# High school teachers' evaluation of argumentative texts in mathematics

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*The goal of this paper is to start an investigation on how high school teachers evaluate students' argumentative texts. In particular, as an argumentation requires dealing with some mathematical content, to produce a verbal text, possibly equipped with diagrams and formulas, and to make clear the links between premises and conclusion, we are interested in finding out whether teachers focus on all of the three aspects or on one or two of them only. We have gathered the comments of 12 high school teachers on five argumentations written by university science students in order to justify their answers to a problem involving formulas and graphs.*

*Keywords: Argumentation, language, evaluation, teachers, mathematics education.*

## Introduction

The importance of argumentative practices in mathematics teaching at all levels has been increasingly highlighted in the last years. In the transition from high school to university these practices are specially important. This holds also for the mathematics classes for science undergraduates, as it is often necessary to teach some basic concepts to large numbers of first-year university students with assorted levels of competence and motivation, in a short span of time. In this context, it might happen that some students try to learn some content by heart or to solve problems through strategies not based on some interpretation of the meanings involved. The practice of asking for argumentations to explain and justify their solution procedures when dealing with mathematical problems has proved effective both to assess students' competence and to help them to adjust their learning processes. Still, a number of students seem not to be acquainted with argumentative practices, regard them as abnormal and cannot tell the difference between a correct and justified procedure and a correct but unjustified one. These remarks induce us to guess that there are some problems about the development of argumentative skills in high school. So we decided to start a study about how high school teachers interpret and assess argumentative text produced by students, in order to see which aspects are, in their opinion, relevant. Throughout the paper we use some terms (such as 'proof', 'argumentation', 'argument', 'explanation', 'justification') in a broad sense, without referring to some distinctions proposed in literature, such as, for example, Duval's distinction between proof and argumentation (2007), or Johnson's distinction between 'argumentation' and 'explanation' (2000), or the distinctions between 'argumentation' and 'argument' proposed in literature (e.g., Johnson, 2000, pos.105; Bermejo-Luque, 2011, pos.68). We regard the proof of a theorem, as well as the justification of a resolution procedure of a mathematical problem, as texts that, in some way, describe and make explicit instances of the semantic relation of *logical consequence*[1]. We shall try to make this point clear in the a priori analysis.

## Theoretical framework

The evaluation of an argumentation involves at least three main aspects: the analysis of the ideas related to the semantic domain of the discourse, in our case mathematics; the linguistic analysis of the text produced and the analysis of how the links between the premises and the conclusion are made explicit. These three aspects are to some extent autonomous, since an argument might happen to be appropriate related to one or two of them and not to the other(s). As remarked by many authors, such as Bermejo-Luque (2011), an argument is, first of all, a piece of text. So language is to be regarded as a relevant factor in the analysis of arguments. We adopt a functional-linguistic perspective according to Halliday (1985) and Hasan (2005). This perspective fits well with frameworks assuming that language plays a major role in the development of thought, and also with a sociocultural, non-platonistic view of mathematics. In particular,we assume Hasan's description of the central role of verbal language (2005) and Halliday's definition of the three metafunctions of language, ideational, interpersonal and textual (1985). In short, the ideational metafunction is related to the representational meaning, i.e. what the text is about. The interpersonal metafunction is concerned with interactional meaning, i.e. what the text is doing as a verbal exchange between people. The textual metafunction regards the organization of the message, i.e., how the text relates to the surrounding discourse, and to the context of situation in which it is produced. We believe that in a context of communication like a classroom, all of these components are involved and so are to be taken into account. In particular, the textual metafunction is closely related to argumentation, since explaining the links between premises and conclusion requires a well organized text. The debate on argumentation theory is wide and complex and involves a number of stances. In our opinion both the specific features of the semantic domain of mathematics and the fact that an argumentation is essentially a text are to be taken into account.

## Goals

The main goal of this study is to begin to understand how high school teachers of different subjects evaluate the written argumentations produced by students, and in particular what weight they assign to each of the three aspects mentioned above, and how. This is important in order to reflect on the transition from a model of learning based on the acquisition of content only to a model aimed at developing the competencies required to take part in a discourse on such content. Arguing on a subject requires a different and somewhat deeper understanding of the subject itself.

More specifically, our main research question is: do the teachers involved also take into account the logical structure of the arguments they are evaluating (i.e., how the parts of the argumentative text are linked each other)?

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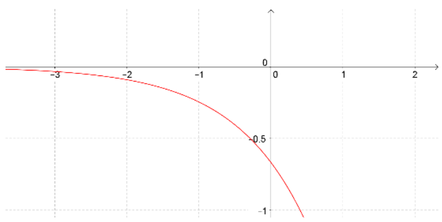
[1] A sentence $P$ is a *logical consequence* of a set $\Gamma$ of sentences if any interpretation satisfying all of the sentences of $\Gamma$ satisfies $P$ as well.

## Methodology

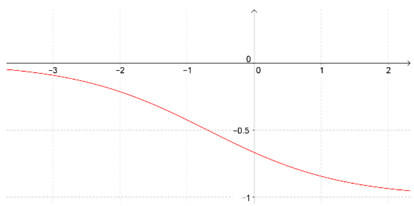
We asked a group of 12 high school teachers of different subjects (9 Mathematics, 2 Philosophy, 1 Science) to evaluate the argumentations proposed by 5 first year science undergraduate students to justify their answers to a problem administered during a tutorial session. The teachers were involved in a long-term in-service training course focused on transversal competences. We asked them to write down their evaluation of the arguments and any comments they felt appropriate. In particular we asked them to state if they regarded the arguments acceptable for students at the end of high school, with regard to mathematical correctness, linguistic adequacy and explicitation of the links between the parts of the argument. Afterwards we interviewed them. The aim of the interviews was not to elicit further information but to be sure of our interpretation of what they meant in their written evaluations, as it was possible that teachers of different subjects might use some words with different meanings. One example of such a word is ‘formal’ [‘formale’], which can be associated to a range of different meanings, such as ‘independent from content’ or ‘conformed to a given pattern’ or even ‘related to structure rather than to function’. The problem under discussion is shown below.

Let *f* be the function defined on real numbers by: $f(x) = -\dfrac{2}{e^x + 2}$. Among the following graphs one is associated to *f*. Mark it and explain in detail.

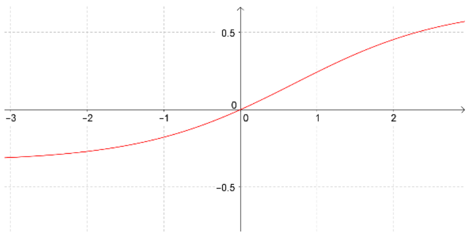
A



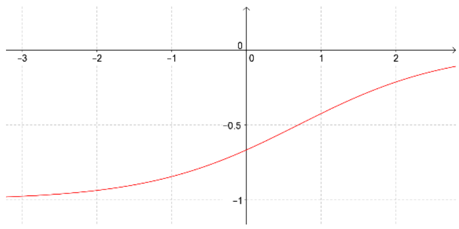

B




C




D




A good translation from one language to another, which is usually based on grammar, often succeeds in rendering the ideational meaning of a text. With interpersonal and textual meanings there are sometimes problems. In particular, the textual organization of Italian is quite different from English. For example, in the text of Student 1 some subjects are missing. In English this is both a grammatical and a textual error, whereas in Italian it is textually inappropriate but not grammatically wrong. In the translation, if we put the subjects, as we have been taught at school, we hide the textual inadequacy of the text, if we do not, we add a grammatical error (which affects the ideational meaning) to a text which was just textually inappropriate. So we have added the Italian original version.

Student 1: I exclude the graphs A and B because we have calculated $f'(0)=2/9$ and 2/9 is the slope of the tangent line to *f(x)* in $x_0$. As is positive, the function is increasing, but in *A* and *B* we see that decreases => I exclude them. Moreover, doing the sign of the function $-2/(e^x+2)>0$ we notice that the function is always negative, because of t$f'(0)=2/9$ he minus in front of the function (otherwise positive). Then we exclude the graph *C*, since from 0 onwards it is positive. The choice then falls on graph *D*, correct.

[Escludo i grafici A e B poiché abbiamo calcolato e 2/9 è la pendenza della retta tangente a *f(x)* in $x_0$. Essendo positiva la funzione è crescente, ma in *A* e *B* notiamo che decresce => li escludo. Inoltre facendo il segno della funzione $-2/(e^x+2)>0$ notiamo che la funzione è sempre negativa, per via del meno davanti alla funzione (altrimenti positiva). Allora possiamo escludere il grafico *C*, poiché da 0 in poi è positiva. La scelta ricade allora sul grafico *D*, corretto.]

Student 2: The exact graph is *D* since by computing $f(0) = -2/3$ I found the point where the function cuts the y axis, that is $-2/3$. Furthermore, by computing $f'(0) = 2/9$ I see that the function in zero is increasing because the derivative in 0 is positive.

[Il grafico esatto è il *D* poiché calcolando $f(0) = -2/3$ ho trovato il punto in cui la funzione interseca l'asse *y* cioè $-2/3$. Inoltre calcolando $f'(0) = 2/9$ noto che la funzione in zero è crescente poiché la derivata in zero è positiva.]

Student 3: Surely graph C does not correspond to *f(x)* as it passes through the origin. Graph *A* does not correspond to *f(x)* since it is decreasing in $x=0$ whereas $f'(0) = 9/2$ which is positive, so *f(x)* for $x=0$ is increasing. Graph *B* does not correspond to *f(x)* since it is decreasing in $x=0$ whereas $f'(0) = 9/2$ which is positive, so *f(x)* for $x=0$ is increasing. The graph that most likely represents *f(x)* is *D*.

[Il grafico *C* sicuramente non corrisponde a *f(x)* in quanto passa per l'origine. Il grafico *A* non corrisponde a *f(x)* in quanto è decrescente in $x=0$ mentre $f'(0) = 9/2$ che è positivo quindi *f(x)* per $x=0$ è crescente. Il grafico *B* non corrisponde a *f(x)* in quanto è decrescente per $x=0$ mentre $f'(0) = 9/2$ che è positivo quindi *f(x)* per $x=0$ è crescente. Il grafico che più probabilmente rappresenta *f(x)* è il *D*.]

Student 4: Graph C does not correspond to *f* as it passes through the origin. I know that in $f'(0)$ the derivative is positive, so its function must be increasing in that point. In *A* and *B* this does not happen. So the graph of the function is letter *D*.

[Il grafico *C* non corrisponde a *f* in quanto passa per l'origine. So che in $f'(0)$ la derivata è positiva, quindi la sua funzione deve essere crescente in quel punto. In *A* e *B* ciò non avviene. Quindi il grafico della funzione è la lettera *D*.]

Student 5: We exclude letter *C* at once since for $x=0$ it passes through point (0; 0) while $f(0) \neq 0$. If we consider that $f'(0)>0$ holds, the function is increasing in 0 so I exclude letters *A* and *B*, as they represent functions decreasing in 0 and a differentiable function decreasing in a point has a non-positive derivative in that point.

[Escludiamo subito la lettera C perché per $x=0$ passa nel punto (0; 0) mentre $f(0) \neq 0$. Considerando che vale $f'(0)>0$ la funzione *f* è crescente in 0 quindi escludo le lettere *A* e *B*, in quanto rappresentano funzioni decrescenti in 0 e una funzione derivabile decrescente in un punto ha la derivata non positiva in quel punto.]

## A priori analysis

Students are requested to find which graph out of a set of four corresponds to *f* and to justify their answer. Teachers are requested to evaluate those explanations. The relation linking the answer to the data is that of logical consequence: the fact that graph *D* corresponds to *f* is a logical consequence of the data. In other words, students have to illustrate a mathematical, semantic relation by means of a verbal text, with the option of adding symbolic expressions and other signs. Basically students had at least three kinds of data: (1) those available from the definition of *f*; (2) those available from the four graphs; (3) the 'closure' condition "Among the following graphs one corresponds to *f*". The sentences that are logical consequence of the definition of *f* (in the context of classical mathematics) are of course a countable infinity. Of course we expect that students will focus on a small choice of them, such as, for example, $f(0) = -2/3$ or $f(0) < 0$ or $f'(0)=2/9$. As regards graphs, the sentences obtainable are infinite too. The process is a bit more complex as the interpretation of graphs anyway requires some conventions. Properties like domain, continuity, differentiability and passage through given points cannot be inferred from the graph alone. At any rate, it is crucial for science students mastering multiple representations. The competencies required to devise counterexamples showing that properties of the kind mentioned above cannot be inferred by the graphs are not included in mathematics curricula for science undergraduates. So there is no alternative to compromise, in relation to the teaching goals. In our classes we usually accept that students, when dealing with a graph, adopt naive interpretations of properties like domain, continuity and differentiability. We usually require a more critical attitude towards properties such as the behavior of the function out of the range represented and the choice of the units of the axes. All of these aspects can be explained by means of examples, problems and activities that are within the reach of our students. So we expect that from the graphs the following properties are inferred.

*Graph A*: the function associated ($f_A$) is negative and decreasing in $(-3, 0)$; $-1 < f_A(0) < -0.5$.

*Graph B*: $f_B$ is negative in $(-3, 2)$; $-1 < f_B(0) < -0.5$.

*Graph C*: $f_C$ is negative in $(-3,-1)$, positive in $(1, 3)$, increasing in $(-3, 3)$; $f_C(0) > -0.5$.

*Graph D*: $f_D$ is negative in $(-3, 3)$ and increasing in $(-3, 3)$; $-1 < f_D(0) < -0.5$.

The closure condition (see above) is perhaps the most important piece of information, as it makes reasoning by exclusion possible. Without this piece of information the problem would not be solvable. As a matter of fact, in many cases (even though not in all) it is possible to decide that a given graph does not correspond to a given formula, but it is never possible to be sure that a graph corresponds to a given formula. A Cartesian graph, for example, represents a function only in a limited range and in an approximate way. For any graph given in a real interval, there are infinite functions that might be associated to it. In a wide range of school problems, and in particular in many of real world ones, data are almost never usable at once, but they need anyway to be identified and extracted and afterwards they require some basic competencies in order to be used. Also, a seemingly primitive property like $f(0) = -2/3$ requires to operate a replacement and apply some arithmetic operations, which implicitly involve a number of properties (and competencies). Even the recognition that the function $f_D$ is negative in the range displayed requires some acquaintance with the conventions usually adopted to draw graphs, and some linguistic competence to master the meaning of words like ‘negative’ without confusing it with ‘decreasing’ or with other words. In the same way, recognizing $f_D$ as ‘increasing’ requires some knowledge of the appropriate definition.

### Criteria

In order to analyze teachers’ comments, we have taken into account the following aspects.

1. The aspect of the argumentation they have focused on (e.g., mathematical, linguistic or logical correctness, communicative adequacy, explicitation level of logical relationships).
2. The overall evaluation of each argumentation, with special care to contrasting evaluations.
3. Any other specific remarks about mathematical, linguistic and argumentative aspects.

## Outcomes

Of the 9 mathematics teachers only one (Flavia) considered the texts from the viewpoint of argumentation, while 7 of them almost exclusively focused on mathematical content. One of them focused on both mathematical content and language. Also Maria, the Science teacher, and Alberto, one of the Philosophy teachers, considered aspects related to argumentation. In particular Maria took into account the *explicitation* of the links among the parts of the texts produced by the students. Sara, the other Philosophy teacher appeared more interested in language. As far as mathematical content is concerned, the two Philosophy teachers did not make any remarks.

The teachers provided different and even contradictory evaluations of the argumentations, especially as far as Student 2 is concerned. Ada, Franca, Oriana and Sara place it among the best, whereas Aldo, Carlo, Fabia, Giulio, Lisa and Maria place it among the worst ones. Claudia and Giorgio considered the five texts substantially equivalent. The first group of teachers appreciated the conciseness of the argumentation of student 2. On the contrary, some of the other teachers found the argumentation incomplete. Giulio commented on this text in a different way compared to the other teachers. He claimed that “it is impossible to find –2/3 on the graph, unless the unit is given divided into three parts, which is not the case, since on the $y$-axis there are multiples of 0,5 only”. He also claimed that, even this would be possible, “it would not be enough to exclude graphs $A$ and $B$”.[2] From the interview, we gathered that Giulio thought that student 2 had stated two relevant facts but had failed to relate them with the goal of the argument. The text of student 3 has been criticized by some teachers as “redundant”. As a matter of fact, the student has explicitly excluded three cases to select the fourth. The structure of his argument is logically much more acceptable than explanation 2, which is compatible with wrong interpretations of logical consequence.

The most common objection raised by some mathematics teachers concerns the association of the sign of the derivative at a point to the monotonicity of the function at that point, whereas speaking of intervals would have been more correct. For example, Lisa, commenting on explanation 4, writes: “Once more we find confusion between pointwise derivative and monotonicity of the function in an interval”. Nonetheless, the same teacher, when summing up, states: “The reference to pointwise

[2] Actually, from $f(0) = -2/3$ one can exclude graph C only. To exclude graphs A and B some more information is needed, such as $f'(0)$.

derivative, which is common to all explanations, could be accepted if one assumes that the students mean to refer to a neighborhood of the point".[3]

Some of the criticism involves aspects related to style, such as the use of plural forms ("We exclude …", "… we consider …") or of expressions like "letter *D*" in place of "graph D". For example Ada comments: "In the conclusion expressions like 'the graph of the function is letter *D*' do not sound well". Some other remarks involve the (linguistic) confusion the between "graph" and "function" or between a point and its *y*-coordinate, labelled as "imprecisions". On the other hand, Aldo, Fabia and Maria spot problems that severely affect the understandability of the text, such as the lack of subjects or of connectives. Fabia speaks of explanations "linguistically inadequate for undergraduates". Student 3 has been criticized for the use of "probably", even though with different motivations. Giorgio thinks it is a mathematical error, for in classical mathematics a sentence is either true or false. Oriana claims that "it cannot be accepted in any case". On the other hand, Ada, Fabia and Lisa interpret it as lack of confidence. For example, Lisa writes "the student is not fully convinced that the others can be excluded".

The specific issue of argumentation has not been dealt with much. Aldo is the only one who explicitly analyzes the structure of arguments, in a classical way, in terms of premises-conclusions, possibly related to the model of Aristotle's syllogisms. In his opinion "In no argumentation are premises and conclusion properly identified", although he regards the arguments overall acceptable. Fabia, Giorgio and Maria underline the topic of the explicitation of the links among statements and remark that the argument of student 3 is the most explicit of the lot.

## Final remarks

Most of the mathematics teachers involved have focused on mathematical content. It seems that their view of mathematics is still oriented to knowledge rather than to competence or reasoning. In other words, their view seems sharply acquisitional rather than discursive. Moreover, some of the objections raised involve aspects (such as the questions related to pointwise differentiability) that are not crucial for students who need a basic mathematical education only. The same teachers seem not regard argumentation as a relevant part of mathematical instruction. This suggests to teacher educators that there is still much attention to pay to teachers' interpretation of mathematics education and to familiarise them with the learning potential of argumentation. It suggests also that teacher formation should focus on argumentation as a component of mathematical competence, rather than as an additional subject to be taught. A nice discussion, at the educational level, of these topics, has been provided by Freudenthal (1988), who clearly related the logical topics in mathematics education to language. The interdisciplinary character of argumentation is damaged if it is narrowly interpreted according to the schemes of some content domain, such as mathematical logic, philosophy or law. Most of them seem to regard linguistic competence as conformity to some pattern rather than adequacy to some functions or goal. This is the case of teachers criticizing the use of plural forms or of expressions like 'letter *D*'. These may not correspond to some stylistic patterns but surely do not severely damage the readability of the text. Moreover, some teachers seem to give value to concise

[3] Actually, this difference is not relevant to this problem as all of the functions involved are continuously differentiable.

texts rather than to more extensive ones. It is possible that high school teachers' expectations of undergraduates' linguistic competence are too high and that they assume that undergraduates can fluently use literate registers, i.e. the more educated varieties of language. This unfortunately does not often happen. The use of plural forms and of 'probably', which has been criticized by some teachers, seem to be related to the interpersonal metafunction of language rather than to the ideational one. In other words the students using such wordings did not mean to express some specific content but to position themselves in relation to the instructors or even to the other students. On the other hand, only three teachers notice some more substantial weaknesses of the texts, such as the lack of subjects or of connectives, which severely damages the accessibility of the text. In our experience of undergraduate teaching, this is a major problem with a number of students. This is not just matter of style, but, in a framework assuming that language plays a major role in the development of thought, it is a major obstacle for students, who lack the means for effectively representing and compacting information and verbally expressing the organization of their thinking process. It seems that some teachers do not recognize the importance of the language demands of a formal proof and do not relate them to the field of mathematical instruction. Verbally expressing argumentation can allow students to become aware of its logical structure, but the language means to do that, such as connectives and their combination within sentences, are crucial (Prediger & Hein, 2017). We have a feeling that some of the teachers involved had too high expectations about the mathematical competence of the students involved, as far as they were at university rather than at high school. In the future we need to improve the methodology to reduce the bias caused by the teachers' view of undergraduate studies, for example by asking them to evaluate argumentations produced by high school students of their or other schools. We propose also to expand the number of teachers involved (including both mathematics and other subjects teachers) and to use these results in teacher education activities.

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