EXTREMAL EIGENVALUES OF THE DIRICHLET BIHARMONIC OPERATOR ON RECTANGLES

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Abstract. We study the behaviour of extremal eigenvalues of the Dirichlet biharmonic operator over rectangles with a given fixed area. We begin by proving that the principal eigenvalue is minimal for a rectangle for which the ratio between the longest and the shortest side lengths does not exceed 1.066459. We then consider the sequence formed by the minimal $k^{\rm th}$ eigenvalue and show that the corresponding sequence of minimising rectangles converges to the square as k goes to infinity.

1. Introduction

The asymptotic behaviour of extremal eigenvalues of the Laplace operator has received some attention in the mathematical literature in recent years, starting with the proof that in the case of rectangles with fixed area and Dirichlet boundary conditions extremal rectangles converge to the square as the order of the eigenvalue goes to infinity [2]. This result has been generalised to rectangular parallelepipeds in higher dimensions and to Neumann boundary conditions [7, 8, 19, 24]. All these results are based on the relation between this eigenvalue problem and lattice point problems, and some generalisations along these lines have also began to appear [4, 23, 25].

A natural question is, of course, whether or not such results also extend to more general domains. That the problem in the most general case of bounded domains is expected to be difficult is a consequence of the result by Colbois and El Soufi which relates this to a statement equivalent to Pólya's conjecture [15]. There are, however, some results under weaker conditions. By imposing a surface area restriction instead of a volume restriction, it is possible to show that in the planar case there is convergence of extremal domains to the disk [10] and, by considering averages instead of single eigenvalues, it then becomes possible to show convergence of such averages or even, in some cases, of the corresponding extremal domains [16, 22].

From a physical perspective, this type of problem may be seen as that of finding the shape for which the number of modes allowed below a given frequency is extremal. Then, the existing results and corresponding proofs indicate that in the high-frequency regime this behaviour is again determined by the classical geometric isoperimetric inequality, just as in the case of the extremal domain for the first eigenvalue. In some sense this is not unexpected, as the first two terms in the Weyl law depend only on the volume and surface measures. On the other hand, it is not

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clear that these two terms should play the dominant role in this setting. Furthermore, it was recently shown that, in the case of Robin boundary conditions with a positive boundary parameter, eigenvalues satisfy (nontrivial) Pólya-type inequalities with lower bounds for the k^{th} eigenvalue, in spite of the fact that the first two terms in the corresponding Weyl law for Robin eigenvalues are the same as those of the Neumann problem – see [3, 17].

In this paper we are interested in studying the above mentioned problem in the case of the biharmonic operator with Dirichlet boundary conditions. Determining extremal domains for the biharmonic operator with Dirichlet boundary conditions is a notoriously difficult problem, as may be seen from the fact that proofs of the corresponding Faber-Krahn inequality exist only in two and three spatial dimensions [5, 26] – see also [6] for a discussion about the limitations arising in higher dimensions. Part of this difficulty stems from the fact that the first eigenfunction is no longer necessarily of one sign. In the case of rectangles this becomes particularly relevant as it is known that the first eigenfunction does indeed change sign, including in the case of the square which is the natural candidate for the minimiser of the first eigenvalue (see e.g., [9, 14, 20]). Indeed, it is not known if the square minimises the first Dirichlet eigenvalue of the biharmonic operator among all rectangles of a given area. Our first result is that there exists one global minimiser for this eigenvalue and that it must be quite close to the square. More precisely, we have the following

Theorem A. *There exists a global minimiser for the first Dirichlet eigenvalue of the biharmonic operator over rectangles of fixed area. Furthermore, the quotient between the lengths of the largest and the smallest sides of the extremal rectangle does not exceed* 1.066459*.*

At the high-frequency end of the spectrum, and since the proofs now do not rely on any such properties, we are able to show that there is convergence to the square.

Theorem B. *Let q_k^* denote the quotient between the lengths of the largest and the smallest sides of a rectangle minimising the k^{th} eigenvalue of the Drichlet biharmonic operator over rectangles of fixed area. Then*

$$\lim_{k\to\infty} q_k^* = 1.$$

Some other results in the spirit of those for the Dirichlet Laplacian mentioned above are more or less straighfrorward consequences of the corresponding original result. They include the case of fixed perimeter and the Colbois-El Soufi results on the sequence of minimisers. For completeness, we collect these in Section 5.

2. Background and notation

Let Ω be a (smooth bounded) domain in $\mathbb{R}^N$, $N \geq 2$. The Dirichlet eigenvalue problem for the biharmonic operator (clamped plate problem) is given by

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

with the corresponding weak formulation being

$$\int_\Omega \Delta u \Delta \phi = \lambda \int_\Omega u\phi, \ \forall \phi \in H_0^2(\Omega).$$

The eigenvalues of the above problem may be characterised by a variational principle of the form

$$\lambda_k(\Omega) = \min_{0\neq u\in V\subset H_0^2(\Omega)} \max_{\dim V=k} \frac{\displaystyle\int_\Omega (\Delta u)^2}{\displaystyle\int_\Omega u^2}$$

and it is known that the sequence λ_k satisfies

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty \text{ as } k \to \infty.$$

Under certain geometric conditions on a piecewise smooth domain Ω, which are satisfied by rectangles (cf. [29]), the corresponding Weyl asymptotics for problem (1) on planar domains may be seen from [28, formulas (6.2.1) and (6.2.2)] to be

$$\lambda_k = \frac{16\pi^2}{|\Omega|^2}k^2 + \frac{16\pi^{\frac{3}{2}}|\partial\Omega|}{|\Omega|^{\frac{5}{2}}}\left(1+\frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt{\pi}\Gamma\left(\frac{5}{4}\right)}\right)k^{\frac{3}{2}} + \mathrm{o}\left(k^{\frac{3}{2}}\right), \tag{2}$$

or equivalently

$$\lambda_k^{\frac{1}{2}} = \frac{4\pi}{|\Omega|}k + \frac{2\pi^{\frac{1}{2}}|\partial\Omega|}{|\Omega|^{\frac{3}{2}}}\left(1+\frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt{\pi}\Gamma\left(\frac{5}{4}\right)}\right)k^{\frac{1}{2}} + \mathrm{o}\left(k^{\frac{1}{2}}\right), \tag{3}$$

where $|\Omega|$ and $|\partial\Omega|$ denote the 2-dimensional measure of Ω and the 1-dimensional measure of its boundary $\partial\Omega$, respectively.

As in the case of the Dirichlet Laplacian, it is also possible to obtain lower bounds of Li-Yau type and we have that the following holds for general smooth domains (see [21, formula (1.9)])

$$\lambda_k \geq \frac{16N\pi^4}{N+4}\left(\frac{k}{\omega_N|\Omega|}\right)^{\frac{4}{N}}. \tag{4}$$

3. The first eigenvalue: the square is (almost) the minimising rectangle

In this section, we focus our attention on the question of determining the minimal possible value for the first eigenvalue of problem (1) among rectangles with a given fixed area. Without loss of generality, we fix the area to be one so that our class of admissible rectangles may be written as

$$\mathcal{R} = \{\text{Rectangles with side lengths } a \text{ and } 1/a, \text{ for } a \in [1,+\infty)\}.$$

We recall that the biharmonic operator appearing in problem (1) is invariant under rotations and translations as is the case for the Laplace operator, and hence the spectrum is the same for any rectangle of given edges. Thus, and due to symmetry considerations, we expect the square to be an extremal point for λ_1, for otherwise there would have to be an infinite number of oscillations for a close to one. However, most other fundamental properties of the Dirichlet Laplacian are not shared by the biharmonic operator. For instance, we know that the first biharmonic eigenvalue is not necessarily simple in general and, although the first eigenvalue of the square is expected to be simple, the only results in this direction are of a numerical nature [30]. Furthermore, some useful properties such as separation of variables are not available for rectangles, and therefore we cannot characterize either its eigenvalues or eigenfnctions explicitly in terms of known functions. This, together with the lack of positivity for the first eigenfunction already mentioned in the Introduction, transforms what is a trivial problem in the case of the Laplacian into a quite hard problem.

Our approach will make use of the sharp estimates provided by Owen [27] in order to narrow down the search of the minimiser to a small neighbourhood of the square ($a = 1$).

Let us denote by R_a the rectangle $R_a = [0, a]\times[0, 1/a]$, and write $\lambda_1(a) = \lambda_1(R_a)$. We recall the following estimate from [30, Table 4].

Lemma 3.1. *The first eigenvalue of problem* (1) *satisfies*

$$1294.933940 \leq \lambda_1(1) \leq \Lambda := 1294.933988.$$

We also recall a lower bound from [27, Theorem 2].

Lemma 3.2. *For any $a \geq 1$ we have*

$$\lambda_1(a) \geq L(a) = \rho(\pi^2 a^4)a^{-4} + \rho(\pi^2 a^{-4})a^4 - 2\pi^4, \tag{5}$$

where $\rho(\alpha)$ is the first eigenvalue of the following problem

$$\begin{cases} y'''' - 2\alpha y'' = \lambda y, & \text{in } (0,1), \\ y(0) = y(1) = y'(0) = y'(1) = 0, \end{cases} \tag{6}$$

and $\rho(\alpha)$ is an increasing function for positive α.

Finally, we will also need the following

Lemma 3.3. *The function $L(a)$ defined in* (5) *is strictly increasing in a for $a > 1$.*

Proof. In order to prove that $L'(a) > 0$ for $a > 1$, we denote by v_t an eigenfunction associated with $\rho(\pi^2 t)$ in (6) such that $\|v_t\|_{L^2} = 1$. Then

$$\rho(\pi^2 t) = \int_0^1 (v_t'')^2 + 2\pi^2 t \int_0^1 (v_t')^2$$

and

$$\rho'(\pi^2 t) = 2\int_0^1 (v_t')^2.$$

Writing

$$F(t) = \rho(\pi^2 t)t^{-1} + \rho(\pi^2 t^{-1})t,$$

we have

$$\begin{aligned} F'(t) &= \frac{2\pi^2}{t}\int_0^1 (v_t')^2 - \frac{1}{t^2}\left[\int_0^1 (v_t'')^2 + 2\pi^2 t\int_0^1 (v_t')^2\right] \\ &\quad -\frac{\pi^2}{t}\int_0^1 (v_{t^{-1}}')^2 + \left[\int_0^1 (v_{t^{-1}}'')^2 + \frac{2\pi^2}{t}\int_0^1 (v_{t^{-1}}')^2\right] \\ &= -\frac{1}{t^2}\int_0^1 (v_t'')^2 + \int_0^1 (v_{t^{-1}}'')^2 \\ &= \frac{1}{t^2}\left[t^2\int_0^1 (v_{t^{-1}}'')^2 + 2\pi^2 t\int_0^1 (v_t')^2 - \rho(\pi^2 t)\right]. \end{aligned}$$

At this point we observe that

$$\int_0^1 (v_{t^{-1}}'')^2 \geq \min_{\substack{v\in H_0^2 \\ \|v\|_{L^2}=1}} \int_0^1 (v'')^2 = \rho(0) = \int_0^1 (v_0'')^2.$$

Moreover, since

$$\rho(\pi^2 t) = \min_{\substack{v\in H_0^2 \\ \|v\|_{L^2}=1}} \left[\int_0^1 (v'')^2 + 2\pi^2 t \int_0^1 (v')^2\right],$$

then, also using the Poincaré inequality

$$\int_0^1 (v')^2 \geq \pi^2 \int_0^1 v^2 \quad \forall v \in H_0^1(0,1),$$

we get

$$\begin{aligned} t^2 \int_0^1 (v''_{t^{-1}})^2 + 2\pi^2 t \int_0^1 (v'_t)^2 - \rho(\pi^2 t) &\geq (t^2-1)\int_0^1 (v''_0)^2 - 2\pi^2 t \int_0^1 (v'_0)^2 + 2\pi^4 t \\ &\geq \pi^2(t^2-2t-1)\int_0^1 (v'_0)^2 + 2\pi^4 t \\ &\geq \pi^4(t^2-1). \end{aligned}$$

Hence $F'(t) > 0$ for $t > 1$, and the result now follows by observing that $L(a) = F(a^4) - 2\pi^4$. □

Our strategy is to find bounds for $a \in [1, +\infty)$ such that

$$\Lambda < L(a). \tag{7}$$

In fact, if $\hat{a}$ is a solution of $\Lambda = L(a)$, then we obtain that the solution of

$$\min_{a\geq 1} \lambda_1(a) \tag{8}$$

has to be a rectangle R_a with $a \in [1, \hat{a})$, the precision of this bound being strictly related to the precision of the bounds Λ and $L(a)$. We remark that there exists at least one solution to problem (8), since $\lim_{a\to\infty} \lambda_1(a) = \infty$.

In order to find the smallest value $\hat{a}$ satisfying (7), we implement a bisection procedure in the software Mathematica™ starting from $L(2) = 9442.68$.

Theorem 3.4. *Problem* (8) *admits (at least) one minimiser* a^* *in the interval*

$$a^* \in [1, \hat{a}),$$

where $\hat{a}$ *is the solution of equation* $\Lambda = L(a)$*, lying in the interval*

$$\hat{a} \in [1.03269, 1.032695).$$

We observe that this method allows us to say that the minimiser has to be very close to the square, but we cannot go any further below the threshold $\hat{a}$. Having some additional information such as convexity of the first eigenvalue with respect to this perturbation, or simplicity of eigenvalues for rectangles close to the square, would allow for a more complete result. Some numerical simulations based on the method of fundamental solutions [1], however, give support to the conjecture that the square is the actual global minimizer among all rectangles of unit area (see Fig. 1). We also note that, even though the general form of the shape derivative for eigenvalues of problem (1) is known (see e.g., [11, 12]), its value is extremely difficult to estimate for the square and for rectangles in general, since, in contrast with the Dirichlet Laplacian and as mentioned above, the explicit form of the eigenfunctions is not known.

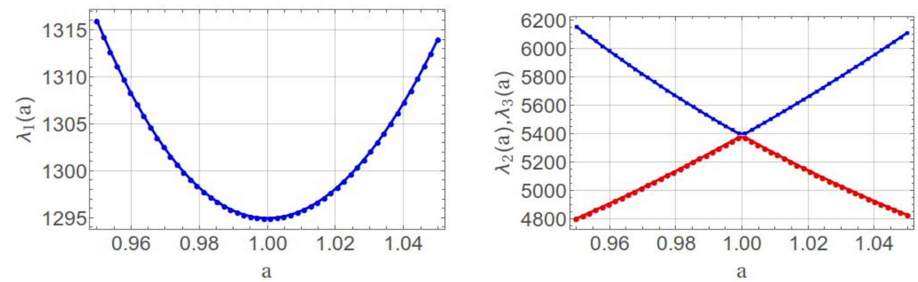

FIGURE 1. On the left, $\lambda_1(a)$ for rectangles with different sides around a square ($a = 1$); on the right, the same for λ_2, λ_3.

Since the bounds obtained by the above methods are not explicit and still require the solution of a transcendental equation in each case, we conclude this section with a simple bound which, although not as accurate, has the advantage that it only requires the determination of one such root.

Theorem 3.5. *The first eigenvalue of problem* (1) *on a rectangle* R_a *satisfies the bound*

$$\lambda_1(a) \geq \omega_1^4 \left(a^4 + a^{-4}\right) + 2\pi^4,$$

where $\omega_1 \approx 4.73004$ *is the first positive root of the equation* $\cos(\omega)\cosh(\omega) = 1$.

Proof. On a rectangle R_a we have

$$\begin{aligned}\int_{R_a} (\Delta u)^2 &= \int_{R_a} u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy} \; \mathrm{d}x\mathrm{d}y \\ &= \int_{R_a} u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 \; \mathrm{d}x\mathrm{d}y.\end{aligned}$$

We thus have

$$\begin{aligned}\lambda_1(a) &= \inf_{0\neq u\in H_0^2(R_a)} \frac{\displaystyle\int_{R_a} (\Delta u)^2}{\displaystyle\int_{R_a} u^2} \\ &\geq \inf_{0\neq u\in H_0^2(R_a)} \frac{\displaystyle\int_{R_a} u_{xx}^2 + u_{yy}^2}{\displaystyle\int_{R_a} u^2} + 2 \inf_{0\neq u\in H_0^2(R_a)} \frac{\displaystyle\int_{R_a} u_{xx}u_{yy}}{\displaystyle\int_{R_a} u^2}.\end{aligned}$$

The first term corresponds to the first eigenvalue of the problem

$$u_{xxxx} + u_{yyyy} = \gamma u$$

in R_a, together with Dirichlet boundary condtions. Note that the operator $\partial_{xxxx} + \partial_{yyyy}$ is strongly elliptic, and therefore admits a purely discrete specrtum accumulating at infinity, with the usual minimax characterization for its eigenvalues (see also [13, Theorem 9, page 176]). For this problem it is possible to separate variables in the usual way to obtain

$$\gamma_1(a) = \omega_1^4(a^4 + a^{-4}).$$

Regarding the second term, we have

$$\begin{aligned}\int_{R_a} u_{xy}^2 \, \mathrm{d}x\mathrm{d}y &= \int_0^{a^{-1}} \int_0^a \left[(u_y)_x\right]^2 \, \mathrm{d}x\mathrm{d}y \\ &\geq \int_0^{a^{-1}} \frac{\pi^2}{a^2} \int_0^a u_y^2 \, \mathrm{d}x\mathrm{d}y \\ &\geq \pi^4 \int_0^{a^{-1}} \int_0^a u^2 \, \mathrm{d}x\mathrm{d}y \\ &= \pi^4 \int_{R_a} u^2 \, \mathrm{d}x\mathrm{d}y,\end{aligned}$$

and putting these two estimates together yields the result. □

4. High-frequency limit: minimising rectangles converge to the square

In this section we are interested in what happens to the minimiser of λ_k as $k \to \infty$. To simplify notation, in what follows we write $\lambda_k(a)$ for $\lambda_k(R_a)$. As in [2], we start with a lower bound for $\lambda_k(a)$ which, for the case of rectangles, improves upon what would be obtained by a direct application of Pólya's bound for tiling domains.

Lemma 4.1. *For any $a \geq 1$ we have*

$$\lambda_k^{\frac{1}{2}}(a) \geq 4\pi k + 2a\lambda_k^{\frac{1}{4}}(a) - \frac{4\sqrt{2\pi}}{3\sqrt{3}} a^{\frac{3}{2}} \lambda_k^{\frac{1}{8}}(a). \tag{9}$$

Proof. From [2, Theorem 3.1] we have

$$\lambda_k^D(a) \geq 4\pi k + 2a \left[\lambda_k^D(a)\right]^{\frac{1}{2}} - \frac{4\sqrt{2\pi}}{3\sqrt{3}} a^{\frac{3}{2}} \left[\lambda_k^D(a)\right]^{\frac{1}{4}}, \tag{10}$$

where $\lambda_k^D(a)$ is the k-th eigenvalue of the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } R_a, \\ u = 0, & \text{on } \partial R_a. \end{cases}$$

At this point we observe that

$$\lambda_k(a) \geq \left[\lambda_k^D(a)\right]^2, \tag{11}$$

since $\left[\lambda_k^D(a)\right]^2$ is the k-th eigenvalue of the Navier problem

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } R_a, \\ u = \Delta u = 0, & \text{on } \partial R_a, \end{cases} \tag{12}$$

and may be characterized as

$$\left[\lambda_k^D(a)\right]^2 = \min_{0 \neq u \in V \subset H^2(R_a) \cap H_0^1(R_a)} \max_{\dim V = k} \frac{\int_{R_a} (\Delta u)^2}{\int_{R_a} u^2}.$$

See also [18, Chapter 2.7] for a discussion about the coercivity of problem (12).

We rewrite (10) as

$$\lambda_k^D(a) - 2a \left[\lambda_k^D(a)\right]^{\frac{1}{2}} \geq 4\pi k - \frac{4\sqrt{2\pi}}{3\sqrt{3}} a^{\frac{3}{2}} \left[\lambda_k^D(a)\right]^{\frac{1}{4}}.$$

Using the fact that $t-2a\sqrt{t}$ is increasing in t for $t \geq a^2$ and that $\lambda_k(a)^{\frac{1}{2}} \geq \lambda_k^D(a) \geq a^2$, we obtain inequality (9). □

Let us now set

$$\lambda_k^* = \min_{a\geq 1} \lambda_k(a).$$

It is clear that the minimum is achieved since $\lambda_k(a) \to \infty$ as $a \to \infty$. We also set a_k^* in such a way that

$$\lambda_k(a_k^*) = \lambda_k^*.$$

We remark that, in line with what happens for the Dirichlet Laplacian [2], a_k^* does not have to be uniquely defined as a function of k. although it would probably be extremely difficult to prove so in this case; however, we can always choose one particular value for each $k \in \mathbb{N}$.

Then we have

Theorem 4.2. *The sequence of optimal rectangular shapes for λ_k converges to the square as $k \to \infty$, i.e.,*

$$\lim_{k\to\infty} a_k^* = 1. \tag{13}$$

As we just noticed, uniqueness of the optimizer may fail for some k; nevertheless, for any possible choice, the limit (13) holds.

Proof. First of all, following the argument used in [2, Theorem 3.5] coupled with the Weyl asymptotic expansion (3) and Lemma 4.1, we get that

$$\limsup_{k\to\infty} a_k^* \leq \frac{6\sqrt{3}}{3\sqrt{3}-2\sqrt{2}} \left[1 + \frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt{\pi}\Gamma\left(\frac{5}{4}\right)}\right],$$

meaning that the sequence $\{a_k^*\}_k$ is bounded. At this point, using the boundedness of $\{a_k^*\}_k$ and the fact that

$$\lambda_k^{\frac{1}{2}}(a) \geq \lambda_k^D(a) \geq \pi^2\left(a^2 + \frac{1}{a^2}\right),$$

and that $t \to t - 2\left(a + \frac{1}{a}\right)\sqrt{t}$ is increasing for $t \geq \pi^2\left(a^2 + \frac{1}{a^2}\right)$, from [2, formula (3.6)] we deduce that

$$\lambda_k^{\frac{1}{2}}(a) \geq 4\pi k + 2\left(a + \frac{1}{a}\right)\lambda_k^{\frac{1}{4}}(a) - C\lambda_k^{\frac{\theta}{4}}(a) - 3\pi,$$

for some $\theta \in (0,1)$. We thus deduce an inequality of the type of [2, inequality (3.7)] and, following the same argument as in [2, p. 8], we obtain the deired result. □

5. Further results

For completeness, we now collect some results whose proofs are similar to their Laplacian counterparts.

5.1. **Perimeter constraint.** The first of these corresponds to the minimisation of the k^{th} eigenvalue under a perimeter restriction. More precisely, let

$$\lambda_k^* = \min\{\lambda_k(\Omega) : \Omega \in \mathcal{R}, |\partial\Omega| = \alpha\},$$

for some fixed value $\alpha > 0$, where $\mathcal{R}$ is a family of bounded domains in $\mathbb{R}^2$. Let also $\Omega_k^* \in \mathcal{R}$ be a minimiser for λ_k, i.e.,

$$\lambda_k^* = \lambda_k(\Omega_k^*).$$

We have the following

Theorem 5.1. *Let $\alpha > 0$ be fixed.*

i) *Let $\mathcal{D}$ be the class of open domains in $\mathbb{R}^2$. Then the sequence of optimal domains Ω_k^* converges to the disk with perimeter α.*
ii) *Let $\mathcal{P}_n$ be the class of polygons having exactly n sides in $\mathbb{R}^2$. Then the sequence of optimal domains Ω_k^* converges to the regular n-gon with perimeter α.*
iii) *Let $\mathcal{T}$ be the class of tiling domains in $\mathbb{R}^2$. Then the sequence of optimal domains Ω_k^* converges to the regular hexagon with perimeter α.*

The proof of Theorem 5.1 goes along the same lines of the correponding results in [10], now using the first term in the Weyl asymptotics (2), and inequalities (4) and (11). We also note that this result can be extended to a general polyharmonic problem of the form

$$\begin{cases} (-\Delta)^m u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0, & \text{on } \partial\Omega, \end{cases} \tag{14}$$

for $m \geq 1$, as formulas (2) and (4) can be generalized to this case as well.

5.2. **Subadditivity.** Let us now set

$$\lambda_k^* = \min\{\lambda_k(\Omega) : \Omega \in \mathbb{R}^N, |\Omega| = 1\},$$

for any k.

Theorem 5.2. *Let $i_1 \leq i_1 \leq \cdots \leq i_p$ be positive integers such that $i_1 + \cdots + i_p = k$. Then*

$$(\lambda_k^*)^{\frac{N}{4}} \leq (\lambda_{i_1}^*)^{\frac{N}{4}} + \cdots + (\lambda_{i_p}^*)^{\frac{N}{4}}.$$

In particular,

$$(\lambda_{k+1}^*)^{\frac{N}{4}} - (\lambda_k^*)^{\frac{N}{4}} \leq (\lambda_1^*)^{\frac{N}{4}}.$$

The proof of this result can be obtained following that of [15, Theorem 2.1]. We also have the following corollary thanks to Fekete's Lemma (cf. [21] for a general statement of the Generalized Polya conjecture).

Corollary 5.3. *The following are equivalent.*

i) (Generalized Polya conjecture) *For any k and for any domain in $\mathbb{R}^N$,*

$$\lambda_k \geq 16\pi^4 \left(\frac{k}{\omega_N |\Omega|}\right)^{4/N}.$$

ii) $\displaystyle\lim_{k\to\infty} \frac{\lambda_k^*}{k^{4/N}} = \frac{16\pi^4}{\omega_N^{4/N}}$.

We again observe that both Theorem 5.2 and Corollary 5.3 may also be stated for the polyharmonic problem (14).

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