Research Article

Mathematical Methods in the Applied Sciences

Received XXXX

(www.interscience.wiley.com) DOI: 10.1002/sim.0000
MOS subject classification: XXX; XXX

Shape deformation for vibrating hinged plates

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We consider the biharmonic operator subject to homogeneous intermediate boundary conditions of Steklov-type. We prove an analyticity result for the dependence of the eigenvalues upon domain perturbation and compute the appropriate Hadamard-type formulas for the shape derivatives. Finally, we prove that balls are critical domains for the symmetric functions of multiple eigenvalues subject to volume constraint. Copyright © 2009 John Wiley & Sons, Ltd.

Keywords: biharmonic operator; intermediate, Steklov boundary conditions; hinged plate; domain perturbation

1. Introduction

Let Ω be a smooth bounded open set in $\mathbb{R}^N$, $N \geq 2$. We consider the eigenvalue problem

$$\begin{cases} \Delta^2 v = \lambda v, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \\ \Delta v - K\frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν denotes the unit outer normal to $\partial\Omega$ and K the mean curvature, i.e. the sum of the principal curvatures of $\partial\Omega$. For $N = 2$ problem (1.1) arises in linear elasticity, for instance in the study of a vibrating hinged plate. We refer to [13] for a detailed discussion concerning hinged plates and to the monograph [12] for a comprehensive study of boundary value problems for polyharmonic operators. We refer also to [4, Appendix] for explicit computations of the eigenvalues of (1.1) on the unit ball.

The boundary conditions in (1.1) are often called Steklov boundary conditions. However, we warn the reader that the eigenvalue problem (1.1) should not be confused with the classical Steklov eigenvalue problem (2.4) where the eigenvalue λ enters the boundary conditions. See e.g., [5], see also Remark 2.3.

Since problem (1.1) involves a fourth order operator, it would be natural to assume that Ω is at least of class C^4. However, the proof of our analyticity result exploits only the weak formulation of (1.1). This allows us to relax the regularity assumptions on Ω and require that Ω is of class C^2. Under this assumption, problem (1.1) admits a divergent sequence of positive eigenvalues $\lambda_j[\Omega]$, $j \in \mathbb{N}$, of finite multiplicity. In this paper, we study the dependence of $\lambda_j[\Omega]$ upon Ω. The presence of the mean curvature in (1.1) requires particular attention and unappropriate considerations in domain perturbation problems may lead to wrong conclusions, as in the case of the celebrated Babuška Paradox (cfr. [3], see also [12, § 2.7]). For this reason, we focus our attention to a class of diffeomorphic open sets of the type $\phi(\Omega)$ where Ω is fixed and ϕ is a diffeomorphism of class C^2. This enables us to avoid paradoxical situations and to prove not only continuity but also analyticity results for the dependence of $\lambda_j[\phi(\Omega)]$ on ϕ. Namely, we prove that simple eigenvalues or the elementary symmetric functions of the eigenvalues splitting from a multiple eigenvalue are real analytic functions of ϕ. Moreover, we compute Hadamard-type formulas for the corresponding derivatives. These formulas allow us to prove that balls are critical domains in isovolumetric perturbations of problem (1.1). It would be interesting to clarify whether balls are solutions to the corresponding optimization problems. Indeed, it is proved in [21] for $N = 2$ and in [1] for $N = 2, 3$ that the first eigenvalue of the biharmonic operator subject to Dirichlet boundary conditions is minimized by the ball in the class of bounded open sets with prescribed measure, and a maximization result is proved in [10] for the case of Neumann boundary conditions. Moreover, it is proved in [2] that the buckling load of a clamped plate admits a minimizer in the class of simply connected open sets in the plane with prescribed measure, and the argument of Willms and Weinberger allows to prove that such minimizer is a ball under the assumption that it is of class C^2. However, the interesting results in [5] point out that shape optimization problems are more involved in the case of Steklov boundary conditions (in particular, it is worth mentioning that the first eigenvalue of the classical Steklov problem (2.4) on a planar square is strictly smaller than the first eigenvalue on a planar disk with the same measure, as it is proved in [15]). We refer to [14] for a comprehensive exposition of extremum problems for the eigenvalues of elliptic operators.

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Our study follows the approach developed in [6, 16, 17, 18, 19, 20] combined with a delicate analysis of complicated boundary terms involved in several computations. We also refer to the survey paper [9] for a general discussion of domain perturbation problems for elliptic operators and to [7, 8] for recent results concerning high order operators.

2. An analyticity result

Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^2. Let $V(\Omega) = H^2(\Omega) \cap H^1_0(\Omega)$ where $H^2(\Omega)$ and $H^1_0(\Omega)$ denote the standard Solobev spaces of real-valued functions. It is easy to see that the weak formulation of problem (1.1) is given by

$$\int_\Omega Hv \cdot H\varphi dx = \lambda \int_\Omega v\varphi dx, \quad \forall\, \varphi \in V(\Omega), \tag{2.1}$$

in the unknown $v \in V(\Omega)$, where Hu denotes the Hessian matrix of a function u and $Hv \cdot H\varphi = \sum_{i,j=1}^N \frac{\partial^2 v}{\partial x_i x_j} \frac{\partial^2 \varphi}{\partial x_i x_j}$. Indeed, if $v \in V(\Omega)$ is smooth enough, then by integrating by parts we get

$$\int_\Omega Hv \cdot H\varphi dx = \int_\Omega \Delta^2 v\varphi dx + \int_{\partial\Omega} \frac{\partial^2 v}{\partial \nu^2}\frac{\partial \varphi}{\partial \nu} d\sigma = \int_\Omega \Delta^2 v\varphi dx + \int_{\partial\Omega} \left(\Delta v - K\frac{\partial v}{\partial \nu}\right)\frac{\partial \varphi}{\partial \nu} d\sigma, \tag{2.2}$$

for all $\varphi \in V(\Omega)$, which shows that a smooth function $v \in V(\Omega)$ is a solution to (1.1) if and only if it is a solution to (2.2).

Remark 2.3 *If in (2.1) the space $H^2(\Omega) \cap H^1_0(\Omega)$ is replaced by the Sobolev space $H^2_0(\Omega)$, we get the weak formulation of the eigenvalue problem for the biharmonic operator subject to the Dirichlet boundary conditions $v = \frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$. Similarly, if in (2.1) the space $H^2(\Omega) \cap H^1_0(\Omega)$ is replaced by the Sobolev space $H^2(\Omega)$, we get the weak formulation of the eigenvalue problem for the biharmonic operator subject to the Neumann boundary conditions $\frac{\partial^2 v}{\partial \nu^2} = \mathrm{div}_{\partial\Omega}[P_{\partial\Omega}[(Hv)\nu]] + \frac{\partial \Delta v}{\partial \nu} = 0$ on $\partial\Omega$. Here $\mathrm{div}_{\partial\Omega}$ is the tangential divergence and $P_{\partial\Omega}$ the orthogonal projector onto the tangent hyperplane to $\partial\Omega$, see also [10]. Recall that the Dirichlet and the Neumann problems arise for example in the study of clamped and free plates, respectively.*

Thus, considering that the space $H^2(\Omega) \cap H^1_0(\Omega)$ is intermediate between the two spaces $H^2_0(\Omega)$ and $H^2(\Omega)$, one may refer to (1.1) as to the intermediate problem for the bihamornic operator.

Note that the weak formulation of the classical Steklov eigenvalue problem for the biharmonic operator

$$\begin{cases} \Delta^2 v = 0, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \\ \Delta v - \lambda \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

is given by

$$\int_\Omega \Delta v \Delta \varphi dx = \lambda \int_{\partial\Omega} \frac{\partial v}{\partial \nu}\frac{\partial \varphi}{\partial \nu} d\sigma, \quad \forall\, \varphi \in V(\Omega),$$

in the unknown $v \in V(\Omega)$.

The space $V(\Omega)$ is equipped with the scalar product defined by the left-hand side of (2.1). By the Poincaré inequality, the corresponding norm is equivalent to the standard norm in $H^2(\Omega)$, hence $V(\Omega)$ is a Hilbert space. Moreover, $V(\Omega)$ is compactly embedded into $L^2(\Omega)$. Clearly, the operator Δ_Ω defined by the pairing $\Delta^2_\Omega[v][\varphi] = \int_\Omega Hv \cdot H\varphi dx$ for all $v, \varphi \in V(\Omega)$, is a linear homeomorphism from $V(\Omega)$ to its dual. Let J_Ω be the standard embedding of $V(\Omega)$ into its dual defined by $J_\Omega[v][\varphi] = \int_\Omega v\varphi dx$ for all $v, \varphi \in V(\Omega)$. It is immediate to see that the eigenvalues $\lambda_j[\Omega]$, $j \in \mathbb{N}$, of problem (2.1) coincide with the reciprocal of the eigenvalues of the nonnegative compact selfadjoint operator $T_\Omega = (\Delta^2_\Omega)^{(-1)} \circ J_\Omega$ defined from the Hilbert space $V(\Omega)$ to itself.

We consider the set of domain transformations

$$\mathcal{A}_\Omega = \{\phi \in C^2(\bar\Omega; \mathbb{R}^N) : \ \phi \text{ is injective and } \min_{\bar\Omega} |\det \nabla\phi| > 0\},$$

where $C^2(\bar\Omega; \mathbb{R}^N)$ is the space of functions of class C^2 from $\bar\Omega$ to $\mathbb{R}^N$ equipped with its standard norm defined by $\|\phi\| = \max_{0\le|\alpha|\le 2} \max_{x\in\bar\Omega} |D^\alpha\phi(x)|$ for all $\phi \in C^2(\bar\Omega; \mathbb{R}^N)$. Note that if $\phi \in \mathcal{A}_\Omega$ then $\phi(\Omega)$ is an open set in $\mathbb{R}^N$ of class C^2 and $\phi^{(-1)} \in \mathcal{A}_{\phi(\Omega)}$. We set $\lambda_j[\phi] = \lambda_j[\phi(\Omega)]$ and we study the dependence of $\lambda_j[\phi]$ upon $\phi \in \mathcal{A}_\Omega$. By using the min-max representation formula as in [7, Lemma 4.1], it is possible to prove that $\lambda_j[\phi]$ depends with continuity on $\phi \in \mathcal{A}_\Omega$. In order to prove differentiability results one has to consider simple eigenvalues or the symmetric functions of multiple eigenvalues (cfr. [18]). Let F be a nonempty finite set in $\mathbb{N}$. It is convenient to set

$$\mathcal{A}_{F,\Omega} = \{\phi \in \mathcal{A}_\Omega : \ \lambda_j[\phi] \neq \lambda_l[\phi], \ \forall\, j \in F, \ l \in \mathbb{N} \setminus F\}, \quad \text{and} \quad \Theta_{F,\Omega} = \{\phi \in \mathcal{A}_{F,\Omega} : \ \lambda_{j_1}[\phi] = \lambda_{j_2}[\phi], \ \forall\, j_1, j_2 \in F\}. \tag{2.5}$$

For $\phi \in \mathcal{A}_\Omega$, the elementary symmetric functions of the eigenvalues with index in F are defined by

$$\Lambda_{F,s}[\phi] = \sum_{\substack{j_1,\dots,j_s \in F \\ j_1<\dots<j_s}} \lambda_{j_1}[\phi] \cdots \lambda_{j_s}[\phi], \quad s = 1, \dots, |F|. \tag{2.6}$$

In the sequel, vectors are thought as column vectors, whilst gradients of real-valued functions are thought as rows. Moreover, by A^t we denote the transpose of a matrix A. Accordingly, $a^t b$ denotes the scalar product of two vectors a, b in $\mathbb{R}^N$.

Theorem 2.7 *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^2, $N \geq 2$, and F be a nonempty finite set in $\mathbb{N}$. The set $\mathcal{A}_{F,\Omega}$ is open in $C^2(\bar{\Omega};\mathbb{R}^N)$ and the real-valued maps which take $\phi \in \mathcal{A}_{F,\Omega}$ to $\Lambda_{F,s}[\phi]$ are real-analytic on $\mathcal{A}_{F,\Omega}$ for all $s = 1, \dots, |F|$. Moreover, if $\tilde{\phi} \in \Theta_{F,\Omega}$ is such that the eigenvalues $\lambda_j[\tilde{\phi}]$ assume the common value $\lambda_F[\tilde{\phi}]$ for all $j \in F$, and $\tilde{\phi}(\Omega)$ is of class C^4 then the Fréchet differential of the map $\Lambda_{F,s}$ at the point $\tilde{\phi}$ is delivered by the formula*

$$d|_{\phi=\tilde{\phi}}\Lambda_{F,s}[\psi] = \lambda_F^s[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l\in F}\int_{\partial\tilde{\phi}(\Omega)}\left(2\Delta_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\right)^2 + 2\frac{\partial v_l}{\partial\nu}\frac{\partial^3 v_l}{\partial\nu^3} - |Hv_l|^2\right)(\psi\circ\tilde{\phi}^{(-1)})^t\nu d\sigma, \tag{2.8}$$

for all $\psi \in C^2(\bar{\Omega};\mathbb{R}^N)$, where $\{v_l\}_{l\in F}$ is an orthonormal basis in $V(\tilde{\phi}(\Omega))$ of the eigenspace associated with $\lambda_F[\tilde{\phi}]$, and $\Delta_{\partial\tilde{\phi}(\Omega)}$ denotes the Laplace-Beltrami operator on $\partial\tilde{\phi}(\Omega)$.

Proof Let Δ^2_ϕ, J_ϕ be the pull-backs to Ω of the operators $\Delta^2_{\phi(\Omega)}$, $J_{\phi(\Omega)}$, i.e. the operators defined by the pairings $\Delta^2_\phi[u][\eta] = \Delta^2_{\phi(\Omega)}[u\circ\phi^{(-1)}][\eta\circ\phi^{(-1)}]$, $J_\phi[u][\eta] = J_{\phi(\Omega)}[u\circ\phi^{(-1)}][\eta\circ\phi^{(-1)}]$ for all $u, \eta \in V(\Omega)$. The proof of the analyticity of $\Lambda_{F,s}$ follows by the abstract results in [18] applied to the operator $\left(\Delta^2_\phi\right)^{(-1)} \circ \mathcal{J}_\phi$. See also [6]. We now prove formula (2.8). Let $u_l = v_l \circ \tilde{\phi}$ for all $l \in F$. By proceeding as in [6, 18], we have that

$$d|_{\phi=\tilde{\phi}}\Lambda_{F,s}[\psi] = -\lambda_F^{s+1}[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l\in F}\Delta^2_{\tilde{\phi}}\left[d|_{\phi=\tilde{\phi}}\left(\left(\Delta^2_\phi\right)^{(-1)}\circ\mathcal{J}_\phi\right)[\psi](u_l)\right][u_l]. \tag{2.9}$$

The proof of (2.8) will follow by combining (2.9) with the following formula

$$\Delta^2_{\tilde{\phi}}\left[d|_{\phi=\tilde{\phi}}\left(\left(\Delta^2_\phi\right)^{(-1)}\circ\mathcal{J}_\phi\right)[\psi](u_l)\right][u_m] = -\lambda_F^{-1}[\tilde{\phi}]\int_{\partial\tilde{\phi}(\Omega)}\left(2\Delta_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\frac{\partial v_m}{\partial\nu}\right) + \left(\frac{\partial v_l}{\partial\nu}\frac{\partial^3 v_m}{\partial\nu^3} + \frac{\partial v_m}{\partial\nu}\frac{\partial^3 v_l}{\partial\nu^3}\right) - Hv_l\cdot Hv_m\right)\mu^t\nu d\sigma, \tag{2.10}$$

which holds for all $l, m \in F$. Here and in the sequel $\mu = \psi\circ\tilde{\phi}^{(-1)}$. We now prove formula (2.10). We note that we shall systematically use the fact that $v_l \in W^{4,2}(\tilde{\phi}(\Omega))$ for all $l \in F$, which follows by classical regularity theory (see e.g., [12, Chp. 2]). By calculus in normed spaces we have

$$\Delta^2_{\tilde{\phi}}\left[\mathrm{d}|_{\phi=\tilde{\phi}}\left(\left(\Delta^2_\phi\right)^{(-1)}\circ\mathcal{J}_\phi\right)[\psi](u_l)\right][u_m] = \left(d|_{\phi=\tilde{\phi}}\mathcal{J}_\phi[\psi](u_l)\right)[u_m] + \Delta^2_{\tilde{\phi}}\left[\mathrm{d}|_{\phi=\tilde{\phi}}\left(\Delta^2_\phi\right)^{(-1)}[\psi]\circ\mathcal{J}_{\tilde{\phi}}(u_l)\right][u_m]. \tag{2.11}$$

Note that

$$\Delta^2_{\tilde{\phi}}\left[\mathrm{d}|_{\phi=\tilde{\phi}}\left(\Delta^2_\phi\right)^{(-1)}[\psi]\circ\mathcal{J}_{\tilde{\phi}}(u_l)\right][u_m] = -\mathrm{d}|_{\phi=\tilde{\phi}}\left(\Delta^2_\phi\right)[\psi]\circ\left(\Delta^2_{\tilde{\phi}}\right)^{(-1)}\circ\mathcal{J}_{\tilde{\phi}}(u_l)[u_m] = -\lambda_F^{-1}[\tilde{\phi}]\left(d|_{\phi=\tilde{\phi}}\Delta^2_\phi[\psi](u_l)\right)[u_m]. \tag{2.12}$$

Moreover,

$$\left[\left(d|_{\phi=\tilde{\phi}}(\det\nabla\phi)[\psi]\right)\circ\tilde{\phi}^{(-1)}\right]\det\nabla\tilde{\phi}^{(-1)} = \mathrm{div}\mu, \tag{2.13}$$

hence

$$\left(d|_{\phi=\tilde{\phi}}\mathcal{J}_\phi[\psi](u_l)\right)[u_m] = \int_{\tilde{\phi}(\Omega)} v_l v_m \mathrm{div}\mu dy.$$

Moreover, we have

$$\begin{aligned}\left(d|_{\phi=\tilde{\phi}}\Delta^2_\phi[\psi](u_l)\right)[u_m] = &\int_\Omega (d|_{\phi=\tilde{\phi}}H(u_l\circ\phi^{(-1)})\circ\phi)[\psi]\cdot(H(u_m\circ\tilde{\phi}^{(-1)})\circ\tilde{\phi})|\det\nabla\tilde{\phi}|dx\\ &+\int_\Omega (H(u_l\circ\tilde{\phi}^{(-1)})\circ\tilde{\phi})\cdot(d|_{\phi=\tilde{\phi}}H(u_m\circ\phi^{(-1)})\circ\phi)[\psi]|\det\nabla\tilde{\phi}|dx\\ &+\int_\Omega (H(u_l\circ\tilde{\phi}^{(-1)})\circ\tilde{\phi})\cdot(H(u_m\circ\tilde{\phi}^{(-1)})\circ\tilde{\phi})d|_{\phi=\tilde{\phi}}|\det\nabla\phi|[\psi]dx,\end{aligned} \tag{2.14}$$

and we note that the last summand in (2.14) equals $\int_{\tilde{\phi}(\Omega)} Hv_l\cdot Hv_m \mathrm{div}\mu dy$. By means of a few computations (see also [20, (3.3)]) we get $H(u\circ\phi^{(-1)})\circ\phi = (\nabla\phi)^{-t}Hu(\nabla\phi)^{-1} + A$, where A is the matrix defined by $A_{i,j} = \sum_{k,l=1}^N \frac{\partial u}{\partial x_k}\frac{\partial\zeta_{k,i}}{\partial x_l}\zeta_{l,j}$ and $\zeta = (\nabla\phi)^{-1}$. This yields the following formula

$$\left(d|_{\phi=\tilde{\phi}}(H(u\circ\phi^{(-1)})\circ\phi)[\psi]\right)\circ\tilde{\phi}^{(-1)} = -Hv\nabla\mu - \nabla\mu^t Hv - \sum_{r=1}^N\frac{\partial v}{\partial y_r}H\mu_r, \tag{2.15}$$

where $v = u \circ \tilde{\phi}^{(-1)}$. We rewrite formula (2.15) componentwise and get

$$\left((d|_{\phi=\tilde{\phi}}(H(u \circ \phi^{(-1)}) \circ \phi)[\psi]) \circ \tilde{\phi}^{(-1)} \right)_{i,j} = -\sum_{r=1}^{N} \left(\frac{\partial^2 v}{\partial y_i \partial y_r} \frac{\partial \mu_r}{\partial y_j} + \frac{\partial^2 v}{\partial y_j \partial y_r} \frac{\partial \mu_r}{\partial y_i} + \frac{\partial^2 \mu_r}{\partial y_i \partial y_j} \frac{\partial v}{\partial y_r} \right). \tag{2.16}$$

To shorten notation, from now on all summation symbols will be dropped. By (2.16) the first summand of the right-hand side of (2.14) equals

$$-\int_{\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_l}{\partial y_i \partial y_r} \frac{\partial \mu_r}{\partial y_j} + \frac{\partial^2 v_l}{\partial y_j \partial y_r} \frac{\partial \mu_r}{\partial y_i} + \frac{\partial^2 \mu_r}{\partial y_i \partial y_j} \frac{\partial v_l}{\partial y_r} \right) \frac{\partial^2 v_m}{\partial y_i \partial y_j} dy. \tag{2.17}$$

In order to compute (2.17), we note that integrating by parts yields

$$\begin{aligned}
\int_{\tilde{\phi}(\Omega)} \frac{\partial^2 v_l}{\partial y_i \partial y_r} \frac{\partial \mu_r}{\partial y_j} \frac{\partial^2 v_m}{\partial y_i \partial y_j} dy &= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_i} \frac{\partial \mu_r}{\partial y_j} \nu_r \frac{\partial^2 v_m}{\partial y_i \partial y_j} d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_i} \frac{\partial \mathrm{div}\mu}{\partial y_j} \frac{\partial^2 v_m}{\partial y_i \partial y_j} dy - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_i} \frac{\partial \mu_r}{\partial y_j} \frac{\partial^3 v_m}{\partial y_i \partial y_j \partial y_r} dy \\
&= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_i} \frac{\partial \mu_r}{\partial y_j} \nu_r \frac{\partial^2 v_m}{\partial y_i \partial y_j} d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_i} \frac{\partial \mu_r}{\partial y_j} \frac{\partial^3 v_m}{\partial y_i \partial y_j \partial y_r} dy - \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_i} \mathrm{div}\mu \frac{\partial^2 v_m}{\partial y_i \partial y_j} \nu_j d\sigma + \int_{\tilde{\phi}(\Omega)} Hv_l \cdot Hv_m \mathrm{div}\mu dy \\
&\quad + \int_{\tilde{\phi}(\Omega)} \nabla v_l (\nabla \Delta v_m)^t \mathrm{div}\mu dy,
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
\int_{\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_r} \frac{\partial^2 \mu_r}{\partial y_i \partial y_j} \frac{\partial^2 v_m}{\partial y_i \partial y_j} dy &= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \nu_j \frac{\partial^2 v_m}{\partial y_i \partial y_j} d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial^2 v_l}{\partial y_r \partial y_j} \frac{\partial \mu_r}{\partial y_i} \frac{\partial^2 v_m}{\partial y_i \partial y_j} dy - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \frac{\partial \Delta v_m}{\partial y_i} dy \\
&= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \nu_j \frac{\partial^2 v_m}{\partial y_i \partial y_j} d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \frac{\partial \Delta v_m}{\partial y_i} dy - \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \nu_r \frac{\partial^2 v_m}{\partial y_i \partial y_j} d\sigma + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_j} \frac{\partial \mathrm{div}\mu}{\partial y_i} \frac{\partial^2 v_m}{\partial y_i \partial y_j} dy \\
&\quad + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \frac{\partial^3 v_m}{\partial y_i \partial y_j \partial y_r} dy = \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \nu_j \frac{\partial^2 v_m}{\partial y_i \partial y_j} d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \frac{\partial \Delta v_m}{\partial y_i} dy - \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \nu_r \frac{\partial^2 v_m}{\partial y_i \partial y_j} d\sigma \\
&\quad + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \frac{\partial^3 v_m}{\partial y_i \partial y_j \partial y_r} dy + \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_j} \mathrm{div}\mu \frac{\partial^2 v_m}{\partial y_i \partial y_j} \nu_i d\sigma - \int_{\tilde{\phi}(\Omega)} Hv_l \cdot Hv_m \mathrm{div}\mu dy - \int_{\tilde{\phi}(\Omega)} \nabla v_l (\nabla \Delta v_m)^t \mathrm{div}\mu dy.
\end{aligned} \tag{2.19}$$

We recall that the eigenfunctions v_l satisfy the boundary conditions $v_l = \frac{\partial^2 v_l}{\partial \nu^2} = 0$ on $\partial\tilde{\phi}(\Omega)$, in particular $\nabla v_l = \frac{\partial v_l}{\partial \nu} \nu^t$ on $\partial\tilde{\phi}(\Omega)$, for all $l \in F$. Thus, by (2.18) and (2.19) we have that (2.17) is equal to

$$\begin{aligned}
\int_{\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \frac{\partial^3 v_m}{\partial y_i \partial y_j \partial y_r} dy + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \frac{\partial \Delta v_m}{\partial y_i} dy - \int_{\tilde{\phi}(\Omega)} \nabla v_l (\nabla \Delta v_m)^t \mathrm{div}\mu dy - 2 \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_l}{\partial \nu} \frac{\partial^2 v_m}{\partial y_i \partial y_j} \nu_j \frac{\partial \mu_r}{\partial y_i} \nu_r d\sigma \\
- \int_{\tilde{\phi}(\Omega)} Hv_l \cdot Hv_m \mathrm{div}\mu dy.
\end{aligned} \tag{2.20}$$

Thus the right-hand side of (2.14) equals

$$\begin{aligned}
\int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_l}{\partial y_r} \frac{\partial \Delta v_m}{\partial y_i} + \frac{\partial v_m}{\partial y_r} \frac{\partial \Delta v_l}{\partial y_i} \right) \frac{\partial \mu_r}{\partial y_i} dy + \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_l}{\partial y_j} \frac{\partial^3 v_m}{\partial y_i \partial y_j \partial y_r} + \frac{\partial v_m}{\partial y_j} \frac{\partial^3 v_l}{\partial y_i \partial y_j \partial y_r} \right) \frac{\partial \mu_r}{\partial y_i} dy \\
- 2 \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_l}{\partial \nu} \frac{\partial}{\partial \nu} \nabla v_m + \frac{\partial v_m}{\partial \nu} \frac{\partial}{\partial \nu} \nabla v_l \right) \nabla \mu_r^t \nu_r d\sigma - \int_{\tilde{\phi}(\Omega)} Hv_l \cdot Hv_m \mathrm{div}\mu dy \\
- \int_{\tilde{\phi}(\Omega)} (\nabla v_l (\nabla \Delta v_m)^t + \nabla v_m (\nabla \Delta v_l)^t) \mathrm{div}\mu dy.
\end{aligned} \tag{2.21}$$

The first summand in (2.21) equals

$$\begin{aligned}
\int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_l}{\partial \nu} \frac{\partial \Delta v_m}{\partial \nu} + \frac{\partial v_m}{\partial \nu} \frac{\partial \Delta v_l}{\partial \nu} \right) \mu^t \nu d\sigma - \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_l}{\partial y_i \partial y_r} \frac{\partial \Delta v_m}{\partial y_i} + \frac{\partial^2 v_m}{\partial y_i \partial y_r} \frac{\partial \Delta v_l}{\partial y_i} \right) \mu_r dy - \int_{\tilde{\phi}(\Omega)} (\Delta^2 v_m \nabla v_l + \Delta^2 v_l \nabla v_m) \mu dy \\
= \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_l}{\partial \nu} \frac{\partial \Delta v_m}{\partial \nu} + \frac{\partial v_m}{\partial \nu} \frac{\partial \Delta v_l}{\partial \nu} \right) \mu^t \nu d\sigma - \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_l}{\partial y_i \partial y_r} \frac{\partial \Delta v_m}{\partial y_i} + \frac{\partial^2 v_m}{\partial y_i \partial y_r} \frac{\partial \Delta v_l}{\partial y_i} \right) \mu_r dy + \lambda_F[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} v_l v_m \mathrm{div}\mu dy \\
= \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_l}{\partial \nu} \frac{\partial \Delta v_m}{\partial \nu} + \frac{\partial v_m}{\partial \nu} \frac{\partial \Delta v_l}{\partial \nu} \right) \mu^t \nu d\sigma + \lambda_F[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} v_l v_m \mathrm{div}\mu dy - \int_{\partial\tilde{\phi}(\Omega)} (\nabla v_l (\nabla \Delta v_m)^t + \nabla v_m (\nabla \Delta v_l)^t) \mu^t \nu d\sigma \\
+ \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial^2 \Delta v_l}{\partial y_r \partial y_i} \frac{\partial v_m}{\partial y_i} + \frac{\partial^2 \Delta v_m}{\partial y_r \partial y_i} \frac{\partial v_l}{\partial y_i} \right) \mu_r dy + \int_{\tilde{\phi}(\Omega)} (\nabla v_l (\nabla \Delta v_m)^t + \nabla v_m (\nabla \Delta v_l)^t) \mathrm{div}\mu dy.
\end{aligned} \tag{2.22}$$

The second summand in (2.21) equals

$$\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial y_j}\frac{\partial^3 v_m}{\partial y_i\partial y_j\partial y_r}+\frac{\partial v_m}{\partial y_j}\frac{\partial^3 v_l}{\partial y_i\partial y_j\partial y_r}\right)\nu_i\mu_r d\sigma-\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial y_j}\frac{\partial^2\Delta v_m}{\partial y_j\partial y_r}+\frac{\partial v_m}{\partial y_j}\frac{\partial^2\Delta v_l}{\partial y_j\partial y_r}\right)\mu_r dy$$
$$-\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial^2 v_l}{\partial y_i\partial y_j}\frac{\partial^3 v_m}{\partial y_i\partial y_j\partial y_r}+\frac{\partial^2 v_m}{\partial y_i\partial y_j}\frac{\partial^3 v_l}{\partial y_i\partial y_j\partial y_r}\right)\mu_r dy=\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial y_j}\frac{\partial^3 v_m}{\partial y_i\partial y_j\partial y_r}+\frac{\partial v_m}{\partial y_j}\frac{\partial^3 v_l}{\partial y_i\partial y_j\partial y_r}\right)\nu_i\mu_r d\sigma$$
$$-\int_{\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial y_j}\frac{\partial^2\Delta v_m}{\partial y_j\partial y_r}+\frac{\partial v_m}{\partial y_j}\frac{\partial^2\Delta v_l}{\partial y_j\partial y_r}\right)\mu_r dy+\int_{\tilde{\phi}(\Omega)}Hv_l\cdot Hv_m\mathrm{div}\mu dy-\int_{\partial\tilde{\phi}(\Omega)}Hv_l\cdot Hv_m\mu^t\nu d\sigma. \quad (2.23)$$

By combining (2.21)-(2.23), we get that the right-hand side of (2.14) equals

$$\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial \nu}\frac{\partial^3 v_m}{\partial y_i\partial y_j\partial y_r}+\frac{\partial v_m}{\partial \nu}\frac{\partial^3 v_l}{\partial y_i\partial y_j\partial y_r}\right)\nu_i\nu_j\mu_r d\sigma-2\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\nu^t Hv_m+\frac{\partial v_m}{\partial\nu}\nu^t Hv_l\right)\nabla\mu^t\nu d\sigma$$
$$-\int_{\partial\tilde{\phi}(\Omega)}Hv_l\cdot Hv_m\mu^t\nu d\sigma+\lambda_F[\tilde{\phi}]\int_{\tilde{\phi}(\Omega)}v_lv_m\mathrm{div}\mu dy. \quad (2.24)$$

Now we claim that

$$\nu^t Hv_m=\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_m}{\partial\nu}\ \text{on}\ \partial\tilde{\phi}(\Omega), \quad (2.25)$$

for all $m\in F$, where $\nabla_{\partial\tilde{\phi}(\Omega)}$ denotes the tangential gradient to $\partial\tilde{\phi}(\Omega)$. Here and in the sequel it is understood that the normal vector field ν is extended to a neighborhood of $\partial\tilde{\phi}(\Omega)$ as a unitary vector field. We have

$$\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_m}{\partial\nu}=\nabla_{\partial\tilde{\phi}(\Omega)}(\nabla v_m\nu)=\nabla(\nabla v_m\nu)-(\nabla(\nabla v_m\nu)\nu)\nu^t. \quad (2.26)$$

Clearly

$$(\nabla(\nabla v_m\nu))_j=\frac{\partial^2 v_m}{\partial y_i\partial y_j}\nu_i+\frac{\partial v_m}{\partial y_i}\frac{\partial\nu_i}{\partial y_j}=\frac{\partial^2 v_m}{\partial y_i\partial y_j}\nu_i+\frac{1}{2}\frac{\partial v_m}{\partial\nu}\frac{\partial(\nu_i)^2}{\partial y_j}=\frac{\partial^2 v_m}{\partial y_i\partial y_j}\nu_i\,,\quad\text{on}\ \ \partial\tilde{\phi}(\Omega). \quad (2.27)$$

Thus

$$\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_m}{\partial\nu}=\nu^t Hv_m-(\nu^t Hv_m\nu)\nu^t=\nu^t Hv_m-\frac{\partial^2 v_m}{\partial\nu^2}\nu^t=\nu^t Hv_m, \quad (2.28)$$

and (2.25) is proved. Now we note that

$$\nabla(\nu^t\mu)=\nu^t\nabla\mu+\mu^t\nabla\nu\quad\text{ hence }\quad\nabla\mu^t\nu=\nabla(\nu^t\mu)^t-\nabla\nu^t\mu. \quad (2.29)$$

By observing that $|\nu|^2=1$ implies that $\nu^t\nabla\nu=0$, by (2.25) and (2.29) we get

$$\frac{\partial v_l}{\partial\nu}\nu^t Hv_m\nabla\mu^t\nu=\frac{\partial v_l}{\partial\nu}\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_m}{\partial\nu}\nabla(\nu^t\mu)^t-\frac{\partial v_l}{\partial\nu}\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_m}{\partial\nu}\nabla\nu^t\mu$$
$$=\frac{\partial v_l}{\partial\nu}\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_m}{\partial\nu}\nabla_{\partial\tilde{\phi}(\Omega)}(\nu^t\mu)^t-\frac{\partial v_l}{\partial\nu}\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_m}{\partial\nu}\nabla\nu^t(\mu_\nu+\mu_{\partial\tilde{\phi}(\Omega)})$$
$$=\frac{\partial v_l}{\partial\nu}\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_m}{\partial\nu}\nabla_{\partial\tilde{\phi}(\Omega)}(\nu^t\mu)^t-\frac{\partial v_l}{\partial\nu}\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_m}{\partial\nu}\nabla\nu^t\mu_{\partial\tilde{\phi}(\Omega)}, \quad (2.30)$$

where $\mu=\mu_\nu+\mu_{\partial\tilde{\phi}(\Omega)}$, μ_ν is the normal component of μ and $\mu_{\partial\tilde{\phi}(\Omega)}$ the tangential one. Hence the second integral in (2.24) equals

$$2\int_{\partial\tilde{\phi}(\Omega)}\nabla_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\frac{\partial v_m}{\partial\nu}\right)\nabla\nu^t\mu_{\partial\tilde{\phi}(\Omega)}d\sigma-2\int_{\partial\tilde{\phi}(\Omega)}\nabla_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\frac{\partial v_m}{\partial\nu}\right)\nabla_{\partial\tilde{\phi}(\Omega)}(\nu^t\mu)^t d\sigma. \quad (2.31)$$

Now we consider the first integral in (2.24), and we recall that

$$\frac{\partial^2 v_m}{\partial y_i\partial y_j}\nu_i\nu_j=0\,,\quad\text{on}\quad\partial\tilde{\phi}(\Omega). \quad (2.32)$$

By differentiating (2.32) with respect to any tangential direction τ to $\partial\tilde{\phi}(\Omega)$ we obtain

$$\frac{\partial^3 v_m}{\partial y_i\partial y_j\partial y_r}\nu_i\nu_j\tau_r+2\frac{\partial^2 v_m}{\partial y_i\partial y_j}\frac{\partial\nu_i}{\partial y_r}\nu_j\tau_r=0,$$

hence

$$\frac{\partial^3 v_m}{\partial y_i\partial y_j\partial y_r}\nu_i\nu_j\mu^t\tau\tau_r=-2\frac{\partial^2 v_m}{\partial y_i\partial y_j}\frac{\partial\nu_i}{\partial y_r}\nu_j\mu^t\tau\tau_r\,. \quad (2.33)$$

By taking in (2.33) vectors τ belonging to a basis of the tangent hyperplane to $\partial\tilde{\phi}(\Omega)$ and using (2.28), we easily get

$$\frac{\partial^3 v_m}{\partial y_i \partial y_j \partial y_r}\nu_i\nu_j\mu_{\partial\tilde{\phi}(\Omega),r} = -2\nu^t H v_m \nabla \nu^t \mu_{\partial\tilde{\phi}(\Omega)} = -2\nabla_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_m}{\partial\nu}\right)\nabla\nu^t\mu_{\partial\tilde{\phi}(\Omega)}. \tag{2.34}$$

Thus

$$\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_l}{\partial\nu}\frac{\partial^3 v_m}{\partial y_i\partial y_j\partial y_r}\nu_i\nu_j\mu_r d\sigma = \int_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_l}{\partial\nu}\frac{\partial^3 v_m}{\partial y_i\partial y_j\partial y_r}\nu_i\nu_j(\mu_{\nu,r}+\mu_{\partial\tilde{\phi}(\Omega),r})d\sigma = \int_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_l}{\partial\nu}\frac{\partial^3 v_m}{\partial y_i\partial y_j\partial y_r}\nu_i\nu_j\nu_r\mu^t\nu d\sigma$$
$$+\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_l}{\partial\nu}\frac{\partial^3 v_m}{\partial y_i\partial y_j\partial y_r}\nu_i\nu_j\mu_{\partial\tilde{\phi}(\Omega),r}d\sigma = \int_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_l}{\partial\nu}\frac{\partial^3 v_m}{\partial\nu^3}\mu^t\nu d\sigma - 2\int_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_l}{\partial\nu}\nabla_{\partial\tilde{\phi}(\Omega)}\frac{\partial v_m}{\partial\nu}\nabla\nu^t\mu_{\partial\tilde{\phi}(\Omega)}d\sigma. \tag{2.35}$$

Hence the first integral in (2.24) is equal to

$$\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\frac{\partial^3 v_m}{\partial\nu^3}+\frac{\partial v_m}{\partial\nu}\frac{\partial^3 v_l}{\partial\nu^3}\right)\mu^t\nu d\sigma - 2\int_{\partial\tilde{\phi}(\Omega)}\nabla_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\frac{\partial v_m}{\partial\nu}\right)\nabla\nu^t\mu_{\partial\tilde{\phi}(\Omega)}d\sigma. \tag{2.36}$$

Finally, by (2.31), (2.36) and by the tangential Green formula (see [11, § 5.5]), we get that the right-hand side of (2.24) equals

$$\int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\frac{\partial^3 v_m}{\partial\nu^3}+\frac{\partial v_m}{\partial\nu}\frac{\partial^3 v_l}{\partial\nu^3}\right)\mu^t\nu d\sigma - \int_{\partial\tilde{\phi}(\Omega)}Hv_l\cdot Hv_m\mu^t\nu d\sigma - 2\int_{\partial\tilde{\phi}(\Omega)}\nabla_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\frac{\partial v_m}{\partial\nu}\right)\nabla_{\partial\tilde{\phi}(\Omega)}(\mu^t\nu)^t d\sigma$$
$$+\lambda_F[\tilde{\phi}]\int_{\tilde{\phi}(\Omega)}v_l v_m \mathrm{div}\mu dy = \int_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\frac{\partial^3 v_m}{\partial\nu^3}+\frac{\partial v_m}{\partial\nu}\frac{\partial^3 v_l}{\partial\nu^3}\right)\mu^t\nu d\sigma - \int_{\partial\tilde{\phi}(\Omega)}Hv_l\cdot Hv_m\mu^t\nu d\sigma + \lambda_F[\tilde{\phi}]\int_{\tilde{\phi}(\Omega)}v_l v_m\mathrm{div}\mu dy$$
$$+2\int_{\partial\tilde{\phi}(\Omega)}\Delta_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\frac{\partial v_m}{\partial\nu}\right)\mu^t\nu d\sigma. \tag{2.37}$$

This combined with (2.11) and (2.14) concludes the proof of (2.10).

3. Criticality of balls in isovolumetric perturbations

We consider the following extremum problems for the symmetric functions of the eigenvalues

$$\min_{V[\phi]=\mathrm{const}}\Lambda_{F,s}[\phi] \quad \text{or} \quad \max_{V[\phi]=\mathrm{const}}\Lambda_{F,s}[\phi], \tag{3.1}$$

where $V[\phi]$ denotes the N-dimensional Lebesgue measure of $\phi(\Omega)$. By the Lagrange multiplier theorem, if $\tilde{\phi}\in\mathcal{A}_\Omega$ is a minimizer or maximizer in (3.1) then $\tilde{\phi}$ is a critical domain transformation for the map $\phi\mapsto\Lambda_{F,s}[\phi]$ subject to volume constraint, i.e., there exists $c\in\mathbb{R}$ such that $\mathrm{d}|_{\phi=\tilde{\phi}}\Lambda_{F,s} = cd|_{\phi=\tilde{\phi}}V$, where V is the real-valued function defined on $\mathcal{A}_\Omega$ which takes $\phi\in\mathcal{A}_\Omega$ to $V[\phi]$. By using (2.8) and (2.13), one can easily see that under the same assumptions of Theorem 2.7, $\tilde{\phi}$ is a critical domain transformation for any of the functions $\Lambda_{F,s}$, $s=1,\dots,|F|$, with volume constraint if and only if there exists $C\in\mathbb{R}$ such that

$$\sum_{l\in F}\left(2\Delta_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\right)^2+2\frac{\partial v_l}{\partial\nu}\frac{\partial^3 v_l}{\partial\nu^3}-|Hv_l|^2\right)=C, \quad \text{on } \partial\tilde{\phi}(\Omega). \tag{3.2}$$

Then we can prove the following.

Theorem 3.3 *Let the same assumptions of Theorem 2.7 hold. If $\tilde{\phi}(\Omega)$ is a ball then condition (3.2) is satisfied.*

Proof Assume that $\tilde{\phi}(\Omega)$ is a ball B of radius R centered at zero. By arguing as in [6, 17], one can easily prove that $\sum_{l\in F}v_l^2$ and $\sum_{l\in F}(\Delta v_l)^2$ are radial functions. By differentiating $\sum_{l\in F}v_l^2$ twice with respect to the radial coordinate r, we get that $\sum_{l\in F}(\frac{\partial v_l}{\partial r})^2$ is constant on ∂B, hence

$$\sum_{l\in F}\Delta_{\partial\tilde{\phi}(\Omega)}\left(\frac{\partial v_l}{\partial\nu}\right)^2=0, \quad \text{on } \partial B. \tag{3.4}$$

The function

$$\frac{\partial^4}{\partial r^4}\sum_{l\in F}v_l^2=\sum_{l\in F}\left(6\left(\frac{\partial^2 v_l}{\partial r^2}\right)^2+8\frac{\partial v_l}{\partial r}\frac{\partial^3 v_l}{\partial r^3}+2v_l\frac{\partial^4 v_l}{\partial r^4}\right)$$

is clearly radial, hence

$$\sum_{l\in F}\frac{\partial v_l}{\partial\nu}\frac{\partial^3 v_l}{\partial\nu^3} \quad \text{is constant on } \partial B. \tag{3.5}$$

Note that $\frac{\partial}{\partial \nu} \sum_{l \in F} (\Delta v_l)^2 = 2 \sum_{l \in F} \frac{N-1}{R} \frac{\partial v_l}{\partial \nu} \frac{\partial \Delta v_l}{\partial \nu}$ on ∂B, hence

$$\sum_{l \in F} \frac{\partial v_l}{\partial \nu} \frac{\partial \Delta v_l}{\partial \nu} \quad \text{is constant on} \quad \partial B. \tag{3.6}$$

Finally, we note that

$$\Delta^2 \sum_{l \in F} v_l^2 = \sum_{l \in F} \left(2\lambda_F[\tilde{\phi}] v_l^2 + 2(\Delta v_l)^2 + 4|Hv_l|^2 + 8\nabla v_l (\nabla \Delta v_l)^t \right) \tag{3.7}$$

is radial, hence by (3.6) the function $\sum_{l \in F} |Hv_l|^2$ is constant on ∂B. This, combined with (3.4), (3.5) implies that (3.2) holds.

Remark 3.8 *It would be interesting to characterize those open sets $\tilde{\phi}(\Omega)$ such that condition (3.2) is satisfied. We recall that in the case of the first eigenvalue of the Dirichlet Laplacian, the corresponding condition is $\frac{\partial u}{\partial \nu} = C$ on $\partial \tilde{\phi}(\Omega)$ in which case it is a classical result that the existence of a positive solution implies that $\tilde{\phi}(\Omega)$ is ball, see the celebrated paper [22]. We also refer to [6, 14] for more references.*

Acknowledgement

The authors are very thankful to the referees for useful comments and references, and they acknowledge financial support from the research project "Singular perturbation problems for differential operators", Progetto di Ateneo of the University of Padova. The second author also acknowledges financial support from the research project PRIN 2008 "Aspetti geometrici delle equazioni alle derivate parziali e questioni connesse".

References

1. Ashbaugh M.S., Benguria R.D. *On Rayleigh's conjecture for the clamped plate and its generalization to three dimensions*. Duke Math. J. 78 (1995), no. 1, 1-17.
2. Ashbaugh M.S., Bucur D. *On the isoperimetric inequality for the buckling of a clamped plate*. Z. Angew. Math. Phys. 54 (2003), no. 5, 756-770.
3. Babuška I. *Stabilität des definitionsgebietes mit rücksicht auf grundlegende probleme der theorie der partiellen differentialgleichungen auch im zusammenhang mit der elastizitätstheorie*. I, II (Russian) Czechoslovak Math. J. 11 (86) 1961 76-105, 165-203.
4. Berchio E., Gazzola F. *Positive solutions to a linearly perturbed critical growth biharmonic problem*. Discrete Contin. Dyn. Syst. Ser. S 4 (2011), no. 4, 809-823.
5. Bucur D., Ferrero A., Gazzola F. *On the first eigenvalue of a fourth order Steklov problem*. Calc. Var. Partial Differential Equations 35 (2009), no. 1, 103-131.
6. Buoso D., Lamberti P.D. *Eigenvalues of polyharmonic operators on variable domains*. to appear in ESAIM Control Optim. Calc. Var.
7. Burenkov V., Lamberti P.D. *Spectral stability of higher order uniformly elliptic operators*. Sobolev spaces in mathematics. II, 69-102, Int. Math. Ser. (N. Y.), 9, Springer, New York, 2009.
8. Burenkov V., Lamberti P.D. *Sharp spectral stability estimates via the Lebesgue measure of domains for higher order elliptic operators*. Rev. Mat. Complut. 25 (2012), no. 2, 435-457.
9. Burenkov V., Lamberti, P.D., Lanza de Cristoforis M. *Spectral stability of nonnegative selfadjoint operators*. (Russian) Sovrem. Mat. Fundam. Napravl. 15 (2006), 76-111; translation in J. Math. Sci. (N. Y.) 149 (2008), no. 4, 1417-1452.
10. Chasman L.M. *An isoperimetric inequality for fundamental tones of free plates*. Comm. Math. Phys. 303 (2011), no. 2, 421-449.
11. Delfour M.C., Zolesio J.-P. *Shapes and geometries. Analysis, differential calculus, and optimization*. Advances in Design and Control, 4. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001.
12. Gazzola F., Grunau H-C., Sweers G. *Polyharmonic boundary value problems. Positivity preserving and nonlinear higher order elliptic equations in bounded domains.* Lecture Notes in Mathematics, 1991. Springer-Verlag, Berlin, 2010.
13. Gazzola F., Sweers G. *On positivity for the biharmonic operator under Steklov boundary conditions*. Arch. Ration. Mech. Anal. 188 (2008), no. 3, 399-427.
14. Henrot A. *Extremum problems for eigenvalues of elliptic operators* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.
15. Kuttler J.R. *Remarks on a Stekloff eigenvalue problem*. SIAM J. Numer. Anal. 9, 1-5 (1972)
16. Lamberti P.D. *Steklov-type eigenvalues associated with best Sobolev trace constants: domain perturbation and overdetermined systems*. to appear in Complex Var. Elliptic Equ.
17. Lamberti P.D., Lanza de Cristoforis M. *Critical points of the symmetric functions of the eigenvalues of the Laplace operator and overdetermined problems*. J. Math. Soc. Japan 58 (2006), no. 1, 231-245.
18. Lamberti P.D., Lanza de Cristoforis M. *A real analyticity result for symmetric functions of the eigenvalues of a domain dependent Dirichlet problem for the Laplace operator*. J. Nonlinear Convex Anal. 5 (2004), no. 1, 19-42.
19. Lamberti P.D., Lanza de Cristoforis M. *A real analyticity result for symmetric functions of the eigenvalues of a domain-dependent Neumann problem for the Laplace operator*. Mediterr. J. Math. 4 (2007), no. 4, 435-449.
20. Lamberti P.D., Lanza de Cristoforis M. *An analyticity result for the dependence of multiple eigenvalues and eigenspaces of the Laplace operator upon perturbation of the domain*. Glasg. Math. J. 44 (2002), no. 1, 29-43.
21. Nadirashvili N. *Rayleigh's conjecture on the principal frequency of the clamped plate*. Arch. Rational Mech. Anal. 129 (1995), no. 1, 1-10.
22. Serrin J. *A symmetry problem in potential theory*. Arch. Rational Mech. Anal. 43 (1971), 304-318.