WORKING GROUP 8
Language and Mathematics

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LANGUAGE AND MATHEMATICS

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Introduction

This collection of papers on Language and Mathematics presented in Working Group 8 at CERME 4 demonstrates some of the complexity of the field and the multiplicity of theoretical approaches taken by researchers. Perhaps we should expect this to be the case, given the field's positioning at the intersection of disciplines of mathematics, education, linguistics, semiotics, psychology and sociology. Nevertheless, the depth of engagement with theoretical ideas and their application to the empirical field that we may observe in the various papers by both new and experienced researchers makes a serious contribution to developments in this area. In this introduction, we will attempt to map out the field, identifying the various areas of interest for mathematics education, the issues with which researchers must grapple and some of the questions that remain.

First it is necessary to elucidate the meanings of our title 'Language and Mathematics'. What is the relationship between language and mathematics (and we should probably add 'and mathematics education' to this list, as educational concerns certainly frame the understandings and the aims of all the authors represented here) or, rather, what are the relationships that we may consider? Without wanting to engage with the sterile argument about whether mathematics 'is' a language (Pimm, 1987), it is nevertheless unavoidable to recognise that doing mathematics at all levels involves making use of language and other sign systems. These include the specialist register or registers of mathematics (conceived and defined rather differently by Duval, 2000; and by Halliday, 1974) but in many circumstances also include what may be termed 'everyday' language. The research reported here starts from questions about the nature and use of those forms of language and other sign systems that are special to mathematics, but also about the roles that 'everyday' language interactions in the classroom play in students' mathematical experiences and learning.

An issue that was problematic in the discussions in the Working Group was the lack of a common, precise language for talking about language in mathematical contexts. The remit of the group was taken to encompass the use of any representational system for doing, learning or communicating mathematics. In its broadest sense, we have used

language to refer to any such form of representation. At the same time, however, we often want to be able to distinguish between the various semiotic systems and thus make a distinction between the language that we speak and write using words and, for example, algebraic notation. Further distinction becomes necessary when considering the differences between mathematical and non-mathematical uses of the same semiotic system. The need to make these distinctions has led to proliferation of expressions such as 'natural language', 'verbal language', 'everyday language', etc., used in ways that are not precisely defined. This problem has not yet been resolved; we hope that the reader will be able to make sense of the various uses of the word *language* in this introduction and in the papers that follow.

What is mathematical language?

Investigating the nature of mathematical language may appear a very abstract enterprise and it has sometimes actually been felt to be unrelated to the goal of designing and studying language-based practices in mathematics classrooms. This may be explained by the deep changes in attitudes towards language that have occurred over a short period of time among researchers and mathematics educators.

In the past, verbal language was generally regarded among mathematicians and mathematics educators as a sort of imperfect, imprecise and ambiguous version of the symbolic systems of mathematics. Learning problems were often addressed with reference to the symbolic systems of mathematics rather than to ordinary language. Difficulties with the interpretation of conditional statements, for example, which now are customarily studied from the perspective of verbal language, a few years ago were dealt with within the frame of symbolic logic, with little or no reference to other linguistic features or to the context, perhaps under the hidden assumption that truth-tables were the main road to creating meaning. In other words, syntax, truth-functional semantics and lexicon were regarded as the characteristic features of mathematical language. Under these assumptions, mathematical language became a very rigid model that people had to conform to rather than something that could be profitably used to do mathematics.

More recently, research on mathematical language has recognized that the crucial properties of mathematical language are related not only to syntax, formal semantics or lexicon, but to use as well. This view has been accompanied by the assumption that mathematical language cannot be thoroughly investigated without taking into account all the linguistic systems adopted in doing mathematics at any level, including written and spoken verbal language, symbolic notations, visual representations and even gestures. This assumption is now widely shared (at least implicitly) in the community of researchers in mathematics education, although it is put into practice in many different ways. One feature of more recent trends in research on mathematical language that

reflects this broader view is the interest in applying theory from, for example, socio-linguistics, semiotics, or pragmatics to mathematics. This trend is clearly represented among the papers in this collection, including in particular a focus on the ways that language (in its broad sense) functions within mathematical activity (also conceived broadly) rather than solely on its representational function.

The transition from a rather formal view of mathematical language to a more pragmatic one has not been a smooth one. For a number of reasons, the idea that 'natural language', with no further specification, is enough to carry out language-based classroom activities has grown popular. This has led to some neglect of explicit development of characteristically mathematical ways of creating meaning. Now a more balanced and fully developed perspective on the relationships between language and mathematics is needed. A basic goal for research is to understand what is specific in mathematical language, going beyond both the wrong ideas that the core of mathematical meaning (or, worse, mathematical truth) is embedded into symbolic expressions and that mathematical language has no specific features relevant from the viewpoint of education.

Among the papers in this collection, some address the issue of the specifically mathematical features of language more or less explicitly. For example, Guidoni *et al.* and Consogno both present analyses of written problem solutions produced by students, identifying ways in which the construction of the texts using various verbal, symbolic and graphical representations appears related to the types of solutions achieved. Morgan's critique of textbook definitions links the linguistic form of the texts to the messages they convey about the nature of mathematics. Starting with signs created by students in order to communicate mathematical ideas with one another, the papers by Schreiber and by Roubíček investigate the representational functions of these signs and their development in the course of mathematical activity. These two studies both draw on and develop aspects of Peircean triadic semiotics. However, there is still a long way to go in developing understanding of which aspects of the various semiotic systems used in doing, learning and teaching mathematics are critical to the construction of specifically mathematical meaning, especially with respect to the oral and informal communication in the classroom. This is needed to lead us to a better understanding of which kind of linguistic competence is appropriate for doing and learning mathematics. It should also provide us suggestions for the design of language-related practices apt to support mathematical thinking at each school level.

The focus on use should open new bridges between research on mathematical language and language based practices. Whereas conformity to a more or less formal linguistic pattern has proved a major obstacle to mathematics learning, a more pragmatic view should ease the interplay between research and practice. In this perspective, the great variety of approaches and of research topics that can be found in the papers presented

here may be regarded as a resource, and the discussions that followed the presentations of the papers in CERME Working Group 8 have supported the emergence of some common ground. To advance in the understanding of linguistic phenomena relevant to mathematics education, the opportunity to compare the studies with each other must be preserved and the proliferation of theories and vocabularies with no actual comparison must be avoided.

The social environment of/and learning

The wider field of research in mathematics education increasingly concerns itself with the social environment within which teaching and learning take place, not only in order to define the context and the domain of generalisability of the research but also, more radically, conceptualising learning itself as a socially situated and structured phenomenon (cf. the 'social turn' identified by Lerman, 2000). Influential in this trend have been the psychological theories of Vygotsky and his successors (in both East and West). Several aspects of these are prominent in current research about language and mathematics.

Interaction and learning

The notion that learning occurs first on the interpersonal and then on the intrapersonal plane has drawn attention to teacher-learner interaction and to groups of learners, both as pedagogic devices for supporting learning and also as objects of study. Thus Ozmanter & Monaghan study the notion of 'scaffolding' of learning by a teacher/researcher in a small group setting while using the Bakhtinian concept of 'voice' as a means of describing students' contributions to group problem solving. Adopting a neo-Vygotskian focus on processes within groups of learners, Edwards examines the occurrence of 'exploratory talk', a form of group collaboration hypothesised to contribute to cognitive change.

Semiotic mediation

In the cases described above, language tends to be seen as a vehicle for communication and as a means for the researcher to gain insight into the dynamics of learning. A stronger conceptualisation of the relationship between language and learning is evident in the use of theories of semiotic mediation. Vygotsky (1986) argues that the signs/ words/ linguistic tools available to an individual shape the meanings they will make. Again, this provides mathematics education with both a pedagogic device and a means for making sense of what occurs in mathematics classes. Douek's study of a primary classroom combines these two aspects. The task with which the children were faced provided them with a tool which, because it was not immediately suited to the task, forced them to adapt it, to form and to express mathematical generalisations. The university students studied by Consogno selected their own semiotic tools as they solved a problem; Consogno's analysis of two students' written solutions tracks how their

choices of different means of expressing their initial ideas appears to shape their subsequent approaches to the problem. The relationship between students' choice of language and other forms of representation and their mathematical problem solving is also explored by Guidoni *et al*. These studies provide evidence for strong relationships between semiotic tools and problem solving activity and mathematical thinking.

Discourse theory

The importance of social organisation of learning and of mathematics is emphasised further in the use of discourse theoretical approaches that make strong links between ways of using language and ways of 'being' (especially 'being mathematical'). This is seen in Morgan's critical discourse analytic critique of the ways textbooks present definitions to secondary school students and in Back's use of Wittgenstein's notions of 'language games' and 'forms of life' as a framework for analysing mathematics classrooms. Ways of being mathematical are linked with other significant ways of being in Partanen's sociolinguistic study of gendered participation in group work.

Methodological issues

As will be apparent from the discussions above, issues related to language arise from a number of different theoretical perspectives and within a variety of research contexts. It is thus not surprising to find a range of methodological approaches adopted by researchers in the area. In a relatively young field of research it is interesting to find that many of the papers also have a theory-building dimension: not simply applying existing frameworks to the empirical data but adapting and building analytic methods and theoretical ways of understanding the phenomena studied. This suggests a lack of established linguistic theory that is easily applicable to mathematical contexts; it may also reflect the exploratory nature of much of the research reported here, compared to the more established research of the articles analysed by Lerman et al. (2002), which predominantly applied theory to empirical data without modification of the theory.

All the papers presented in the Working Group involve some empirical data, most of which are in the form of transcripts of student talk, with or without a teacher. As in many other areas of mathematics education, however, there are choices to be made between studying talk within a naturalistic classroom setting and setting up 'experimental' settings in order to answer specific questions. Where data includes written texts, similar issues arise: are the texts used and produced in 'ordinary' classrooms adequate to address the research aims or must situations be created in which texts are produced? In the case of writing, the question of students' linguistic competence emerges particularly acutely, especially if we attempt to infer mathematical understanding from their written productions.

Innovative pedagogies

The limited nature of traditional classroom discourse, characterised by the I-R-F/E (Initiation-Response-Feedback/Evaluation) sequence (Mehan, 1979; Sinclair & Coulthard, 1975), is well known and it has been generally accepted as common sense that this form of discourse is unlikely to give rise to currently desirable forms of knowledge and understanding. More recent studies of teacher-student interaction in mathematics classrooms have identified similarly restrictive patterns of interaction, such as 'funnelling' (Bauersfeld, 1980), and linked these to procedural views of mathematics and of learning (Wood, 1998). It is perhaps unsurprising, therefore, that researchers interested in understanding relationships between language and learning mathematics should choose to study classroom situations in which the social norms and patterns of interaction are hypothesised to be more conducive to desired forms of learning. Much of the research reported here makes use of transcripts and/or students' written work produced in classrooms with innovative pedagogies. These pedagogies are generally theoretically based and, in some cases, very carefully designed with social and didactic aims, whether primarily intended as a curriculum development (as in Misfeldt's study of undergraduates working in collaborative groups or the "emancipatory learning environment" of Edwards' research) or with equally prominent research aims (as in the development of 'experience fields' as used in the primary classroom setting studied by Douek or the 'emblematic' tasks designed by Guidoni *et al*.). An important feature of such studies is that, as Consogno notes, the tasks in which students and teachers are engaged are "within the didactic contract". In other words, though the didactic setting itself may be experimental, from the point of view of the participants the setting is their everyday classroom and, from the research point of view, the data can be considered 'naturalistic'.

Non-interventionist research

An entirely non-interventionist approach to studying classroom discourse is seen in just two of the papers, both of which use their analyses in order to develop theoretical ways of describing and understanding classrooms. Schütte introduces the notion of "monolingual habitus" to explain apparent gaps in communication between teacher and students learning mathematics in their second (or third) language. Adopting Wittgenstein's notions of 'language games' and 'forms of life', Back uses transcripts of teacher-student interactions in several classrooms to develop and test a two-dimensional framework with social and mathematical components for describing how classroom talk may induct students into mathematical 'forms of life'. Morgan's analysis of extracts from textbooks may also be seen as using a non-interventionist sample, though in this case the classroom in which the textbooks might be used is absent.

The use of classroom data has important advantages for research related to language. Especially for those (many) researchers working within a theoretical framework that sees mathematics, teaching and learning as social in some sense, the social setting in which interactions take place is an important factor in interpreting the meanings and functions of the interactions. Nevertheless, unstructured classroom data also has certain disadvantages. Most significantly it is very messy and there is far too much of it. The researcher inevitably has to make choices about which parts to transcribe and then which parts to analyse at what level. For example, most lessons will include episodes in which teacher-student or student-student interactions are dealing with matters apparently unrelated to mathematics teaching and learning, yet these are seldom included in transcripts that appear in reports of research (however, see Zevenbergen's (1998) analysis of how such interactions relate to differential access to mathematics for students of different social groups). The bases of such selections are seldom made explicit, leaving open questions about the reliability and generalisability of the analyses.

Experimental settings

For some types of research question, classroom data may not be appropriate because there are too many things going on to allow the focus of interest to be visible to the researcher. An 'experimental' or 'laboratory'-based study may enable the researcher to design a constrained setting that makes visible the phenomena that are of interest. This is the approach taken by Ozmanter & Monaghan, studying 'scaffolding', and by Roubíček, who constructed a situation designed to elicit very explicit oral communication between pairs of students in order to be able to deduce the nature of their representations of mathematical objects. A similarly constrained setting, in this case mediating all communication between pairs through an "internet-chat" interface, is used by Schreiber to study the development of shared inscriptions as students collaborate in problem solving.

Analytic tools

The various types of questions addressed and the variety of data used make it unsurprising that the research reported here makes use of a range of approaches and analytic tools to make sense of written texts and transcripts of speech. These analyses address different levels, from interaction analyses of whole conversations to a focus on key words or other signs within a text.

In the space afforded by 10 pages for each paper, it is unfortunate but perhaps inevitable that the detailed methods of analysis and interpretation are often omitted or abbreviated in favour of providing a coherent theoretical framework and displaying the most interesting results. As a consequence, the reader may find that, while the interpretations of data offered by each author appear feasible and even persuasive, the tools are not always provided in a sufficiently explicit form to enable us to see the direct route from

data to interpretation. In order for the field of research to develop greater maturity, analytic tools need to be published in a form that is explicit enough to allow them to be applied by others. This would enable researchers to build productively on earlier work and to make meaningful evaluations and comparisons between the results of studies undertaken by different researchers in different contexts. The use of analytic tools from various branches of linguistics helps in this respect, though these are often inadequate without some adaptation or extension to allow them to be applied to mathematical contexts.

An issue that arose during discussions in the Working Group was the question of translation. Although the examples of data given in the papers had originally been collected in a number of different languages, it was all presented in English (apart from a small number of Finnish words used as 'plausibility shields', given in Partanen's paper). While translated data is likely to be sufficient to support the arguments and conclusions made in many other areas of mathematics education research, it presents some problems for the adequate communication of research into language. This is especially critical when there is detailed grammatical or lexical analysis of transcripts or written texts. There are strong similarities between the grammars of many European languages, but to what extent is it valid to draw conclusions from a grammatical analysis of a translated text? In reporting analyses of non-English texts in the context of an English-medium conference or publication, should the original language text and its analysis be provided – alone or with a translation?

Future directions

An important area for research in language and mathematics education that has not been substantially addressed by the papers in this collection is bilingual or multilingual learning and teaching of mathematics. Schütte's analysis of interaction in multilingual classrooms is the one exception, his interpretation indicating a teaching culture that appears alarmingly neglectful of linguistic issues. The study of multilingual classrooms clearly involves important research issues that go beyond purely linguistic ones. Moreover, the study of language in such classrooms is perhaps inevitably related to cultural and societal factors. We must consider it a step forward that the mathematics education research community no longer treats the problems faced by multilingual learners as solely a matter of language differences. Nevertheless, there is a good case for greater interaction between researchers whose primary focus is on cultural issues and those with a focus on language in order to develop our awareness of relevant theoretical perspectives and methodological tools across the field.

A related area that emerged during discussion of the papers during the conference was that of the development of learners' linguistic competences in mathematics. This has not been addressed directly by any of the research reported in this collection, though the

need to consider it is identified explicitly or implicitly by many of the studies. Significant steps are being made in describing forms of language that are appropriate for expressing mathematical ideas or for engaging in mathematical 'forms of life' and that function effectively for learners engaged in mathematical problem solving. These descriptions involve not only identification at a lexical level of vocabulary, notational or graphical elements but also the choice, combination and manipulation of these in texts that are functional in producing and/or communicating mathematics. However, the question of how learners may acquire knowledge of these forms and competence in using them remains largely unanswered. The study of this issue seems likely to require the development of longitudinal research methods in order to study the evolution of learners' language and relate this to the evolution of their mathematical thinking. In engaging in this area, we should also benefit from drawing upon the existing knowledge and methodologies of our colleagues in linguistics and language education.

Related to the question of the development of learners' linguistic competence is, of course, the question of teachers' awareness, knowledge and competence: awareness of the ways in which language is important in doing and learning mathematics; knowledge about the range of forms of language that may be used in the mathematics classroom, about the forms of language that may be effective for mathematical problem solving and learning, and about learners' language development; competence in using and developing practices that promote the use and development of mathematical language, including classroom organisation, task design and forms of interaction. The research in this collection offers some suggestions of relevant classroom practices but a challenge still remains for research and for teacher education and professional development.

It is evident that the research presented in this collection and the wider field of research related to language and mathematics is very diverse. Discussion within the Working Group identified cognitive, social, epistemological, historical, cultural and emotional dimensions with specific relevance to language in mathematics education, in addition to the linguistic and semiotic perspectives. These dimensions certainly co-exist and it seems likely that they interact significantly, yet individual researchers tend to focus on just one dimension at a time. This raises a number of problems for the field. Differences between research paradigms clearly lead to a lack of coherence in the field as a whole and a tendency towards horizontal rather than vertical development (that is, proliferation of alternative theories and interpretations rather than building upon and extending existing work). More importantly perhaps, such diversity may lead to misunderstanding and miscommunication between researchers. Not only may we fail to appreciate the subtle differences in the meanings intended by researchers who, apparently using the same language to report their work, are operating within different conceptual frameworks, but we must ask whether those working within different paradigms are even looking at the same objects. This is not necessarily an argument for uniformity or

convergence of approaches. Different paradigms can offer us different, yet equally interesting and relevant insights. It is, however, an argument for greater clarity about the assumptions underpinning our research in order to facilitate meaningful and productive communication between us.

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TALKING TO EACH OTHER: PUPILS AND TEACHERS IN PRIMARY MATHEMATICS CLASSROOMS

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Abstract: *The paper will outline the key findings from a small scale study of talk in primary mathematics classrooms. It will describe the vital and complex interrelationship between the social purposes and mathematical foci of the interactions and examine the characteristics of conversations that succeed in involving children in expressing and developing their mathematical thinking and understanding. A framework was developed which analyses talk in terms of its mathematical and social dimensions with the mathematical dimension varying from low to high and the social dimension varying from closed to open. The principal argument of the paper is an assertion that if talk contributes to the induction of pupils into mathematical thinking and reasoning then it must be highly mathematical and socially open.*

Keywords: primary mathematics; classroom talk; discourse analysis; mathematical thinking; reasoning

Introduction

The study was based on participant observation of a large number of lessons with a small number of teachers and classes from three schools situated in a market town within commuting distance of London. The analysis focused on the detailed study of the transcripts of five lessons. As such this constitutes a fine-grained analysis of talk in classrooms. Examples are presented to illustrate the application of the framework to transcript data.

Connections between social and mathematical aspects of talk

In my study I focused on the talk of the primary mathematics classroom and paid considerable attention to social and mathematical aspects of this talk. I based my analysis on ideas about language games and forms of life outlined by Ludwig Wittgenstein (1968). Wittgenstein's writings were presented in a series of short disconnected paragraphs so that his work is difficult to interpret and hard to find coherence in. I found Paul Ernest's (1998) exploration of the relationship between Wittgenstein's thinking and a constructivist approach to mathematics education helpful in developing my theoretical framework. Wittgenstein's philosophy was essentially radical, although it has now been adopted into the mainstream, and is social in its perspectives on language, meaning and necessity. Wittgenstein suggests that mathematics is based on social agreement about the rules and procedures that govern it and as such lacks the infallibility and objective truth often ascribed to it.

There are several different aspects of mathematics that can be addressed: doing mathematics, thinking mathematically and 'pieces of mathematics' or mathematical problems. Wittgenstein argues that all mathematics is socially based. 'Pieces of mathematics' or mathematical problems are socially based: they are constructed for people with other people in mind. Doing mathematics could be undertaken as an individual on one's own but it necessarily involves working within socially agreed practices. Thinking mathematically involves using one's own reasoning in ways that can be characterised as mathematical but again the dialogue with oneself involved in the thinking is essentially socially constructed.

Key concepts in Wittgenstein's philosophy are the notions of language games and forms of life and I have interpreted language games as the use of language together with the actions that are woven into it. I have taken a form of life to be an established human social practice, established within a community and involving its own purposes, rules and behaviours as well as its own special language games. This means that mathematics can be considered to be a form of life but may not be just one single form of life since it varies in its interpretation between different groups of mathematical practitioners. For instance the mathematics which is established as a form of life in a primary school classroom is likely to be very different from the mathematics established in a community of research mathematicians.

Mathematical forms of life are characterised by thinking and reasoning that emphasise exemplifying, specialising, changing, varying, altering, completing, deleting, correcting, generalising, conjecturing, comparing, sorting, organising, explaining, justifying, verifying, convincing and refuting about number, data, shape and space. These characteristics of mathematical forms of life are based on the work of Anne Watson and John Mason (1998). Mathematical forms of life also involve making connections between mathematical ideas and concepts in a variety of contexts as part of the process of generalising mathematically. This process of generalising comprises conscious mathematical thinking and reasoning and the development of mathematical argument and includes notions of proof.

I examined the language games of the classrooms in which I was working and identified a number of them in the talk of primary mathematics classrooms. These included the use of patter and templates and variations to the IRF (Initiation, Response, Feedback) sequence of interaction (Sinclair and Coulthard 1975). I also observed the importance of the listeners' interpretations of the talk in establishing discursive foci (Back 2001) and the important roles of symbols and metaphors in establishing mathematical meanings. I also explored the ways in which teachers engaged pupils with generalising and participating in mathematical reasoning and argument. All these findings were derived from qualitative analysis of the data which was subjected to rigorous scrutiny. The findings were triangulated with a group of researchers to ensure the authenticity of the analysis.

From these findings I concluded that both social and mathematical factors are key to the development of children's ability to participate in mathematical forms of life. At this stage I sought to find a framework to take account of mathematical and social aspects of the talk that would give some indication of the extent to which the talk might be enabling children to participate in mathematical forms of life. I will begin outlining my framework by considering a brief excerpt from a lesson with 8 and 9 year old pupils about the factors of large numbers in order to illustrate the qualities that the framework seeks to identify:

Example 1: Factors and Multiples: Lines 40 – 44: A factor of 132

40	T:	Right. Number one. Give me a factor of one hundred and thirty two then. Just give me one of the ones that you've chosen. Neil?
	N:	Two
	T:	How do you know that two is a factor of a hundred and thirty two, please Ryan?
	R:	Because a hundred and thirty two is an even number.

In this exchange the teacher asks two questions: firstly she asks for a factor of the given number and then she asks for a reason for choosing that factor. The two questions are very different. The first question is asking for a claim to be made whereas the second is asking for a reason why the claim is valid, or a warrant. It would have been possible to pursue this further and ask for a backing, or a further warrant, to support the first warrant but in this lesson the teacher rarely did so. The evidence from this excerpt of an argument based on mathematical reasons and justifications is strong.

This excerpt illustrates the different functions that questions can have in the talk of the classroom and shows how the teacher can be involved in helping to structure the argument through the course of the lesson. She is asking her pupils to think of reasons 'why?' in the case of each number offered as a factor rather than just to focus on the factor given as 'the answer'. Such an approach is indicative of intentions on the part of the teacher very different from an approach that focuses just on the right answers. This in turn implies the valuing of different forms of life. A teacher focusing on right answers and structuring her lesson on IRF sequences is not asking for engagement with mathematical reasoning to the same extent as this teacher. In offering feedback immediately after the answer is given, the emphasis is placed on the 'right answer' rather than any mathematical reasoning that might lie behind it. In contrast the adjustment to the normal IRF sequence which asks for reasons or justifications before offering feedback shifts the emphasis of this lesson onto mathematics that is focused on reasons and justifications rather than correct answers. By the same token it would be possible for the teacher to pursue the mathematical reasoning and ask for substantive backing to support the warrant. This would be indicative of an even stronger focus by the teacher on mathematical forms of life that reflect mathematical thinking and reasoning. In developing the dimensions as

outlined below, I am seeking to construct an analytic tool that helps to consider how these differences can be identified.

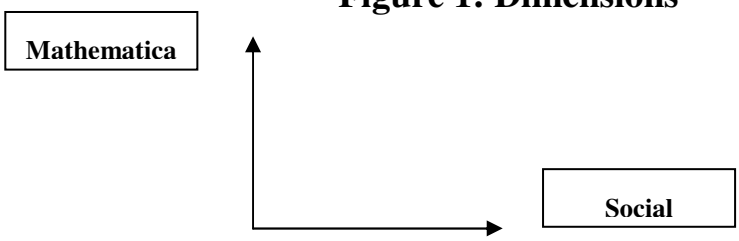
These findings suggest that the management of the talk of the classroom is complex as well as the talk itself and, in the management of the talk, social aspects play a crucial role. I suggest that the central language games of the primary classroom that contribute to pupils' participation in mathematical forms of life are those that involve generalising, reasoning and argument. I will use these ideas to develop a framework to help to analyse the social and mathematical components that contribute to this participation.

In devising this framework I am trying to raise the level of analysis to explore how the language games in mathematics classrooms might help to induct pupils into mathematical forms of life. There seem to be components of the social aspects of the interaction and also of the mathematical aspects of the interaction that can work in quite distinct ways, sometimes working against each other and sometimes in conjunction to facilitate the induction of pupils into mathematical thinking and expression.

Dimensions of classroom talk

My model suggests that every utterance in the talk can be analysed in relation to its social and mathematical components and I want to suggest that these components can be viewed as dimensions of the talk as illustrated by the following diagram:

Figure 1: Dimensions



The social dimension of talk is connected with building and maintaining the social relationships within the class, between teacher and pupils and between pupils. There is a sense in which all the talk is social: it involves social interaction between the participants. However I am interested in the contribution of the talk to the social contexts of the learning environment that the teacher and pupils are creating.

The mathematical dimension is concerned with the mathematical component of the talk and relates to the way in which the talk contributes to mathematical forms of life, particularly those that support mathematical thinking and reasoning. I am interested in the contribution of the talk to the mathematical contexts of the learning environment that the teacher and pupils are creating. I will now consider the dimensions in turn and suggest the development of a scale along each of them.

The social dimension

In separating the social from the mathematical, I am seeking to draw attention to the effects that different approaches to teaching and learning have on classroom talk and the ways in which the talk serves to induct pupils into mathematical forms of life. I hope that this will help to illuminate the ways in which social and mathematical components of the talk interact to facilitate or inhibit participation by the pupils in mathematical forms of life. The pedagogical relationships between teacher and pupils are built up over time through the course of the 'long conversation' that they share. The social dimension is affected by the contexts of the school as a whole. The personalities, both of the teacher and her pupils, will influence the social dimension. Some aspects of the social component of the talk are connected to organisational aspects of classroom phenomena.

I would suggest that the social dimension can vary from open to closed depending on the emphasis of the utterance in terms of its contribution to the social relationships within the class. Openness on the social dimension would suggest contributing to open relationships that encourage pupils and teachers to view themselves as joint participants in the learning and teaching processes. Closedness would be linked with rigid interpretations of the participants' involvement and force them to follow predetermined patterns of contribution to the talk.

Both the teacher and her pupils can and do have an effect on the nature of the social interaction that takes place in the classroom. In settings in which the social dimension is open, there are opportunities for teacher and pupils to negotiate the exchanges that take place. Pupils are able to ask questions and to challenge the assertions that are being made, even the teacher's assertions. From my transcripts I have a number of examples in which pupils take the initiative in negotiating a discussion or even possibly wrest control from the teacher. For example, Yaseen in the lesson described above about factors of large numbers contests the teacher's control of the talk. He even asks her what question is under discussion, which runs counter to standard interaction patterns, in which teachers ask questions and pupils supply answers. I would describe this context as open socially. However in the same lesson the teacher's control over the mathematics and what counted as mathematics in the talk was strong so that talk that is open socially may occur with different levels of control by the teacher of the mathematics under discussion. My conception of the dimensions is capable of analysing the social and mathematical components separately in order to explore the effect that one has on the other.

The mathematical dimension

The mathematical dimension can vary from low to high depending on the emphasis of the utterance in terms of its contribution to the mathematical contexts of the learning environment in the classroom. A high mathematical dimension would suggest that the utterance was closely linked with mathematical forms of life that take account of mathematical thinking and reasoning. A low mathematical dimension

would suggest little relationship to these forms of life and might possibly reflect an instrumental understanding of mathematics. The teacher may ask a question of a pupil that has a high or low mathematical component. 'What is three times four?' is a question with mathematical content but it has a limited or a low mathematical component. It might be described as simple recall. 'How would you find all the factors of twelve?' is a question with a higher mathematical component as it is seeking to elicit mathematical thinking and reasoning and has some element of problem solving. 'Do you have the answer to question eight?' is not a mathematical question at all and so has no mathematical component. However it is possible that the response to the question might be a statement with a high mathematical component and show evidence of mathematical thinking and reasoning as well as problem solving. Utterances with high mathematical components show evidence of mathematical thinking and reasoning and various other characteristics of mathematical forms of life.

The mathematical dimension of a sequence of utterances can be considered to be high when the teacher and pupils extend the mathematical component beyond the recall of procedures toward participation in mathematical argument, mathematical thinking and reasoning. This cannot be characterised on the basis of the teacher's questions alone but needs to take account of the responses of the pupils. The mathematical thinking and questioning needs to reflect the characteristics described earlier in order to be characterised as high level. These link closely with mathematical problem solving and conceptual learning. However I would also stress the importance of the social dimension of the process of inducting children into mathematical forms of life. In exploring this connection between social and mathematical I hope to enhance the understanding and analysis of classroom talk.

In considering the mathematical component of questions and statements some interpretation of the expectations of the person posing the question or making the statement needs to be made in terms of the response they intend to elicit. This also applies to considering the social dimension of utterances. There is a place in mathematics lessons for statements and questions with both high and low mathematical dimensions but I suggest that an exploration of the mathematical dimensions of utterances will be fruitful in exploring the induction of pupils into mathematical forms of life.

The mathematical dimension or social dimension could be different in the same utterance. For example the question in the first example quoted above: 'How do you know that two is a factor of a hundred and thirty two, please Ryan?' has a fairly high mathematical component as it asks for a mathematical justification for an answer. However it is closed socially because it requires a response from one nominated pupil who is expected to offer the justification in response to the question.

Dimensions and 'real' classrooms

It is clear that my identification of mathematical and social dimensions of classroom communication is not unique and other researchers have investigated their relationship. My findings differ from those of some researchers in that they were gathered from 'normal' classrooms rather than in an experimental setting. The perspectives offered by a number of different researchers have informed the development of my framework. These include Basil Bernstein's concept of framing(1971), Margaret Brown's analysis of levels of mathematical learning (1979)and Barbara Jaworski's ideas about the teaching triad of sensitivity to students, management of learning and mathematical challenge (1994). Further information about the links to the work of these researchers can be found in my thesis (Back 2004).

I feel that there is some gain to be made by separating the social and mathematical dimensions. This enables one to explore those strategies that are common across teaching and learning situations generally and those that are special to teaching and learning mathematics and that may be related to mathematical forms of life. In identifying the social and mathematical dimensions of the talk I hope to disentangle some of the complex issues involved in classroom talk and present them more clearly in the contexts of primary school mathematics classrooms generally.

The contribution of the framework to consideration of talk in classrooms

At this stage I am putting forward the suggestion that it is impossible to have a lesson in which the talk is closed socially but has a high mathematical component. This would lead to the conclusion, if it were true, that inducting pupils into forms of mathematical life that emphasise mathematical thinking and reasoning is dependent upon communication that is open socially and challenging mathematically. It would also imply that children need to be active participants in the discourse and engage in the mathematical tasks with interest if they are to be inducted into mathematical forms of life. I will now consider this suggestion in relation to the data that I have gathered in the course of my research.

To summarise the ideas behind the framework: I have focused on mathematical and social dimensions of the talk as reasonable key notions that are central to classroom talk. The mathematical dimension is considered to be high if the talk is strongly related to mathematical thinking and reasoning. This contrasts with talk with a low mathematical dimension that would be limited to recall of knowledge about facts or algorithms. By identifying talk that is high in the mathematical dimension and examining its social dimension, I will be able to explore the relationship between these two components. I am interested in talk that shows strong evidence of reasoning: answers from pupils that show mathematical thinking or questions from teachers that elicit or provoke thinking and reasoning. I will also examine the social dimension and look for evidence of talk that is open socially so that pupils are given opportunities to make contributions with some degree of autonomy.

Applying the framework to an episode from a lesson

In this section I will illustrate the value of the framework by applying it to an episode from a different lesson which I would suggest is a 'telling case'. The intention is to use this as an illustration of the potential of the framework. The following excerpt is taken from the lesson on 'Triangular Walls' which was a worksheet involving filling in missing numbers in a pattern that involved addition.

Example 2: Triangular Walls: Lines 189-200: The biggest number

	L:	Oh no! Oh no!
190	T:	Don't worry about these ones down here, these are **really** difficult.
	L:	Yes I'd need about a [thousand square
	T:	[A two hundred square I think!
	L:	A two hundred square, yes I would! (...)
	T:	The biggest number is one hundred and fifty.
195	J:	The biggest number in **this** is one hundred and fifty.
	(...)	
	T:	Just carry on till you finish.
	J:	The biggest number is a trillion
	T:	Is it? **Is it?** What about a trillion and one? That's bigger.
200	J:	I don't think about it! (laughter)

At this stage in the lesson the pupils were moving on to solving some triangular walls that involved larger numbers and were finding that their facility with arithmetic was being taxed. To start with they had been using rulers as number lines to help to add the numbers (they were not familiar with the empty number line). They went on to using a "hundred square". The question that Lenny was considering was the answer to 76 + 74 and he started by speculating about how big his number square would need to be to solve such a problem.

The exchange had elements of many of the characteristics that I explored in the course of my study. There was use of indexical expressions that were only clear to the participants from the contexts of which they were aware. The teacher revoiced the pupils' utterances and there was the development of an argument. Lenny seemed put off by the size of the numbers and suggested that he would need a bigger number square to cope: he suggested a thousand square but the teacher countered this by suggesting that a two hundred square would do. Lenny agreed. This involved a suggestion and a counter suggestion. The teacher then made a claim: the biggest number is 150 which another pupil, John, countered again by modifying the suggestion with the proviso 'in this'. The argument was then developed further between John and the teacher with a generalisation that took it beyond the limits of the task on which we were working.

I will now consider the mathematical and social dimensions. In the first utterance the pupil's 'Oh no oh no!' there was no explicit mathematical element although he was responding to concern about the difficult numbers. Throughout the excerpt there was

a sense of social banter between the teacher and the pupils and a sense of fun about what they were doing. This was evident from the tones of voice of the participants on the audio tape. I would like to suggest that an element of playfulness can play a key role in taking the pressure off children, when they are involved in mathematical thinking that is actually difficult for them. The utterance 'Oh no, oh no!' appears to show openness on the social dimension as the pupil was willing to express his anxiety. The teacher responded with 'Don't worry about these ones down here, these are **really** difficult' which might have served to acknowledge that the anxiety was real but that they could cope with it. This has a social dimension showing the teacher understanding her pupils' anxiety about the difficult numbers and closing down the anxiety. It also shows evidence of the teacher's sensitivity to her students and of awareness of the mathematical challenge of the activity. This awareness of the difficulty of the mathematics makes the utterance quite high on the mathematical dimension but the sensitivity about the potential anxiety also make it open on the social dimension. At the same time there is interplay between the mathematical and social which is important in terms of recognising the social and mathematical aspects involved in learning mathematics even though the mathematics remains implicit.

The next three lines go on to focus on mathematics and reveal a high mathematical content. The pupil started by wondering how big the number square would need to be to solve the problem. He suggested 'a thousand square' and the teacher suggested a 'two hundred square' as big enough for this problem. It would have been interesting to explore the pupil's reasons for suggesting a 'thousand square' but the opportunity to do so was ignored by the teacher. These three utterances are all high on the mathematical dimension as they are related to the mathematics in the task not just the procedures surrounding its completion: the discussion is about the relative sizes of numbers as well as the 'hundred square' tool. This relates back to the key elements of mathematical thinking and reasoning which are associated with the ability to generalise both within and between mathematical contexts and also to develop mathematical arguments involving conjectures, exemplifications, justifications and reasoning. In suggesting a thousand square Lenny was predicting the limit of the largest number that can be made with three digits and the teacher's comment restricted the discussion to the limit of the largest number that can be made by the addition of two two-digit numbers. The exchange is also fairly open as the pupil is not limited to a predictable answer to a set question. Not only are the utterances mathematical in content, the exchange also demands engagement with mathematical thinking from the child.

The social relationships in the excerpt are relaxed and the pupils are free to make unsolicited contributions in this small group of six pupils. The social openness in this example is shown in the ways in which the teacher makes suggestions and also in the response she receives from the pupils. The pupils make spontaneous contributions and comments about their work and are not restricted to answers to the teacher's questions. There is a high level of mathematics in the utterances. There are shifts in

the teacher's control of the exchanges both socially and mathematically which demonstrate flexibility rather than rigid structures. The pupils also exert control over the flow of the exchange and do so quite strongly.

This excerpt can only serve as an illustration of the quality of talk in this classroom and there is insufficient space in this paper to offer contrasting data. The evidence indicates that what the teacher brings to the talk of the classroom in terms of subject knowledge is equally important to their knowledge of the processes of coming to know mathematics. An open social setting seems to be essential for children's mathematical voices to be heard. If the children are exposed to highly mathematical talk but do not have the opportunity to express their mathematical thinking or voice their mathematical ideas they are unlikely to develop as participants in mathematical forms of life. If children are able to engage in discussion freely and express their ideas but are not exposed to talk that is highly mathematical they are again unlikely to become participants in mathematical forms of life. This lends further support to the hypothesis that inducting pupils into mathematical forms of life requires not only talk with a high mathematical dimension but talk that is open socially.

My thesis is therefore that classroom talk must be socially open if pupils are to be inducted into mathematical forms of life. However although this is a necessary condition it is not sufficient. The mathematical dimension of the talk must be high as well. Similarly talk with a high mathematical dimension is not in itself sufficient to ensure the induction of pupils into mathematical forms of life. The social dimension must be open as well. This suggests that there is a dialectical relationship between the mathematical and the social dimensions of the talk and that the successful induction of pupils into mathematical forms of life is dependent on talk that is both high in its mathematical dimension and open socially.

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THE SEMANTIC-TRANSFORMATIONAL FUNCTION OF WRITTEN VERBAL LANGUAGE IN MATHEMATICS

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Abstract: *This paper deals with the role of written verbal language[1] in the development of "creative" thinking in mathematical activity. The central idea discussed in this report is that written verbal language can play a semantic transformational function during the process of production/interpretation of a mathematical text: such dynamic may be interpreted as a two-way interplay between a subject and his/her written linguistic outcomes. One's awareness of content becomes enriched by the expansion of its linguistic expressions: such development suggests associative links that enable the transformation of the perception of content itself. This paper describes how the semantic-transformational function may work in the development of some of the creative and informal processes in mathematical activity.*

Keywords: creative thinking; interplay between thinking and language; semantic-transformational function; association of ideas; linguistic expression.

Introduction

The "creative" side of mathematical activity is highlighted by several mathematicians and mathematics educators (together with the rigour necessary for the achievement of products to be included in the mathematical knowledge system). A recurrent hypothesis (more or less explicit) in recent educational literature, especially in that of Vygotskian inspiration (see Lerman, 2001; Zack and Graves, 2001), is that written verbal language can play an important role in the implementation of creativity in the field of Mathematics. (See also Sfard, 2000.) The topic I deal with in this paper concerns the ways in which written verbal language intervenes within open mathematical problem solving (particularly, but not only, within the production and management of conjectures, towards the construction of the related proofs).

Starting from recent results concerning the role of verbal language in the construction of conjectures and proofs (Robotti, 2002) and the relationships between argumentation and proof (Boero, Douek & Ferrari 2002; Duval, 1995; Pedemonte, 2001), I have looked at written verbal language (alone or in synergy with the use of other languages) as a possible means of the implementation of creativity in that domain. Through analysis of some University students' texts produced during problem solving activities, I have progressively elaborated a research hypothesis

[1] By "written verbal language" I mean word language in its written form.

concerning the potential of written verbal language in performing creative thought within mathematical activity through the psycho-linguistic process that I have called "the Semantic Transformational Function (STF) of written verbal language". It consists in the action of one's written linguistic outcomes towards one's own cognitive processes, in the two-way dynamics of production/interpretation. The STF may be accomplished through different mechanisms belonging to the psycho-linguistic sphere, particularly thanks to the linguistic expansion carried out through the association of ideas on the basis of the key words of the text. The plausibility of this research hypothesis was derived through the analysis of written texts produced by students in very particular situations (in which the use of written verbal language in open problem solving activities had been set up in the didactic contract). Such analysis also suggested further improvements of the hypothesis (see Discussion).

Theoretical framework

The theoretical framework involves different contributions derived from psycho-linguistic and semiotic areas. In particular, Vygotskij (1934); Eco (1984); Plantin (1990) offer a wide range of perspectives for the analysis of language and its functions (respectively: the relationships between thought and language; the semiotic analysis of language; and the functioning of argumentation). Thus the framework develops into a sort of synthesis of those authors' complementary outlooks as usually interpreted, and the variety of views suggests important hints for a deeper understanding of the functions of verbal language in mathematical activities.

The central idea discussed in this report relies upon the Vygotskian hypothesis that verbal language does not have just an important role in communicating thoughts or as a meta-cognitive tool, but definitely intervenes in structuring thinking (see Vygotskij, 1991). In particular, I will consider its key role (when used in written form) in transforming the perception of content in one's thinking process during mathematical activities. This is what I have called the STF of written verbal language; the analysis of such function is aimed at better understanding language's potential in performing creative thinking and developing mathematical reasoning.

The STF-hypothesis states that when people write down the results of their thinking (a sentence, eventually accompanied with drawings, formulas, etc…) just one part of the potential of their thought is expressed with intent, the on-going conceptions that they are completely conscious of in that precise moment, and they communicate only their aware intentions. On stopping and reading the production again, they can perceive new interpretations, which are the result of the action of written language itself towards their thinking dynamics. So the text becomes a hint for the further development of the solving process as well. It is implicitly assumed that written verbal language is more powerful than oral language in performing the STF, when one communicates with oneself during the process of production/interpretation of his/her written text. Indeed it is in its written form that verbal language concretely becomes other than thinking, more independent from one's intentions and consciousness. (See the Discussion for a widening.)

The question is how written verbal language exercises its transformational power, and what concrete mechanisms allow accomplishment of the STF. I have identified some of the theoretical constructs, which may describe the cognitive processes that develop during the problem solving activity and how the STF intervenes in promoting them. I have also highlighted a possible track of the STF process.

The main theoretical constructs I propose to describe the STF process are:

- Linguistic expansion

i.e. a gradual transformation in the written verbal expressions, starting from particular elements of the text (key-words), whose result is a new formulation (for instance: more general; more complete; with different words carrying new meanings; etc.) that can suggest new interpretations or a wider comprehension of content. I have hypothesized that linguistic expansion may play the counterpart-role of the transformation of algebraic formulas. The substantial difference between the two processes is that everyday language facilitates the comprehension of the meaning beneath the formulas, whereas the algebraic transformations just develop the form of formulas, and formulas must be translated into verbal language to restore the contact with meaning.

Linguistic expansion is thus helpful in leading thought from the early stages to progressively more developed ones, in which one has a deeper and wider awareness of the content of one's linguistic outcomes. The idea is that there might be a sort of parallelism between the developmental stages of the written text and the developmental stages of thought. Linguistic expansion on the written outcomes thus suggests a wider perspective of understanding: the (external) transformations on the written text promote a transformation (an increase) in cognitive awareness of content.

- Association of ideas

that is, a psycho-linguistic process that guides the development of discourse, by means of personal links between cognitive and linguistic sides; in other words, it carries out the connection between the exterior[2] plane and the interior[3] one through a complex dynamic relationship with linguistic expansion: the association of ideas can be both the promoter of linguistic expansion and the result of such a process; what is more significant for the transformational power of written verbal language is the latter relation, as it shows the reflexivity of written linguistic outcomes on thinking. This new perspective, in which roles seem to be exchanged in the relationships between thought and language, is the core of the STF: thinking guides written linguistic production just to some extent; from then on, functions are inverted and what guides the development of thinking are the written verbal outcomes themselves

[2] By "exterior plane" I mean, in this context, the written text that one produces and interprets.

[3] By "interior plane" I mean, in this context, the pure mental one, when thinking has not been made explicit yet in a written form.

that suggest new associative links. This association process is not fixed or determinate, but can be strongly different from one person to another, and also for the same person at different stages of his/her knowledge development. This is one of the ways by which the STF, through the cooperation of the association of ideas and the use of written verbal language, can perform the creative side of mathematical activity. In this perspective, I have tried to revalue the role of association processes, considering them not as simple reflex mechanisms, but showing their importance as constructive and helpful steps in the evolution of mathematical reasoning.

- Key/concept-words

These are the linguistic tools that suggest individual intuitive links with other terms or concepts belonging to the individual's personal cultural background. They guide reasoning through associative mechanisms, which start on the basis of such key-words. One key-word can become a concept-word in different ways during the process of interpretation of the written text. According to Frege as interpreted by Arzarello, Bazzini & Chiappini (1995, pp. 120-122), more than one sense may correspond to a linguistic expression and more than one conceptual interpretation is possible (see the analysis of protocols for an example). Concept-words play a functional role in the development of mathematical thinking, as they evoke particular concepts of the knowledge system and guarantee the recall of theorems and specific properties of concepts that are useful for the progress of reasoning.

In synthesis, the process of the STF of verbal language can be summarized as follows: starting from key-words in the initial written statement one makes a discursive expansion on the linguistic plane that has repercussions in the psycho-linguistic sphere through a process of association of ideas; the final result is an actual transformation of the perception of content on the cognitive plane: after the result of such transformational process has been written down, a new cycle may start.

Methodology

The written texts that I have examined were produced by a sample of fourth year university students, who were preparing to become mathematics teachers. Such students worked under a one-year didactic contract characterized by rather unusual requests - like writing explicitly every passage, thought, and doubt in their solution strategies and processes (that is a non-standard situation at school or university, where students are usually asked to present just the formal passages and transformations, and the final result). I have chosen, among about fifty texts read, those in which I have found linguistic sequences most suitable for showing a possible track of the accomplishment of the STF. These texts are not the outcomes of a teaching experiment or a test planned to prove the validity of the research hypothesis, but rather concrete examples of how the hypothesized constructs might work. I have thus used these texts as a means of showing the potential inherent in the theoretical construction, and improving it (see the Discussion).

With this perspective, I have chosen to deal with problems from different areas (algebra, geometry, probability, applied mathematics) and analyse problem solving activities in which different skills and competencies were requested (making conjectures, proving, generalizing, modelling…). This choice allowed me to provide some evidence about the plausibility of my hypotheses - especially the relevance of the STF in enhancing creativity and the fact that the STF is a characteristic of cognitive processes and needs, thus independent from the particular mathematical content.

The texts have been analysed on the basis of criteria, guided by the research on those elements of speech (key-words) and mechanisms (association of ideas; linguistic expansion) that I have hypothesized as the actual promoters of the STF. From a methodological point of view, what shows the potential of the theoretical construction is the effective possibility of describing and explaining the examined reasonings through the use of such elements and mechanisms. In other words, the method consists in showing that the STF and its mechanisms and constructs successfully fit the written outcomes and thus may contribute to describe and interpret the actual dynamics of production/interpretation during creative reasoning. As a limit inherent in the work done, I must recall that, in general, it is rather narrow to classify a psycho-linguistic phenomenon just through written linguistic outcomes. Another difficulty is the impossibility of bypassing the problem of subjectivity in the analyses of behaviours, due to the necessary process of interpretation of written texts. (In our case, while linguistic expansion is effectively/objectively detectible in a written product, association processes are not; they can only be hypothesized to explain the progress of the reasoning that stands behind the written passages, and their actual presence is often likely but never certain.)

Analyses of behaviours

In most texts that I have examined, I detected possible signs of the STF through the identification of likely productive association processes in the problem solving strategies; in a few of them I found a "non-transformational" (ritual or purely descriptive) use of verbal language; in a small number written verbal language was nearly absent.

Under the specific didactic contract described above, written verbal language seems to emerge as a privileged tool for managing the complexity of the requested tasks and keeping a close contact between the internal plane of reasoning and the external plane of working and vice versa. In those texts in which I did not find a written verbal approach I observed more frequent difficulties for students in being successful in understanding and managing the problem situation, in particular the preparation of the subtle passage from argumentation to proof; this could be a first confirmation for the STF-hypothesis which gives written verbal language an effective role in the development of reasoning that passes through the interpretation of the produced text.

In order to exemplify concretely how the STF may work, I am going to analyse an excerpt from a representative written text related to an open problem solving activity, in which I could detect clear signs of the STF and a track of associative processes. The text of the problem is the following:

"Generalize the following property: "The sum of two consecutive odd numbers is divided by 4". Prove the property obtained through the generalization."

I report and focus on the key-steps of the construction of the conjecture:

The sum of two consecutive odd numbers is divided by 4.

From this true (and easy to be proved) statement I will try and prove that:

The sum of an even number of consecutive odd numbers is divided by 4.

Let's see if it works with 4 simple numbers:

1+3+5+7 = 16 ok…

(3+5+7+9, 9+11+13+15 …)

*even with other examples it seems it works…but I see something more: if I sum "two couples" of consecutive odd numbers I get a number which is divided by 8 = 4*2*

*Is it true that with 3 couples I get a number divided by 4*3 = 12?*

1+3+5+7+9+11…it seems it works!

⇒ I will try and prove this:

the sum of 2n (n are the couples) consecutive odd numbers is divided by 4n

Let's see in details a possible track of the associative processes that may be detected in the student's protocol:

The sum of two consecutive odd numbers is divided by 4.

From this true (and easy to be proved) statement I will try and prove that:

The sum of an even number of consecutive odd numbers is divided by 4.

Let's see if it works with 4 simple numbers:

1+3+5+7 = 16 ok…

(3+5+7+9, 9+11+13+15 …)

even with other examples it seems it works…

*"The sum of **two** consecutive odd numbers is divided by 4"*

Two is an even number

*"**The sum of an even number** of consecutive odd numbers is divided by 4"*

In this excerpt the student writes down the property to be generalized and analyses it; this starting point (the joint analysis of the text and the task) seems to suggest an association process in the student's cognitive dynamics: attention is focused just on the term "two" which becomes a key-word in the process of interpretation of the request, and thus read as one of the possible bonds to be broken in order to get a first possible generalization. The interpretation of the key-word "two" as a particular instance of "even" is led by the final aim of getting a generalization and accomplished through a mechanisms of linguistic expansion that makes the new sense ("even number") explicit; so the final goal is the motor and the expansion is the way of performing the shift from the word "two" to the concept "even number".

Considering the order of the steps in the protocol, we should note that the numerical exploration on examples follows the linguistic exploration and analysis; the student uses the examples a second time just to control the plausibility of his supposition.

So this first step leads the student to a first correct generalization without a passage to examples.

In the second part of the excerpt, an interplay between the use of verbal language and numerical examples emerges:

*…but I see something more: if I sum "two couples" of consecutive odd numbers I get a number which is divided by 8 = 4*2*

*Is it true that with 3 couples I get a number divided by 4*3 = 12?*

1+3+5+7+9+11…it seems it works!

⇒ I will try and prove this:

the sum of 2n (n are the couples) consecutive odd numbers is divided by 4n

The sum of an even number of numbers is the sum of couples of numbers

*"The sum of **one couple** of consecutive odd numbers is divided by **4×1**"*

Two = one couple; divisibility by 4 = 4×1

Four = two couples; divisibility by 8 = 4×2

*"**2 couples** ⇒ divided by **8 = 4×2**"*

*"The sum of an even number (**2n**, where n are the couples) of consecutive odd numbers is divided by **4n**, that is 4 times the number of couples"*

The passage from the first step of exploration to full understanding and awareness of the relationships between the particular property and the general one begins on the

basis of the linguistic development of the former outcomes; at this point, the text effectively becomes the tool for understanding through its words and symbols: the expression "an even number of odd numbers", together with the numerical examples, have an important reflexivity in the direction of thinking dynamics, revealing functional features of the internal structure of the property of the sum of consecutive odd numbers. The numerical exploration induces the choice of the word "couples", which replaces the idea of "sum of even numbers". In its turn, the choice of this word seems to lead to think in terms of couples and to suggest a new interpretation of the examples. The further reformulation in terms of "couples", suggested by the analysis of the results of the numerical exploration of examples, exploits the linguistic expansion which acts on the cognitive dynamics and allows structural links to be made between the particular statement and the general one.

This analysis makes me suppose that, in general, transformations on the text are accomplished through a sort of dialogue between different kinds of transformational powers: the semantic-transformational function by means of written verbal language, and the transformational function of algebraic language (in this case); see the "Discussion" for details.

The last step in the process of generalization is the result of a process of associative links and explicit written linguistic outcomes.

The strategy followed by the student in building the generalization seems to confirm the constructive role of the association processes, and gives an idea about the fact that informal and creative components may represent useful counterparts of rational thinking. The excerpt also suggests a refining of the STF hypothesis in the direction of a possible dialogue between verbal language and other symbolic systems.

The expansion processes related to key-words are not only oriented by the task, but also strictly connected with individual cognitive dynamics (there can be different successful associative processes); just to give an idea about this, let's briefly consider a second text, focusing on the differences in the two associative dynamics that seem to accomplish the STF:

"The sum of two consecutive odd numbers is divided by 4.

$2n+1;\ 2(n+1)+1$

$2n+1+\ 2(n+1)+1=2n+1+2n+2+1=4n+4=4(n+4)$

Now I can pass from 2 consecutive odd numbers to 3 consecutive odd numbers:

$2n+1+2(n+1)+1+2(n+2)+1=6n+9=3(2n+3)$

Then from 3 to 4:

$2n+1+2(n+1)+1+2(n+2)+1+2(n+3)+1=8n+16=8(n+2)$

Then from 4 to 5:

2n+1+2(n+1)+1+2(n+2)+1+2(n+3)+1+2(n+4)+1=10n+25=5(2n+5)

So, if I sum an odd number, a, of consecutive odd numbers I find out that the sum is divided by a; if I sum an even number, b, of consecutive odd numbers, I find out that the sum is divided by 2b"

The word "two" represents for this student a key-word that gives the hint for a numerical exploration. The associative link is not: "two → even number" as in the previous text examined, but the chain is "two, then three, then four, then five… then odd numbers and even numbers". So this student's likely process of association of ideas, developed on the basis of the word "two" interpreted as a key-word, might be the following:

*The sum of **two** consecutive odd numbers is divided by 4*

↓ numerical examples

Three, four, five…

↓ recognition of different behaviours of even and odd numbers

Two, four three, five

↓ generalization

*The sum of an **odd** number, **a**, of consecutive odd numbers is divided by **a**; the sum of an **even** number, **b**, of consecutive odd numbers is divided by **2b**.*

This was a simple example to show how different the processes that lead to the same product (the conjecture) may be; in this sense, personal approaches become important means to the accomplishment of creativity.

Discussion

I have tried to show how written verbal language may help to develop creative thinking. In the sample of written outcomes that I have examined, I have found some evidence of the relevance of the STF in cognitive processes.

Further research is needed both on the theoretical and the experimental side, specially to find limits or possible obstacles to the STF.

The functioning of the association of ideas (particularly the possible interplay between different registers in the dynamics between verbal language, formulas, schemas, etc.) should be investigated. The outcomes examined suggest that everyday language and algebraic language act in synergy for the development of reasoning; the relationships between the different transformational functions of verbal and algebraic languages should be a crucial issue for further research: it would be interesting to examine how verbal language intervenes in restoring semantic awareness after an algebraic transformation on formulas, which occurs on a pure syntactic plane (during which the meaning seems to be suspended).

From a methodological point of view what must still be done is to find out more precise and objective criteria for analyses of written linguistic products, in order to bypass the problem of subjectivity in the interpretation of the thinking dynamics that stand behind the written product. Moreover the criteria of evidence used in examining the sample of texts chosen is not suitable for analysing protocols of students who do not work under the same kind of didactic contract.

Another important question arising during this research is whether the STF can also work in the case that verbal language is not written but oral. The idea is that it might: it is likely that oral language can play a transformational function when one communicates with others. In that case, the transformation may arise through the shift from the communicated meaning to the one interpreted by the other interlocutors (the latter can be, and very often actually is, different from the former). A general methodological problem is how to detect the plausibility objectively and test the validity of these hypotheses. One possibility for dealing with both questions could be the analysis of the strategies in a situation of interaction, in which the person who produces and interprets the solving strategies is induced to communicate both with him/herself and with others (eventually the researcher).

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THE ROLE OF LANGUAGE IN THE RELATION BETWEEN THEORISATION AND THE EXPERIENCE OF ACTIVITY

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Abstract:*In the expert's development of mathematical knowledge, we can recognise phases in which knowledge, procedures and competencies are organised into a theoretical construction. We wish to describe, from a developmental point of view, elements of such an evolution at its early stages, and to illuminate the role of various language and semiotic activities to favour similar movements between "activity" and "theorisation" in primary school, in the experience field's didactical setting. The description of the dynamic is inspired by our interpretation of the Vygotskian dialectic of scientific knowledge/ everyday knowledge; we will use it both to clarify what we intend by "theorisation" and as a model to describe the dynamic.*

Keywords: experience fields; didactics; activity; theorisation; Vygotsky

Introduction

Theory and theoretical thinking are difficult for students to access, especially in mathematics (see Boero, Douek, Ferrari, 2002). Because they are part of mathematical activity and structure, we need to better understand the nature of transition from what could be described as common knowledge and ordinary thinking processes, to theories and theoretical thinking. The aim is to discuss a way to introduce very young students to some of their relevant aspects through the analysis of an example. We assume that such a transition implies a complex and demanding cultural and cognitive evolution. In order to frame the transition, we will mainly consider the Vygotskian dialectic of scientific knowledge/everyday knowledge as a model to detect some features of the movements of theorisation in student's behaviour, which can become object of intentional mediation by the teacher. After a description of "activity" and "theorisation", we will consider a didactical sequence where some early steps of transition to theorisation were accomplished, and we will put the "signs" of theorisation movements into evidence. Their linguistic and semiotic aspects will be analysed, both as signs and means of theorisation. Conclusion will concern the means and didactical choices that favoured such movements.

Our interpretation of the vygoskian dialectic of scientific knowledge/ everyday knowledge

We will interpret Vygotsky's description of "scientific concepts" and "everyday concepts" as describing a tension between tendencies rather than determined static objects. Hence, concerning school uses, we will rather speak of a "scientific use of a

concept" and an "everyday use of a concept". For further elaboration about this issue, see Douek (2003).

Vygotsky (1985, Ch. 6) characterises scientific concepts by the fact that they are consciously, voluntarily and intentionally used. They are explicitly handled to allow some aimed action or result. They are *used as* related to systems of concepts and are somehow general. It is expected that they concern delimited objects in a rigorous manner, so that a definition can represent them. Everyday concepts are heavy with the child's rich personal experience. But the systems they are related to remain generally unconscious. Children develop them spontaneously in relation to their experience in their cultural environment. Their use may be only locally meaningful, and there is generally no need to define them. According to Vygotsky, in school context, the teacher introduces "scientific concepts" using the "everyday" ones to build their meaning, but he may also develop the latter ones. The uses of concepts evolve in school, and particularly their systemic links, their explicitability and their generality. The systemic and conscious characteristic of scientific concepts are not given from outside children's sphere of concepts, their mediation needs, in fact, the existence of rich enough everyday concepts in this sphere. The non-conscious character of everyday concepts is not due to children's egocentrism, but to the non-systematisation of the spontaneous concepts. Thus, the criteria to distinguish scientific from everyday uses of concepts are:

- The extent to which the child develops awareness and explicitness concerning the objects of conceptualisation he deals with through various activities;
- The way the use of concepts evolves in relation to the subject's knowledge (in terms of generality and conscious, systemic links with other concepts).

By elaborating these criteria, I will try to provide some keys to describe "theorisation" and detect signs of its development at early stages in school context.

Activity/ theorisation dynamic

Mathematical problem solving in the experience field's didactical setting (see Boero, 1994) relies on various developments of student activity, including a "reality"[1] component in most cases. By observing students, I realised that, from the early grades, they can produce something that in this paper I will try to present (within a suitable theoretical framework) as a "theoretical elaboration" related to their activity. Such observation led me to question the theoretical elaboration developed within wider experienced activity, and the conditions that favour such a dynamic. This dynamic involves evolution in the use of concepts.

[1] We will avoid defining "reality" and will only use its familiar meaning.

"Theorisation"

A description (useful as a first perspective, but not operational) is that by theorising one tends to build a self-consistent system of propositions, held by rules of rationality (e.g. inner coherence, derivation from explicit assumptions, etc.). In order to follow the early evolution of students' skills involved in theorising, we will refer to Vygotsky's elaboration concerning scientific concepts, particularly the idea of generalisation. Indeed, theorisation (as usually considered) implies the development of a scientific use of concepts. We need to detect early movements towards theorisation (and related inherent skills) in young students' school practice, and to recognise evolution towards theorisation out of a given activity. Inspired by Vygotsky's description of the scientific use of concepts, various cognitive processes can characterise movements towards theorisation. From an epistemological perspective, they all have in common a "surpassing" movement:

I. A detachment or a distance taken from the particular object of study by considering only some aspects of it, and/or producing a schematic representation of it;

II. A "transversal" change of the object of study: a new situation is recognised as similar. This means that some characteristics have been detached from the first one, and they become independent of the first particular case. Balacheff's generic examples imply such potential theorisation: *the child gives an example to explain. He is aware that he could have chosen another one. He uses the particular case to explain a phenomenon that he implicitly considers general.* Analogies too are transversal changes, as well as generalisation to a wider set. A change of context or a transfer of rules to another system are also transversal movements. An extension of a systemic construction of a structure or a representation system on a wider domain, is as much horizontal as vertical (see the following type).

III. A "vertical" change of object of interest, as R. Douady's tool/object dialectic describes it. In this case, the "theorisation" becomes explicit: *a property of a concept or a procedure is elaborated as a tool to solve a particular problem, but then the property or the procedure itself is singled out independently of its late context to become an object of study.* An explicit reference to a systemic construction or to a theory to which corresponds the particular case at study or with which one finds analogies. A general description is also a vertical change. Systematisation of representation or evolution of the system of representation (typical of category theory), as well as an elaboration of a systemic construction, a structure or a representation system, are vertical changes.

IV. A speculative reflection following shared rules of rationality and aims : comparing procedures, discussing the limits of validity of a given procedure, or conjecturing and drawing conclusions.

In order to both recognise the processes in students' activities and favour their development in the school context, it is useful to point out some of their semiotic features: verbal precision in descriptions and/or schematic representations and/or

symbolic representations are means to perform the first type of process; the use of specific expressions is a means to express and recognise the second *("for instance if you take 3…"* is a typical expression to introduce a generic example, *"it is as when…"* may express analogies); specific terms and/or symbolic representations are needed to make explicit the "object" that becomes an object of study in the third type; specific linguistic constructs are needed to develop reflexive process *("the first method is more suitable…", "this method does not work if (such condition…)", "if I imagine to push…, then (such consequence)…"*). A change of semiotic system ("registers" in Duval, 1999) can be a means for the considered processes to take place, as well as a sign that they are taking place.

Finally, we assume (following Vygotsky's description of scientific concepts) that theorisation needs students' awareness that they overstep the particular situation and go beyond it in a critical way to approach some generality. If not, links between the particular case and the more general (or theoretical) are not likely to develop, nor systemic links. Pupils would have no means to return to it consciously (for interpretation, for instance). Therefore, scientific use of concept(s) would not develop either. And most probably it will not be possible to speak of theorisation for such students from the cognitive point of view.

"Activity", in relation to possible involved "theorisation"

Activities, in the primary school context, can be more or less practical, as growing plants, simulating to buy sweets, drawing, investigating family history, etc. They can be performed through experiments, descriptions, organisation of data, conjecturing, argumentation. Under teacher's guidance, they can involve processes like those considered in the preceding subsection.

An example : in order to buy sweets (or simulate doing it in the classroom) one needs to combine coins to pay the correct price and not confuse the quantity of coins with their total value. In first grade, reflecting on the conventional value of a coin and the combining procedures can be steps towards theorisation, given the expression of the procedures and the degree of generality. For instance, concerning procedures, a statement such as "*to pay this 4c. sweet, I have to choose coins such as they make 4c. I can give 1 and 1 and 1 and 1 or I can give 1 and 1 and 2"* is a movement towards theorisation of the activity of paying with the correct coins for a 6 years child. We note that he states, indirectly, that there are many ways to combine numbers for a given sum.

General activity by itself is not sufficient to allow the development of explicit knowledge, which can often be compared to vertical theorisation. As was argued in Douek & Scali (2000), language activity and, more particularly, argumentation play a crucial role for this evolution to take place. *A student can develop a strategy to solve a problem that may implicitly use a property, or she/he can use a concept in a situation that illustrates it in a meaningful way. Such experiences are not sufficient for the student to become aware of (and stabilize) an operational aspect of their*

activity, nor to be able to recognise a similar situation as meaningful for the same concept. Linguistic and semiotic developments concerning activity are needed to favour this development and to allow them to become available as references and allow their conscious and voluntary use. Argumentation, appropriately oriented by the teacher, favours scientific use of concepts, in particular appropriate use of the available systemic links, or establishment of new links and, more specifically, theorisation movements.

The educational setting and the methodology

As a member of the Genoa university research group leaded by P. Boero, I followed Ezio Scali and Nicoletta Sibona's class in Piossasco (near Torino, Italy) for a few years. My aim was to analyse teaching/learning situations in the experience fields didactical setting: students work systematically for a rather long time (up to 100 hours in two years) on a single subject familiar to students (for instance: sun shadows). Under the teacher's guidance and mediation, they progressively develop knowledge about that subject, and develop mathematical concepts needed to deal with it. Mathematical subjects (as far as they become familiar to students) can become subjects of investigation for them. Individual, written productions (if necessary, supported by the teacher through 1-1 interaction) functionally alternate with classroom discussions orchestrated by the teacher. Concerning the cognitive relevance of individual, written productions, see Duval (1999).

I observed several didactical sequences belonging to various experience fields. In the case reported here, I worked on the transcripts of all the oral interactions (1-1 student – teacher interactions and classroom discussions) that took place, and on students' individual written productions.

Mathematical problem solving, in the experience field's context, develops with an intentional theoretical dimension on teacher's side. As we will see in the example, the teacher enhances the theoretical dimension of the activity by suitable tasks and appropriate mediation. I will try to highlight some movements towards theorising that I have detected by analysing students' behaviours (as well as some levers that favoured them) according to the processes that, in my view, characterise theorising. I considered actions and the semiotic features of oral and written productions to track both their effects on the movements of theorisation (spontaneously initiated by the pupil or favoured by the teacher) and the theorisation they reflect.

An example of a problem solving activity in the plant culture experience field

Measuring the Height of a Plant in a Pot with a Ruler

We shall analyse a five phase class sequence of a problem-solving activity. Second year students (7-8 years old) were studying wheat plants growing, and had already measured plants taken off the ground in a field. Now they had to follow the increase over time of the heights of plants in the classroom pot. The problem was posed as

follows: "*we need to record the heights of the wheat plants in the pot. Here is a ruler. How would you measure their height?"*

Rulers do not have the zero at the edge. The teacher did not allow pushing them into the earth (pretending to avoid harming the roots, though the didactical aim was to deal with a complex additive problem). The question was meant to lead children to find a general solution (not concerning a specific plant). It was expected that they could use the idea of translating the numbers written on the rulers ("translation solution": measure is invariable through translation). They could also use the idea of reading the number at the end of the plant, then add the measure of the length between the edge of the ruler and zero ("additive solution", measure is additive).

1) The first phase of the problem solving was an individual discussion guided by the teacher. At its end the students had to express the result they arrived at, dictate it to the teacher, then copy down their produced text. Teacher's argumentation had as effects to help the student to:

- focus on the problem;
- consider that the procedure has to be general, not measuring a particular plant;
- realise that reading the number at the top of a plant does not give its measure, as it usually did, thus separate measuring activity from the conceptualisation of length.
- imagine a new procedure, and, given the constraints the teacher put up, a fictional one, thus a theoretical solution.
- transform the status of the ruler from a tool to measure, to an objet to be measured or transformed: the object disappears behind its structure.
- change the meaning of the measure: The known practical procedure (identified with measuring till then) is questioned so that measure gradually acquires additive properties that were only familiar with numbers. This will imply an extension of their mastery of the conceptual field of additive structures (Vergnaud, 1990).
- make explicit some of the systemic links involving the concept of measure.
- reconstruct the whole reasoning. The teacher discussed the problem of representing students' own ideas.

The teacher systematically pays attention to student's verbal expression and guides the discussion so that it remains coherent enough. This attitude favours students' awareness of their past measuring activities, as well as a mental representation of the problem situation, and the movements of theorisation in response to adequate requests.

Here are two of the produced texts:

Rita's solution: "*In order to measure the plant we could imagine that the numbers slide along the ruler, that the zero goes to the edge, one goes where zero was, two goes where one was, and so on. When I read the measure of the plant I must*

remember that the numbers have slid: if the ruler gives 20 cm, I must consider the number coming after 20, namely 21".

Alessia's solution: "*We put the ruler where the plant is and read the number on the ruler, which corresponds to the height of the plant, and then add a small piece, that is the piece between the edge of the ruler and zero. But before we must measure that piece behind zero*."

2) The second phase took place next day. The teacher presented a photocopy of Rita's "translation" solution and Alessia's "additive" solution. The task was an individual written production to say whose solution was like one's own, and why. One of its aims was to provide all students (including, possibly, those who had not reached the solution) with an idea about possible solutions.

3) The third phase followed immediately and was a classroom discussion. The teacher worked at the blackboard and the students worked in their copybooks (with photocopies of a drawn pot with a plant) using a paper ruler similar to the teacher's. They effectively put into practice (through schemas) the two proposed solutions, first the "translation" then the "additive solution". Meanwhile they discussed several problematic points that emerged, for example the fact that the "translation" procedure was easy to perform only in the case of a length (between zero and the edge of the ruler) of 1 cm (or eventually 2 cm), while the "additive" solution was a easy to use in any case. They also discussed the interpretation of the equivalence of the results of two solutions (*"why do we get the same results?"*).

4) The fourth phase was an individual written production where students had to *"explain why Rita's method works, and explain why Alessia's method works"*.

With three exceptions (unclear presentation, revealing nevertheless an "operational" mastery of the two procedures) all the students produced the demanded explanations. Half of them commented on the comparison of the two methods, and often explained in clear terms the limitation of the "translation" solution.

5) The fifth phase was a collective dictation of a synthesis to the teacher, to be copied on students' copybooks. The quality of the majority of students' individual productions allowed the production of an exhaustive synthesis.

Some movements towards theorisation in relation to semiotic and linguistic features in the example

In phase one discussion, linguistic activity was the means the teacher used to help the student grasp the generality of the posed problem, and, as a matter of fact, introduce her/him to generality. At the same time, it was intended to root the reasoning in "reality", which means to enrich, dialectically the representation of the pragmatic situation. The teacher could rely on the visible variety of plants (the pot was there) to make clear the necessary surpassing reflection. When a student was unable to consider the general aspect of the problem, the teacher said something like *"we will*

choose the plant to be measured all together later on, but now tell me how you would do it". He referred to the possible practice and at the same time postponed it.

The setting combining teacher's argumentation, referring to "reality" and to past measuring practice, with the inhibition of practical solution, played an important role in the movements towards generalisation:

- It enabled students to conjecture trials and means of verification. For instance, when the teacher replies : *"do you think that if we measure the way you say, we would measure the whole of the plant?"*, he drives the student to verify his proposition, by considering the procedure and the known "reality", and its possible effects, without returning to the practical activity he was used to. The student is guided towards a theoretical reasoning of type IV.

- It favoured "virtual experiments". Some students decided to imagine that they push the ruler in the earth (they were not allowed to do it). More argumentation brought them to more virtual solutions like shifting the graduation to get the zero (and no more the whole ruler) at the level of the earth. The teacher encouraged them to express their ideas as conjectures, and to imagine their consequences, approaching again the fourth type of theorisation. He obtained propositions such as *"I will pretend that the numbers on the ruler slide"… " so when I will read 11 at the top of the plant, I will know that the plant measures 12"*. The expression of this virtual solution enhances its systemic character, extending additive properties to measure; the example also works as a generic one – second type of theorisation. This movement goes with a change of register. After describing pushing the ruler in the earth in natural language, some students describe the sliding of the graduation, then gradually move towards numerical relations, using the arithmetic-symbolic register. At that stage, they perceived the common system functioning for measure and for numbers (that were not yet identified with measure). This extension reflects another aspect of the second type of theorisation.

During the collective discussion (the third phase) semiotic activities were essential both to root reasoning in experienced activity, and to surpass it towards theorisation.

Variety of registers and changing from one to another reflected movements of theorisation and stimulated new reasoning, developing a theoretical solution of the problem. The solution discussed in terms of shifts reflected a systematisation (type I as well as III of theorisation) attained by most students. As a conclusion of a precise description such as *"zero goes to the ground, 1 takes the place of zero…etc…"* a student said *"that makes +1",* introducing addition to the whole classroom, and the arithmetic-symbolic register (already familiar to the students, but not in this specific context). This change came naturally with the change of level of reasoning. It then inspired another student who proposed the schematisation of the additive operator with an arrow, turning to another register. This reflects theorisation of type two, since he transferred this schematisation from the experience field of thermometer by analogy, surpassing the context. And this was not just resemblance: in the

thermometer context, the arrow represented the movement of the red line of mercury along the number line, but in this case it represents shifts of graduation. The system the schema represents becomes valid in a wider context. The arrow schema (exploited through an important linguistic activity of description, explanation, and argumentation) in its turn allowed another student to create a new virtual procedure that would have been impossible if she only relied on "reality". She said: *"it is as if Alessia turned the plant upside down and transported the little unmeasured bit on its top… as if the plant was hanging … and this way it becomes easy to measure all of it"*. She separates the length from the plant, and proceeds to its partition to deal with the different segments abstractly. She somehow illustrates the theoretical procedure developed by her mates, as if considering it from the point of view of the third type (vertical).

Conclusion about some levers of the dynamic activity/ theorisation

Referring to characteristic processes inherent in theorisation, we present, in a synthetic and general way, some levers of the dynamic activity/ theorisation and some didactical choices that can favour it which emerged from the analysis of our example.

The particular use of context and decontextualisation*:* We can speak of decontextualisation when the solution of a problem is expressed with reference to a wider context than the original one (horizontal theorisation) or a more systematised one (vertical). In general, in the experience fields setting, the problem-situations are contextualised in a familiar environment belonging to the experience field. Usually discussions, observations, etc. have already been developed before the problem is posed in order to further increase familiarity. The expected solution generally involves some decontextualisation. The context helps the students support their reflection and reasoning on an activity they already master, and for which they do not need to practice anymore to understand the meaning of the problem. The example shows how students used their experience of measuring and the available reality of the plant pot to develop virtual solutions and gradually approach some aspects of theoretical reasoning. Schematization is also an instance of decontextualisation, and (as we have seen in our example) can be a step in theorisation (first or third type). It allows a system of representation to be put into light while using it. From the didactical point of view we observe that this "game" of contextualisation and decontextualisation is favoured by the variety of registers put into relation together, as in verbal description of representational conventions.

Virtual situations: as in hypothetical reasoning, one imagines that some conditions hold and builds further reasoning or draws consequences, or imagines some actions on the situation, given these conditions. In our example, it was favoured by the combination of argumentative activity referring to experienced activity and inhibition of practical activity. Various language and semiotic activities (i.e. descriptions and schemas) replaced practical activity. The virtual situation gave rise to general,

conscious and explicit (i.e. "theoretical") solutions and their validation. This illustrates the fourth type of theorisation.

"Speculative discussions": Another situation favouring theoretical reflection is to justify or criticise a conjecture, reasoning or procedure produced by some other subject. This took place in the comparison of Alessia's and Rita's solution.

And last, but not least, teacher mediation played an important role as a means to ensure the above mentioned levers of theorisation to function.

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EXPLORATORY TALK IN PEER GROUPS – EXPLORING THE ZONE OF PROXIMAL DEVELOPMENT

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Abstract: *This paper reports on a study which examined the occurrence of 'exploratory talk', as defined by Barnes (1976) and Mercer (1995), amongst peers in collaborative small groups in secondary school mathematics classrooms (11 to 16 years) in a UK school. This form of talk is thought to contribute to mathematical reasoning. The classroom learning environment in which the study was undertaken is based on sociocultural theories of learning and emancipatory pedagogic practices. The study was undertaken in naturalistic settings during the normal activity of each of the classes. Findings support the neo-Vygotskian view of social dialogic amongst peers as a means of generating talk which culminates in cognitive change.*

Keywords: collaborative groups; cooperative groups; friendship groups; exploratory talk; zone of proximal development.

Collaborative groups

The focus of this research is learning in *collaborative* small groups. Much of the research into *cooperative* learning has not made the necessary distinction between cooperative and collaborative, indeed many studies interchange the terms. Since the distinction between collaborative groupings and cooperative groupings is rarely made, little has been reported about a range of issues such as how the composition and dynamics of *collaborative* groups affect their ability to function effectively in relation to cognition. Studies that do so include that reported by Cobb and Bauersfeld (1995) and Barnes (1998). For the purposes of this study, *collaborative* learning is defined as that which is constructed amongst student peers working together in self-selected groups. The *process* involved in mathematical endeavour is considered as important a focus to the group as the end outcome. This contrasts with *cooperative* learning in which participants are assigned, or take on, particular roles within the group and work mutually towards an outcome.

Research on talk in peer groups

The benefits to learning of working in groups have been known for some time. In 1981 an influential meta-analysis by Johnson *et al* of more than 120 research studies indicated that group work in learning situations was considerably more effective than competitive or individualistic goal structures. In a comprehensive review, Good, Mulryan and McCaslin (1992, p167) describe "clear and compelling evidence that small group work can facilitate student achievement as well as more favourable attitudes towards peers and subject matter". They advocate a future focus for research on the socially situated learning which occurs in small groups. These authors argue that research on small groups has gone beyond a need to justify its overall benefit

through improved learning outcomes. They emphasise the need for work on the factors which affect discourse *processes* as well as factors which affect achievement outcomes. Research has suggested that the composition of the groups and the form of tasks the groups tackle are important factors in determining the quality of learning achieved through such group work (Barnes & Todd, 1977; Cohen, 1994).

Problem-solving tasks appear to provide a productive forum for generating mathematically effective talk in small groups (see, for example, Gooding and Stacey 1993, Mulryan 1995, Pirie 1991, and Whicker, Nunnery and Bol 1997). This remains problematic, though, despite an apparent similarity in approach. Problem-solving can take a variety of forms. In all the studies cited, the 'problems' consisted of closed activities. Such use of closed problems is more reminiscent of psycholinguistic analyses of children solving arithmetic word problems than an investigation of socially constructed knowledge.

Much of the research on peer talk in classrooms has been undertaken with young children (three to eleven year olds). Most of the curriculum contexts studied are not mathematical, though there are a few significant examples of the study of mathematical talk (for example, Cobb and Bauersfeld *op cit*, Lyle 1996). Other studies, such as Maher (1991), are undertaken outside of naturalistic classroom environments and therefore raise questions about the applicability of the findings for secondary classrooms in the UK. Studies of peer talk in secondary *mathematics* classrooms are particularly rare, the most influential of these being Pirie (*op cit*).

Longitudinal studies of small group work for longer than a few months are rare in classroom research, though such studies are more common in research on cooperative work at computers, usually with a pair of students rather than a larger group (see, for example, Hoyles and Sutherland 1989). This research on small group activity in the classroom provides evidence of the need for studies in a naturalistic setting at secondary school level reflecting the use of more open activities for problem-solving and a longer time scale for group interaction. It is in such settings that an examination of the necessity for a 'more learned other' can be undertaken.

The zone of proximal development

Vygotsky (1978) described the social construction of knowledge within a 'zone of proximal development'. This is defined as "the distance between the actual developmental level as determined by independent problem-solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers" (p86). Thus, in a classroom situation, the actual developmental level can be determined by traditional question-response-evaluation sequences and, therefore, described. The potential development can only be explained rather than described because it is a process observed in relation to working with others. Achieving the potential is usually described in relation to a 'more learned other'. Some of the findings from research on small group talk

challenge the need for this 'more learned other' (see, for example, Lyle *op cit*, Wegerif 1998).

Much of the theoretical basis for a pedagogic approach using small group work in classrooms comes from the sociocultural, neo-Vygotskian field. Collaborative group work (and research in this field), in which students work jointly on the same problem, is linked with ideas such as situated cognition, scaffolding, and the zone of proximal development. As Coles (1995, p165) describes "The social interactions developed in this kind of enquiry stimulate members of the group to think together; from a psychological point of view this pushes forward the level of thinking of each child and 'scaffolds' his or her cognitive processes". Although a Vygotskian view of learning encompasses a broad spectrum from social institutions and cultural influences to group interactions and individual cognition, it focuses on the individual outcome via an interpersonal process. Neo-Vygotskians (for example, Mercer and Fisher 1997, Wegerif *op cit*) have shifted this focus to an understanding of the process of learning within groups of individuals in specific social contexts. The focus here is on the interpersonal relations and their effect on intrapersonal learning to achieve a group objective. These new units of analysis support a means of interacting which involves the whole self and a view of the interactions of a group as a means of cognitive development. The basis for this approach is reasoning as a dialogical activity. The shift is from a Vygotskian framework of self-identity to a neo-Vygotskian assumption of intersubjectivity. This change in focus demands a new methodology - one that moves from description to explanation. The 'exploratory talk' evidence from the study described here, is used to identify this dialogic as it appears in classrooms.

Evidence from sociocultural research

Barnes and Todd (*op cit*) performed an in-depth qualitative analysis of group discussions amongst 13 year olds in the classroom. The standpoint of these researchers was that teachers, rather than learners, traditionally do most of the talking in classrooms, taking "responsibility for the content, pacing, and style of pupil contributions" (p ix). Believing that the teacher does not have to be present for learning to take place, they argued that "children are underestimated", and that "they possess skills and competencies which are rarely called upon in a conventional classroom" (p ix). They hoped to prove that when students work in small groups, without the aid of an authoritative adult, they could take responsibility for knowledge gained and management of the group, because they needed to make judgments, monitor situations, resolve conflicts, and cope with the opinions of others.

In analysing the dialogue amongst the groups of students, Barnes and Todd considered types of speech and their impact on the construction of meaning during group interactions. This necessitated an analysis of both the *social* and *cognitive* functions of conversation. They proposed a system describing speech acts that has been useful subsequently in the analysis of talk sequences. This system is based on

two levels. Level one consists of a) discourse moves (such as initiating, eliciting, extending and responding) and b) logical processes (such as proposing a cause, advancing evidence, negating, suggesting a method, evaluating). Level two is comprised of a) social skills (such as supportive behaviour, competition and conflict), b) cognitive strategies (such as setting up hypotheses, constructing new questions), and c) reflexivity (such as monitoring one's own speech, evaluating one's own and others' performance). They identified 'exploratory' speech characteristics such as hesitation and changes of direction, tentativeness in voice intonation, assertions and questions made as hypotheses rather than direct assertions, invitations to modify or surmise, and self-monitoring and reflexivity. Barnes and Todd proposed conditions for *collaborative* work amongst groups in classrooms, based on this empirical evidence. Further analysis (Barnes and Todd, 1995) provides descriptive examples of the "... four categories of collaborative moves: initiating, eliciting, extending and qualifying".

Several authors have suggested that children's facility in collaboration may relate to the social structure of particular classrooms that do or do not support collaborative interaction. For example, Forman and McPhail (1993) speculated that fourth graders' difficulty in collaboration on mathematical problems may have been because their traditional classrooms provide little support for engagement in the sort of dialogue involved in collaboratively solving problems. It is important to note that the emancipatory classroom environment, in which the children in the study reported in this paper were immersed, provided a setting which allowed collaboration to occur and for it to be controlled and monitored by the groups themselves.

Following ten years experience of supporting collaborative group work in primary school classrooms, Lyle (*op cit*) studied classroom organisation and task structure related to the use of small group activity and the composition of small groups. Working in a theoretical perspective of a Vygotskian 'zone of proximal development', Lyle challenged the necessity for a 'more learned other' and cited a group of four boys studied as evidence that cognitive growth can occur amongst participants of equal status. This is borne out in the study reported here.

The study

Students in this study (Edwards, 2003) attended an inner-city girls' comprehensive secondary school (11–16 year-olds) of approximately 1070 students in the south of England. The school population represented a very wide social mix, with the majority of students of white background, though there was a significant minority of 22 per cent Asian students and a total non-white ethnic minority of 25 per cent. Students in the classes studied experienced an open-ended problem-solving mathematics curriculum. The sociocultural and emancipatory learning environment involved students taking considerable responsibility for their mathematics learning. Small group organisation within classes was on a self-selection (usually friendship) basis. Classes undertook normal mathematical activity throughout the study. At two points

over a period of a year, small group talk was audio-recorded for all of the sessions relating to a particular problem-solving activity for each class. This involved from three to seven consecutive lessons for each class. One class from Year 7 (11-12 years), one class from Year 8 (12-13 years), two classes from Year 9 (13-14 years) and one class from Year 10 (14-15 years) were studied. Transcripts of lessons were made following the completion of the problem-solving activity. These were analysed, in conjunction with the audiotapes, for evidence of reasoning activity, or 'exploratory talk'. Episodes which represented evidence of shifts in conceptual understanding were identified.

Findings

To place the following findings in context, I present an example of a transcript for a Year 9 (13-14 years) group of five girls, recorded during the first of seven lessons investigating the logarithmic scale through an open-ended activity. This episode is taken at approximately 30 minutes from the start of their work during which time they have generated some data. At this point, the group is drawing on previous knowledge about patterns in differences between numbers and attempt to reconcile this knowledge with the evidence they have in their data.

	S	Maybe it's because, you know, the differences are getting smaller, maybe they've got so small they're actually the same
301	**K**	Yeah, I know that
	S	Do you know what I mean ... cos here the differences ...
	K	S ...
	S	… between 16 and 16 .. 32 ... ahhhh .. hang on, that's 16, 16, 32, that repeats itself and then you've got another table 48 and 64, what do they belong to?
306	**C**	They're all times 8
	S	6 times 8, yeah?
	K	Hey, hang on S. I say, S. Put a star by the repeat pattern of 16 and 32, cos they're coming up mostly every column, and every C number, do you get what I mean, every C section they're coming in. I'm just going to put a star by some of the C numbers
312		[shuffling for 20 seconds]
	K	S, S, listen
	C	shhhh [to the others]
315	**K**	I just went through, yeah, and ...
	C	shhhh [to the others]
	K	Every C number which has got a 16 and a 32 in it, they come really close together
319	**S**	Yeah
	K	Cos they're next to each other, cos you know they've got a gap in between here ...
	S	Yeah
	K	You can work it out, do you know what I mean?
	S	Yeah, Yeah
325	**K**	I think it's because you half that number or you put double that one
	J	And plus she's right
	S	She's more right than anyone else

	P	OK, so if you had 48, that's 24 .... then you've got C24
	S	I've got C24
330	**J**	458
	P	Oh yeah, they are actually
	K	See!
	C	It's only because they're getting so close together
	K	It's not, its not, cos look,.....458 and C ... no, you haven't got C23, so …

This episode appears to be mainly an interchange between **S** and **K**, but, in fact, all five group members are very much involved. The initiation made by **S** (lines 299-300) is first taken up by **K** and, during the discussion, there is a shift in influence between these two participants. **S**'s musings and questioning (lines 304-5) prompt **K** to follow her line of reasoning. She attempts to address **S** directly on several occasions (lines 303, 308, 313) and needs **C**'s support (lines 314 and 316) to gain the attention of the rest of the group who have lapsed into inaudible muttering. **S** accepts the direct address (indicated by the intervening acknowledgements to **K**) but the other three members in the group remain involved and supportive of **K**'s explanation. Both **P** and **J** indicate an acceptance of **K**'s explanation (lines 326, 328, 330 and 331). However, **C** provides a challenge to her explanation (line 333) and **K** attempts to justify her explanation by example. She finds that she is unable to do so because her evidence relies on having data for a prime number (not able to be generated). **C**'s reasoning may be moving her towards the idea of a mathematical limit.

All the groups studied demonstrated such evidence of exploratory talk, though to varying degrees. There was a direct relationship between the length of time groups had worked together and the amount of exploratory talk identified. Similarly, the length of time a group had experienced a sociocultural and emancipatory learning environment had a direct relationship to the amount of exploratory talk evident. The class which had experienced the pedagogy for the longest period of time (Year 10) demonstrated the highest levels of exploratory talk activity.

Some groups demonstrated a 'follow-on' means of connecting everyone's talk. The metaphor of a thread traced through this discussion comes to mind. This method served to keep everyone engaged with the task and perhaps served as a means of maintaining cognitive cohesion. Even groups which exhibited little exploratory talk during a lesson, had a 'way of working' together that was positive and evident in the way interactions occurred. For the Year 10 group (14-15 year olds), who had worked together for almost two years, findings indicate that a 'way of working' based on co-constructed 'norms' had evolved. This enabled each member to function in an atmosphere of trust and a familiarity of 'unwritten rules'. Similar patterns emerged in different ways for different groups.

A Year 9 group (13-14 year olds) used strategies of 'holding back, supporting affective or emotional aspects of learning and an acceptance of 'talking aloud' as a 'way of working'. Another Year 9 group demonstrated 'polite turn-taking' as a means of working together. The variation in modes of developing what Yackel (1995) calls

"sociomathematical norms" reflects the variety of ways each group used to engage with mathematics learning and the subsequent maintenance of the group's progress in this.

Much of the research on peer interaction in small groups has focused on giving and receiving explanations in relation to student achievement. Webb (1991) provides a review of such studies, some of the findings of which are supported by the analysis of episodes in this study. One of the less productive classes, in terms of developing exploratory talk, is the Year 7 class (11-12 year olds). In one episode, a peer tutoring relationship develops between two students in which one student gave answers without explanation. Webb (*op cit*) found that received help was most effective when accompanied by an elaborated explanation rather than just a given answer. The existence of such peer tutoring relationships in some groups may limit the opportunities for exploratory talk to develop between members of a group. In these cases, the role of the teacher may be to intervene to encourage explanation. In contrast, other groups elaborated constructively on their explanations, generating an improved cognitive learning environment. However, Webb's description of elaborated explanations does not encompass the socially constructed knowledge evident in the episodes in this study. This further highlights the differences between Webb's review of cooperative learning and the *collaborative* learning explored in this study and raises questions about the difference between action-performance outcomes in cooperative group studies and interaction-cognitive development outcomes in collaborative group studies.

In an analysis of talk within small groups in a Year 10 class solving closed problems related to finding the equation of a graph, Barnes (1999) identified exploratory talk in the transcripts. She was observing students working on closed, closely defined problems and it is not made clear whether the learning environment was sociocultural. The lesson fitted a more traditional model of teacher exposition followed by student activity during which students were expected to work in small groups on the assigned problems. This situation contrasts with the study described here, in which problem-solving groups are the normal mode of working and learning. The lack of commonality between the categorisation of particular episodes of talk as 'exploratory' in Barnes' study and that described here may support Cohen's (*op cit*) findings. She identified the need for interactions amongst group members to be more critical with a more mutual exchange of ideas and speculations if *conceptual* learning is to take place. In the episodes analysed in the study reported here, the conceptual shifts are evident in some of the exploratory talk described and indicate a higher level of reasoned thinking than that described by Barnes (*op cit*). This may be directly related to the difference in openness of the respective tasks and the consequent opportunities offered for conceptual learning. However, another factor is the familiarity of group members with group work as a mediator for learning and, more specifically, in a sociocultural learning environment. This comparative evidence is not available in the description of Barnes' study.

Discussion

Sociocultural models of learning are promoted through collaborative groups, the use of open-ended activities for learning situations and an encouragement of active participation in learning. The longer the experience of these modes of learning and the longer the students work as a group, the greater the authority students have over their learning. Episodes of 'exploratory talk' in this study provide evidence that cognitive growth can happen within collaborative groups without the presence of a particular 'more learned other'. This raises questions about the model which necessitates such a person in the learning context and how this is reflected amongst group members.

One of the factors in this study which separates it from almost all other studies of small group work in mathematics education is the study of self-selecting groups on the basis of friendship. I propose that this factor impinges on other factors already discussed – length of time engaged with the pedagogy, length of time working together and establishing sociomathematical 'ways of working'. The findings in this study support those of Zajac and Hartup (1997) who found that friends were better co-learners than non-friends. They suggest reasons for this include the fact that knowing each other well means that they know their similarities and differences, so that suggestions, explanations and criticisms are more likely to be more appropriately directed to each other. Their mutual commitment generates particular expectations which support collaborative means of working. They feel more secure with friends so become more active in novel problem-solving situations.

This sense of trust is also supported by Wegerif (*op cit*) who claims that trust can be conceptualised as a prop for cognitive development. Being able to trust others facilitates being able to take the risks involved in learning new concepts. These findings and those from the study reported here contradict other evidence which suggests that group composition needs to be altered regularly for effective working relationships. The findings also support evidence that group members do not like having membership of a group pre-assigned by others.

Forman and Cazden (1985) found that partners can require several sessions to develop an effective problem solving style. Forman and McPhail (*op cit*) highlighted the need for students to develop joint perspectives over time to achieve shared goals and Laborde (1994) found that one of the factors of effective small group learning was the time the members of the group had worked together. Friendship groupings appear to negate the necessity for teaching group skills and accelerate the rate at which effective reasoning can develop (Edwards, 2004).

A feature common to all the groups studied is the extent to which group members 'talk aloud'. There may be many reasons for this. One possible reason is that it acts as another level of cohesion for the group, enabling thinking to become public knowledge so that the group's *thinking* is bound together. If so, it may be evidence for shared cognition in which knowledge is co-constructed through socially shared

images, experiences, and, in this case, acts. Gooding and Stacey (*op cit*) similarly found more heightened levels of 'talking aloud' than other studies on small group interactions. They suggest reasons for this may include the level of difficulty of the task. More difficult tasks generate a higher level of 'talking aloud' (which they term 'thinking aloud').

If Vygotsky's model of a 'zone of proximal development' is to accommodate the evidence from this study, it needs to be redefined as a social space as well as a cognitive space. This social space would encompass the learning environment, the specific learning context (the task), the affective and emotive attributes of learning and the dialogic (Wegerif *op cit*) which binds them together in socially constructed knowledge. Friendship groups, in particular, support dialogical reasoning, which is based on differences and challenge, because of the assumed level of trust among participants. Friendship groups also explain high levels of 'exploratory talk', in which participants question, hypothesise, challenge, explain and justify, because the assumed basis of this type of talk is the complete acceptance of the offered statement in the spirit of moving thinking and learning forward.

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MULTIMODAL LANGUAGE STRATEGIES ACTIVATED BY STUDENTS IN UNDERSTANDING AND COMMUNICATING MATHEMATICS

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Abstract: *Many language-based theories are currently being employed extensively to analyze students' and teachers' mathematical behaviours. In this framework, on the basis of a long term research experience in classrooms and teacher-training, we claim that when students are free to select their own "linguistic" tools for specific goals (e.g. to understand or to communicate), the functions of different language components vary considerably and their effectiveness increases, with a particular role always played by the figural component. Our evidence is supported by the analysis of a protocol from one of the emblematic classroom activities according to which we structure our teaching strategies at all levels. Finally, starting from the observed cognitive behaviours of our students, we argue for the necessity of refining and enlarging the commonly used theoretical-interpretative settings.*

Keywords: mathematical language; representation: cognitive dynamics

Introduction

It is widely accepted in teaching/learning research that a very *general linguistic metaphor* can be quite helpful to discuss mathematical activities at school[i], mainly in order to characterize interventions and interpret correlated difficulties and errors. Several *points of view* can contribute to articulating and particularizing potential meanings for such a metaphor: from Vygotskij-inspired stressing of the support of natural language to (actual or possible, concrete or mental) actions, to the rooting of culture-driven cognitive processes on sensory-motor ones as stressed by *embodied-cognition* (Lakoff & Núñez, 2000); from neuroscience's initial experimental evidences of deep interference between number/space specialized brain sub-areas and linguistic ones (Dehaene, 1997), to the accent put (Duval, 1995; Radford, 2000) on the semiotic processes always active within mathematical performances; and so on.

It therefore appears essential to analyze carefully, from a largely linguistic perspective, both the resources students bring into their learning activities and the criteria able to shape our teaching strategies most effectively. This is what we have been doing for several years, on all content planes and at both levels of our teaching-research activities: classroom research, to optimize the transmission criteria of

productive *mathematical thinking*, and university-course research, to share such criteria with future or active teachers.

In the next section we discuss a few aspects of a *linguistic framework* for mathematics teaching, concentrating on some issues raised by two authors, Sfard and Ferrari, which offer us a good opportunity to focus our point of view. Both theories are well integrated with recent proposals from *embodied cognition* supporters, as well as with ones which refer to the social nature of learning. Then we briefly describe some cognitive behaviours of a group of future-elementary-school teachers, involved in one of our *emblematic teaching activities*. In conclusion, we briefly comment, starting from an analysis of these behaviours, about the need to specify and articulate such a linguistic framework better, in order to improve its interpretation potentialities.

The theoretical landscape and definition of the research problem

In the "discursive" research framework proposed by Sfard (2000), mathematics knowledge is seen as an aspect of culture-driven, discursive human activities. We recall some basic assumptions and issues in her work as a starting point for our discussion. Re-assuming the point of view expressed in a phrase by the French philosopher De Condillac, "Une science bien traitée n'est que une langue bien faite", Sfard proposes to regard different human knowledge fields, among which mathematics, as different discourses – all with their own specific characteristics, structures and goals. The term "discourse" is intended in a very general linguistic sense: it includes not only syntactic/semantic aspects, namely propositions and rules describing the content and structure of a text considered in itself, but also, at a pragmatic level, communication rules peculiar to any particular specialized (action driving) discourse.

In particular, in mathematical discourse one can identify two kinds of rules: *object-level rules,* which govern the *content* of the discourse, i.e. *mathematical entities and their relations*, and *meta-discursive rules,* which frame and support, almost always implicitly, the structure of the action-communication processes.

> ...Meta-rules are only implicitly present in discourses, and their learning occurs spontaneously, without being deliberately planned by teachers, without being intended by the students, and without being consciously considered by anybody. And yet, these invisible rules are responsible not only for the ways we do things, but also for the very fact that we are able to do them at all. (Sfard, 2000, p.161)

This way the evolutionary nature of knowledge is also directly taken into account, as

> …discourses are dynamic and ever-changing entities: thus determining their exact identities and mapping their boundaries is not a straightforward task, as any researcher would hope. (ibid., p.160)

A possible vagueness/ambiguity in the very definition of *mathematical discourse* as a totality of communication activities involving mathematical entities is noticed by Sfard herself.

More directly referring to the logical tradition, Ferrari (2003, 2004) too, in order to generalize the description of mathematical language as given e.g. by Frege and Russell, places traditional syntactic and semantic aspects side by side with pragmatic ones, necessary to account for the socio-historical nature of any learning. (In this sense Ferrari also refers to Vygotskijan positions). More specifically, we wish to underline three basic points (Ferrari, 2003):

- *mathematical language* can be seen as a *multimodal* (including verbal texts, symbolic expressions and figural representations) and *multivariated* (a wide range of different registers are always involved) *linguistic system*;
- in this frame the notion of *register,* imported from functional linguistics, plays a very important role:

  …Register is a construct which links the situation simultaneously to the text, to the linguistic system and to the social system; a register is formed by selecting the linguistic resources of a particular subject, so different registers are determined by different criteria for selecting these resources (Ferrari, 2003, part III, p.8);

- finally, again from linguistics, Ferrari borrows a classification of *registers* into *colloquial* and *literate* ones: colloquial registers are peculiar to conversational uses, fulfil communication functions and are characterized by context/situation induced *implicatures*; literate registers, an extreme example of which is given by the language of formal mathematics, appear as independent from any context/ situation, referring to a culturally defined meta-context: In fact such registers are used to represent by stable and coherent patterns a whole area of knowledge, according to structures which are at the same time compatible and codetermined by the knowledge field itself. In particular, mathematical language pursues the aim of "an effective, well organized picture of mathematical knowledge" (ibid., part III, page8), ruled by the syntactic structure according to a dominating *treatment function*.

In both theories we have schematically evoked, the focus appears to be on verbal and algebraic components of mathematical discourse, as they are developing along the paths of mathematical learning. In particular, both theories acknowledge the role fulfilled by *natural language in its functions of universal cognitive* ***frame*** *and universal cognitive* ***mediator***: this aspect is in fact well represented in terms both of "metarules" by Sfard, and of functions within the "colloquial register" by Ferrari.

Both theories, furthermore, stress the negative interference by which cultural constructs associated to the natural thought/language can affect (do affect) any learning path converging to culturally validated cognitive habits. Coherently, both theories reveal, though softened and mediated by "evolutionary continuity" nuances, a *basic duality/discontinuity between "natural" and "scientific" thinking*, maybe ascending to epistemological and/or to Piagetian psychological assumptions.[ii]

In our opinion, their common choice to concentrate attention on verbal and algebraic components does not allow them to fully recognize and exploit the semiotic value of

the *natural language* itself[iii]. Natural language, indeed, often also *evokes and integrates* patterns belonging to the *imaging/figural components of natural thought* – in turn directly *correlated to action components* (concrete as well as virtual actions). Moreover, we think that a complete isomorphism (reduction) of *mathematical thinking* to a network of discourses finalized to (and ruled by) communication could oversimplify the complexity of cognitive dynamics.

So, assuming Sfard's general discursive framework, and on the basis of our research evidence, we interpret Ferrari's colloquial/literate registers as omnipresent *dimensions* of common human cognition processes, always mixed and interfering in their continuous evolution. At all ages/levels such processes range from flexible everyday strategies, implementing natural thought/language structures, to more and more stabilized, systematized, "conventional mathematics" (among others) strategies, validated and effective within well controlled *mathematical-style games* (in Wittgenstein's sense).

So, while it is certainly useful, for the sake of simplicity, to schematize inherently multidimensional and continuous teaching-learning processes according to a discrete initial-vs-final-state dichotomy, nevertheless we find it necessary to emphasize the following points, that could be hindered by an excess of schematization:

1. actual, explicit thought processes are the emergent part, *evoked and constrained by a specific context*, of a *space of cognitive possibilities*, defining, with all its redundancies and multiplicities of available approaches, the evolving status of a subject's cognition: in this precise sense the mathematics-as-a-discourse and the mathematics-as-a-language metaphors are extremely powerful, and should be taken seriously;
2. the "colloquial" register, as contrasted to the "literate" one, is undoubtedly responsible for several dissonant outcomes which at any step hinder cultural transmission; but, much more significantly, it acts as the richest resource upon which a *resonant interaction* between explaining and understanding, between teaching and learning can be soundly rooted, to grow effectively and evolve convergently;
3. the potentially constructive interference between different registers, characterizing successful cultural transmission, has however to be evoked, triggered and driven by careful *teaching mediation*, which takes into account and productively employs the multidimensionality of cognition, that extends well beyond a reduction to a mere communication process, as could be suggested by a literal recourse to discourse-language metaphors.

Such a view, also in good agreement with "embodied cognition" in a wide sense, asks for an extension of the linguistic metaphors themselves: such a possible extension will be sketched in the conclusions, and will be supported by the presentation, in the next section, of emblematic teaching-research evidence.

A problematic situation and the analysis of a protocol

According to the philosophy of our long term research activity, we propose to students at every school level – from primary classes to university and postgraduate courses for would-be teachers –emblematic activities, in order to explore and to analyze the cognitive resources activated within the context. We systematically observe an interesting phenomenon, namely a substantial invariance with respect to age of some features of students' significant behaviours and strategies. Though in our opinion this fact in itself deserves a deeper discussion, we intend here only to present an example of students working at a mathematical task in a particular but typical case, and to interpret their behaviours in the light of the above theoretical frame.

Two kinds of mathematical activities, sharing common cognitive features, are suitable for our purpose: modelling activities starting from real life situations never artificially simplified (see, for instance, Tortora, 2001); or problematic situations rich enough to encompass several questions of different cognitive values. The following is one of the problems that we propose both to children and to future teachers.

The Egg problem

A peasant woman goes to the market with a basket of eggs for sale. To the first buyer she sells half of the eggs in her basket plus one half of an egg. To the second buyer she sells half of the remaining eggs plus one half of an egg. To the third buyer she sells half of the remaining eggs plus one half of an egg. Since in the basket there are no more eggs, she happily goes back home.

Notice that only whole eggs can be sold. Now try to answer to the following questions, utilizing any kind of representations and reasoning you have at your disposal:

1. *how many eggs there were in the basket at the beginning?*
2. *can you generalize the problem, supposing that the eggs and the buyers are more, provided that to each buyer the woman sells always half of the remaining eggs plus one half of an egg?*

As can be seen, we explicitly encourage our students, in their attempts to grasp the problem, to utilize any kind of representation they can manage, that is any component (verbal, figural, symbolic) of language. In this we totally agree with Duval's (1995) assumption about the importance of using different *registers*, in his sense.

We believe that symbols and signs fully play their cognitive role when they are freely chosen for a determined, explicit and possibly shared goal. In this sense no particular component of language (e.g. the symbolic one) has an exclusive pragmatic function (like stabilization or treatment), but any usage function is contextually assigned by individual or collective choices: in other words, the focus is on the choosing subject more than on the sort of language chosen.

The protocol from which we present some excerpts comes from the logbook of a group of university students: future elementary school teachers, about half way through their mathematics training. In (Guidoni *et al*., 2003)) we give more details about our work with these students, together with other examples of activities; in this

paper we limit ourselves to underline a few aspects, in order to understand the context in which the teaching activity takes place.

Every activity is presented for the first time at the end of a lesson, and the students are required to explore autonomously the particular situation in a small-group as homework. Then their various attempts are presented and discussed in classroom in various steps and for several lessons, until generalizations, formalizations and related questions are examined and shared by everybody. As teachers we never directly suggest strategies or "correct" answers or procedures, but let every group speak about its attempts and proposals, governing the thread of discussions and only suggesting changes in perspective and links to other arguments. At the end of the activities each group summarizes personal achievements and comments in his logbook.

The emblematically recursive structure of the *Egg Problem* situation is obviously already incorporated in the verbal formulation of the problem. However such a structure appears unnoticed in a first phase of the solving process, where the search for a numerical solution directly induces the use of equations. We faithfully reproduce here the steps presented in the "algebraic section" of the protocol.

We have utilized an algebraic procedure, where the unknown x is the initial number of eggs in the basket. Let's see:

first sale: $\frac{1}{2}x+\frac{1}{2}=\frac{1}{2}(x+1)$

remaining eggs: $x-\frac{1}{2}(x+1)=x-\frac{1}{2}x-\frac{1}{2}=\frac{1}{2}x-\frac{1}{2}=\frac{1}{2}(x-1)$

second sale: $\frac{1}{2}\left[\frac{1}{2}(x-1)\right]+\frac{1}{2}=\frac{1}{2}\left[\frac{1}{2}x-\frac{1}{2}\right]+\frac{1}{2}=\frac{1}{4}x-\frac{1}{4}+\frac{1}{2}=\frac{1}{4}x+\frac{1}{4}=\frac{1}{4}(x+1)$

remaining eggs: $\frac{1}{2}(x-1)-\frac{1}{4}(x+1)=\frac{1}{2}x-\frac{1}{2}-\frac{1}{4}x-\frac{1}{4}=\frac{1}{4}x-\frac{3}{4}=\frac{1}{4}(x-3)$

third sale: $\frac{1}{2}\left[\frac{1}{4}(x-3)\right]+\frac{1}{2}=\frac{1}{2}\left[\frac{1}{4}x-\frac{3}{4}\right]+\frac{1}{2}=\frac{1}{8}x-\frac{3}{8}+\frac{1}{2}=\frac{1}{8}x+\frac{1}{8}=\frac{1}{8}(x+1)$

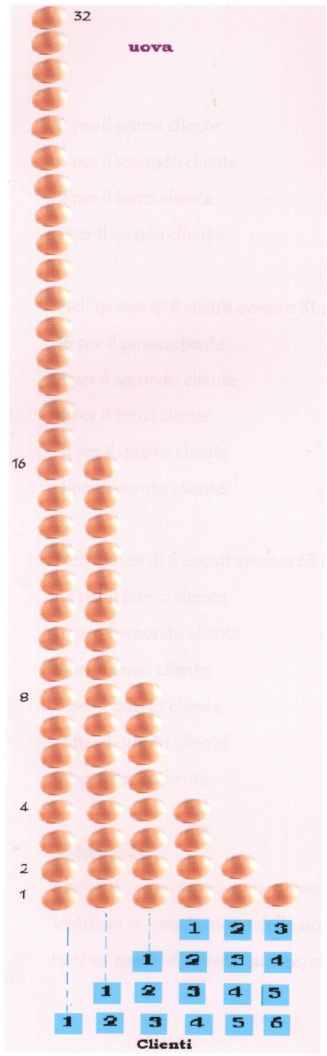
remaining eggs: $\frac{1}{4}(x-3)-\frac{1}{8}(x+1)=\frac{1}{4}x-\frac{3}{4}-\frac{1}{8}x-\frac{1}{8}=\frac{1}{8}x-\frac{7}{8}=\frac{1}{8}(x-7)$.

Since after the third and last sale there are no more eggs, to find out the number of eggs, we have to solve for x.

$\frac{1}{8}(x-7)=0 \qquad \frac{1}{8}x-\frac{7}{8}=0 \qquad x=\frac{7}{8}\cdot\frac{8}{1} \qquad x=7$

Therefore the initial number of eggs in the basket is 7.

In this work phase the algebraic language, correctly used, still exhibits some characters of a colloquial register. However, it is evidently dominated by a "procedural" intentionality, preventing the students from grasping the recursive structure of the problem (though such a pattern might be noticed as implicit in the parentheses game, throughout the algebraic transformations[iv]). In other words, it is possible to recognize a metarule (originated in school practice, but shared also by professional mathematicians) according to which the "transforming power" of



algebraic language is assumed both as an aim and an instrument, sufficient in itself. The request to produce a figural representation tends to destabilize such a way-to-look-at, by better focusing attention on structural aspects of the problem.

In reply to our request, the protocol presents the picture in the figure, accompanied by the following text:

2nd question. In our picture there are some rows of eggs. According to the number of buyers, represented below as blue squares, we can calculate for each buyer the number of eggs bought, by considering every time the total number of buyers. For space reasons, in the picture we have limited the number of buyers to six, but, as we will see after, the procedure extends to an infinite number of buyers.

In the case of three buyers, the total number of eggs is seven, divided this way:

4 to the first buyer, 2 to the second buyer, 1 to the third one.

In the case of 4 buyers, the total number of eggs is 15, divided this way...”

It can be seen that the logical-syntactic structure incorporated in the figural language, though initially unconsciously evoked for illustrative/narrative purposes, at some point directly induces a change of register. The graphic sign, in fact, actually re-introduces, also by their perceptual values, the space-time categories that necessarily support cognitive actions: variation, generalization and symbolization. The relationship between the two discrete variables finally emerges as a powerful Gestalt, though its representation still keeps track of the colloquial register. Quoting from the protocol:

> *We have noticed that each buyer gets one half of the eggs of the preceding buyer, and that each total number is two times plus one the number of eggs sold to a number of buyers corresponding to one buyer less. In fact, for 3 buyers we have 7 eggs, if the buyers become 4 we have to double the number 7 and to add another egg, that is 15 eggs to sell, then 31 for 5 buyers, 63 for 6 buyers, and so on… We see that the rule of eggs sold is exponential: when the number of buyers increases by one, the number of eggs doubles, and moreover we note that 1, 2, 4, 8, 16, 32 are powers of 2. Therefore we can generalize the procedure, writing:*
>
> $7 = 2^3 - 1$; $15 = 2^4 - 1$; $31 = 2^5 - 1$; $63 = 2^6 - 1$,
>
> *where the exponents stand for the numbers of buyers.*
>
> *From this argument, we have derived the following formula:* $x = 2^n - 1$

In this formula x, the total number of eggs, follows an exponential path, and the exponent n is any number of buyers

The algebraic language is reintroduced by the students in the phase of representing their "discovery", as a stable, time-independent relationship – but still partially linked to the process of "generating" it. The same symbolic component is now enriched, endowed with a different use and sense: the transition through the figural component of language allows re-construction of a symbolic value for the algebraic sign[v]. The "local" isomorphism between the verbal description of the situation and the final formula assumes the role of a non-arbitrary, graphically reified constraint; where the formula itself appears as the result of an autonomous, object-driven choice process within available linguistic resources, which can be connoted (in agreement with Ferrari) as complying with a literate register.

The above analysis shows that, among various linguistic components, a special cognitive role is played by graphical representation, inasmuch as it can be used for different purposes accomplishing different functions even more than other language components. This has already been argued from another point of view in (Iannece & Tortora, 2003). Furthermore, in a constructive cognitive process the figural component, ruled and controlled in its production by metarules typical of the "colloquial" register, also embodies the logical structure of the problematic situation, and shares its symbolic content with the written language, according to (Duval, 2000). It therefore has the advantage of better objectifying important aspects of what has been thought, making this accessible to further analysis: in this sense, a "literate" register is also potentially at work.

In this situation teacher mediation has the crucial role of making explicit the continuities and discontinuities with other language components, in particular with the symbolic. The semiotic function of language's figural component has been extensively recognized and analyzed also at research level. In the protocol we make in fact a relatively new observation: the connection between semiotic and pragmatic aspects of the "solving" process, elicited by students' possibility/capability to utilize different registers at different steps, according to their individual cognitive needs. At the same time a systematic (induced) recourse to graphic representations fulfils a meta-cognitive role, fostering students' awareness of their available cognitive resources.

Finally we notice that repeated reference both to the context of experience and to the colloquial register, not only to validate the conclusive algebraic formula but also to construct step-by-step and validate the thought process, can be seen as confirmation of a general Wittgenstein-like approach to this kind of cognitive/cultural game.

Conclusions and hints for further developments

We believe that linguistic metaphors are a crucial tool in interpreting and driving mathematical cognition, but they deserve more careful analysis and integration when confronted with the complexity of cognitive dynamics as a whole. Moreover, we

believe that learning contexts mainly based on problem-solving and modeling activities gradually lead specific successful strategies to coherent patterns throughout the learning process. In this sense the protocol analysis we have sketched above is intended only as an illustration, to clarify our general approach.

About the linguistic metaphors as applied to elucidate mathematical activities we return to two widely discussed research tenets. On one side, Sfard claims that in mathematical discourse metarules are substantially unconscious, and that for this reason they unavoidably and strongly condition any observed cognitive behaviour. However, we have systematic evidence of how much the simple suggestion to introduce a parallel graphic representation of the problem structure, supported and addressed by teaching mediation, can allow the metarules themselves (normally subsumed from natural language) to be made explicit, and, at the same time, to compare/integrate them to more specific ones associated with algebraic representation. On the other side, Ferrari claims that it is necessary to develop the ability to switch from one to the other of the two main language registers, the natural-colloquial and the literate-symbolic ones, always keeping strategic control of their specific meanings. Again we have systematic evidence that other thought/ action/ perception modalities – starting from graphic representations of all kinds – do play crucial roles in understanding and learning processes: therefore, we believe that new "bridging registers", with their specific cognitive functions, should be introduced and that the meta-control suggested above should extend to all them.

If we accept the evidence of a contextual interaction of at least three "registers" even in a simple mathematical activity like the one discussed above, an attempt to track down their reciprocal roles becomes significant. As an example, we might correlate the clarifying outcomes following the introduction of a "bridging register", actually mediating between the verbal-colloquial and the literate-symbolic ones, to the recognition of *different cognitive approaches*, as suggested in (Bruner, 1966), and *cognitive phases*, as suggested in (Watzlawick *et al.*, 1967). Rephrasing this last contribution, we might in fact easily recognize in any significant mathematical activity *three kinds of references*: a *reference to fact*, seen as a redundant source of raw materials evoking potential "cognitive objects" and correlations among them; a *reference to meaning*, active at the level of individuation and choice of potentially stable correlations among the initially accessible ones; a *reference to sense*, leading to the stabilization of selected correlations within a linguistic system, codified in its syntactic and semantic rules.

With the introduction of "bridging registers", it is also possible to notice, to interpret and to exploit a substantially "circular" dynamics ruling the use of semiotic tools, at work as soon as the evoked registers happen to be more than one or two. For instance in a dynamics driven by explicit purposes and pivoting on the figural component, like the one in the protocol, the experience context plays two different roles at different moments in the activity: at the very beginning it evokes/ generates the mathematical model; while at the conclusion of the "fitting" cycle it acts as a cognitive support to

validate the model itself, and even as a warrant of the interpretation of the symbolic writing as a language of its own.

Finally, we always worry about how such research "subtleties", indeed very effective and efficient to interpret and to mediate the actual evolution of learning and understanding, might be significantly transferred to heal ever-present student and teacher "difficulties".

NOTES

[i] The same metaphor can also be applied to give account of higher levels of mathematics; but that is another question.

[ii] For both contrasting terms, however, Piaget was acknowledging to language a secondary role– in both logical and temporal perspectives.

[iii] The crucial role of all semiotic aspects in the process of constructing effective mathematical thinking has been underlined in recent years by many authors. In particular, our research results are in agreement with many of the points of view extensively presented by Duval (1995) and by Radford (2000).

[iv] By the way, we believe that it would be of great interest to carefully analyze the semantic role of parentheses in representing the hierarchical structure of a situation, and correspondently to implement students' better awareness of their usage.

[v] For Piaget "a *symbol* is a "motivated" signifier, which means that the signifier bears a certain resemblance to the signified. A *sign*, in contrast, bears an arbitrary non-motivated relationship to its signified" (quoted from Radford, 2003)

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CONVERSATIONS IN UNDERGRADUATE STUDENTS' COLLABORATIVE WORK

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Abstract: *In this paper I describe an investigation into undergraduate students' collaborative work. The investigation emphasizes the informal collaborative situation, by video recording out of class meetings among students working on projects. I present a qualitative analysis of a conversation between three students, and discuss implications for the design of collaborative activities and teaching of collaborative techniques.*

Keywords: collaboration; undergraduate level; ethnography

Introduction

In this paper I will report on an investigation into students' out of class activities in connection to project based collaborative learning activities in an undergraduate course in mathematics. The aim of the investigation is to get an understanding of the students' collaborative working and writing process.

Student Collaboration and Collaborative Writing

Group work can be an important working style for university students, this has long been acknowledged as a pedagogical mantra and student collaboration is central to learning at some universities. For instance the project Cooperative Learning in Undergraduate Mathematics Education (CLUME) has been concerned with promoting and evaluating strategies for student group work. Nevertheless, Keith Weber (2004) argues that most research in undergraduate mathematics education has to do with either students' acquisition of mathematical concepts, the nature of these concepts or with proposing and evaluating novel teaching strategies. What is lacking, according to Weber, is research on actual teaching of mathematics. Weber focuses on one teacher lecturing using a 'definition-theorem-proof' style of instruction. Complementary to this approach I attempt to describe the students' work outside class. This purpose is different from that of the CLUME project that develops and assesses group-work methods to be used inside the classroom. I have found the analytic and critical approach taken by Anna Sfard and Carolyn Kieran (2001) a relevant method. Their interaction analysis method was developed for another age group, but it has the advantage of systematically going beyond a yes or no to group work. Sfard and Kieran describe an instance where the conversation between two 13 year old boys working with mathematics can be very

counter-productive and discuss to what extent the competent involvement of teachers in students' conversations can help such conversations stay on a learning track.

The large amount of collaboration that goes on outside the classroom in most undergraduate settings represents an interesting problem. Teachers are obviously unable to interact directly in such collaborations but this does not mean that they cannot attempt to affect the collaboration through designing the students' conditions for collaborating. The development of such designs can obviously benefit from knowledge about how collaborative activities, under these designed conditions, goes about.

Context

The context of the investigation is an undergraduate course in mathematics taught at the university of Copenhagen (for a detailed description of the course see (Grønbæk & Winsløw, 2004)). The list of topics includes metric spaces; continuity; Hilbert spaces; Fourier analysis; partial differential equations. The course uses a book written by N. L. Carothers (2000) and has as prerequisites courses on advanced calculus (Adams, 1995) and linear algebra (Messer, 1994). It is considered a 'tough' course and traditionally quite a large number of students quits the course or fails the exam.

Currently the course is going through a research-based revision and has been affected in a number of ways (Grønbæk & Winsløw, 2003a, 2003b, 2004). One of the most significant changes is the introduction of a new working and evaluation format: 'thematic projects'. The formal format of the examination is unchanged as a three-hour written examination combined with a 30-minute oral presentation of a randomly chosen question. But the oral exam has been changed so that students no longer present theory from the book but instead present one of five 'thematic projects'. A 'thematic project' is a short (5-10 pages) note prepared by students in groups of 2-4 in response to a task or set of instructions given by the lecturer.

Methodology

I followed two groups of students and investigated their work using a combination of informal interviews, video observations, and diaries kept by each of the students (see Hyldegård, 2003 for a detailed description of a diary methodology).

This paper is concerned with video observations of student meetings. I recorded a total of seven meetings of various lengths (about six hours of video in total). Furthermore I was in frequent contact with the group participants, mainly by finding them in an area where they often worked (a large open space in the university). This frequent contact served several purposes. The contact was necessary to gain access to the students' working meetings. Because the students typically saw each other several times a day, they were able to change schedule for meetings at very short notice – and often did so. The contact also gave me a better longitudinal picture of the activities. Finally it helped

to build trust between me and the group, which was crucial in order to participate with a camera in these intimate meetings.

The videos were summarized, interesting parts identified and transcribed using the Transana program for video analysis (Fassnacht & Woods, 2003).

Interaction Analysis

Inspired by Anna Sfard and Carolyn Kieran (Sfard & Kieran, 2001) I have used flowcharts to perform an interaction analysis. The idea in this form of analysis is to visualize the flow of the conversation and hence give another perspective on the conversation by showing how the participants move between personal and interpersonal channels of communication (Sfard & Kieran p. 192). In a conversation between two people both of these channels can be activated. The interaction analysis attempts to reveal what channels the participants intend to activate.

Sfard & Kieran have developed a flowchart tool to perform this analysis. The basic idea is to look at relations between utterances. An utterance that is a part of a conversation will typically to some extent be an "answer" to an earlier utterance, though not necessarily the previous one. The task in creating the flowchart is to uncover which of the previous utterances any utterance refers to. This tool enables us to see if the participants mainly use the interpersonal channel or the personal channel when they communicate.

In the flowchart shown later in the paper, communication in the personal channel is represented by the vertical arrows and the interpersonal channel is represented by arrows going from one column to another.

Data

The frequent contact together with the data in the diaries gave a good picture of the students' working style. The students in general met once or twice a week to discuss their projects. The meetings were to some extent concerned with dividing the labor, and quickly discussing strategies for solving parts of the task given, and in interviews the students point to these logistic matters as an important reason for having these meetings. But in the majority of the meetings I have looked at the students also engage in mathematical problem solving during the meeting, and it is this type of activity that is the focus of the present investigation.

Working papers, drafts and books are often referred to during these conversations and the amount of pointing is very substantial, mainly through deictic words or phrases such as "this one here" supported by indexing gestures such as pointing with a hand, finger or pencil. Another general remark is that in many cases the pieces of paper on the desk seem to function both as personal tools and as conversation tools pointed at during the conversations. Switching between these functions can be a delicate matter.

Flowchart of a Conversational Episode

In this episode we are half an hour into a meeting between three male students. The students work on a task about double sequences. They are given a number of hypotheses and are asked to determine if they are true. Furthermore they should either prove the statement or give a counter example.

The task that they discuss is to determine whether the following five propositions hold for a double sequence (x_{nm}) with the additional property that $\lim_n \lim_m x_{nm} = \lim_m \lim_n x_{nm} = \lambda$.

(7) $\sup\left\{x_{nm} \middle\| n,m \in N\right\} < \infty$

(8) for all m,n we have that $\lim_k x_{nk} = \lim_k x_{km}$

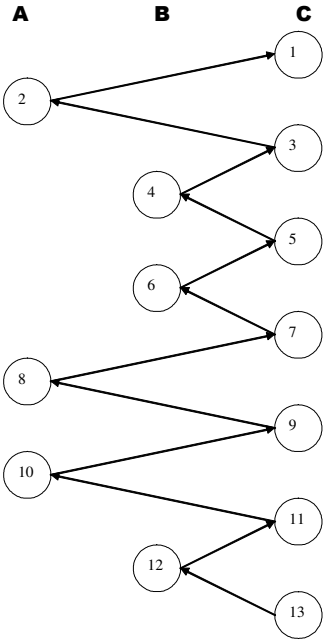
(9) the diagonal sequence converges towards $\lim_i x_{ii} = \lambda$

(10) there exists a sequence of double indices (n_k, m_k) such that $\lim_k x_{n_k,m_k} = \lambda$

(11) there exists a sequence of double indices (n_k, m_k) such that $\lim_k x_{n_k m_k} = \lim_k x_{m_k n_k} = \lambda$

In the transcript below I have included a flowchart of the conversation. The flowchart visualizes what an utterance refers back to, and I will use the flowchart to analyze the interaction between the students. In Sfard and Kieran's version of the flowchart they distinguish to what extent an utterance invites an answer, I have not done that because of the complexity in applying the method to a conversation with three parties rather than two. Hence invitations for answers are not visualized. Nevertheless this simple flowchart reveals an interesting structure to the conversation.

Each column in the flowchart represents a student and the arrows represent which one of the previous utterances refers to. The strength of the flowchart is that it provides a visualization of what lines of conversation each student takes up, if a student is kept out of the conversation by the others or is isolating himself from the others.



1 student C: I sad an messed around with this one and was very confused, I might be wrong: the supreme, this should be right?

2 student A: well it is right, because it converges for all these, so we should be able to…

3 student C: for every coulomb and row we should be able to say that that … this is true

4 student B: are we sure, couldnt we imagine that, it so to speak has a hill in the middle almost, in some way. If that hill lays askew in the sceme sso that it does not really messes with anything? What if we have the example from before, your example: one, two, three, in the diagonal so the supreme is infinity. But then chooses zeros down here.

5 student C: that is the same question

6 student B: so do we have a counter example

7 student C: yes that is a counterexample

8 student A: yes it is

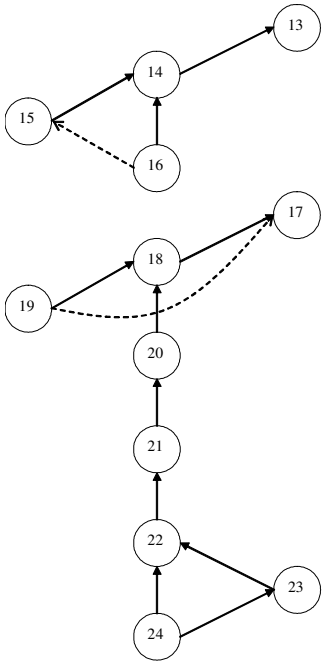
9 student C: but that is what this matrix explains, just that

10 student A: yes

11 student C: but the next must be very straightforward ,

12 student B: yes

13 student C: It can not be right that just because you have convergence in the to [not able to hear]



13 student C: It can not be right that just because you have convergence in the to [not able to hear]

14 student B: we should be able to construct such a sequence

15 student A: oh yes you could have one that goes from minus something towards zero and something that goes from the positive side and…

16 student B: one with the number one here and two there and then zeros for the rest.

17 student C: well the diagonal sequence, isen tit convergent in…???

18 student B: Shouldn't any of this be true?

19 student A: yes

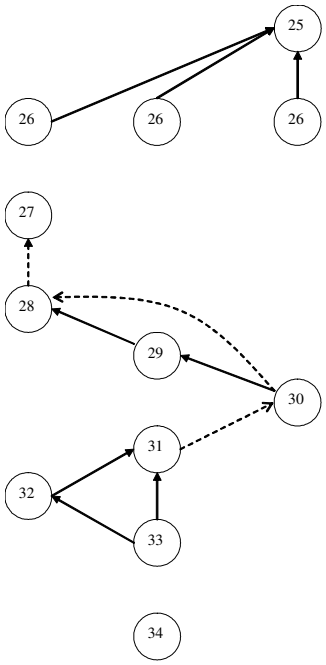
20 student C: this can't be right, I think this is a little to easy.

21 student B: we where sure about this one, yes that was the argument from before

22 student B: can it be that these are the ones Jens Christian (instructor) talks about?

23 student C: hmm I don't think so

24 student B: yes it must be them



25 **student C**: here I had myself convinsed that the diagonal sequence actually would work, an then ten and elleven where very easy. Laughs

26 All Laughs

27 **student A**: there has been found a convergent subsequence…

28 **student A**: i would say that number ten was false too

29 **student B**: Actually i dont think so

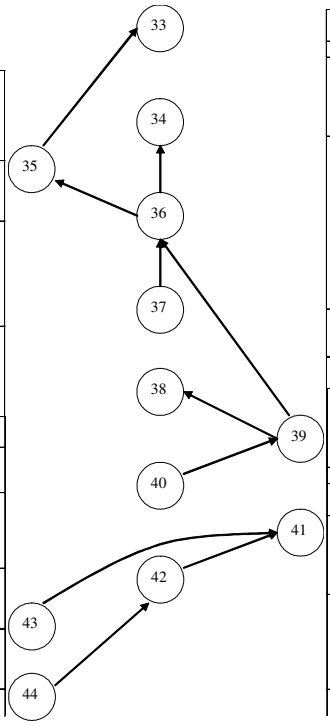
30 **student C**: none of the examples we have worked with so far points it out as false

31 **student B**: coulnt you imagine that you choses something that is very close to the values down here? If we have infinety down here and then chooses something that is close, and then leaves them to go to the right

32 **student A**: i have to remember it. We assumes this to be true. Yes ofcourse you can do that, thats what i thougt of. Of course it will converge, yes it must be true

33 **student B**: wll how are we goin to cunstruct it?

34 **student B** calculates



33 **student B**: well how are we going to construct it? øhm

34 **student B** calculates

35 **student A**: what if we,, couldn't you just take one for each, we have a lot, for one and to we have convergent subsequenses, we have some convergent sequences for every n and m, if you assumes,…

36 **student B**: I think that if you take this and puts a-n equal the limit down here and then constantly chooses something that is epsilon close to a5 there then you can chose something in here, epsilon close. An then for every a_n you can chose such epsilon intervals and then chose a number t inside, then you have a sequence that converges and then, i dont know, well you can of course chose epsilon as one divided by n, then you should get it.

37 **student B**: because the sequence will get very (vilkårligt) close to every of the a_n's that converges towards our limit.

38 **student B**: let me write it down so we see what we get.

39 **student C** but the essens of it is that you say: Ok first we go downwards as long as needed in the one direction, we have n_1, 2, 3 , a billion. Then we start to move to the right. That is what you think about right?

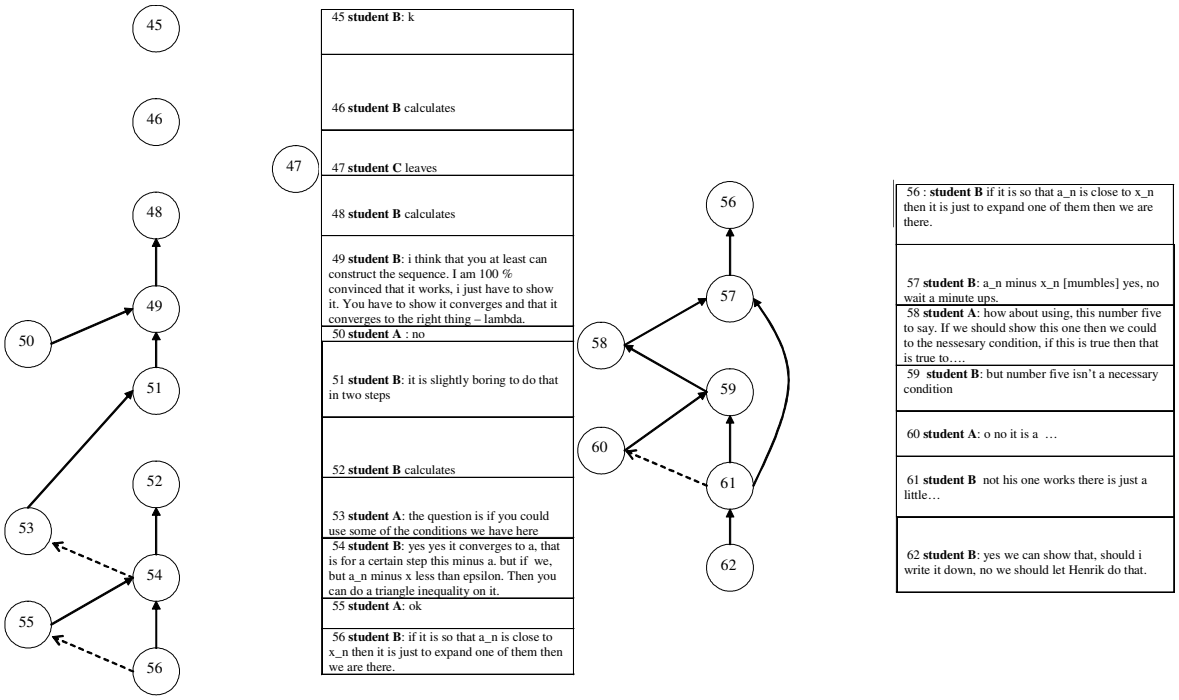
40 **student B**: well yes

41 **student C**: it is down there and then straght ahead

42 **student B**: thats fine

43 **student A**: or we can go and say now we can find an n, we should be able to find some n large enough, and gives an epsilon, the distance… well hmm...

44 **student A**: these chi's are they x's



If we look at the interaction flowchart and transcript we can identify several periods in the conversation. From node 1 until node 9 the students discuss question seven, and identify a counter example in the form of a matrix representation:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 4 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}$$

In nodes 13 to 17 they agree to seek a counter example to proposition eight. Student A explains a vague idea supported by large gesturing, but is more or less ignored by the other two. Again Student B finds a counterexample consisting of the double sequence represented by the matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 2 & 2 & 2 & 2 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Student C then realizes that the counterexample to question seven is also a counter-example for question nine and that makes all of them worry since this is the third negative result in a row. This gives rise to a longer discussion of a meta-discursive

character (as the term is used by Sfard & Kieran (2001), that is to signify a discussion about the form of the question rather than about its mathematical content) (nodes 18 to 24).

The students then start to discuss whether proposition number ten is true. Student B starts constructing the sequence whose existence is proposed. Student B retreats to long and quiet calculation (nodes 34, 38, 46, 48, 52). Several times Student A attempts to approach with an idea but Student B is very involved in his own calculation. Unlike Student B, Student A doesn't calculate on paper, and he is unable to get through to Student B with his ideas. Even though Student B is extremely involved in his own calculations he does not entirely forget the other participants. Several times he approaches them and explains his ideas referring to his calculations in an ostensive way (nodes 31, 36 and 49), pointing towards the paper while he talks. Despite his attempt to involve the others in his calculations, the social mathematical process ends as Student C leaves (node 47) and Student A's comments get less relevant (node 44). The conversation ends in a non-mathematical discussion about who should type their solution (node 62).

Discussion: how to support a collaborative writing process?

So what does this episode tells us about the collaborative writing process of mathematics students? Why does the communication break down, and is it possible to overcome these problems? It seems clear from the flowchart that from a certain point one of the students increasingly refers back to his own utterances. But it is not apriori obvious to what extent this breakdown is connected to the students' interest in maintaining a fruitful collaboration, to the task, or to the semiotic representations that the students work with. It is not even clear if such breakdowns in general should be avoided.

The students meet in order to discuss and develop their collaborative thematic project and they are in general interested in collaborating. The only indications that they do not show interest in collaboration occurs very late in the process (nodes 44 and 62-64) and might very well express a frustration caused by their lack of ability to communicate, rather than a lack of interest in communication. So while Sfard and Kieran's work with the flowchart methodology shows that students' different interests can create an unhealthy collaborative environment (Sfard and Kieran, p. 201), this does not seem to be the reason here.

It is interesting that on the occasion when fruitful communication is threatened by one student communicating mainly with himself, the students are engaged in working with a task (task number 10) that on the structural level is different from the ones before.

Task number ten is:

given that $\lim_n \lim_m x_{nm} = \lim_m \lim_n x_{nm} = \lambda$

(10) there exists a sequence of double indices (n_k, m_k) such that $\lim_k x_{n_k, m_k} = \lambda$

The very obvious reason why the students' work with this task is less fruitful than their work with the prior tasks is that the students, for various reasons, are not able to solve this task as easily as the rest. Nevertheless it is still interesting to see that the way the task is structured makes the students' approach it differently than they approached the prior ones.

Task ten is specifically about *the existence* of a specific sequence, and it is the attempt to construct this sequence that leads Student B to neglect the other students and work by himself. The tasks seven, eight and nine are all about determining the validity of a proposition, in all cases by constructing a counterexample, and not about proving the existence of a general mathematical object. It could be the case that the *construction* of an abstract mathematical object is more difficult to discuss than the very concrete construction of a counterexample.

When the students work with tasks seven, eight and nine they refer again and again to concrete matrix representations used as candidates for counter examples. When talking about task number ten Student B uses an algebraic register to denote the double sequence he attempts to construct. From looking at the video it is obvious that the algebraic register is very difficult for the other students to access, partly because the students do not use a blackboard but only sketchpads. You can share a matrix or figural representation on a piece of paper, but it is not possible to follow somebody else's algebraic formulas while he is writing them on a piece of paper.

The data presented here are of course not conclusive but it would be relevant to investigate the relation between types of tasks and fruitful collaborative strategies. If future research can point to types of tasks that are better suited for some collaborative strategies than for others this can be of valuable when designing tasks for collaborative learning.

One can ask if such conversational breakdowns necessarily are to be avoided. Is it not a natural thing to need a little time alone while working on a mathematical problem? It might be a good idea to think about how to teach collaborative strategies to students. Knowing how and when to interact and retreat in order to use conversation partners effectively when doing collaborative writing or problem solving is of course extremely valuable, though such knowledge is far beyond the scope of this study. It might be possible to teach students about typical problems and dynamics in collaborations on mathematics, and thereby create awareness among students about the challenges that are inherent in communications in relation to mathematical collaborations.

It might also be of value to look at the working environment for students. Creating an environment where it is possible to move flexibly back and forth between group discussion and personal work while keeping focused is always a challenge, but, at least at the university where these observations where conducted, this has been given very little, if any, attention.

Conclusion

In this paper I have presented an episode where three students' out of class collaboration on a writing task is challenged because they are not able to keep a conversation going since one of the students retreats from interacting with the others, exclusively referring to his own utterances. More research could be valuable in order to inform the interplay between didactical choices in the design of tasks for collaborative learning, the collaborative strategies taught and the physical environment that students are offered for their collaborative writing activities.

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WHAT IS A DEFINITION FOR IN SCHOOL MATHEMATICS?

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Abstract: *This paper discusses the place of definitions in school mathematics, considering official UK curriculum guidance, literature related to definitions in advanced mathematical thinking and to experimental teaching focused on student development of definitions. A two dimensional framework is suggested for considering their functions, the ways in which students are expected to relate to them and their didactic purposes. Two contrasting examples of definitions from textbooks are analysed using systemic-functional linguistic tools.*

Keywords: definitions; systemic-functional linguistics; textbooks; student positioning; discourse analysis.

Definition of mathematical concepts has been a topic of interest in mathematics education research for some years. This interest arises primarily from the commonly observed difficulties met by students entering advanced levels of study as they are asked to use definitions in formal mathematical reasoning. Yet students also encounter definitions of mathematical concepts much earlier in their educational experience. Recent government guidance for teachers in English primary and secondary schools recommends classroom use of mathematical dictionaries by teachers and students (DfES, 2000, 2001). This guidance, including a list of 'key words' for each of Years 1 to 9, constructs an official curriculum discourse that privileges vocabulary over other characteristics of mathematical language. Central to this discourse is the notion that mathematical words are unambiguous and that their meaning can be clarified by using a dictionary definition. This and other assumptions about the nature of mathematical language and approaches to learning it are discussed more widely in a critique of this official guidance by Barwell, Leung, Morgan and Street (2005). In this paper I consider critically the roles played by definitions in school mathematics, in the light of curriculum guidance and the place of definition in mathematical activity, presenting analyses of some examples of definitions occurring in secondary school textbooks.

Are mathematical definitions 'special'?

In discussing the characteristics of mathematical definitions, Borasi identifies two functions they must fulfil. A definition of a given mathematical concept should:

> 1. Allow us to discriminate between instances and non-instances of the concept with certainty, consistency, and efficiency (by simply checking whether a potential candidate satisfies *all* the properties stated in the definition).

2. "Capture" and synthesise the mathematical essence of the concept (*all* the properties belonging to the concept should be logically derivable from those included in its definition). (Borasi, 1992, pp.17-18)

The first requirement does not seem peculiar to mathematics; though definitions of everyday concepts may be 'fuzzy', precision characterises the definition of scientific concepts in many specialist domains.[1] Borasi's second criterion, however, hints at a role for definitions within mathematical practice that goes beyond both the record of usage of standard dictionaries and the technical taxonomising of common-sense phenomena in natural and social sciences (Wignell, 1998). Definitions in mathematics form a basis for logical derivation not only of properties already known (perhaps in a common-sense way) to belong to the concept but of new properties.

Vinner (1991) claims that, while definitions in everyday contexts have little relationship to development of concepts (Fodor et al., 1980), they are essential for technical concepts. By providing examples of mathematical situations in which use of a formal definition appears vital to overcoming the limitations of students' intuitive 'concept images', he distinguishes advanced mathematics as a technical context. This seems uncontroversial. Definition is distinguished from description by a number of mathematics education researchers working in the area of advanced mathematical thinking (e.g., Barnard, 1995; Tall, 1991), with the use of definitions presented as characteristic of advanced mathematics. Alcock and Simpson (2002) identify this distinction between the functions of 'dictionary definitions' and of mathematical definitions as a root cause of breakdown in communication between lecturers and undergraduate students:

what eludes the students is the distinction between a dictionary definition as a *description* of pre-existing objects and a mathematical definition as the chosen basis for deduction, one which serves to *determine* the nature of the objects. (p.33, original emphasis)

Here Alcock and Simpson also hint at another characteristic of the ways mathematicians use definitions – the element of choice. While dictionary definitions describe the ways a word is actually used in practice, mathematical definitions are chosen *in order that they may be used* for deduction and proof of theorems.

The research mathematician may come to his results starting from special cases, which will appear as corollaries in the final version, from which he gets his ideas, which is worked with until he has a proof. Then the theorem is what has been proved. At this point he formulates his definitions so as to make the theorem and proof as neat as possible. (Burn, 2002, p.30)

At first, the concepts the mathematician works with may be more or less intuitive, derived from special cases. The construction of the formal definition and consequent

[1] Leung (2005) argues that some mathematical concepts also have core and non-core meanings and hence some 'fuzziness'.

creation of a technical term is thus purposeful and creative, aiming not simply to describe or "capture" a pre-existing concept but to shape that concept in a way that lends itself to particular purposes. Of course, this definition may subsequently be used to generate deductive sequences leading to the discovery of further theorems.

The idea of choice and purposeful formulation of definitions constructs an active role for the mathematician him/herself, not simply as a user of correct mathematical vocabulary but as one who chooses between alternative definitions or creates new ones. This role is very different from that constructed for school students by the official discourse of the English curriculum. Here the booklet *Mathematical Vocabulary* focuses on students' development of understanding of the meaning of words, "using the correct mathematical terminology" and "learning to read and write new mathematical vocabulary" (DfES, 2000, p.2), using a dictionary "to look up the meaning of words" (p.36).

Rather more active student roles in relation to definition are proposed elsewhere. In particular, activities that engage students in forming and critically evaluating their own definitions have been described with middle school (Keiser, 2000; Lin & Yang, 2002) and high school students (Borasi, 1992). Keiser's students developed their own definitions of 'angle'. While the discussions she describes seemed to support the students' development of the ability to distinguish between examples and non-examples of the concept, the notion of 'definition' in this case was descriptive of an independently existing object rather than purposeful design of a definition for theory building. Lin & Yang's study involved a problem solving activity in which students were encouraged to develop minimal definitions of rectangle and square. In this case, some of the students were able to make logical connections between the two, suggesting that their understanding of the nature of definition was going beyond the purely descriptive. At a higher level, Borasi's students, as well as working with the idea of minimal definition, explored the consequences of using alternative definitions of the same object (e.g. the different approaches to solution of a problem that might arise when using metric or analytic definitions of a circle), thus being introduced to the idea of choice and purposeful definition.

A framework for curricular approaches to definition

For learners of mathematics, definitions function in several ways. On the one hand, using a definition to distinguish between instances and non-instances of the defined concept is one approach to developing awareness and understanding of the concept itself as well as learning correct application of the language. This is the purpose of definition assumed by English curriculum guidance for teachers at primary and secondary level. At the same time, however, if one of the aims of mathematics education is to develop participation in the discipline of mathematics itself and in mathematical ways of thinking, then negotiation of definitions, choice between

alternative definitions, deduction from agreed definitions and arguments traceable back to definitions also need to feature in the experiences offered to students.

The two functions of definitions for learners of mathematics outlined above may be characterised as a content-process dichotomy. Most current curriculum thinking recognises the need for aims related both to the learning of specific content and to more general processes of using and applying mathematics, though there may be differences in emphasis (and in implementation). A second dichotomy relates to the positioning of the mathematician/ student in relation to mathematics in general and to definitions and the act of defining in particular. This may be characterised as the opposition between seeing mathematics as a 'given' body of knowledge to be discovered or acquired and allowing that mathematicians (in general and students in particular) themselves play an active part in constructing mathematical knowledge. Table 1 suggests a framework for thinking about the ways in which definitions may feature in mathematics classrooms; the four cells identify the types of activity that might utilise definitions and the didactic purposes these might have. (The types of activity and purpose suggested here are indicative rather than exhaustive.)

Table 1: Framework for definition-related activity in the classroom

		nature and function of definition	
	Definitions …	… distinguish between instances and non-instances of a concept	… are used as the foundation of logical argument
positioning of user in relation to definition	… are pre-existing/ given by authority	A: to apply criteria to test examples or to create examples that match criteria *purpose: develop the concept itself*	C: to deduce further properties and to construct proofs *purpose: develop connected knowledge within the domain of study; develop proof skills; engage in mathematical deductive reasoning*
	… may be constructed by the user	B: to 'pin down' the user's concept image (and through debate, counter-examples etc. refine the concept image to become closer to that of the mathematical community) *purpose: develop the concept; engage in mathematical reasoning and debate*	D: to create a new concept that yields interesting or useful results *purpose: engage in 'authentic' mathematical practice*

Cells B and D might be further sub-divided according to whether the active agent of construction is the student him/herself or whether any such creative mathematics is the activity of a more distant mathematician. The examples described above suggest that several of these cells can be identified with school curricular discourses involving definition in mathematics. The official discourse of the English curriculum

is clearly located in cell A; the examples offered by Keiser and Lin & Yang fall within cell B, constructing definition as primarily descriptive but positioning the student actively and powerfully. Borasi's course included elements within both cells B and C.

Analysis of textbook definitions

Recent curriculum developments in the UK have paid considerable attention to the need to develop students' understandings and capabilities in relation to mathematical proof, but little has been said about the nature or function of mathematical definition at primary or secondary school level beyond the simplistic assumptions of the DfES booklet already mentioned. While students certainly encounter definitions throughout their mathematical education, the difficulties reported at university level suggest that their earlier experiences may not provide a basis for using definitions in ways that go beyond the development of concepts.

Table 2: Analytic Tools.

Descriptive questions:	Grammatical tools:	*Illustrative interpretations**
Who or what are the actors and where does agency lie?	What objects and humans are present? How are active or passive voice used?	*Human agency, especially in mental processes (e.g. think, decide), tends to position mathematicians more actively in relation to definition. (Cells B/D)*
What are the processes?	Relational, material, mental/behavioural?	*A preponderance of relational processes (e.g. be, have) tends to characterise definitions used to distinguish between instances and non-instances. (Cells A/B)*
What are the roles of the author and reader and what is the relationship between them?	How are personal pronouns used? In what kinds of processes are author and reader actors?	*This can distinguish the way in which the student is positioned or not as a potential creative mathematician. (further sub-dividing cells B and D)*
Is the modality absolute or contingent?	Modal verbs, adverbs, adjectives	*Contingent modality allows the possibility of alternative definitions and choices (distinguishing between cells A/C and B/D).*

**These illustrations refer to the framework presented in Table 1. Further illustration is provided in the analysis of Examples 1 and 2 below. The illustrations should not be interpreted deterministically as any analysis has to take into account the broader text and the context of its use.*

In this section, I use the framework outlined in Table 1 to consider examples of definitions taken from secondary school textbooks published in the UK, analysing the nature and function of the definition as it is presented in the text and the

positioning of the student/ mathematician in relation to it. The analysis uses tools drawn from systemic functional grammar (Halliday, 1985) selected to illuminate the ways in which the nature of mathematics and mathematical activity may be constructed through the texts presented to students. A fuller discussion of this approach and its applications in mathematics education research may be found in (Morgan, 1998; in press). Table 2 identifies the questions used to interrogate each text and the grammatical tools that operationalise the resulting description. These are a subset of the tools described and used in (Morgan, 2005). The first two questions in the table are related to the ideational function of language, concerned with the nature of our experience of the world, the next two to the interpersonal function, concerned with the identities of the participants and relationships between them. The description thus constructed[2] allows us to address critical questions that help to locate each occurrence of definitions within the framework presented above, in particular: *What is the function of definition?* and *How is the student/ mathematician positioned in relation to definition?*

Example 1: (extract from Bostock & Chandler, 1978, pp.134-135)

For any acute angle θ there are six trigonometric ratios, each of which is defined by referring to a right angled triangle containing θ. …

Since we are now regarding an angle as the measure of rotation from a given position of a straight line about a fixed point, it is clear that the size of an angle is unlimited, as the line can keep on rotating indefinitely. The meaning of the six trigonometric ratios is, as yet, restricted to acute angles, since the definition used so far for each ratio refers to an angle in a right angled triangle. If we wish to extend the application of trigonometric ratios to angles of any size, they must be defined in a more general way.

Actors & Agency	Human actors "we" are present as decision makers. However, at other points, agency in the process of definition is obscured by use of the passive voice: *meaning … is … restricted*; *they must be defined …* As well as more or less concrete objects such as angles and lines, *meaning* and *definition* are themselves actors in this text. This produces a meta-discourse about definitions in addition to introducing a new definition of trigonometric ratios.
Processes	Mental processes *regard* and *wish* construct mathematics as an intellectual activity involving choices Trigonometric ratios are to be applied, a material process, although agency in this is obscured by the nominalization *application*.
Author & Reader	It is not clear whether the use of *we* is exclusive or inclusive, though it could certainly be read as an invocation of solidarity, calling upon the reader to share in the new way of thinking about angles and the desire to extend the application of

[2] Only partial descriptions are presented here, focusing on those aspects most relevant to definition.

	trigonometric ratios to take account of this.
Modality	There are several temporal modifications: "we are *now* regarding"; "The meaning … is, *as yet*, restricted"; "the definition used *so far*". These emphasise the contingent nature of definition and, further, suggest progression for the student-reader from an earlier, basic or elementary, understanding of angle and trigonometric ratio, to a more advanced one. The high modality of "Since we …, *it is clear*" and "If we …, they *must* be defined" ascribes authority to the argument rather than primarily to the author as each occurrence appears as the consequence of a premise that the reader has been called into sharing.

There is only space here to present two examples, taken from texts for university-bound (though not necessarily intending to study mathematics at university) students (aged 17-18). Elsewhere (Morgan, 2005) I have presented analyses of examples from texts aimed at intermediate and higher attaining students aged 15-16, showing marked differences between the ways in which definitions were presented to the different groups of students. The intermediate text constructed definition simply as naming pre-existing objects while the higher text demonstrated the purposeful construction of an alternative definition, opening up the possibility that the student-reader would make active choices about the usefulness or applicability of alternatives.

Example 2: (extract from Martin et al., 2000, pp.89-90)

Right-angled triangles are used to define the three basic trigonometric functions for some acute angle θ; sine, cosine and tangent.

$$\sin\theta = \frac{a}{c} = \frac{\text{side opposite } \theta}{\text{hypotenuse}} \quad \cos\theta = \frac{b}{c} = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} \quad \tan\theta = \frac{a}{c} = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta}$$

This principle can be used to define the sine, cosine and tangent of any angle θ.

Draw perpendicular axes Ox and Oy, and a circle centred on the origin, with radius 1 unit. Then θ will fix some point P on the circle.

[diagram]

The coordinates of P (x,y) are then $(\cos\theta, \sin\theta)$. Now adopt the convention that θ is measured anti-clockwise from the positive x-axis. …

Actors & Agency	The passive voice is used, obscuring agency, especially in the act of definition, though a human agent is implicitly present in the imperative instructions to *draw* and to *adopt the convention.* In addition to concrete objects, *principle* and *convention* are included as mathematical objects.
Processes	The mental process of defining is presented as a mathematical activity, yet, as its

	agency is obscured, it is distanced from the student-reader. Material processes (*draw*, *fix*, *measure*) construct a mathematics which is about practical activity.
Author & Reader	The imperative constructs an active role for the student-reader – but this role involves material activity (drawing) and following conventions (whose origins are obscured) rather than decision making. The author is absent from the text, again distancing them from the reader and placing authority in mathematics rather than in human mathematicians.
Modality	The modality is generally absolute, presenting the content as unquestionable. The temporality (then… now…) sequences the argument rather than suggesting contingency.

The two examples both address the issue of re-defining trigonometric ratios (previously defined for acute angles only) to apply to general angles. The idea that mathematical definitions can be changed seems likely to be new or at least unusual for students at this level and the extracts of text considered introduce this notion. This context gives us a particularly good opportunity to consider how the nature of definition itself and the role of mathematicians in the construction of knowledge are presented to the students, though it may not lend itself to considering other aspects of definition, such as its use in constructing proofs.

As these examples involve the extension of definition of terms to new contexts, it might be considered that they should be located in cell D, creating a new concept. However, neither text contains a clear purpose for this extension. Example 2 merely states that it can be done "This principle can be used to define …"; thus the extended definition is derived from the original but there is little sense of why it might be worthwhile doing so. Example 1 suggests that "we" might "wish to extend the application …", hinting at some motivation for doing so but still not stating an explicit purpose. The function of definition in both examples, therefore, seems to be located in the left-hand column of Table 1, allowing instances of the concept to be distinguished.

There are, however, significant differences between the two texts in the positioning of the student/ mathematician in relation to the definition and to mathematical activity more generally. Example 1 constructs an important role for human mathematicians in making decisions. The student may consider him/herself to be invited to share in this intellectual activity and to be engaged in and persuaded by argument (though, as Pimm (1984) suggests, there are alternative ways the use of *we* might be interpreted by the student reader). In contrast, Example 2 constructs a less powerful student role. Rather than being invited to share in decision-making activity, the student is instructed to carry out material tasks; rather than being persuaded by argument, s/he is presented authoritatively with a procedure to follow. Example 2, therefore, may be located in cell A of the framework with the limited didactic

purpose of developing the new or extended concept, while Example 1 is located in cell B with the additional purpose of engaging the student in mathematical reasoning.

Discussion

I do not wish to claim too wide a scope for the results of the analysis presented in this paper. The examples clearly represent a very limited sample of the texts, both written and oral, that students encounter during their school mathematics experience. In textbooks we will find definitions of different kinds of mathematical concepts, some of which lend themselves more (or less) fully to the various activities and purposes identified in the proposed framework. We will also find definitions making use of a wider range of semiotic systems, especially algebraic notation, that have meaning potentials not immediately addressed by the analytic tools used in this paper. It may further be argued that students' experience is affected more by their teachers' practices than by their textbooks. While agreeing that this is so, I would also argue that teachers themselves are strongly influenced by the resources available to them in textbooks and curriculum guidance. Such texts provide ways of structuring and sequencing the subject matter and also construct emphases and values that, while they may be resisted and revised by some teachers, are nevertheless likely to be influential in shaping classroom practices.

In preparing students to study advanced mathematics, I suggest that they not only need to have opportunities to learn to appreciate the roles definitions play in mathematical reasoning but also to begin to see that doing mathematics involves more than following procedures or reproducing standard arguments. Neither of the examples presented here, nor the examples from school texts discussed in (Morgan, 2005)[3], hints at the function of definitions as a basis for logical deduction. It may be that the topic does not lend itself to this function, particularly as it is primarily about extending an existing conceptual structure rather than creating and using a new concept. On the other hand, if a clear reason were identified for needing to extend the concept of trigonometric ratios to be applicable to general angles then the activity of creating a definition suitable for such a purpose would involve logical reasoning and could be located in cell D of the proposed framework.

The analysis of the two examples displays a sharp contrast in the ways in which the student-reader is positioned in relation to mathematics: as a potentially active participant in decision making and reasoning or as a rule follower. Both of these roles may be necessary parts of learning and doing mathematics. However, students whose predominant experience constructs definitions as dictionary entries – authoritative but author-less– seem likely to find more difficulty in adapting to the

[3] An example from a research paper discussed in (Morgan, 2005) demonstrates the purposeful creation of a new definition for an existing concept.

demands of advanced mathematics. The discourse of vocabulary in the UK curriculum thus needs to be addressed critically. More generally, the framework of types of definition-related activity suggested here, while no doubt incomplete, provides a starting point for thinking about the purposes and effects of various approaches to definitions in the classroom. The analysis of textbook extracts provides concrete tools for anticipating the meanings, both substantive and positional, that students may construct from interacting with such texts. This analytic method could be developed to offer guidelines for writing or choosing texts for students. It has potential to be applied more widely beyond the study of definition to inform critique of other aspects of students' experiences of mathematical discourse.

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VOICES IN SCAFFOLDING MATHEMATICAL CONSTRUCTIONS

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Abstract:*This paper asks 'who is talking in scaffolded discourse?' Protocols of two students' scaffolded construction of a method of drawing absolute value graphs of linear functions are considered. Our analysis identifies 'other voices' not physically present in the scaffolded discourse. We argue that formation of this construction involves an interaction of 'pedagogically resonant' voices of the participants and of others voices and that the appropriation of other voices reflects certain value judgements.*

Keywords: construction; scaffolding; voice; zone of proximal development.

Introduction

In this paper we consider students' mathematical constructions, scaffolding and the zone of proximal development. These considerations, however, are merely background issues for the main focus of the paper: who is talking (what voices are present) in scaffolded discourse? Clearly the scaffolder and the scaffoldees' voices can be heard. We argue, however, that other voices are present and important in such discourse. We examine this through an analysis of protocol excerpts in which two students are scaffolded in tasks concerned with the graph of linear absolute value functions. There are four sections below. The first briefly outlines the research study our data is taken from. The second positions our work in the literature. The third presents protocol excerpts. The paper ends with a discussion which considers the 'other voices' within the scaffolder's and students' utterances and which focuses on the importance of 'pedagogical resonance' of voices in such scaffolded discourse.

The context of the study

This paper is a by-product of research which set out to investigate social interaction with regard to an activity theoretic model of abstraction proposed by Hershkowitz, Schwarz and Dreyfus (2001). The research focused on an investigation of the validity of Hershkowitz et al.'s model and aimed to relate it to the aspects of human interactions including scaffolding and peer interactions. We now give a brief account of this model and summarise relevant findings with regard to scaffolded abstractions.

Hershkowitz et al. (ibid.) propose a dialectical materialist account of abstraction which develops from an undeveloped initial entity, through the use of mediational means and social interaction. The development consists of vertically reorganising previously constructed mathematical knowledge into a new 'structure'. The new

structure is produced, in an activity, from three epistemic actions: recognising – identifying a mathematical structure; building-with – using the recognised structure to fulfil a goal; and constructing – assembling knowledge artefacts to produce a new structure. This model of abstraction is primarily concerned with the construction of new knowledge and *a priori* assumes the importance of its consolidation. Elsewhere (Monaghan & Ozmantar, in press) we propose that a mathematical abstraction is a consolidated construction which can be used to create new constructions. In this paper we focus on construction in the sense of Hershkowitz et al's (ibid.) model but avoid the term 'abstraction', as consolidation is not a focus of this paper.

In Ozmantar & Roper (2004) we examined the verbal protocols of two students who were scaffolded in the formation of a new construction. Their development was assisted by a scaffolder who regulated their work by organising the main goal of the activity into subgoals. This analysis was extended in Ozmantar (2004) where it was argued that the emergence of subgoals were contingent upon at least four dialectically interrelated parameters: the task, scaffolder's interventions, students' interpretation of the task and of the scaffolder's interventions and prior emergent goals. In this paper we further argue that the achievement of a construction in a scaffolded discourse involves voices of others, who are not physically present, in the interactions of the scaffolder and scaffoldees. The next section briefly outlines the literature with regard to 'voices' in relation to scaffolding and the zone of proximal development (ZPD).

Voice, scaffolding and the zone of proximal development

There is a sense that considerations of mathematical discourse, scaffolding and the ZPD have gone through a developmental process, from a focus on teacher-learner dyads to a "view of the ZPD as the nexus of social, cultural, historical influences [which] takes us far beyond the image of the lone learner with the directive and determining tutor" (Daniels, 2001, pp.67). We have, at least, experienced this in our own development. The term 'scaffolding' was coined by Wood, Bruner & Ross (1976) to describe the role of an adult in enabling "a child or novice to solve a problem … beyond his unassisted efforts … the adult 'controlling' those elements of the task that are initially beyond the learner's capacity…" (ibid, p.90). Bruner (1985) later linked this to Vygotsky's ZPD which refers to a metaphorical 'distance' between the "actual developmental level" and "the level of potential development as determined through problem solving under adult guidance" (Vygotsky, 1978, p.86). Vygotsky appears to say very little about the forms of assistance that an adult may provide, but others have offered their views (see Chi, Siler, Jeong, Yamauchi & Hausmann, 2001). A focus on forms of assistance or the role of the adult may suggest that scaffolding proceeds from the scaffolder to the scaffoldee. Whilst there may be scaffolded scenarios where this is the case it seems presumptive to assume this is always the case and we, like Daniels (2001), question whether scaffolded supports are "produced by 'the more capable partner' or are they negotiated?" (ibid, p.59).

In the light of extant literature (e.g., Mercer, 1995) and our earlier findings, we view scaffolding as an asymmetrical collaboration between 'more knowledgeable'

(scaffolder: teacher/tutor/peer) and 'novice' (scaffoldee: student/learner) towards successful task completion within the novice's ZPD. The more knowledgeable provides assistance augmented or reduced depending on the novice's progress.

Although this view of scaffolding is primarily concerned with the interaction of the parties involved in such a collaboration, protocol analysis in this study has convinced us of the 'voice' of others in scaffolded discourse, which resonates with ideas inspired by Bakhtin (1981, 1986). In Bakhtin's view voice is the speaking personality, the speaking consciousness and always has a desire or will behind it (Bakhtin, 1981, p.434). By coming into contact with other voices, this desire (to affirm, reject, object and so on) reflects itself in the creation of an utterance which is "the real unit of speech communication" (Bakhtin, 1986, p.71) and is "a link in a very complexly organised chain of other utterances" (ibid, p.69). Bakhtin (1986) argues that a personal voice in producing particular utterances is shaped and developed in continuous interaction with others' individual utterances. This is what he calls the process of *assimilation* (ibid, p.89), which suggests that one's voice takes on and reproduces other people's voices and is saturated with the words and voices of others. Due to the value-laden nature of language (Bakhtin, 1981), a voice and its particular utterances always express a point of view and enact particular value judgements (Cazden, 1993). We now turn to our protocols and examine if they are saturated with the voices of others and if they enact particular values.

The task and protocol data

The protocol data excerpts, below, are taken from the protocols of two 17-year-old girls, H&S, working on an absolute value of linear functions task. H & S were two of 20 students selected for this work from a much larger sample who sat a test. The purpose of the test was to find students who had the prerequisite knowledge structures necessary to carry out the task but were not familiar with the task content. H&S worked for four consecutive days on four tasks without any time limitations. The first two tasks focused on sketching the graphs of, respectively, $|f(x)|$ and $f(|x|)$ by using the given graphs of $f(x)$. The third task was designed to consolidate the constructions made in the initial tasks. The protocol data excerpts below are from H&S's work on the fourth task which focused on sketching the graphs of $y=|f(|x|)|$: students were expected to develop a method to obtain the graph of $|f(|x|)|$ from the graph of $f(x)$. This task had five questions. The first question asked the students to draw the graph of $y=|(|x|-4)|$ and to comment on any patterns in the graph. The second question asked about the relationships between the graph of $y=|(|x|-4)|$ and the graph of $y=x-4$. In the third question the graph of $f(x)=x+3$ was given and the students were asked if they could draw the graph of $y=|(|x|+3)|$. In the fourth question four linear graphs without equations were given and the students were asked to obtain the graph of $y=|f(|x|)|$ for each of these graphs. In the fifth question, students were asked to explain how to obtain the graph of $y=|f(|x|)$ from a general linear function $y=f(x)$.

Protocol data

In this section we provide excerpts from the verbal protocols of H&S on the fourth task where the interviewer (I, the first author) scaffolds their work. The excerpts are presented in four episodes. In the first episode, over halfway through the task, H&S are 'stuck'. We insert words in square brackets in a few instances to assist the reader.

In questions 1 and 2 H&S sketched the graphs of $|f(|x|)|$ and $f(x)$ by substituting (see Figure 1-A and B). They commented on symmetries and possible reflections to obtain these symmetries. They then moved on to the third question and again sketched the graph by substituting (correctly, see Figure 1-C). They were, however, confused by the different shapes of the two $|f(|x|)|$ graphs, as can be seen in Episode 1 below.

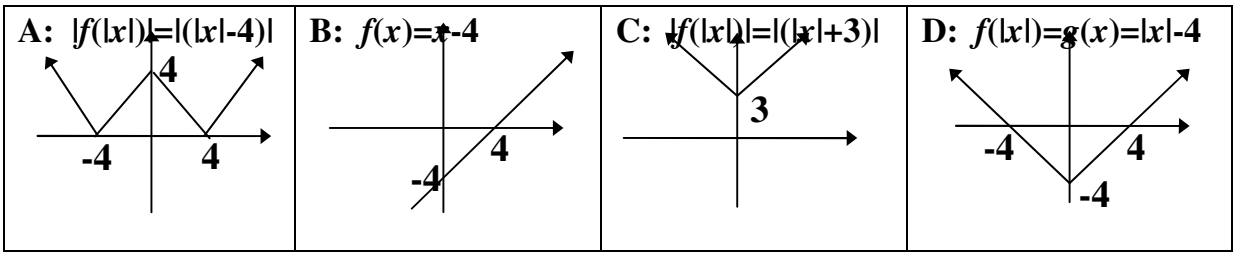

Figure 1: The graphs obtained by the students.

Episode 1

141H: I don't think we can ever understand how to use $f(x)$ to draw the graph of $|f(|x|)|$.

142S: The first graph was something like W-shaped… but this graph is V-shaped.

143H: They are totally different! How can we speak in a general way? Even this question made things worse rather than helping us.

144S: I think we better stick to substituting; we can answer the next questions by substituting.

145H: Yeah after all it is definitely working to draw this [$|f(|x|)|$].

In this episode the students, at this point in their development, failed to achieve the expected construction. They decided to "stick to substituting" because, for them, this method was "definitely working". Until this moment the scaffolder intentionally limited his assistance in order to observe how far H&S could go without his assistance. He now felt, however, that it was time to intervene and suggested (146I – not shown) returning to the first question and working on the task together. He then pointed out features of $|f(x)|$ and $f(|x|)$ that H&S constructed in the first two tasks and suggested that they keep these in mind in this task. We resume the protocol at 165I.

Episode 2

165I: Ok, if you pay a closer attention to the equation… I mean look at the expression itself, $|f(|x|)|$, it is a combination of these two [of $|f(x)|$ and $f(|x|)$]. Do you see that?

…

170H: … look, if $|f(|x|)|$ is a combination of $f(|x|)$ and $|f(x)|$, can we think about it like a computation with parentheses?

171I: Computation with parentheses?

172H: I mean for example when we are doing computations with some parentheses like… let's say for example, (7-(4+2)), then we follow a certain order…

173S: Right, I understood what you mean… we need to first deal with the parenthesis inside of the expression, is that what you mean?

174H: Yeah, I think it is somehow similar in here, I can sense it but I am unable to clarify…

175S: I know what you mean but how could we determine the parenthesis in here?

176I: You both made an excellent point. Ok, let's think about it together! In the expression of $|f(|x|)|$, can we think about the absolute value sign at the very outside of the whole expression as a larger parenthesis, which includes another one just inside?

Following the scaffolder's prompt in 165, H (170) suggested an analogy with arithmetic (computational order and parentheses; NB this was not consciously in the scaffolder's mind when he mentioned "combination" in 165). As in many countries computational order is emphasised in Turkish junior high school classes. S shared this recognition (173) which provided them with a starting point to build a method. They were not, however, clear how they could "determine the parentheses in here". The scaffolder intervened (too soon?) and explained how computational priority might work with absolute value signs. This provoked an immediate 'aha' from H.

Episode 3

177H: Aha, I got it… I know what we will do.

178I: Could you please tell us?

179H: We can consider $f(|x|)$ as if it was the smaller parenthesis!

180I: Smaller parenthesis?

181H: I mean it should be the first thing that we need to deal with

182S: Yeah, I agree… I think we should begin with the graph of $f(|x|)$ and first draw it

183H: But what next?

184S: Then we can use the absolute value at the outside… in the similar way of doing computations.

185H: But we will be drawing graphs! Can we really do this?

186S: I am not too sure if we can… but it sounds plausible…

187I: What you are doing here is not computation of course… but you are making an analogy, (…) and I see no problem with that… let's draw the graph by considering what we've just talked about and then decide if it will work or not, huh?

H&S collaboratively built a strategy (H initiating and then S leading from 184) about how to use the structures of $|f(x)|$ and $f(|x|)$ in order to sketch the graph of $|f(|x|)|$. The scaffolder intervenes (too soon?) and suggests drawing the graph. H&S (not shown) construct the graph of $|f(|x|)|$ in two steps: first drawing the graph of $f(|x|)$ (see Figure 1-D) and then applying their earlier method to draw the graph of $|f(x)|$ to the graph of $f(|x|)$. H&S see that this gives the same graph as that obtained by substituting.

Episode 4

H&S proceeded to draw the graph of $|f(|x|)|$ for the third question using their two-step method and concluded that the method was correct. In response to the fifth question, H&S gave an account of their new method which is reproduced below. Their account undoubtedly shows that they constructed a new method which was not available before (see episode 1).

243S: First using the graph of $f(x)$ we obtain the graph of $f(|x|)$ and then obtain $|f(|x|)|$…

244H: To do this, first when drawing $f(|x|)$, part of $f(x)$ at the positive [values of] x remains unchanged… umm then this part is taken symmetry in the y-axis and err and also part of $f(x)$ at the negative [values of] x is cancelled. After that, we apply absolute value to this graph and for this… umm… negative values of y are taken symmetry in the x-axis and thus we obtain the graph of $|f(|x|)|$.

Discussion

Two things are clear to us in these protocol excerpts: three voices, scaffolder and scaffoldees, interact; the dialogue takes place in the ZPD (H&S constructed a method for drawing $|f(|x|)|$ which they were not able to do at the beginning of the fourth task, and the help of the scaffolder assisted them in this construction). We take these two things as evident and do not discuss them further. Instead we discuss three things that are, perhaps, less evident but noteworthy: that there are other voices which reside in the participants' utterances which convey value judgements; that the voices have what we call 'pedagogic resonance'. i.e. the dialogue takes place within an implicit pedagogy; the discourse assists the formation of the construction. Please note that the discussion below sometimes focuses on scaffoldees' and scaffolder's utterances separately as if they were isolated utterances. This artificial separation is designed to aid clarity and should not be construed as a theoretical division of the discourse.

Voices within the students' utterances

Repeated readings of the verbal protocols in the light of Bakhtin's writings led us to identify several voices existent within the students' utterances. We focus here on two instances of those voices present in H&S's utterances: substituting (see episode 1) and the computational priority rule (see episode 2). At the outset of their work on this task H&S immediately began to draw the graphs of $f(x)$ and $|f(|x|)|$ by substitution. A common practice in Turkish mathematics classroom is to draw graphs by substituting. This is usually first introduced in Grade 7 (13-14 years old students) and although other methods are used later, e.g. using the gradient and the y-intercept in the case of linear functions, teachers tend to instruct students to draw the graphs by substitution. H&S's immediate use of, and apparent preference for, substitution over another method, e.g. breaking the equation into x-axis interval cases, is indeed related to their earlier experiences as mathematics students. Given this, H&S's teachers' voices are present in their utterances and their actions (using substitution to sketch the graphs). This is an example of what Bakhtin called 'ventriloquation' which is the "process whereby one voice speaks through another voice or voice type in a social

language (Wertsch, 1991, pp. 59).” H&S’s use of substitution also enacts particular values; they considered substitution as a ‘definitely working’ method (145H) and did not question the validity of this method although it gave them two very different V- and W-shaped graphs of $|f(|x|)|$ (see Figure 1-A and C). When they failed to construct another method, they both decided to stick to substitution (144S&145H); this certainly suggests that H&S attribute value judgements on substitution as a method.

Teachers’ voices were also, we believe, present when the computational priority rule was invoked (see episode 2). The order of precedence of arithmetic operations and brackets is the focus for a great deal of student work in Turkish junior high school mathematics classrooms and is tested in the high-stakes university entrance exam. Students are taught that if a computation involves nested parentheses, then they should work ‘from the inside to the outside’. When the scaffolder suggested considering $|f(|x|)|$ as a combination of $|f(x)|$ and $f(|x|)$ (165I), H recognised the computational priority rule (170H). Later S also recognised and further elaborated how to deal with the computational priority (173S). Although there are no parentheses (other than the ones enclosing x) in $|f(|x|)|$ and the scaffolder did not have parentheses in mind in his 165I utterance, H ‘heard’ parentheses; it was, we posit, again a ventriloquated teacher’s voice that she heard and repeated.

Voices within the scaffolder’s utterances and actions

A host of voices are present in the scaffolder’s utterances and actions. The main ones, in our opinion, are the voices of academics: authors he read and tutors whom he had discussions with as he started his PhD studies. We focus on authors rather than tutors below as it is, arguably, more objective to link scaffolder protocol utterances to printed statements than it is to link them to unrecorded verbal discussions.

Extensive reading is an important and formative activity for many novice researchers. In Fatih’s (first author) case, reading significantly contributed to his developing understanding of what scaffolding is and how to provide it. He studied scaffolding literature on human tutoring (e.g., Graesser and Person, 1994) and developmental research (Rogoff, 1990) and learnt of potential hazards involved in tutoring. Leseman and Sijsling (1996), for example, argue that tutoring may cast students into an essentially passive role and tutors may ignore the learner’s perspective. In addition strong value judgements that scaffoldees should be actively involved in the learning task (e.g., Mercer, 1995) were implicit, and often explicit, in Fatih’s reading, e.g. Chi et al., (2001), that students’ active responses play a crucial role in enhancing learning. Further to this Chi et al. (ibid.) state that tutors tend to give unnecessarily extensive explanations to the students even when they do not need it and Leinhardt (2001) argues that good explanations are those which are targeted at the students’ confusion, lack of understanding and misunderstanding. Fatih appropriated these value judgements and the voices of these researchers reside in the scaffolder’s actions and utterances, and shaped his approach to scaffolding H&S’s work: to support students’ autonomy; to obtain the active involvement of students in the tasks; to give ‘good’

explanations; to avoid unnecessary explanations and interventions; and to take the students' perspectives into account in the course of scaffolding their work.

We now turn to the protocol data and provide examples that reveal the voices of other researchers in Fatih's work with H&S. In order to support H&S's autonomy, Fatih gave them almost complete freedom until 145H, the point at which H&S were clearly having problems, were 'sticking to substituting' and were not developing a 'better' method. After this point the frequency of Fatih's regulative interventions increased considerably. He tried, however, to support H&S's autonomy and active involvement in the task by inviting them to develop their own insights rather than telling them how to work out things. For example, in the crucial 165I intervention he prompted H&S to see that $|f(|x|)|$ is a combination of $|f(x)|$ and $f(|x|)$ but left it to the students to take it further. In addition, he delayed giving further assistance until he felt such assistance necessary, e.g.: 176I comes after H "I am unable to clarify" and S "how could we determine"; 187I comes after H "Can we really do this?" and S "I'm not too sure". Finally, we feel that Fatih tailored the type and extent of assistance he gave to support H&S's developing insights and perspectives. For example, when H & S discussed the computational priority rule and related it to $|f(|x|)|$ in episode 3, he followed up their perspective rather than forcing them to follow a certain path that he had in his mind.

Pedagogic resonance

Other voices do not simply enter one's own voice, they are implicitly or explicitly appropriated by the individual. Further to this dialogic interaction assumes 'a tradition of discourse' between the agents. In a scaffolding process this tradition of discourse involves pedagogic discourse. We see pedagogy in the plural – there are pedagogies. Although it may be possible to make a case for more or less 'effective' pedagogies, this is not our interest in this paper. We view pedagogic practice as "the fundamental social context through which cultural reproduction-production takes place" (Daniels, 2001, p.69). Pedagogic resonance in a scaffolding process concerns scaffolder/scaffoldee(s) mutual understanding of the social context of cultural reproduction-production. Our histories as learners/teachers instil us with expectations regarding learning/teaching. If a scaffolder has, say, a particularly 'open' approach to teaching, e.g. tries to avoid 'leading' the student, and an adolescent scaffoldee has been taught in a 'didactic' manner from early childhood, then the scaffoldee may find the scaffolding experience frustrating and/or unproductive. We call this situation one of 'low pedagogic resonance'. In the scaffolded protocol excerpts used in this paper all parties directly involved were educated in Turkish state schools; they shared a common pedagogic basis. We explore their pedagogic resonance by examining the two 'too soon?' queries in the commentaries following episodes 2 and 3.

The first 'too soon?' refers to 176I. H immediately responded "Aha, I got it". We do not have a precise way to determine whether a scaffolder's intervention was appropriately timed but we feel that a scaffoldee's response of "Aha, I got it" indicates that the intervention was apposite. The second 'too soon?' refers to 187I; Fatih suggests that they draw the graph and they set about drawing it. There is not

space in this paper to include the protocol of this (see Ozmantar & Roper (2004) for 188H – 205S) but they: draw $f(|x|)$ without any problem; it is not what they expected and they are unsure of the next step; Fatih suggests renaming $f(|x|)$ as $g(x)$; H says "Aha, I can see it now" and explains to S they should draw the absolute value graph for $g(x)$; with some hesitancy and discussion of how to do this, they succeed in drawing $|f(|x|)|$. As with 176I we cannot be certain that the scaffolder's intervention was appropriately timed but the 'aha', their initial hesitancy and their eventual success make us believe that this was an apposite intervention for these students.

We are not denying that these two interventions were 'leading'; we are simply saying that leading interventions were appropriate for these two Turkish students because there was pedagogic resonance between the scaffolder and the scaffoldees.

Interaction of voices in the formation of the construction

H&S' performance can be viewed as having two distinct stages: (i) from the outset of their work until the end of the first episode; (ii) from 165I, the point from which the scaffolder regulated H&S actions and foci of attention. With regard to the ZPD, H&S's performance in the first stage might be considered within their actual development level as they worked independently with little assistance from the scaffolder. In the second stage, however, they appear to be within their 'potential development level'; they constructed, with the help of scaffolder, a new method. During this construction H&S recognised, used and reorganised mathematical features of $|f(x)|$ and $f(|x|)$. This reorganisation took place following H&S's recognition and utilisation of an analogical form of computational priority rule. H&S's efforts for this reorganisation were supported by the scaffolder's assistance and consequently they constructed a two-step method as an alternative to substitution.

In the formation of the new construction, the interaction of voices, from physically present participants and from the others not present in the activity but whose voices ventriloquate through the utterances and decisions of the participants, can be heard. Further to this the ventriloquation of these voices gives direction to the unfolding interaction. For example, teachers' voices of substitution and computational priority were utilised in the activity and a great deal of interaction evolved in and around these ideas. In a similar vein, the scaffolder's actions determined by the voices from the relevant literature which influenced his decision as to how to intervene in H&S's work. Could H&S have achieved the construction of a method without these voices being involved in the interaction? We do not feel we can answer this question as the construction was achieved with the involvement of these voices but, we believe, interaction and utterances always involve words and voices of others in one form or another. As Maybin (1993, p.132) states, "We have no alternative but to use the words of others, but we do have some choice over whose voices we appropriate and how we reconstruct the voices of others within our own speech" and, in mathematics, we find it hard to imagine making a construction without the voices of others.

Our final words concern value judgements. The appropriation of specific voices inevitably involves appropriating value judgements – other voices always express a point of view and pedagogic discourse is not an exception. But noting that there are value judgements and making value judgements on noted value judgements are two different things; this is akin to the comment in the 'pedagogic resonance' section above of noting that there are pedagogies and making a case for more or less 'effective' pedagogies. Fatih made an implicit value judgement that the graphical '$f(|x|) \Rightarrow g(x) \Rightarrow |g(x)| \Rightarrow |f(|x|)|$' was 'better' than H&S's substitution method, but is it?

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UNCERTAINTY AND DIFFERENT STYLES OF FRIENDLY CONVERSATION

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Abstract:*I conducted a teaching experiment with two of my upper secondary classes in Northern Finland. We studied the basics of calculus, using an investigative approach and small group setting. In this paper I shall discuss what kinds of cultural features could be seen in the interaction of four small groups. I found that the girls in the groups could be interpreted to express more uncertainty than the boys, through language and other means. Differences between the style of interaction of boys and girls can be described by the concept of sociolinguistic subculture. What I interpreted as signs of uncertainty in the girls, can also, at least partly, be seen to be a typical way of girls to talk in friendly conversations.*

Keywords: gender and language; modality; self-efficacy beliefs; self-confidence; sociolinguistic subcultures; teachers as researchers; uncertainty.

1. Theoretical framework

1.1 Uncertainty, beliefs and language

Students' self-beliefs in mathematics can be divided into self-concept and self-efficacy beliefs (Pajares and Schunk, 2001). These represent different views of oneself. Self-concept is a description of one's own perceived self accompanied by an evaluative judgement of self-worth. Self-efficacy beliefs refer to a student's beliefs about her or his competence in the domain of mathematics. One perspective in this is the expectancy of future success in mathematics, and the other is the evaluation of one's own performance. Positive self-efficacy beliefs are often called self-confidence. In this research, I see uncertainty as the lack of self-confidence. According to Laine et al. (2004), experiences gathered during studying mathematics influence self-beliefs, and on the other hand self-beliefs, especially self-confidence, influence actions while studying.

In linguistics, modality means that a sentence, in addition to the content of the proposition, always conveys information about the relationship of the speaker to the content (Häkkinen, 1998). Through suitable sentence structure, mood, tense, negation or vocabulary people express beliefs and attitudes or distance themselves from the propositions they make. Epistemic modality enables the speaker to indicate her or his commitment to the truth of a proposition (Rowland, 2000).

Hedges are a class of words and phrases which turned out to be central in Rowland's study on vague aspects of mathematics talk. He divides them according to a taxonomy developed by E.F. Prince et al (1982). Markers such as 'I think',

'probably' and 'maybe' are called Plausibility Shields. They implicate a belief to be discussed as well as some doubt that it will be fulfilled by events, or stand up to evidential scrutiny. Among the other types of hedges are the Adaptors, which are a category of Approximators. Phrases like 'a little bit', 'somewhat' and 'sort of' are used to attach vagueness to nouns, verbs or adjectives associated with class membership. Rowland reports that in his study, in which primary school children were asked to make mathematical predictions and generalizations, when the pupils hedged it was more often than not in order to implicate uncertainty of one kind or another. Rowland reports (2000, p.126) that question intonation was also used in his data to hedge statements. Changing a statement to a question can work like Shields.

Many studies (Merenluoto, 2001; Soro 2002) report that girls in Finnish schools at secondary level are less confident with mathematics than boys. My research question is: Do girls and boys in the four studied small groups express uncertainty in their interaction? How do they do it, and what is the role of language in this?

1.2 Interpretation

Deborah Tannen (1993) argues that the "true" intention or motive of any utterance cannot be determined from examination of linguistic form alone. The same linguistic means can be used for different, even opposite, purposes and can have different, even opposite, effects in different contexts. Interpretation needs looking at the language, the cultural and social context and the effects of what is said.

Feminist epistemologies emphasize that there are no objective researchers (Ronkainen, 1999). One cannot separate what is known from the knower, and what is observed from the observer. A researcher is allowed to utilize her or his personality when analysing and interpreting. What is important, is that the knower and the status of the knowledge should be explicitly reflected. Anna Sfard (2001) has similar ideas about assessing what she calls effectiveness of communication:

> First we must always keep in mind that it is an *interpretive* concept: any assessment of communication is based on personal interpretations of the discourse. The speaker compares her intentions to the effects her statement had on an interlocutor; an observer-a passive participant-compares the intentions evoked *in him* by the different interlocutors he is watching and listening to. Different participants-and this includes the observer-may have differing opinions on the effectiveness of the same conversation. Thus, when it comes to the evaluation of communicative efforts, it is important to be explicit about whose perspective is being considered. (p. 49)

1.3 Different styles of friendly conversation – sociolinguistic subcultures

Daniel Maltz and Ruth Borker (1982) write about male-female miscommunication. Based on a wide range of research they argue that American women and men have differences in their conceptions of friendly conversation, by which they mean talk in informal, familiar settings. The rules for friendly conversation are learnt from peers at the age of 5 to 15, the time when boys and girls interact socially mostly with members of their own sex.

In their intimate and cooperative play in small groups, girls seem to develop friendships involving closeness, equality, mutual commitment and loyalty. Malz and Borker suggest that girls learn to do three things with words: 1)to create and maintain relationships of closeness and equality 2)to criticize others in acceptable ways and 3)to interpret, accurately, the speech of other girls. In order to maintain relationships girls need to learn to give support, to recognise the speech rights of others, to let others speak and acknowledge what they say. In activities they need to learn to create cooperation through speech. Girls also learn to criticize other girls without seeming overtly aggressive, without being thought to be "bossy" or "mean".

According to Maltz and Borker, boys play in larger, hierarchically organized groups. What is important is the relative status. Hierarchies fluctuate over time and over situation. The social world of boys is one of posturing and counterposturing. In the world of boys, speech is used in three major ways: 1)to assert one's position of dominance, 2)to attract and maintain audience, and 3)to assert oneself when other speakers have the floor.

Robin Lakoff (1975) describes typical features of "women's language" in the America of her time. Among other things, she mentions that women spoke with question intonation where one might expect declaratives: for example tag questions ("It's hot, isn't it?"). Women's speech also seemed to include hedges of various kinds. Lakoff argues that girls in America were taught to speak in a friendly way, to talk like ladies. They were socialised to believe that asserting themselves strongly isn't nice or ladylike. At the same time, they were forced to talk as if they were lacking self confidence, and, as a consequence of this, they were not considered persons to be taken seriously. Lakoff can be criticized about her methods, but, for example, Lindroos (1997) emphasizes that many of her observations still seem to be valid.

2. Empirical research

This study is part of the Teachers as Researchers tradition (Kincheloe, 1991), the critical element being less prominent, and it has many ethnographic features. I have been teaching in the upper secondary school Lyseonpuiston lukio since 1995. Before the teaching experiment took place during the term 2001-2002, arrangements were made to let me become acquainted with the students. I was the teacher in many of their courses, and I was the form teacher for one of the classes. There was a big difference between my status as a teacher and the status of the students. I felt, however, that learning mathematics was the area in my students lives that I naturally had access to because of my being their mathematics teacher.

I studied limits and the concept of derivative with two of my second year classes (students aged 16-17 years), using an investigative approach. One class had more structured questions and the other had questions as open as I dared to let them study with. I have restricted my data to include lessons connected to the concept of derivative only. The students worked in groups of three or four, and they were

allowed to choose their partners by themselves. It happened that almost all the groups consisted of girls or boys only. For video recordings, I chose one group of girls and one group of boys from each class so that the groups were as similar as possible. I shall refer to the group of girls of the open approach as GO (Anni, Jenni and Veikko), group of girls of the structured approach GS (Heidi, Leena and Maaria) and groups BO (Juha, Mika, Pekka and Reijo) and BS (Matti, Oula and Tapani) correspondingly. Although there was one boy, Veikko, in the group GO, the interaction of the students seemed originally to me rather girl-like.

This paper is part of a broader study, where I shall investigate the interaction of the students and how meanings connected to the concept of derivative developed in the small group interaction. I have got video recordings of discussions of students in the four small groups, introductions and finishing of the lessons, learning diaries of all the students and pre and post tests as my data. The results reported here are obtained by analysing the videotaped discussions in the small groups.

Because of the ethnographic nature of my research, I did not make hypotheses of the possible differences between girls and boys before working with the data. While writing the transcripts, I observed that the girls in the group GS very often gave short laughs, which I interpreted as a sign of uncertainty. I decided to try to find out whether the students in the four small groups expressed uncertainty, and how they did it. I also noticed that the style of interaction was somehow different in the groups of boys and in the groups of girls. Maltz and Borker's (1982) concept sociolinguistic subculture seemed to describe what I saw very well. The context where I discuss these questions is investigative school mathematics in small groups.

First I classified the ways I thought the students expressed uncertainty. Then I systematically looked for these ways through my data. While doing this analysis, I, no doubt, used my familiarity with the situations and students, and utilized my own ability to interpret discussions as a person accustomed to human interaction. How much self-efficacy beliefs are involved in behaviour that conveys uncertainty is not clear. I am not trying to state what the students really experienced or felt, but I am describing and interpreting their behaviour. In my research, it is the female mathematics teacher of the students, with an additional role as teacher researcher, who made the interpretations.

3. Results

It could be seen, in the peer interaction, that girls, more often than boys, talked and behaved in ways that could be interpreted to be expressions of uncertainty.

3.1 Short laughs

Girls in the group GS very often gave short laughs, when the content of their speech or the context conveyed the possibility of them feeling uncertain. Starting a new investigation may be a threatening situation. The students have no clear idea of the topic; it is their task to discover important ideas. When beginning an investigation

about the derivative function as the limit of the difference quotient, the girls had a discussion:

Maaria: Here should be zero per zero. How should it be? ... I am a bit lost. (Heidi and Maaria laugh a little).

Heidi: (Turns the page and the paper cuts a wound in her hand) Oh, no!

Maaria: Perhaps it should be divided something like this.

Heidi: Mm-m.

Maaria: But that makes it anyway.

Heidi: So how come this be zero? Or

Maaria: Yes. ... (Leena takes the calculator) But ... let me see. Why is here now zero, f(0)? (turns the previous page, so does Heidi) Perhaps it is so, but now I do not realize at all. (silence) But it must be zero (the girls laugh a bit).

Leena: [indistinct]

Maaria: (takes her calculator) If we should check perhaps. How did we do it?

(Heidi and Leena laugh a bit).

The first laugh occurred after Maaria said she can't find the limit of the difference quotient because the substitution gives zero per zero, and expressed herself to be a bit lost. The second time the girls gave a short laugh was after Maaria said: "I do not realize at all", and continued after a silent moment with a sentence meaning: It must be zero, because I can't see any other possibilities. Finally Maaria wanted to check the limit with her symbolic calculator, but she didn't remember how to do it and laughed together with Leena. Seen in their context, all the laughs occurred in situations where the girls didn't understand something or didn't know what to do. There is nothing funny in that. Giving a short laugh may be a way to release tension, which point is supported by the laughs of the girls elsewhere. I interpreted these laughs to be signs of uncertainty.

In addition to the girls in the group GS, Anni in the group GO very often gave short laughs in situations where one might possibly feel uncertain, and so did Veikko, the male member of the group. Laughs of this type were much more rare for other boys.

3.2 Questions instead of declaratives

Making propositions in a question form was more common among the girls than among the boys. This kind of utterance was particularly typical of Anni in the group GO and Maaria in the group GS. Interestingly Heidi from the latter group had a low frequency of this kind of utterance. The girls in the group GO, and the one boy, were calculating gradients of secants for different intervals in a time-distance graph starting from $x = 1$ by making the time interval longer and longer. Anni made a suggestion:

Anni: Should we put a little smaller intervals, or should we try in a way with integers?

Her idea of making the intervals shorter was not noticed by the others, maybe because it was not expressed very forcefully. After I had discussed with the group and advised them to make the intervals shorter instead of longer, the students began to do so. This made sense to Anni, and, after a while, she again proposed as a question that they should find the limit, an idea which was not taken up either.

In Finnish question intonation is not used to change a statement into a question, like in English. But the meaning of the propositions in question form of the girls in my data is very similar. Tim Rowland (2000) writes that question intonation is one of the prosodic hedges that are effectively Shields, which in his data normally conveyed uncertainty of some kind.

What is striking is that the greatest number of suggestions in a question form were made by the male member, Veikko, in the group GO. There is evidence in my data that this is connected to his losing his authority in the group. In the first few episodes of the data, Veikko was the leader of the group. He did not make suggestions in question form. He had to be absent for a few lessons due to a minor operation. When he returned the girls did not let themselves to be led by him as easily as before. From then on he made statements and suggestions as questions to a great extent, although every now and then the old Veikko was also there.

3.3 Tag questions

Related to propositions in question form is the way in which the girls in my data ask support for their suggestions by asking a question afterwards. This is very close to what Robin Lakoff (1975, p.15) defines to be a tag question, which in English is midway between an outright statement and a yes-no question. It is less assertive than the former, but more confident than the latter. The girls in the group GS were differentiating a polynomial function.

Heidi: Well, if you do it like this, then you can find it out. Now do it like this, one third times three x squared just here.

Maaria: Mm. Mm.

Heidi: Can´t I do it like this? (Maaria nods).

Heidi explained how to differentiate the function. By a question following her statement, she asked the others to confirm her method. Lakoff explains the standard use of tag-questions: “A tag being intermediate between a statement and a question, is used when the speaker is stating a claim, but lacks full confidence in the truth of the claim”. This is what Plausibility Shields are used for. Lakoff suggests that the usage of tag questions was especially a feature of “women’s language” in the American English of that time.

3.4 Denial of statements

Sometimes girls in my data totally denied their statements or suggestions. I did not see this happen among the boys. Anni in the open group of girls did so most often. All the other girls except Jenni from the group GO did so sometimes. Anni, Jenni and Veikko were discussing the meaning of the gradient of a secant in a time-distance graph.

> Veikko: Now what did that gradient of secant mean? (looks at Anni triumphantly) So the time and [indistinct].
>
> Anni: (doesn´t notice the expression of Veikko) I wonder if it is like something average? ... I don´t know.

Saying "I don't know" after her suggestion didn't mean that Anni cancelled what she just said. She continued with her idea until Veikko interrupted her and directed the discussion in his way. For me, Anni's sentence gives an impression that she lacked confidence in constructing mathematical knowledge in this situation. Oula from the group BS and Juha from the group BO showed something similar to Anni. But they expressed their uncertainty in a realistic way like *"Maybe."* or *"I am not quite sure."*

3.5 Hedging

Maaria in the group GS made extensive use of Plausibility Shields and Adaptors. Finnish words like *"varmaan", "kai", "luultavasti"* and *"vissiin"* meaning something like *"sure", "I guess.", "probably" and "I think"* were frequent in her talk. Another student who very often used Shield-like expressions was Jenni in the group GO. She supported Anni, but with uncertainty by expressions like *"Obviously.", "Perhaps it is good like this.", "Shouldn´t we?"* and *"I suppose so."*

Lakoff (1975) suggests that women's speech in general (in the America of that time) seemed to contain more hedges than the speech of men.

3.6 Other ways

All the girls in both of the groups studied, referred to lack of ability at least once; Leena from the group GO even 5 times. From the 8 boys only Reijo and Juha from the group BO did so (twice and once). In this category, I classified comments referring to the competence of the speaker or the group in mathematics, more generally than just about the task at hand.

The girls in the group GS simply did not answer many tough but essential questions on the investigations, and continued following the worksheet. Examples of such questions were the relationship between average velocity and gradient of secant, the meaning of gradient of tangent and the usefulness of the gradient function f´(x). Merenluoto (2001) interpreted the greater number of girls' missing answers in her study, to show that Finnish upper secondary school girls were less sure than boys in answering questions about real numbers and limits.

4. Discussion

Findings of Lindroos (1997) and Staberg (1994) support the view that boys are more assertive than girls in classroom discussions in Scandinavian cultures. Further analyses of my own data also point to the direction that the Maltz and Borker's theory can be applied in the Finnish context, too. I have chosen samples of typical talk (207 - 397 turns for each group) in my data. In these samples, boys brought new ideas into the discussions mostly as direct declarations and girls by asking a question. Boys interrupted the speech of their peers more often than girls, they gave more orders to each other and did more name calling and (more or less playful) teasing with words. In addition to the features described in the previous chapter, girls more often than boys, used positive back-channels like "mm" and "hmm".

In how many of the situations above was what I have interpreted as an expression of uncertainty, actually this – or just a different style of girls' talk?

Making suggestions and statements in question form instead of declaratives, which was two or three times as common in groups of girls than in groups of boys, may be giving space to others to express their ideas and thus, in Maltz and Borker's terms, recognising the speech rights of others. It may be inviting the partners to comment on the proposal or it may be emphasizing equal relationships, not being "bossy". Asking support by a question after a suggestion may be seen as a way to maintain co-operation. Maaria, who was the most able girl in the group GS made extensive use of hedges. By her talk she maybe avoided giving an expression of being superior to others and thus emphasized equal relationships. Giving short laughs might be releasing tension and expressing support to others in difficult situations.

In line with Lakoff's (1975) argument it seems to me that, because of not being expressed strongly enough, many high quality proposals by Anni in group GO and Leena in group GS were not noticed in the interaction of the groups. Brigid Barron (2000) described the problem solving activities of a successful team and an unsuccessful team of 6th-grade students. In the less successful team, a boy, Chris, faced difficulties in trying to get his correct idea through. Barren reported that Chris phrased his solution as a question and spoke softly. These characteristics served to mitigate the strength of his proposal. It has been documented in other real-world, problem-solving contexts too, that mitigated responses are less likely to be taken up by others (Linde, 1988). Is it harder for girls than for boys to get their ideas accepted in mixed gender small groups because of the way they talk and express themselves?

As a teacher who wants to promote and support girls' studies in mathematics I find it difficult to cope with the claim (L. Hoffman, lecture, 2003) that teachers have different expectations for boys and girls and that they treat them in different ways. Do I give more challenge to boys in mathematics and do I demand less of girls? There must be unconscious cultural behaviour that I am not aware of. But what I know is that I treat in a different way students who, I think, express anxiousness and uncertainty than students who seem to be sure and confident with mathematics. The

former I try to convince that mathematics is not that difficult and the latter I urge to develop their thinking. Walkerdine (1989) has shown that teachers do make judgements about their pupils' mathematical potential which is influenced by their judgement about the pupils' level of confidence.

The girls' ways of expressing themselves are by no means deficiencies. Giving space to others to express their ideas, cooperation, supportiveness and equal relationships are important, desirable qualities in small group interaction. Furthermore Rowland (2000) describes The Zone of Conjectural Neutrality to be a space between what we believe and what we are willing to assert. The forms of linguistic shielding have the effect of reifying the ZCN and locating the conjecture in it, thus distancing the speaker from the assertion that she or he makes. In that case testing of conjectures happens more on a cognitive level than on an affective level. If girls feel more uncertain than boys, then they may have a stronger need for this and the language described above most probably helps them. But on the other hand, we can think that because of their way of talking, girls may be better equipped, than boys, with linguistic tools suitable for studying in an investigative way. They have the possibility of expressing tentative ideas to be discussed.

5. Summary

It can be interpreted that the girls, in the small groups studied, expressed more uncertainty than the boys in the peer interaction. They gave short laughs and doubted their ability more often than boys. In the group GS the girls left some difficult but important questions unanswered. The girls used linguistic strategies which worked like Shields more frequently than the boys. They asked questions and tag questions instead of direct statements. They could totally deny their suggestions. Some of the girls made an extensive use of Plausibility Shields. In the general trends, however, there were individual differences. The male member, Veikko, of the group GO, talked and behaved very much like the girls, and Heidi, in the group GS, had a low frequency in many of the features typical for the girls' talk in the data.

Differences in the girls' and the boys' interaction and talk may also be signs of sociolinguistic subcultures. What I interpreted as uncertainty, may also be seen as giving space to others to express their thoughts and maintaining cooperation. The girls may have emphasized equal and supportive relationships.

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GEOMETRIC REPRESENTATIVES IN PUPILS' COMMUNICATION

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Abstract: *The analysis of representations which pupils use in the teaching of geometry helps us describe their understanding of mathematical concepts. One of the ways to investigate pupils' representations is a semiotic analysis of their dialogues. The method of semiotic analysis is based on the identification of representatives and on observations of relationships among them. The term representative, which denotes an element of the semiotic representation system or a partial product of the representation system, is defined as a triad consisting of the representing, represented and representational components. The use of semiotic analysis is illustrated by examples from geometrically oriented experiments.*

Keywords: representation; representative; semiotic analysis; communication; understanding.

Research problem

Representations play an essential role in the process of mathematics teaching and learning because they help us grasp and understand abstract notions. The term representation is well known in mathematics education, however, its definition is considerably wide. The term representation means:

1. the expression or designation by some terms, characters, symbols, or the like, i.e. the semiotic system of representation;
2. the act of representing, i.e. a process in which the semiotic system takes part as a product or means;
3. the state of being represented, i.e. the mode in which the representation process is realised.

Duval (1995, 1998) states that semiotic representations are tools of expressing mental representations (to make them visible or accessible), therefore, they are indispensable for communication and the development of mathematical thought. He says that it is necessary to see objects of mathematical cognition neither as rationally independent matters nor as contents of mental representations, but rather as invariants in relation to several semiotic representations.

Halford (1993) speaks about representations in connection with understanding a concept. He states that "to understand a concept entails having an internal, cognitive representation or mental model that reflects the structure of that concept". A way to understanding leads through mental models, that is groups of interconnected

representations, therefore, we can assess the level of understanding according to features of representations used. However, the investigation of representations is only limited to external, sensorially perceptible representations.

The scheme in Fig.1 denotes relationships among perception (i.e. the state of perceiving something), representation (i.e. being represented) and understanding (i.e. concept grasping). Real objects or actions become vehicles of perceptible representations of a concept by means of semiotic systems. Mental representations are formed by processing perceptible representations in the mind and are structured to create mental models. The existence of mental models and their quality testify to the level of understanding.

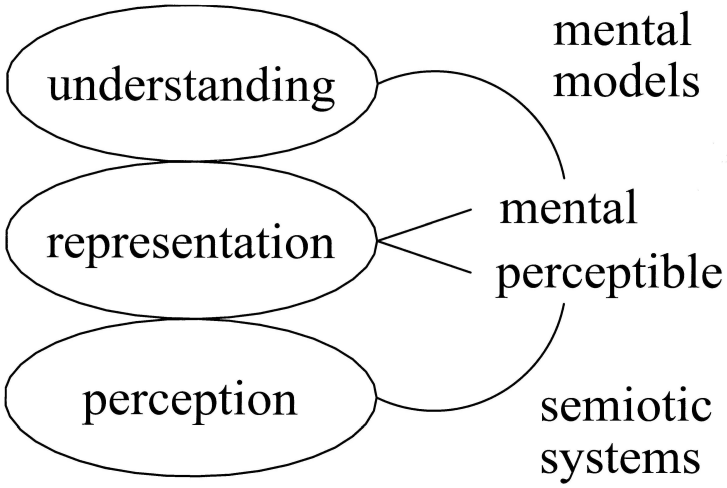

Fig. 1

The question how to investigate pupils' understanding of mathematical concepts by means of their representations has not been answered sufficiently yet. It seems that the investigation of representations in the frame of communication (for example, pupils' dialogues) is one of the possible ways. Sierpinska (1998) states that "language in mathematics education has always been an issue, but now the attention has shifted from the study of texts to the study of language in action ... the focus has moved from language to discourse". For the investigation of this domain of mathematics education, the tools of semiotics are applied increasingly often. The semiotic approach has become a new theoretical starting point of research in the didactics of mathematics (Roubíček, 2003). Winsløw (2004) considers semiotics to be an analytic tool for the didactics of mathematics which is applicable in cognitive, social or cultural levels of investigation. The main reason why the semiotic approach finds its use in didactic research is probably connected to the relation between semiosis and communication.

Methodology

In the analysis of pupils' works or statements, we seldom work with a complete system of representation. Most of the time, we deal with individual elements of this

system. To differentiate between a system of representation and its elements I have defined a new term *representative*. It denotes an element of the semiotic representation system or a partial product of the process of representation. I used the parallel between the didactic term *representation* and the semiotic term *sign*. The term sign is often understood in a narrow meaning as a mark (i.e. something independent of a subject). I, therefore, do not use this term either.

A representative (see the scheme, Fig.2) is a triad formed by three components: representing, represented and representational.

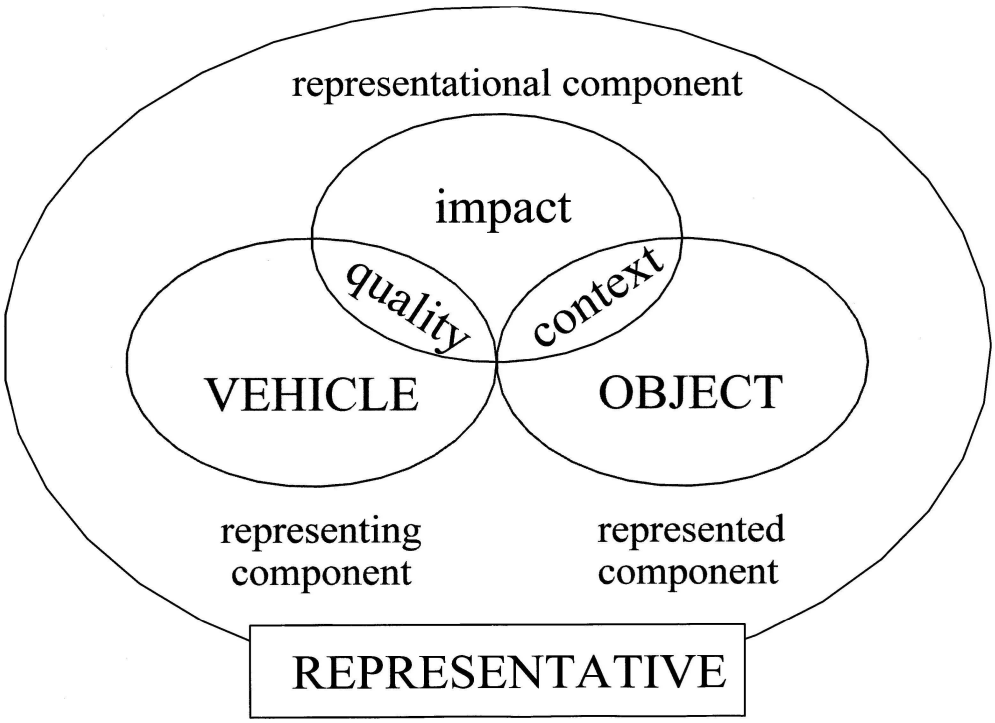

Fig. 2

The representing component or vehicle is something which represents the object being represented. Characters, lines, sounds etc. may be vehicles of representation. The represented component or the object is what is being represented. In mathematics education, the object of representation is most often a mathematical concept. The representing component and the represented component of the representative correspond to the signifier and the signified in Saussure's dyadic sign conception.

The relation of 'representing' between the vehicle and the object is determined by the representational component of the representative, which includes:

1. a qualitative property of the vehicle identical with the property of the object;
2. a context accompanying the process of representation and defining the object being represented;
3. an impact of the vehicle-object relation on the interpreter.

The kind of impact which the vehicle-object relation will have on the interpreter is influenced by his/her semiotic experience and knowledge. In his triadic sign conception, Peirce uses the term interpretant to denote this component.

The method of semiotic analysis which I use in my research is based on the identification of representatives and on the observation of relationships among them. On the syntactic level of the semiotic analysis, representing components of representatives are explored, as well as their mutual relationships, that is syntax. The vehicle-object relations, i.e. the meaning of representatives, are analysed on the semantic level. The pragmatic level of the analysis focuses on the exploration of the representational component of representatives, i.e. the usage of representatives. The definition of the semiotic analysis levels, which corresponds to Morris' division of semiotics into syntactics, semantics and pragmatics, is rather theoretical. In practice, it is not possible to apply it fully, because the description of the observed situation requires us to consider all the above aspects.

To apply the semiotic analysis as a research method, it is important to choose an experimental environment which enables us to identify representatives in their three components. I have designed a method called 'The Telephone' for the investigation of the semiotic representation of geometrical objects. This method is based on the fact that two pupils verbally exchange information on the shape of a geometric model or figure. They are separated by a screen (or perhaps sit facing in opposite directions, see Fig.3) in order to hear each other well but not to see each other. Both work with the same building block set, formed by a set of models of solids or figures. One pupil builds a model or figure using the set and describes it to the other one. The second pupil builds a model or figure based on the description provided. The pupils change roles in the next turn.

There are several modifications of this method. They concern the roles of the communicating persons and the means used. Relatively free communication rules and the same tools are suitable for younger pupils. More strict rules (i.e. with certain limitations) and varied means can be used for older pupils; for example, one pupil can work with a building block set, while the other one draws (Roubíček, 2002). The following text includes two illustrations of the semiotic analysis applied on a record of pupils' description of a geometric situation.

Illustration 1 (experiment *Mosaic*)

This illustration concerns two dialogues of two pupils (9-year-old girls, see Fig.3), describing mosaic figures. The figures were set up using eight identical right-angled isosceles triangles of two different colours and had the shape of a square (see Fig.4).

Fig. 3

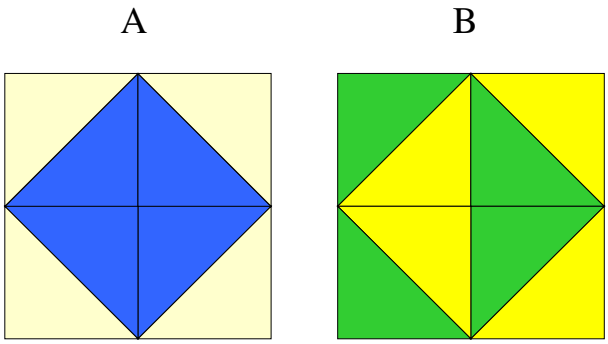

Fig. 4

The description of the mosaic figure A (see Fig.4 on the left)

A-1	Take four blue and four white ones.
B-1	Got it.
A-2	And then with the blue ones make this…rhombus.
B-2	Ahem.
A-3	Then put the white ones around the rhombus as if it was arranged into a square.
B-3	What?
A-4	Surround it with the white ones, so that it is around the rhombus… so that it is arranged into a square.

The description of the mosaic figure B (see Fig.4 on the right)

B-1	Make a rhombus. One half green, the other yellow.
A-1	And four of each (colour)?
B-2	Take two colours, green and yellow.
A-2	And four of each?
B-3	Yeah. Use the green ones to make one half of the rhombus and the yellow ones to make the other one.
A-3	And I use all four?
B-4	Well. See… One half… Take two green ones and two yellow ones. Use them to make a rhombus so that the yellow touches the edge of the other yellow.
A-4	And two?
B-5	What?

A-5 Shall I use two?

B-6 Put the yellow one on the left and the green one on the right.

A-6 Oh, this way.

B-7 And then out of the two that you still have put the yellow ones to the green ones so that it is arranged into a square. And again put the green ones to the yellow ones.

In both descriptions, the word "rhombus" was used to denote the rotated square inside the figure. From the point of view of communication, the word "rhombus" is a vehicle of representation and the square made from four triangles is a represented object. The word "rhombus" evokes an image in the mind of the second pupil and she models this image by means of four triangles. Although this representative is at variance with the mathematical terminology, in the pupils' communication it worked well and did not lead to misunderstanding. The "rhombus – rotated square" relation was apparently a convention for both the pupils, for it had the same effect on them. However, no convention existed for the expression "to arrange into a square". It was necessary to specify the property of the object using the words "so that it is around the rhombus". Afterwards, the expression "to arrange into a square" became a representative of the same meaning for both pupils.

The second dialogue shows symptoms of a communication incongruity. This incongruity was caused by an unclear context. The information on the size of "rhombus" was missing, namely, whether it consisted of four or only two triangles. In the first dialogue, it was mentioned already at the beginning that the "rhombus" is formed by four triangles. In the second dialogue, the number of triangles was only specified after repeated asking. Although there was a convention in using the representative "rhombus", the context had to be specified in greater detail for the given task, i.e. its representational component.

Illustration 2 (experiment *We build a house*)

This is a part of a dialogue between two pupils (14-year-old boys) playing the roles of a client and an architect. The client (C, see Fig.6) describes a model of the house (see Fig.5) and the architect (A, see Fig.7) makes a drawing (see Fig.8) based on the client's verbal description.

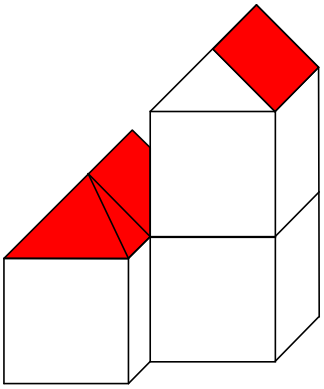

Fig. 5

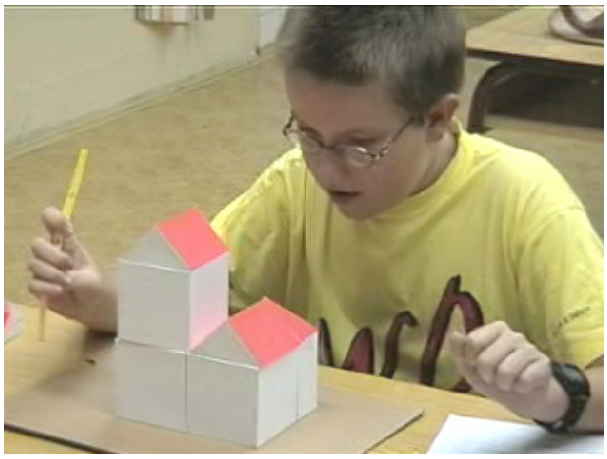

Fig. 6

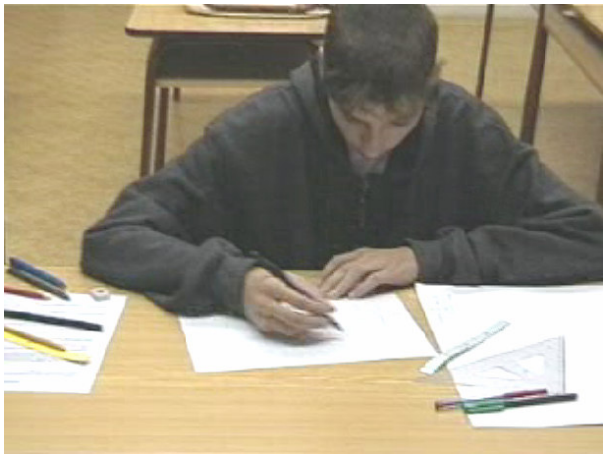

Fig. 7

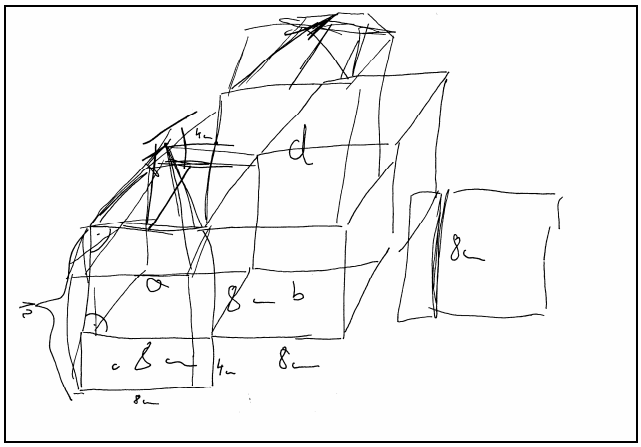

Fig. 8

C-19	Put a roof on the cube “a”. Now I will try to describe it. It is a completely normal roof, one that you can see on houses. Actually, its gable has the shape of a triangle. OK?
A-19	Wait. Once again. Repeat it.
C-20	It is a completely normal roof. Do you know what a roof looks like? Actually its gable, the side, actually the base...
A-20	Well.
C-21	…it is a triangle.
A-21	The base is a triangle?
C-22	Well, if you know what a base is?
A-22	No, I don’t. (Laughing.)
C-23	The base is something which has three sides... Do you know what I mean?

A-23	No.
C-24	Let's try another way. Try to divide the cube "a" as if optically into its upper part, the upper square... in two parts. OK? ... Just divide it in two parts.
A-24	But how?
C-25	To obtain a rectangle eight by four...
A-25	Yes.
C-26	...on the left and on the right, not towards you and in the back. Clear?
A-26	Yeah. I've got it. (Drawing the middle transversal of the upper face of the cube.)
C-27	And now actually raise the line you have...
A-27	Well, I raised it. (Drawing a line parallel with the middle transversal of the upper face of the cube.)
C-28	... actually up to the height of four. Yes? Do you understand? And now, you have actually one as if above, slide it down from its ends and you have a roof.
A-28	Yeah, I've got it now. (Sketching edges of a trilateral prism; putting down the dimension "4 cm".)

The client's task was to represent the verbally described object so that the architect recognises the object being represented and can represent it in a graphic form. The geometric object described by the pupil was a prism with the base of a right-angled isosceles triangle. In the illustration provided, the client represented the object "triangular prism" using the words "normal roof". This vehicle apparently did not evoke a single object in the architect's mind but a set of objects. In further description of the object, the client gave a qualitative property, "the base is a triangle". This information confused the architect because it did not correspond to the context: A triangular face of a prism cannot be put on a square face of the cube. The architect's image of the base was that it always was in the horizontal position. The vehicle "base" thus evoked a qualitatively different object in the architect's mind, differing from that represented by the client. Because of this communication incongruity, the client changed the description strategy. Based on an imaginary construction of the solid edges, the architect did identify the represented object in the class of objects "normal roof". This identification was allowed by the knowledge of the context: The architect worked with the building block set from which the model of the house was built. Also, thanks to this experience, it was not necessary to specify the shape of the triangular base.

Conclusion

In both illustrations, it is possible to register some pupils' incorrect conceptions or lack of comprehension. In the dialogue of the girls, it concerns the confusion of the terms "square" and "rhombus", in the dialogue of the boys, an incorrect interpretation of the term base by one of them appears. Thus, the above method is a tool for diagnosing the extent of pupils' understanding. On the basis of this analysis it is possible to diagnose not only pupils' misconceptions but also possible causes and remedies.

The investigation of representatives which a pupil uses in describing geometric objects allows to characterise his/her semiotic systems of representations and to obtain certain information on his/her mental representation of these objects, i.e. on conceptions the pupil has formed about them. Based on these indications, we can assess the quality of mental models which the pupil uses in geometry and thus evaluate the level of his/her understanding concerning geometric terms.

The investigation of pupils' understanding by means of semiotic analysis of representatives used in their discourses seems to be beneficial. Partial findings also indicate that the definition of the representational component of a representative as a component determining the relation between what it represents and what is being represented is necessary for the description of the investigated situation. However, in the interpretation of the representational component of a representative we have to distinguish the subject to which we relate it. To verify these hypotheses, it is necessary to carry out further investigations.

Acknowledgement

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SEMIOTIC PROCESSES IN A MATHEMATICAL INTERNET-CHAT

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Abstract: *In mathematical problem-solving situations via internet-chat, students are confronted with the fundamental issue of presenting their solution steps in written or graphic form. This circumstance provides the opportunity to study the genesis and use of inscriptions as defined by Latour and Woolgar (1986). In my paper, I present the results of a pilot study, in which primary school students allocated to two separate rooms solve mathematical problems by means of an internet-chat. Based on the research methods of Interpretative Classroom Research, Peirce's semiotic approach is applied to analyse the inscriptions emerging during the chat sessions.*

Keywords: inscriptions; semiotics; mathematical internet-chat; representations.

Introduction

The project "Mathematical Internet-Chat"[1] is about the genesis of 'mathematical inscriptions' in primary education: In an experimental situation, an internet-chat-setting, communication between pupils solving word-problems together is dependent on use of written/ graphical representations. This setting offers new insights into fundamental problems in the teaching and learning of mathematics, because mathematics depends on written forms of communication (Pimm 1987). It has been argued that students' understanding would benefit if they were asked to fix their solutions in a written form and reflect upon them (e.g. Pimm 1987; Morgan 1998; Fetzer 2003). Fixing ideas in a written form changes their status and makes them more explicit and conveyable (s. Bruner's "externalization tenet" 1996, 22-25). The focus of this paper is the written form of language in problem-solving situations in mathematics.

According to Latour & Woolgar, the interactively evoked chat products are called 'inscriptions' (Latour & Woolgar 1986; Latour 1987; 1990). Vocal interaction between the chat partners on either side of the setting is not possible, therefore it is necessary to externalise questions, hints, and solving-attempts in the chat-box or the whiteboard. This process is based on the 'chat-interactive' development of shared inscriptions: In an internet-chat-based dialogue, the pupils externalise their ideas by means of alphanumerical and/or graphical notations. They receive the reactions of their chat-partners whereby, step by step, the inscriptions evolve into a shared

[1] This study was supported by Müller-Reitz-Stiftung (T009 12245/02) entitled „Pilotstudie zur Chat-unterstützten Erstellung mathematischer Inskriptionen unter Grundschülern". Final report Krummheuer & Schreiber (2005)

inscription. Other publications have also focused on the interactive development of inscriptions (Roth & McGinn 1998; Lehrer et al. 2000; Sherin 2000; Meira 1995; 2002; Gravemeijer 2000; 2002), however in original face-to-face situations. In the mathematical internet-chat project, the difference is the focus on the exclusively inscription-based communication between the two poles of the chat-setting, which prevents vocal communication by means of the experimental design. It must be explicitly stated, that this experimental setting is not a suggestion for a new problem-solving, instructional, or learning environment, but a setting for research objectives.

In this project a semiotic instrument for analysing inscription-based mathematical problem-solving processes has been developed. Within these processes there are vocal utterances on each side of the chat-setting, which can also be analysed. This makes the analytic instrument even more powerful. For the broad field of language in mathematics education an analytic tool is provided, not only for inscription-based communication processes but also for vocal communication.

In the following sections, the term 'inscription' will be introduced, the triadic relation of signs by Charles S. Peirce will be presented as a tool of semiotic analysis with reference to Hoffmann's and Presmeg's work and finally, an example will be described and analysed based on the developed instrument in order to reconstruct semiotic aspects of chat-situations.

About inscriptions

Latour and Woolgar examined the development and evolution of knowledge in laboratory surroundings. The different kinds of models, pictures, icons, and notations used in the laboratories are classified by Latour and Woolgar as "inscriptions" (Latour & Woolgar 1986, 51f; Latour 1990, 22ff; Schreiber 2004c, 180). Inscriptions are seen by Latour and Woolgar as a very flexible means of representation that is continuously changing and improving. Thereby these inscriptions represent aspects of conceptual development during the research process. Also, in the mathematical internet-chat project, the focus is on the development of inscriptions in the interactional course of internet-chatting between pupils. Theoretically this process is seen as part of a chat-based interaction process, which produces, among other things, "taken as shared" meanings (Cobb & Bauersfeld 1995). Gravemeijer assumes that this development is a "cascade of ever more simplified inscriptions" (2002, 18). He describes a tendency in such cascades to move in the direction of a greater merging of figures, numbers, and letters towards even simpler and more meaningful inscriptions.

Roth & McGinn allude to the fact that using inscriptions is closely connected to the social practice in which they are produced:

> Inscriptions are pieces of craftwork, constructed in the interest of making things visible for material, rhetorical, institutional and political purpose. The things made visible in this manner can be registered, talked about and manipulated. Because the relationship between inscriptions and their referents is the matter of social practice … students need to appropriate the use of inscriptions by participating in related social practices. (1998, 54)

The project focuses in particular on the genesis of specific inscriptions. The use of the internet-chat setting - the chat-dialog box and the whiteboard frame - enables the evolution of shared inscriptions through an interactive exchange based on inscriptions. Internet-chatting supports the process of creating a text, as the distinction between the writer and the reader evaporates and becomes replaced by a process of collaborative production of a text.

Semiotic framework

For the analysis of the jointly created inscriptions in the chat-based solving-processes, I refer to Peirce's sign model (for a brief discussion on the choice of this semiotic approach see Schreiber 2004c, 186). The Peircean sign-relation consists of "a triple connection of sign, thing signified and cognition produced in the mind" (Peirce, 1.372). The three correlates in this triadic relation are specified in an elaborated definition (image 1):

> A sign, or representamen, is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign, or perhaps a more developed sign. That sign which it creates I call the interpretant of the first sign. The sign stands for something, its object. It stands for that object not in all respects, but in reference to a sort of idea, which I have sometimes called the ground of the representamen. (Peirce, 2.228)

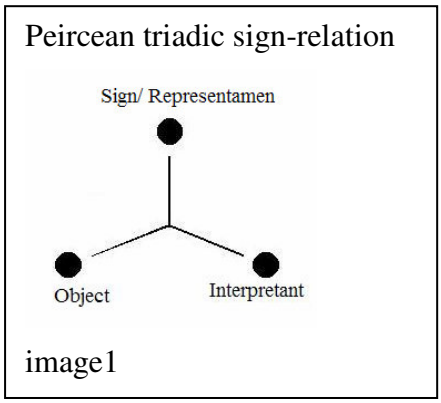

image1

Peirce subdivides each of the three correlates into three subgroups. In the chat-session presented below, 'sinsigns' ("an actual existing thing or event which is a sign" Peirce, 2245) and 'legisigns' ("a law that is a sign" Peirce, 2246) occur. According to Peirce, every conventional sign, a word in a language or a number, is a legisign. But in a single utterance, it is a sinsign. The sinsign is called a replica of its legisign. In my example, the relation of the different interpretants is in some cases 'rhematic' ("any sign that is not true nor false, like almost any word or number" Peirce, 8.337) and 'argumentative' ("a Sign of law" Peirce, 2.252) in others. All the objects are embedded in the sign triads in a 'symbolic' manner (s. Nöth 1990, 45).

Because of the potential confusion caused by using the word 'sign' for two of the three correlates (the representamen and the interpretant) and on other occasions for the whole triad, I will use in the context of the research project the term 'representamen' for one of the correlates and the term 'sign-triad' for the whole 'triple connection' described above.

Peirce describes signification as an on-going process, in which the interpretant of one sign-triad becomes a representamen of another. "Anything which determines something else (*its interpretant*) to refer to an object to which itself refers (*its object*) in the same way, the interpretant becoming in turn a sign, and so on *ad infinitum*" (Peirce, 2.303; italics by Peirce). For Peirce, each interpretant of a sign-triad can be

interpreted in a subsequent sign-triad. This continuous process of semiosis is potentially endless; it can not be "ended", but "interrupted" (Peirce, 5.284). An example of this 'chaining-process' is described by Sfard (2000, 45). Presmeg compares the chaining-process to "Russian dolls" (2001, 7). In her examples, one sign-triad is the *object* of the next triad. Gravemeijer (2000, 262) illustrates this chaining-process with a dyadic relation between the signifier and the signified. In contrast, in my example (s. 5.2), interpretants appear which serve as representamen in the next triad, and groups of triads which serve as one representamen in a further triads. Furthermore, there are triads which are linked together because they refer to the same representamen.

Applying and developing the Peircean approach, Hoffmann (e.g. Hoffmann 1996) focuses on the 'idea' or 'ground' in the Peircean sign model. He uses the term "das Allgemeine" (the "general", translated by Schreiber) instead. As examples for the 'general' Hoffmann mentions concepts, theories, habits, competences etc., which are given mentally or physically. The concept of the 'general' seems to be crucial for the analysis of the examples from the mathematical-internet-chat project. Therefore I integrate certain aspects of Hoffmann's approach (1996) with the classical Peircean triadic sign-relation: the Peircean triad is underlain with Hoffmann's 'general' (image 2). The interpretant is determined by the observer's concepts, theories, habits, competences etc.

The `general` as ground of the Peircean triadic sign-relation

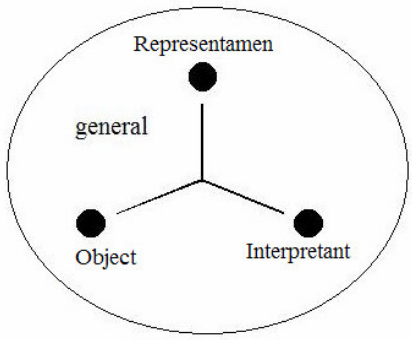

image 2

In order to reconstruct aspects of the inscriptions which evolve during the chat-sessions, semiotics appears to be an appropriate analytical theory. The communication between the two parties working jointly on word-problems is based on inscriptions. The initial analysis of interaction is supplemented by a semiotic approach. This analysis is the basis for the application of the semiotic analysis. With regard to the reconstruction of the 'general', the analysis of interaction is of particular help. It provides stable ground to build on concerning interactional aspects and the negotiation of meaning (Schreiber 2004b).

Organizational aspects of the pilot study

In order to offer the pupils an appropriate setting to communicate via chat, we use two Tablet-PCs with touch-screens and wireless connection. Using the software NetMeeting (Microsoft) the pupils can write in the chat-dialog-box and draw in the whiteboard-frame. All these activities on both computers are recorded as a screen video by the software Camtasia – Studio (Techsmith). Furthermore, the verbal communication of the pupils working together at the same computer is saved with a

digital voice recorder which is embedded in the computer. The three following chat-constellations have been realised: 1 pupil ⇔ 1 pupil; 2 ⇔ 2; 1 ⇔ 2.

The pupils are aged between 9 and 10 years and attend public primary schools in Frankfurt am Main (Germany). In 5 series of chat sessions, from October 2002 to December 2004, more than 80 sessions, each of approximately 40 minutes duration were recorded as a screen video. During this period, the setting, hard- and software, and the word-problems were improved continuously. Twenty-eight scenes have been transcribed and analysed. Early examples are described in Schreiber 2003a, 2003b.

An example

As space does not allow otherwise, only one example of my analysis using Peirce's triads is presented in this paper. The following excerpt concerns four pupils, two on either side of the setting (chat-setting: 2 pupils "Flippers" ⇔ 2 pupils "Sleepers") solving the following word problem jointly via internet-chat: *"A snail is at the bottom of a 3.2 m deep well. Each day it climbs 80 cm up the wall of the well. Each night it slides 20 cm down the wall. How many days does it take the snail to crawl out of the well?"* (see also Schreiber 2004a and 2004c).

In the first step, the scene is described, focusing on the perspective of the Flippers. In the second step, it is analysed using the approach described above. A transcript of this scene is presented in Schreiber 2004c, 189f.

Context and description of the scene

The following section is described from the perspective of the Flippers. They are the recipients of the inscription produced by their chat-partners, the Sleepers.

Whiteboard 0:15 min (Sleepers ⇔ Flippers)

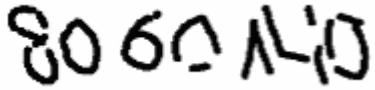

image 3

The digits 8 and 0 appear on the Flippers' screen. As they are fourth grade students with knowledge of the decimal system and implicit knowledge of the coherence of digits and numbers, they name this sequence of numbers "eighty." When the inscription is continued with the digits 6 and 0, they choose two alternatives with which they refer to the four digits: "eighty, sixty", as two two-digit numbers, and "eight thousand and sixty", as one four-digit number. In conjunction with the digits that follow, 1, 4 and 0, they read the entire inscription, as developed up to this point (image 3) as three distinct numbers "eighty, sixty, one hundred and forty-three". The final digit - zero - is read by the Flippers as three. The successively appearing digits 1 and 2 are first called "twelve", and as the next digit, 0, appears, "one hundred and twenty". But the Flippers do not refer to the seven

Whiteboard 0:30 min (Sleepers ⇔ Flippers)

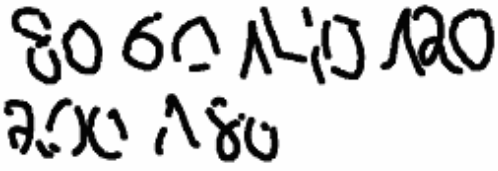

image 4

digits given before. Correspondingly, when the digits 2, 0, 0 and later 1, 8, 0 appear (see image 4), they do not refer to the digits given before either.

At this point an important change occurs on a conceptual level (with regard to the 'general' as described in 5.2): The Flippers refer to the entire inscription as developed to this point (image 4). Now the Flippers interpret the sequence of these numbers in terms of an arithmetic pattern, alternating plus 80 and minus 20. They also recognise that the additions and subtractions are related to the snails' path up and down the wall.

They come to this conclusion through abductive inference. This conclusion is now verified (image 5) by means of the following digits (260 240 320). While observing the completed inscription, the number 320 in particular causes the children to repeat part of the question in the given task: "how many?" This number is identified as the destination on the way up, whereas the other numbers are seen as stages and the gaps are seen as turning points. The Flippers continue to refer to the whole inscription in the context of the task and they use it to count the days: "one, two, three, four, fifth". They count pairs of numbers as one day and the last day as just one number.

Whiteboard 0:55 min (Sleepers ⇔ Flippers)

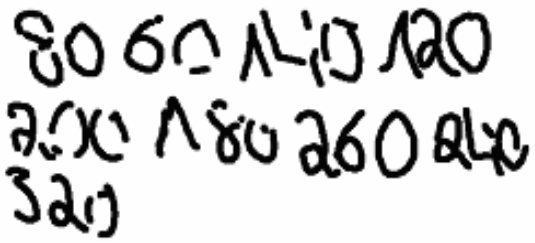

image 5

Analysis based on the developed sign-relation

I will now present an analysis using semiotic triads of the Peircean sign-relation, taking into consideration the concept of the 'general' and the chaining-process described above. The depiction that can be given with the developed instrument can be seen in image 6.

In this analysis, the representamen in triad no. 1 is the beginning of the inscription as described above. It is a rhematic sinsign creating two different interpretants: "eighty, sixty" and also "eight thousand and sixty". The first stands for its object, the numbers 80 and 60, the second for its object, the number 8060; both in reference to the same general, 'general I'. When the inscription is continued, the representamen in triad no. 3, also a rhematic sinsign, evokes in reference to the same general the interpretant "eighty, sixty, one hundred and forty-three", standing for the object 80, 60, 143 as three distinct numbers. Afterwards, the Flippers refer only to the part of the inscription shown in triad no. 4. This rhematic sinsign evokes the interpretant "twelve" in reference to the 'general I' standing for its object, the number 12. When a zero is added to the inscription, the representamen in triad no. 5, also a rhematic sinsign evokes the interpretant "one hundred and twenty" standing for its object, the number 120. Also in the triads no. 6 and 7, there are rhematic sinsigns evoking interpretants in reference to the 'general I', the knowledge of the decimal system and

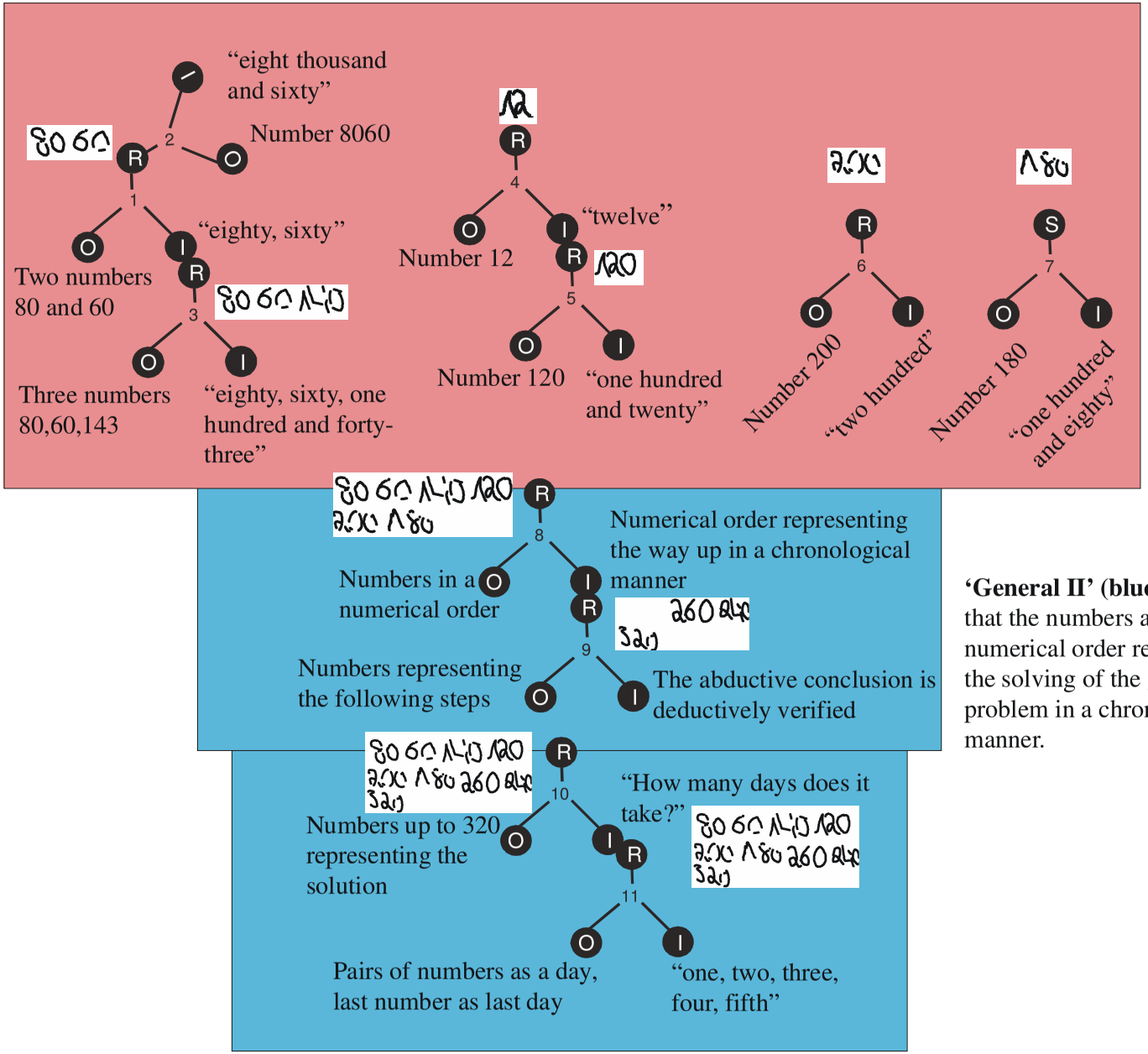

image 6

'General I' (red): knowledge about digits and numbers and the decimal system.

'General II' (blue): idea that the numbers are a numerical order representing the solving of the given problem in a chronological manner.

the implicit knowledge of the correlation of digits and numbers. As it is the same 'general', I use the same colour to underlie the triads. In the following triad, the change described above can be seen: the Flippers refer to the entire inscription as created to this point, in particular to the interpretants evoked by the inscription just before. The sign here is an argumentative legisign, because the relation between the numbers is satisfying a law. The 'general' ('general II') is an idea of these numbers as a numerical sequence representing the task in a chronological manner. The evoked interpretant, numbers signifying steps on the way up, is an abductive conclusion. In triad no. 9, the representamen is the continuation of the argumentative legisign. The evoked interpretant is the deductive verification of the abductive inference. The 'general' is also the idea of these numbers as a numerical sequence and the idea that it will represent the task in a chronological manner ('general II'). In triad no. 10, the representamen, the whole inscription with the evoked interpretants, evokes the rephrasing of the question in the task, referring to 'general II'. Its object is the representation of the solution by the numbers up to 320. In triad no. 11, the inscription which has so far been produced, together with the question of the task with regard to 'general II' evoke the interpretant "one, two, three, four, fifth". The represented object is the representation of days by pairs of numbers (80 and 60; 140 and 120; 200 and 180; 260 and 240) and the last day by one number (320) only.

Conclusions

As described above, the signs up to triad no. 7 are rhematic sinsigns. The signs in the triads no. 8 to no. 11 are argumentative legisigns. This change is linked with the change of the 'general'. While the 'general' is the differentiated knowledge of the correlation between digits and numbers and the decimal system ('general I'), the signs evoke possible numbers composed of the appearing digits. When the Flippers refer to the 'general II' numbers as a numerical sequence representing the solving of the given problem, they recognize the relation between the numbers satisfying a law, and they can verify the abductive conclusion, when the inscription is completed. All the sinsigns occur in a rhematic manner, whereas the legisigns occur in an argumentative manner. In either case, the object is related in a symbolic way.

Regarding the chaining-process one can see the structure of the on-going problem-solving process. The first two triads refer to the same representamen. The third triad is related to the first. The fifth is related to the fourth, while the sixth and the seventh seem to be independent. In image 6, the triads 1–7 can be seen as the representamen of triad no. 8. In the next triad, the representamen is the interpretant of triad no. 8. These two triads are the representamen of triad no. 10, which is the representamen of triad no. 11. The structure of image 6 reflects the problem solving process.

In the example, the Sleepers solve the given task. For them, the inscription which is produced is used to varying degrees either as a tool for solving the problem, or in order to communicate the steps taken and the solution to the Flippers. For the Flippers, the emerging inscription is a kind of developing representation of an attempt

to solve the problem by other pupils in their class. It is possible to compare the task with this representation and to recognize whether the latter is useful and perhaps advantageous, whether it is legible, and what information is apparent. Without promoting their own attempt to solve the problem, it does however provide them with a useful representation. Because they are able to read the individual steps taken, and also the solution, they verify the representation themselves.

The Peircean sign-model, enriched by focusing on the underlying 'general' and taking into consideration the chaining process, seems to be appropriate to analyse the genesis and use of inscriptions in the chat-sessions of the project presented here. In particular, the concepts, theories, habits, and competences of the participants are decisive for the emergent problem-solving and learning process. This process is well depicted as the described chaining-process. The 'general' in the internet-chat examples can be clearly recognized by carrying out an interaction-analysis on the basis of the prepared transcriptions. In further examples it is possible to compare various problem-solving processes by means of these analyses regarding the structure of the process in general, and steps, obstacles and coherences in particular.

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INTERACTION STRUCTURES IN PRIMARY SCHOOL MATHEMATICS WITH A MULTILINGUAL STUDENT BODY

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Abstract: *Inferior educational opportunity and school achievement among pupils with a migration background are, in international research, predominantly attributed to the socio-economic background of the pupils. This relationship seems plausible, but neglects aspects within the design of lessons. This paper shows initial results of a video-based empirical study[1], founded on intercultural and interpretative approaches. The initial analysis uses Gogolin's (1994) concept of the 'monolingual Habitus in German schools'[2]. I will present interim hypotheses of reasons within the lessons that explain inferior school achievement among the children in the German school system who have a multilingual background.*

Keywords: intercultural approach; qualitative approach; video-based; primary school; multilingual students; interaction structure; monolingual Habitus; oral language.

Introduction

In my research, I combine two research directions: intercultural education research (Gogolin, 1994) and interpretive classroom research in mathematics education (Krummheuer / Voigt 1991; Cobb/ Bauersfeld 1995; Krummheuer/ Naujok 1999). My goal is to make statements about learning processes in classroom teaching interactions in primary school mathematics. Within the scope of empirical research, I will describe aspects of a "classroom teaching culture" in primary school mathematics in which the student body lives and learns in two or more languages. For this purpose I will first have a closer look at the teacher's verbal actions[3] used to configure the lesson. This will be the main focus of the present contribution. I will then analyse the effects of the verbal actions on the interaction patterns (cp. Bauersfeld 1978; Voigt 1984) in the mathematics class, and complete my research with the exploration of the pupils' 'active participation'[4] within the lesson. Concerning the last two steps of my analysis, in the following, I will only clarify the theoretical background and give a brief outlook on the first cognisable features.

Theoretical position

Intercultural Education Research

In reference to intercultural research I refer to Gogolin's concept of monolingual habitus (1994) in German schools and the teaching profession. Within the teacher body there is to be found a historically grown basic attitude about organizing German schools monolingually and an assumption that schooling is best carried out through

the medium of German language. It can be argued that the German education system formed the habitus of a monolingual self-conception, which was supposed to produce citizens loyal-to-the-state. This self-conception has a lasting effect in German schools, especially as the origins and cause have sunk into oblivion. The aim of my research is to create a basis for modifying the present monolingually orientated action of the teachers. This foundation, according to Bourdieu, is to be achieved through the realisation of the habitus. For this purpose 'internalised structures, annexed worldviews, which rule the action have to be wrested from the subconscious [time changed]...' (Gogolin 1994, p.35 f.).

Interpretive Classroom Research in Mathematics Education

Interaction Patterns (Voigt, 1984)

Voigt (1984) developed a description system, which allows one to analyse short sequences of everyday (mathematics) lessons. The basis of this system is represented by the term 'interaction pattern' (cp. Voigt 1984, p.46 ff.). This traces back to Bauersfeld's term 'communication pattern' (cp. Bauersfeld 1978, p.159). Bauersfeld reconstructs a communication pattern with five phases, which he calls 'funnel pattern' (cp. Bauersfeld 1978, p.162). Based on this communication pattern, with respect to classroom teaching phases, Voigt (1984, p.128) reconstructs an interaction pattern, the so called 'elicitation pattern'. This refers to phases in which "new subject matter" or novel acquisition of familiar subject matter are supposed to be gathered (cp. Krummheuer/ Fetzer 2004, p.54). Furthermore, Voigt reconstructs the 'pattern of staged everyday occurrence' (1984, p.177). Within this, new unfamiliar subject matter is embedded in an everyday context by the teacher. There will be a first outlook on the presence of this pattern at the end of this contribution. The phases of the patterns of elicitation and of staged everyday occurrence are:

	Elicitation Pattern	Pattern of Staged Everyday Occurrence
Phase 1	'Task Constitution'	'Tie up to the pupils' out-of-school everyday images by the teacher'
Phase 2	'Fixation of the Solution'	'Pupils establishing relationships to out-of-school everyday images'
Phase 3	'Procedure Interpretation'	'Avert from out-of-school images of one pupil'
Phase 4		'Pupils signal comprehension'

(cp. Voigt 1984, S. 128/ 177f; Diagram changed in comparison to original.)

The Participation Model of Learning Mathematics in Primary School (Krummheuer/ Brandt 2001)

The participation model (Krummheuer/ Brandt 2001) allows analysis of dialogical learning processes in primary school mathematics classes. Through the model, and by means of the pupils' participation procedure, statements about possibilities of enabling mathematics learning may be made. Krummheuer/ Brandt (2001) designate learning advancement modes of participation with Bruner's term 'format'. According to Bruner (1983), a format is a '[...] standardised, initially microcosmic interaction pattern between an adult and an infant that contains demarcated roles that eventually

become reversible' (p.120 f.). This term contains the conception of learning as step-by-step accumulating autonomy of action within the frame of stabilized interaction structures. The accumulation of autonomy is evident within the adjustment of roles within the participation in collective arguments in mathematics class (Krummheuer/ Brandt 2001, p.20 f.). The methodology in this model is founded on interaction analysis, based on conversation analysis and Toulmin's functional argument analysis (1969). The participation analysis builds on this and can be subdivided into the reconstruction of the 'production design' and the 'recipient design' (cp. Krummheuer/ Brandt 2001, p.39). In relation to the speaker, the form of authenticity, originality and responsibility through the production design is determined; in relation to the recipients, the level of "being involved" through the recipient design is determined. In my research I especially focus on pupils speaking, evident in the production design.

The production design of a comment consists of decomposition of an utterance into acoustical realisation (speakers' function), verbal formulation (formulation function) and function of the content (content function) (Krummheuer/ Brandt 2001, p. 42). These three elements can be allocated to several persons. Through this there develops a category system of four different "participation types" and their responsibilities, to which different amounts of conducive to learning potential can be attributed.

My Position

One of my assumptions is that German monolingual pupils from "educated social milieus" have internalised language abilities and speech norms that are definitely more similar to the German teacher's norms than those of pupils stemming from lower socio-economic surroundings or different linguistic-cultural backgrounds than their teachers. The teacher's verbal action in a class with a lot of pupils with migration backgrounds is thus of great importance. Pupils with different migration backgrounds get the chance to tie their internalised language abilities and practices in with the patterns and routines utilized by the teacher.

Methodology

Methods of Data Acquisition

The methods used for this research take a qualitative approach. The empirical data consists mainly of transcripts of video recordings of "everyday" Mathematics and German lessons triangulated with complimentary, partly quantitative, background data on the participating students. Data collection took place in three classes in the 4th grade of two Hamburg primary schools where approximately 80% of the pupils had a migration background.

Methods of Data Evaluation

Interaction analysis is the basis of my evaluation methods. It originally dates from work in conversation analysis. In the 80s a lot of work in mathematical didactics accrued, utilizing this methodology. Among others, the work of Voigt (1984) and

Bauersfeld/ Krummheuer/ Voigt (1986) should be especially mentioned. With interaction analysis, the ways that individuals' meaning negotiations are constituted in interaction can be reconstructed. This means that interaction analysis can give information about how the teacher's verbal actions influence the meaning negotiations in the interactions between pupils and teacher as well as among the pupils themselves. The outcome of this is that it can be discovered how the teacher's verbal actions influence pupils' active participation and hence learning through autonomous achievement while participating in collective arguments in the mathematics class. (cp. Krummheuer/ Brandt 2001).

I am using Naujok's and Brandt's interaction analysis procedure (cp. Naujok 2000, p.43-46 und Brandt 2004 p.49-53) modified in two places. These modifications are briefly described below in order to explain the analysis of example scenes.

Interpretation Hypothesis of the Scene

Following Naujok's procedure (2000, p.43), the summary of the interaction analysis is illustrated at the end. Within this, interpretations that turn out to be coherent in the course of a detailed interpretation through sequential analysis of single comments and through turn-by-turn analysis are summarised and related to the focus. I call this step of my interaction analysis 'interpretation hypothesis of the scene' Brandt (2004, p. 55). Brandt (2004) also talks about 'interpretation hypothesis of the individual case' (p.55). Later, in the comparison process, different interpretation hypotheses of the scenes can be compared and thus, through their consolidation, unite the establishment of theory elements to theory development.

Analysis of the Teacher's Verbal Actions:

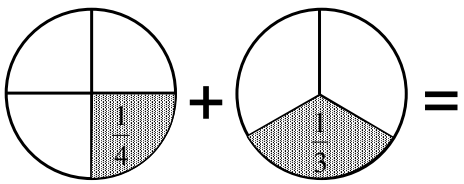
Within the analysis of teachers' verbal actions, I analyse teachers' language self-conception and the linguistic strategies they utilize to design their lessons. The aim is to get to know how teachers' language self-conceptions and linguistic strategies influence pupils' interaction pattern and active participation in the mathematics class.

In the following, I present results of the analyses of two example scenes. I embed the two scenes in the frame of the lesson, reproduce excerpts from the transcript and partially summarize the happenings. In the first scene 'LCM', I illustrate the interpretation hypothesis of the scene and the analysis of the teacher's verbal action. In the second scene, "mirror-game", I only analyse the teacher's verbal actions. In conclusion, I consolidate the analysis results of both scenes.

Analysis of results: 'lcm' and "mirror-game"

Embedding the Scene LCM in the Lesson's Frame and Transcript

The scene LCM originates from a conversation in a mathematics class. At the beginning of the lesson, the teacher (L) asks about the meaning of the abbreviation 'LCM', which the pupils cannot answer. The teacher then writes the term 'least common multiple' on the blackboard and finally lets the



pupils orally calculate multiples. Afterwards, she draws the diagram on the blackboard and writes ‘1/4’ and ‘1/3’ in the marked segments of the circles after the pupils name the values.

85 L: so\ well what comes out here/
86 Sm: Two sevenths\
87 L: (shakes her head) no no you are not allowed to do that\
88 S1: This is with minus seven
89 L: that ehm you are not allowed\ it would be nice if you were allowed then you would not have the problem\
90 ..any other idea/
91 S: Nine
92 L: why nine/ (discomposure / murmur)
93 S: hä/ one eighth\
94 so\ now tell me- Umit\ now tell me what is the least common multiple of three and four\ I have to search for
95 a number in which the three fits as well as the four\ (murmur)
96 Sm: Twelve

The teacher draws two circles, each with twelve segments, and marks three segments in the left and four segments in the right. Afterwards she asks for the number of marked segments. The pupils answer with ‘twelve’, ‘both are the same’, ‘twenty-four’, and ‘two fourths’. The teacher says:

117 L: don’t be overhasty\ (draws on the blackboard) I say I subdivide the cake into twelve pieces \ right/ I did
118 that (writes a twelve underneath the fraction stroke as the denominator next to the left circle) numerator
119 denominator (points at each) denominator says how many pieces there are in the cake\ and here I will do
120 the same\ (writes the same next to the right circle)..also twelve pieces right/ ehm...(goes over to the left
121 circle and wipes in it) how many pieces are in this big piece of pizza or piece of cake here in this third/ how
122 many are there

The pupils answer ‘four’ and ‘three’, whereupon the teacher writes a 4 and a 3 in the respective fraction’s numerator. Then she asks what happens if she shifts the four to the other circle. Most of the pupils answer ‘seven’. The teacher reacts:

149 L: right/ now you can write this down/ (writes right next to the equals sign a fraction stroke with a twelve in the
150 denominator)
151 S: seven\
152 L: twelfths\ and of this one two three four five six
153 Sm: seven ([L writes a seven in the numerator)
154 L: right/ you mustn’t- a big piece of pizza (points at the left circle) and a small one (points at the right circle)
155 add together that is unequal right/ you practically have to chop them into such pieces so they are all the same\
156 (does a chopping movement with her hand).. right/ these pieces are all the same\

		(points at the left circle) here
157		are just less\ right/ here are three and there are four pieces
158	S:	ah now I get it\
159	L:	and therefore you need this\. so you can sum up fractions- these pieces of cake after all\right/ you cannot just
160		say three and four is seven and above we take two so I have then I have two sevenths\ two sevenths is some
161		thing completely different\ right/ that doesn't work/

Interpretation Hypothesis of the Scene LCM

In the illustrated scene, the teacher tries to explain LCM to the pupils. For this purpose she chooses a circle and the example of pieces of cake/ pizza as an illustration. She asks for the sum of 1/4 and 1/3, to which the pupils give her different answers. The teacher seems to assume that the solution nine refers to a multiple of three and four, just miscalculated. From my analysis it emerges that this is to be doubted. Thus you obtain nine through summing all the numbers in the numerators and denominators or through summing the respective numerator and denominator of the solutions 2/7 and 1/8. Acting on her assumption, the teacher continues. She subdivides each circle into twelve segments and almost casually establishes denominator and numerator. As the teacher narrows down the answers, in a way consistent with the funnel pattern (Bauersfeld 1978, p.162 f.) and elicitation pattern (Voigt 1984 p.128.), the course of the lesson does not face any more barriers. Finally the pupils only have to add three and four.

Analysis of the Teacher's Verbal Action in the Scene LCM

The teacher seems to attribute "erroneous" answers (lines 86, 91, 93) to the lack of mathematical skills among the pupils. It is not apparent at any time that the teacher checks her understanding of what the pupils mean. It seems as if she does not take into consideration difficulties in comprehension or other interpretations of the task by pupils as a result of their language abilities. Furthermore she explains en passant 'numerator' und 'denominator' (line 118 f.) and 'fractions' (line 159). This means she trusts that these terms are self-explanatory. You can find a rudimentary explanation of one term in her comment 'denominator says how many pieces there are in the cake\' (line 119).

One of the teacher's strategies seems to be reduction of the degree of difficulty of her verbal action. Instead of using technical terminology, she starts utilizing everyday terms with partial technical didactical formulations (line 94 f., 119, 121 und 159 f). Within this, she does not mark linguistically the speech level she is using, which seems to be due to her language self-conception. Both levels seem to merge. She does not change back to the mathematical concept with mathematical technical terms.

Embedding the Scene Mirror-Game into the Lesson's Frame

The scene mirror-game takes place two minutes after the beginning of the third lesson. Beforehand the teacher talked to the class about some organizational things, which are not linked to the following scene as regards contents.

1 L: you all know a mirror\..and. you also what to do with it right/. soo. then I want to have
2 Rahim (R) here\ [(R. comes to the front) so Rahim you are my mirror now\.
3 L: (L lifts her hands up to her chest) that means everything I do. you do too\
4 R: okaaay\
5 L: so\. (L takes down her hands and back up) so\
6 R: (R takes up his hands)
7 >L: (moves her right hand to the right side)
8 >R: (R moves his left hand to the left side)
9 <L: (L takes up her right arm)
10 <R: (R takes up his left arm)
11 >L: (L takes up her left arm)
12 >R: (R takes up his right arm)
13 L: what does the mirror do/
14 <L: (L moves her left hand to the left)
15 <R: (R moves his right hand to the right)
16 S1: after\
17 >L: he repeats everything\. but get a closer look\. If I lift up my right hand
18 >L: L. lifts up her right hand)
19 >R: (R lifts up his left hand)
20 L: what does the mirror do/
21 Sm: the left\
22 L: the left

The teacher repeats the question 'what does the mirror do if I...' several times, asking for different parts of the body. The pupils are in each case able to give the correct answer. Then the teacher invites the pupils to line up in twos and try out the mirror-game themselves. Many do not do the reflections correctly in relation to mirror inversion. First the teacher helps; after a short amount of time though, she stops the sequence and says:

23 L: so today we want to learn with the mirror\ that means we want to have a look how to deal with a mirror
24 <L: Something
25 <S4: Deal
26 L: can double\ (.....) and we do have different things for that\. first you all know that there is a
27 mirror writing\. on the table (points at table 3) there are texts in mirror writing\. these you should
28 first try to figure out yourselves\. if that doesn't work you should look with a mirror
29 whether you can read it in the mirror\ then you should try yourselves to write some words on third table
30 >L: (L points at table 3) in mirror writing \. on this table-...
31 >L: (L. walks towards table 4 and takes a piece of
32 paper with letters from the table and shows it to Sa) are letters\. you have to write down letters
33 L: yourselves big block letters\ and then you should have a look whether there is any line on which
34 the letter reflects\. this line where the same ehm reflects is called
35 axis of symmetry\. well if you for example (L turns to the blackboard) have an „A" (L writes down big
36 „A" onto the blackboard)

37	S5:	then it is one half\
38	L:	(L turns back towards the class) then there is a line\ and if you hold this line against the mirror\
39	L:	you can see the „A" (turns towards the blackboard) although you don't see one half but
40	L:	if you here the line (L draws the Axis of symmetry inside the „A")
41	<L:	if you put the mirror over here, then you can see how this point here gets reflected\
42	<L:	(L draws dashed line left down side of the „A" to the right down side)
43	L:	this here\ (L draws dashed line from the right middle of the „A" to the left)
44	>L:	this remains unreflected
45	>L:	(points at the top of the „A")
46		hence you see the entire „A" although the mirror stands in front of it\ yes/ this line is called
47	L:	axis of symmetry....
48	Sa:	axis of symmetry \
49	<L:	ax..is of sym..me.try

Analysis of the Teacher's Verbal Actions in the Scene Mirror-Game

In the scene, the teacher seems to try to convey the inversion of a mirror through hands-on learning, as she performs reflections with Rahim. Within this a verbal explanation of the concept is omitted. That means the link between the action and the concept with its verbal demands must be established by the pupils themselves. After the presentation with Rahim, the teacher invites the pupils to try out the mirror-game themselves. She quickly notices that the pupils do not incorporate the concept in their actions. This is when she stops and starts explaining the stations for the following work. In this phase, the teacher introduces, among others, the term axis of symmetry verbally. She does not link the explanation to the previous action. This means within the itemised stations the pupils need the concept of mirror inversion, but they do not obtain an explanation about how the stations are linked to the mirror-game. This could be due to the teacher's monolingual self-conception. It seems as if the teacher assumes a causal connection between the pupils' acting and understanding[5], probably even in terms of being able to verbalize the underlying concept. This might explain her strategy of establishing content through action without verbal explanations.

Summary and outlook

If you add the two scenes' analyses together, it stands out that in the scene LCM, a mixture of elicitation pattern and pattern of staged everyday occurrence can be reconstructed, whereas in the scene mirror-game only the pattern of staged everyday occurrence can be reconstructed. I am interested in the extent to which pupils are given possibilities to learn through participation in collective argument formats within these patterns. For this a more detailed observation of the scenes with an intercultural perspective is required.

In both scenes a monolingual habitus is indicated. In the scene LCM my analysis shows that the teacher only introduces the terms en passant and does not check to what extent the pupils understand what she says. Furthermore, the teacher switches between technical terminology and everyday language without marking this. Thus, a

pattern based on monolingual habitus can be reconstructed. I entitle it: 'From Speaking to Understanding'. In the scene mirror-game, my analysis shows that the teacher performs actions to the pupils or lets them do the actions themselves. The teacher does not give verbal explanations about how the actions are linked to the concept the pupils were supposed to learn. Here too, a pattern based on monolingual habitus can be reconstructed, which I entitle: 'From Acting to Understanding'.

Both patterns contain the implicit thought that, through action or speech, pupils come to an "intuitive" conclusion and understanding of the terms or concepts respectively and furthermore are able to use them correctly. This may vary, but it is a school's everyday demand to make pupils verbalise what they understood. Surely they will not acquire this ability of comprehending texts through the teaching procedure described. It seems to be part of the classroom teaching culture in the observed classes not to explicitly teach linguistic features needed for verbalising (comprehended) terms or concepts, but instead to assume that pupils are able to conduct this autonomously. According to the insights of speech acquisition theory, this should not be expected of a child who lives and learns in two or more languages.

Within the described interaction pattern, considering the patterns of the teachers' verbal actions, enabling of learning through participation in collective arguments is restricted. The corpus of observed active participations in the classroom teaching periods considered is limited to pupil comments which can be assigned to imitation or mostly to "guessing", creator, status. According to Krummheuer and Brandt's category system they are to be classified as minor conducive. Further analyses for this are in hand.

Glancing ahead, one of my study's results could be to deliver helpful suggestions for modifications in primary school mathematics. I proceed from the assumption, following Gogolin (1994), that the described phenomena are shaped by habitus. This means the phenomena elude the observed teachers' consciousness. A basis for changing phenomena shaped by habitus can, according to Bourdieu, only be achieved by the realisation of the structures shaped by habitus. My analysis could contribute to greater "linguistic awareness" among the teacher body. This could lead to greater enabling of learning in German primary schools for all pupils, not just those having a migration background.

Notes

1 This concerns an ongoing doctoral study in the graduate research group on 'educational experience and learner development' of the University of Hamburg.

2 Throughout, quoted text is "framed" by single quotation marks. Terms used for the first time are also framed with single quotation marks. If a word such as "framed" is used in a metaphorical sense it is marked by double quotation marks.

3 In this case, I am talking about "the teacher's verbal actions with which she designs her lesson", as my study is within the paradigm of symbolic interactionism. Symbolic interactionism assumes that negotiations about meanings take place in interaction between individuals. This is not to be combined with the imagination of the teacher conveying the subject matter to the pupils.

4 The term “active participation” goes back to Krummheuer/ Brandt (2001, p. 38). Participation describes the “involved/ participating aspect of own action” (Krummheuer/ Brandt 2001, p. 18). Active participation focuses on the part of pupils’ participation in class that can be described by the term production design. (cp. Krummheuer/ Brandt 2001, p. 39)

5 I use the term “understanding” always to include the meaning of textual comprehension, as this is a typical everyday requirement of pupils in school, namely to verbalise what they understood.

Transcription conventions

Sm	several students/ pupils
Sa	all of the students/ pupils
Fat	emphasised speech
[note] note, commentarial remarks	
/ ; - ; \	raise voice; voice in the balance; lower voice
. .. ...	pause of speech in seconds
<L: The house is smaller\	“score spelling”
<S: is smaller	(partial) speaking at the same time
>M:	change of arrows’ directions indicates a new
>L:	a directly connected “score block”

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