convergence is understood with respect to the Skorokhod topology on the space $D[0,1]$ and W is a Wiener measure on $D[0,1]$.

Theorem 5 (Integral Functional Convergence Theorem (Potscher, 2004, Lemma A.1))**.** *Suppose that the process $X_n(t)$ converges with respect to the Skorokhod topology on the space $D[0,1]$ to a Brownian motion $W(t)$ on [0,1]. Also, suppose that $J : \mathbb{R} \to \mathbb{R}$ is continuous. Then,*

$$\frac{1}{N}\sum_{n=1}^{N} J(X_n(1)) \xrightarrow{d} \int_0^1 J(W(t))dt.$$

Theorem 6 (Continuous Mapping Theorem (Billingsley, 1995, Theorem 29.2))**.** *Let Ω be the unit interval [0,1], $\mathcal{B}$ consist of the Borel sets* *in [0,1], and P be Lebesgue measure on $\mathcal{B}$, so that $(\Omega, \mathcal{B}, P)$ is a probability space. Suppose that X_n and X are random variables with values in $\mathbb{R}^n$ defined on $(\Omega, \mathcal{B}, P)$. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a measurable function and that the set of its discontinuities $D_f \subset \mathbb{R}^n$ is measurable. If $X_n \xrightarrow{d} X$ and $P(X \in D_f) = 0$, then $f(X_n) \xrightarrow{d} f(X)$.*

Theorem 7 (Convergence of mean (Billingsley, 1968, Theorem 5.4))**.** *Suppose that X_n and X are random variables defined on $(\Omega, \mathcal{B}, P)$. If $X_n \xrightarrow{d} X$ and X_n are uniformly integrable, then*

$$\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X).$$

Appendix E. Reference prices for Figs. 1, 2 and 3

Table E.9
The top panel of the table reports reference prices of European, continuously monitored Asian call options with fixed strikes K in the lognormal model computed using the double transform method of Fusai (2004). Model parameters: $\sigma \in \{0.05, 0.4\}$; additional parameters: $S_0 = 100$, $T = 1$ year, and $r = 0.09$ per annum. The bottom panel reports option prices in the continuous-time (cts.) Merton jump diffusion model. The price estimates are calculated by control variate Monte Carlo (CVMC) simulation with the maximum lower bound of Fusai and Kyriakou (2016), corresponding to the same option specification, used as control variate (CV cts. MLB), with standard errors (std. err.) also reported. Model parameters are from Hilliard and Schwartz (2005) : $\sigma \in \{\sqrt{0.05}, 0.05\}$, $\lambda_J = 5$, $\sigma_J = \sqrt{0.05}$, and $\mu_J = -\sigma_J^2/2$; additional parameters: $S_0 = 40$, $T = 1$ year, $r = 0.08$ per annum, and 100 Monte Carlo time steps.

K	σ	Fusai		K	σ	Fusai	
90	0.4	16.49997		90	0.05	13.37821	
95	0.4	13.51071		95	0.05	8.80885	
100	0.4	10.92377		100	0.05	4.30824	
105	0.4	8.72994		105	0.05	0.95839	
110	0.4	6.90349		110	0.05	0.05214	
K	σ	CVMC CV cts. MLB	Std. err. $\times 10^{-4}$	K	σ	CVMC CV cts. MLB	Std. err. $\times 10^{-4}$
30	$\sqrt{0.05}$	11.4808	2.392	30	0.05	11.3290	2.103
35	$\sqrt{0.05}$	8.0241	2.126	35	0.05	7.7162	1.835
40	$\sqrt{0.05}$	5.3866	2.100	40	0.05	4.9386	2.094
45	$\sqrt{0.05}$	3.5533	2.762	45	0.05	3.1063	2.342
50	$\sqrt{0.05}$	2.3458	2.861	50	0.05	1.9792	2.628

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