

A few shape optimization results for a biharmonic Steklov problem

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Abstract: We derive the equation of a free vibrating thin plate whose mass is concentrated at the boundary, namely a Steklov problem for the biharmonic operator. We provide Hadamard-type formulas for the shape derivatives of the corresponding eigenvalues and prove that balls are critical domains under volume constraint. Finally, we prove an isoperimetric inequality for the first positive eigenvalue.

Keywords: biharmonic operator, Steklov boundary conditions, eigenvalues, isovolumetric perturbations, isoperimetric inequality.

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1 Introduction

Let Ω be a bounded domain (i.e., a bounded connected open set) in \mathbb{R}^N of class C^1 , $N \geq 2$. We consider the following Steklov problem for the biharmonic operator

$$\begin{cases} \Delta^2 u - \tau \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \operatorname{div}_{\partial\Omega}(D^2 u \cdot \nu) - \frac{\partial \Delta u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

in the unknowns u (the eigenfunction), λ (the eigenvalue), where $\tau > 0$ is a fixed positive constant, ν denotes the outer unit normal to $\partial\Omega$, $\operatorname{div}_{\partial\Omega}$ denotes the tangential divergence operator and $D^2 u$ the Hessian matrix of u . For $N = 2$, this problem is related to the study of the vibrations of a thin elastic plate with a free frame and mass concentrated at the boundary. The spectrum consists of a diverging sequence of eigenvalues of finite multiplicity

$$0 = \lambda_1(\Omega) < \lambda_2(\Omega) \leq \dots \leq \lambda_j(\Omega) \leq \dots,$$

where we agree to repeat the eigenvalues according to their multiplicity. We note that problem (1.1) is the analogue for the biharmonic operator of the classical Steklov problem for the Laplace operator, namely

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

which models the vibrations of a free membrane with mass concentrated at the boundary. Problem (1.2) was first considered by Steklov in [39], where the author provided a physical derivation (see also [34]). We refer to [30] for related problems, and to [20] for a recent survey on the subject.

In this paper we are interested in the dependence of the eigenvalues $\lambda_j(\Omega)$ of problem (1.1) on the domain Ω . Domain perturbation problems have been widely studied in the case of the Laplace operator subject to different homogeneous boundary conditions (Dirichlet, Neumann, Steklov, etc.), in particular for shape optimization problems. We recall for instance the celebrated Faber-Krahn inequality, which says that the ball minimizes the first eigenvalue of the Dirichlet Laplacian among all domains with fixed measure (see [17, 29]). Similar results have been shown also for other boundary conditions (see e.g., [10, 42, 44]). As for the biharmonic

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operator, much less is known. Lord Rayleigh conjectured that the ball minimizes the fundamental tone of the clamped plate (i.e., the first positive eigenvalue of the biharmonic operator with Dirichlet boundary conditions) among open sets with the same measure. This has been proved by Nadirashvili [37] for $N = 2$, and soon generalized by Ashbaugh and Benguria [6] for $N = 3$, while the general case remains an open problem (see also [36, 40]). Regarding Neumann boundary conditions, Chasman [14] proved that the ball is a maximizer for the fundamental tone. We refer to [27] for a general approach to domain perturbation problems (see also [25]), and to [26] for a comprehensive discussion on eigenvalue shape optimization problems for elliptic operators. We also refer to [12, 13] where the authors prove analyticity properties in the spirit of [31] for Dirichlet and intermediate boundary conditions respectively, and show that balls are critical domains for all the elementary symmetric functions of the eigenvalues.

Problem (1.1) should not be confused with other Steklov-type problems already discussed in the literature. For example, in [11] the authors consider the following problem

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \Delta u = \lambda \frac{\partial u}{\partial \nu}, & \text{on } \partial\Omega, \end{cases}$$

which has a rather different nature. We note that, broadly speaking, one may refer to Steklov-type boundary conditions for those problems where a spectral parameter enters the boundary conditions.

The aim of the present paper is to discuss the Steklov problem (1.1) as the natural fourth order version of problem (1.2). We derive problem (1.1) starting from a physical model and study the relationship with the Neumann eigenvalue problem for the biharmonic operator considered in [14]. Then we adapt the arguments used in [31, 32] in order to show real analyticity of the elementary symmetric functions of the eigenvalues of (1.1) and compute Hadamard-type formulas, which are used to prove that balls are critical domains. For completeness, we do the same also for the Neumann problem as stated in [14]. Finally, we study problem (1.1) when Ω is a ball and identify the fundamental tone and the corresponding modes (the eigenfunctions). By following a scheme similar to that used in [42] we prove that the ball is a maximizer for the first positive eigenvalue of problem (1.1) among all bounded domains of class C^1 .

The paper is organized as follows. In Section 2 we derive problem (1.1) providing a physical interpretation. In Section 3 we characterize the spectrum and show that problem (1.1) is strictly related to the Neumann eigenvalue problem as described in [14]. As a byproduct, we provide a further physical justification of (1.1). In Section 4 we compute Hadamard-type formulas and prove that balls are critical domains under measure constraint for the elementary symmetric functions of the eigenvalues of (1.1) and of the corresponding Neumann problem (3.10). In Section 5 we prove the isoperimetric inequality for the fundamental tone. Finally, in Section 6, we provide some remarks on problems (1.1) and (3.10) when $\tau = 0$.

2 Formulating the problem

In this section we provide a physical interpretation of problem (1.1) for $N = 2$, which arises in the theory of linear elasticity, in particular in the study of transverse vibrations of a thin plate. Actually, in Sections 2 and 3 we will consider a slightly more general version of problem (1.1), namely

$$\begin{cases} \Delta^2 u - \tau \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \operatorname{div}_{\partial\Omega}(D^2 u \cdot \nu) - \frac{\partial \Delta u}{\partial \nu} = \lambda \rho u, & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where a positive weight $\rho \in L^\infty(\partial\Omega)$ appears in the boundary conditions. The weight ρ has the meaning of a mass density. We shall always assume that τ is a fixed positive real number. We recall that the tangential divergence $\operatorname{div}_{\partial\Omega} F$ of a vector field F is defined as $\operatorname{div}_{\partial\Omega} F = \operatorname{div} F|_{\partial\Omega} - (DF \cdot \nu) \cdot \nu$, where DF is the Jacobian matrix of F .

As usual, we assume that the mass is displaced in the middle plane of the plate parallel to its faces. When the body is at its equilibrium it covers a planar domain Ω with boundary $\partial\Omega$ in \mathbb{R}^2 . We describe the vertical deviation from the equilibrium during the vibration of each point (x, y) of Ω at time t by means of a function $v = v(x, y, t)$. We suppose that the whole mass of the plate is concentrated at the boundary with a density which we denote by $\rho(x, y)$.

Moreover, we assume that $\rho(x, y)$ is bounded and positive on $\partial\Omega$. Under these assumptions, the total kinetic energy of the plate is given by

$$T = \frac{1}{2} \int_{\partial\Omega} \rho \dot{v}^2 d\sigma,$$

where we denote by \dot{v} the derivative of v with respect to the time t , and by $d\sigma$ the surface measure on $\partial\Omega$. Now we obtain an expression for the potential energy of the plate. By following [43, §10.8], under the assumption that the strain potential energy at each point depends only on the strain configuration at that point and that the Poisson ratio of the material is zero, we have that the strain potential energy is given by

$$V_s = \frac{1}{2} \int_{\Omega} (v_{xx}^2 + v_{yy}^2 + 2v_{xy}^2) dx dy.$$

Besides V_s , we have another term of the potential energy due to the lateral tension

$$V_t = \frac{\tau}{2} \int_{\Omega} (v_x^2 + v_y^2) dx dy,$$

where $\tau > 0$ is the ratio of lateral tension due to flexural rigidity. The Hamilton's integral of the system is given by

$$\begin{aligned} \mathcal{H} &= \int_{t_1}^{t_2} T - V_s - V_t dt \\ &= \frac{1}{2} \int_{t_1}^{t_2} \int_{\partial\Omega} \rho \dot{v}^2 d\sigma dt - \frac{1}{2} \int_{t_1}^{t_2} \int_{\Omega} (v_{xx}^2 + v_{yy}^2 + 2v_{xy}^2) + \tau(v_x^2 + v_y^2) dx dy dt. \end{aligned} \quad (2.2)$$

According to Hamilton's Variational Principle, the actual motion of the system minimizes such integral. Let $v(x, y, t)$ be a minimizer for \mathcal{H} . By differentiating (2.2) it follows that v satisfies

$$\begin{aligned} - \int_{t_1}^{t_2} \int_{\partial\Omega} \eta \rho \ddot{v} d\sigma dt - \int_{t_1}^{t_2} \int_{\Omega} \eta (\Delta^2 v - \tau \Delta v) dx dy dt \\ - \int_{t_1}^{t_2} \int_{\partial\Omega} \frac{\partial \eta}{\partial \nu} \frac{\partial^2 v}{\partial \nu^2} - \eta \left(\tau \frac{\partial v}{\partial \nu} - \operatorname{div}_{\partial\Omega} (D^2 v \cdot \nu)_{\partial\Omega} - \frac{\partial \Delta v}{\partial \nu} \right) d\sigma dt = 0, \end{aligned}$$

for all $\eta \in C^2(\Omega \times [t_1, t_2])$ such that $\eta(x, y, t_1) = \eta(x, y, t_2) = 0$ and $(x, y) \in \Omega$. Since η is arbitrary we obtain

$$\begin{cases} \Delta^2 v - \tau \Delta v = 0, & \text{in } \Omega, \\ \frac{\partial^2 v}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \rho \ddot{v} + \tau \frac{\partial v}{\partial \nu} - \operatorname{div}_{\partial\Omega} (D^2 v \cdot \nu) - \frac{\partial \Delta v}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases} \quad (2.3)$$

for all $t \in [t_1, t_2]$. We remark that we wrote $\operatorname{div}_{\partial\Omega} (D^2 v \cdot \nu)$ instead of $\operatorname{div}_{\partial\Omega} (D^2 v \cdot \nu)_{\partial\Omega}$ since $(D^2 v \cdot \nu)_{\partial\Omega} = D^2 v \cdot \nu - \frac{\partial^2 v}{\partial \nu^2} \nu$ and $\frac{\partial^2 v}{\partial \nu^2} = 0$ on $\partial\Omega$.

We separate the variables and, as is customary, we look for solutions to problem (2.3) of the form $v(x, y, t) = u(x, y)w(t)$. We find that the temporal component $w(t)$ solves the ordinary differential equation $-\ddot{w}(t) = \lambda w(t)$ for all $t \in [t_1, t_2]$, while the spatial component u solves problem (2.1).

3 Characterization of the spectrum. Alternative derivation of the problem

In this section we prove that the spectrum of the eigenvalue problem (2.1) is discrete. In particular, each eigenvalue is non-negative and has finite multiplicity and there exists a Hilbert basis of the standard Sobolev space $H^2(\Omega)$ of eigenvectors. Then we provide a further derivation of problem (2.1). Namely, we show that this problem can be seen as a limit of eigenvalue problems for the biharmonic operator with Neumann boundary conditions and mass density ρ_ε which concentrates in a neighborhood of the boundary as ε goes to zero. We refer to [5, 34]

for similar discussions concerning second order problems. We observe that the asymptotic analysis of mass concentration problems for second order operators has been performed by several authors by exploiting asymptotic expansions methods, see e.g., [23, 24] and the references therein. We also mention the alternative approach based on potential theory and functional analysis proposed in [15, 35].

Note that here and in the sequel we shall not put any restriction on the space dimension. Thus Ω will always denote a bounded domain in \mathbb{R}^N of class C^1 , with $N \geq 2$.

3.1 Analysis of the spectrum of problem (2.1)

Let $\rho \in \mathcal{R}^S$, where $\mathcal{R}^S := \{\rho \in L^\infty(\partial\Omega) : \text{ess inf}_{x \in \partial\Omega} \rho(x) > 0\}$. We consider the weak formulation of problem (2.1),

$$\int_{\Omega} D^2 u : D^2 \varphi + \tau \nabla u \cdot \nabla \varphi dx = \lambda \int_{\partial\Omega} \rho u \varphi d\sigma, \quad \forall \varphi \in H^2(\Omega), \quad (3.1)$$

in the unknowns $u \in H^2(\Omega)$, $\lambda \in \mathbb{R}$, where

$$D^2 u : D^2 \varphi = \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$$

denotes the Frobenius product. Actually, we will obtain a problem in $H^2(\Omega)/\mathbb{R}$ since we need to get rid of the constants, which generate the eigenspace corresponding to the eigenvalue $\lambda = 0$. We denote by \mathcal{J}_ρ^S the continuous embedding of $L^2(\partial\Omega)$ into $H^2(\Omega)'$ defined by

$$\mathcal{J}_\rho^S[u][\varphi] := \int_{\partial\Omega} \rho u \varphi d\sigma, \quad \forall u \in L^2(\partial\Omega), \varphi \in H^2(\Omega).$$

We set

$$H_\rho^{2,S}(\Omega) := \left\{ u \in H^2(\Omega) : \int_{\partial\Omega} \rho u d\sigma = 0 \right\},$$

and we consider in $H^2(\Omega)$ the bilinear form

$$\langle u, v \rangle = \int_{\Omega} D^2 u : D^2 v + \tau \nabla u \cdot \nabla v dx. \quad (3.2)$$

By the Poincaré-Wirtinger Inequality, it turns out that this bilinear form is indeed a scalar product on $H_\rho^{2,S}(\Omega)$ whose induced norm is equivalent to the standard one. In the sequel we will think of the space $H_\rho^{2,S}(\Omega)$ as endowed with the form (3.2). Let $F(\Omega)$ be defined by $F(\Omega) := \{G \in H^2(\Omega)' : G[1] = 0\}$. Then, we consider the operator \mathcal{P}_ρ^S as an operator from $H_\rho^{2,S}(\Omega)$ to $F(\Omega)$, defined by

$$\mathcal{P}_\rho^S[u][\varphi] := \int_{\Omega} D^2 u : D^2 \varphi + \tau \nabla u \cdot \nabla \varphi dx, \quad \forall u \in H_\rho^{2,S}(\Omega), \varphi \in H^2(\Omega). \quad (3.3)$$

It turns out that \mathcal{P}_ρ^S is a homeomorphism of $H_\rho^{2,S}(\Omega)$ onto $F(\Omega)$. We define the operator π_ρ^S from $H^2(\Omega)$ to $H_\rho^{2,S}(\Omega)$ by

$$\pi_\rho^S[u] := u - \frac{\int_{\partial\Omega} \rho u d\sigma}{\int_{\partial\Omega} \rho d\sigma}. \quad (3.4)$$

We consider the space $H^2(\Omega)/\mathbb{R}$ endowed with the bilinear form induced by (3.2). Such bilinear form renders $H^2(\Omega)/\mathbb{R}$ a Hilbert space. We denote by $\pi_\rho^{\sharp,S}$ the map from $H^2(\Omega)/\mathbb{R}$ onto $H_\rho^{2,S}(\Omega)$ defined by the equality $\pi_\rho^S = \pi_\rho^{\sharp,S} \circ p$, where p is the canonical projection of $H^2(\Omega)$ onto $H^2(\Omega)/\mathbb{R}$. The map $\pi_\rho^{\sharp,S}$ turns out to be a homeomorphism. Finally, we define the operator T_ρ^S acting on $H^2(\Omega)/\mathbb{R}$ as follows

$$T_\rho^S := (\pi_\rho^{\sharp,S})^{-1} \circ (\mathcal{P}_\rho^S)^{-1} \circ \mathcal{J}_\rho^S \circ \text{Tr} \circ \pi_\rho^{\sharp,S}, \quad (3.5)$$

where Tr denotes the trace operator acting from $H^2(\Omega)$ to $L^2(\partial\Omega)$.

Remark 3.6. We observe that the pair (λ, u) of the set $(\mathbb{R} \setminus \{0\}) \times (H_\rho^{2,\mathcal{S}}(\Omega) \setminus \{0\})$ satisfies (3.1) if and only if $\lambda > 0$ and the pair $(\lambda^{-1}, p[u])$ of the set $\mathbb{R} \times ((H^2(\Omega)/\mathbb{R}) \setminus \{0\})$ satisfies the equation

$$\lambda^{-1}p[u] = T_\rho^{\mathcal{S}}p[u].$$

We have the following

Theorem 3.7. The operator $T_\rho^{\mathcal{S}}$ is a non-negative compact selfadjoint operator in $H^2(\Omega)/\mathbb{R}$, whose eigenvalues coincide with the reciprocals of the positive eigenvalues of problem (3.1). In particular, the set of eigenvalues of problem (3.1) is contained in $[0, +\infty[$ and consists of the image of a sequence increasing to $+\infty$. Each eigenvalue has finite multiplicity.

Proof. For the selfadjointness, it suffices to observe that

$$\begin{aligned} \langle T_\rho^{\mathcal{S}}u, v \rangle_{H^2(\Omega)/\mathbb{R}} &= \langle (\pi_\rho^{\#, \mathcal{S}})^{-1} \circ (\mathcal{P}_\rho^{\mathcal{S}})^{-1} \circ \mathcal{J}_\rho^{\mathcal{S}} \circ \text{Tr} \circ \pi_\rho^{\#, \mathcal{S}}u, v \rangle_{H^2(\Omega)/\mathbb{R}} \\ &= \mathcal{P}_\rho^{\mathcal{S}}[(\mathcal{P}_\rho^{\mathcal{S}})^{-1} \circ \mathcal{J}_\rho^{\mathcal{S}} \circ \text{Tr} \circ \pi_\rho^{\#, \mathcal{S}}u][\pi_\rho^{\#, \mathcal{S}}v] \\ &= \mathcal{J}_\rho^{\mathcal{S}}[\text{Tr} \circ \pi_\rho^{\#, \mathcal{S}}u][\pi_\rho^{\#, \mathcal{S}}v], \quad \forall u, v \in H^2(\Omega)/\mathbb{R}. \end{aligned}$$

For the compactness, just observe that the trace operator Tr acting from $H^1(\Omega)$ to $L^2(\partial\Omega)$ is compact. The remaining statements are straightforward. \square

As a consequence we have that the spectrum of (3.1) is of the form

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots$$

Note that the first positive eigenvalue is λ_2 as proved by the following

Theorem 3.8. The first eigenvalue λ_1 of (3.1) is zero and the corresponding eigenfunctions are the constants. Moreover, $\lambda_2 > 0$.

Proof. It is straightforward to see that constant functions are eigenfunctions of (3.1) with eigenvalue $\lambda = 0$. Suppose now that u is an eigenfunction corresponding to the eigenvalue $\lambda = 0$. Then we have

$$\int_\Omega |D^2u|^2 + \tau|\nabla u|^2 dx = 0,$$

where $|D^2u|^2 = \sum_{i,j=1}^N \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2$. Since $\nabla u = 0$, it follows that u is constant. Then the eigenvalue $\lambda = 0$ has multiplicity one. \square

Thus λ_2 is the first positive eigenvalue of (3.1) which is usually called the fundamental tone. Note that we can characterize λ_2 by means of the Rayleigh principle

$$\lambda_2 = \min_{\substack{0 \neq u \in H^2(\Omega) \\ \int_{\partial\Omega} \rho u d\sigma = 0}} \frac{\int_\Omega |D^2u|^2 + \tau|\nabla u|^2 dx}{\int_{\partial\Omega} \rho u^2 d\sigma}. \quad (3.9)$$

3.2 Asymptotic behavior of Neumann eigenvalues

We consider the following eigenvalue problem for the biharmonic operator subject to Neumann boundary conditions

$$\begin{cases} \Delta^2 u - \tau \Delta u = \lambda \rho u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \text{div}_{\partial\Omega}(D^2 u \cdot \nu) - \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \quad (3.10)$$

where $\rho \in \mathcal{R}^{\mathcal{N}} := \{\rho \in L^\infty(\Omega) : \text{ess inf}_{x \in \Omega} \rho(x) > 0\}$ (we refer to [14] for the derivation of the boundary conditions). It is well known that this problem arises in the study of a free vibrating plate whose mass is displaced on the whole of Ω with density ρ .

Let us denote by Ω_ε the set defined by

$$\Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}.$$

We fix a positive number $M > 0$ and choose the family of densities ρ_ε defined as follows

$$\rho_\varepsilon(x) = \begin{cases} \varepsilon, & \text{if } x \in \Omega_\varepsilon, \\ \frac{M - \varepsilon|\Omega_\varepsilon|}{|\Omega \setminus \overline{\Omega}_\varepsilon|}, & \text{if } x \in \Omega \setminus \overline{\Omega}_\varepsilon, \end{cases} \quad (3.11)$$

for $\varepsilon \in]0, \varepsilon_0[$, where $\varepsilon_0 > 0$ is sufficiently small. If in addition we assume that Ω is of class C^2 , ε_0 can be chosen in such a way that the map $x \mapsto x - \nu\varepsilon$ is a diffeomorphism between $\partial\Omega$ and $\partial\Omega_\varepsilon$ for all $\varepsilon \in]0, \varepsilon_0[$. We note that $\int_\Omega \rho_\varepsilon dx = M$ for all $\varepsilon \in]0, \varepsilon_0[$. We refer to the quantity M as the total mass of the body.

We prove, under the additional hypothesis that Ω is of class C^2 , convergence of the eigenvalues and eigenfunctions of problem (3.10) with density ρ_ε to the eigenvalues and eigenfunctions of problem (3.1) with constant surface density $\frac{M}{|\partial\Omega|}$ when the parameter ε goes to zero (see Corollary 3.22). This provides a further interpretation of problem (3.1) as the equation of a free vibrating plate whose mass is concentrated at the boundary in the case of domains of class C^2 .

Problem (3.10) has an increasing sequence of non-negative eigenvalues of finite multiplicity and the eigenfunctions form a Hilbert basis of $H^2(\Omega)$. We consider the weak formulation of problem (3.10) with density ρ_ε ,

$$\int_\Omega D^2u : D^2\varphi + \tau \nabla u \cdot \nabla \varphi dx = \lambda \int_\Omega \rho_\varepsilon u \varphi dx, \quad \forall \varphi \in H^2(\Omega), \quad (3.12)$$

in the unknowns $u \in H^2(\Omega)$, $\lambda \in \mathbb{R}$. In the sequel we shall recast this problem in $H^2(\Omega)/\mathbb{R}$ since we need to get rid of the constants, which generate the eigenspace corresponding to the eigenvalue $\lambda = 0$. We denote by i the canonical embedding of $H^2(\Omega)$ into $L^2(\Omega)$. We denote by $\mathcal{J}_{\rho_\varepsilon}^{\mathcal{N}}$ the continuous embedding of $L^2(\Omega)$ into $H^2(\Omega)'$, defined by

$$\mathcal{J}_{\rho_\varepsilon}^{\mathcal{N}}[u][\varphi] := \int_\Omega \rho_\varepsilon u \varphi dx, \quad \forall u \in L^2(\Omega), \varphi \in H^2(\Omega).$$

We set

$$H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega) := \left\{ u \in H^2(\Omega) : \int_\Omega u \rho_\varepsilon dx = 0 \right\}.$$

In the sequel we will think of the space $H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega)$ as endowed with the form (3.2). Such form defines on $H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega)$ a scalar product whose induced norm is equivalent to the standard one. We denote by $\pi_{\rho_\varepsilon}^{\mathcal{N}}$ the map from $H^2(\Omega)$ to $H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega)$ defined by

$$\pi_{\rho_\varepsilon}^{\mathcal{N}}[u] := u - \frac{\int_\Omega u \rho_\varepsilon dx}{\int_\Omega \rho_\varepsilon dx},$$

for all $u \in H^2(\Omega)$. We denote by $\pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}$ the map from $H^2(\Omega)/\mathbb{R}$ onto $H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega)$ defined by the equality $\pi_{\rho_\varepsilon}^{\mathcal{N}} = \pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}} \circ p$. As in (3.3), we consider the operator $\mathcal{P}_{\rho_\varepsilon}^{\mathcal{N}}$ as a map from $H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega)$ to $F(\Omega)$ defined by

$$\mathcal{P}_{\rho_\varepsilon}^{\mathcal{N}}[u][\varphi] := \int_\Omega D^2u : D^2\varphi + \tau \nabla u \cdot \nabla \varphi dx, \quad \forall u \in H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega), \varphi \in H^2(\Omega).$$

It turns out that $\mathcal{P}_{\rho_\varepsilon}^{\mathcal{N}}$ is a linear homeomorphism of $H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega)$ onto $F(\Omega)$. Finally, let the operator $T_{\rho_\varepsilon}^{\mathcal{N}}$ from $H^2(\Omega)/\mathbb{R}$ to itself be defined by

$$T_{\rho_\varepsilon}^{\mathcal{N}} := (\pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}})^{-1} \circ (\mathcal{P}_{\rho_\varepsilon}^{\mathcal{N}})^{-1} \circ \mathcal{J}_{\rho_\varepsilon}^{\mathcal{N}} \circ i \circ \pi_{\rho_\varepsilon}^{\sharp,\mathcal{N}}. \quad (3.13)$$

Remark 3.14. We observe that the pair (λ, u) of the set $(\mathbb{R} \setminus \{0\}) \times (H_{\rho_\varepsilon}^{2,\mathcal{N}}(\Omega) \setminus \{0\})$ satisfies (3.12) if and only if $\lambda > 0$ and the pair $(\lambda^{-1}, p[u])$ of the set $\mathbb{R} \times ((H^2(\Omega)/\mathbb{R}) \setminus \{0\})$ satisfies the equation

$$\lambda^{-1} p[u] = T_{\rho_\varepsilon}^{\mathcal{N}} p[u].$$

As in Theorem 3.7 it is easy to prove the following

Theorem 3.15. Let Ω be a bounded domain in \mathbb{R}^N of class C^1 and $\varepsilon \in]0, \varepsilon_0[$. The operator $T_{\rho_\varepsilon}^{\mathcal{N}}$ is a compact selfadjoint operator in $H^2(\Omega)/\mathbb{R}$ and its eigenvalues coincide with the reciprocals of the positive eigenvalues $\lambda_j(\rho_\varepsilon)$ of problem (3.12) for all $j \in \mathbb{N}$. Moreover, the set of eigenvalues of problem (3.12) is contained in $[0, +\infty[$ and consists of the image of a sequence increasing to $+\infty$. Each eigenvalue has finite multiplicity.

We have the following theorem on the spectrum of problem (3.12) (see also Theorem 3.8).

Theorem 3.16. *The first eigenvalue $\lambda_1(\rho_\varepsilon)$ of (3.12) is zero and the corresponding eigenfunctions are the constants. Moreover, $\lambda_2(\rho_\varepsilon) > 0$.*

Now we highlight the relations between problems (3.1) and (3.12) when Ω is of class C^2 . In particular we plan to prove the following

Theorem 3.17. *Let Ω be a bounded domain in \mathbb{R}^N of class C^2 . Let the operators $T_{\frac{M}{|\partial\Omega|}}^S$ and $T_{\rho_\varepsilon}^N$ from $H^2(\Omega)/\mathbb{R}$ to itself be defined as in (3.5) and (3.13) respectively. Then the sequence $\{T_{\rho_\varepsilon}^N\}_{\varepsilon \in]0, \varepsilon_0[}$ converges in norm to $T_{\frac{M}{|\partial\Omega|}}^S$ as $\varepsilon \rightarrow 0$.*

We need some preliminary results in order to prove Theorem 3.17. We remark that $\pi_c^{\#,S} = \pi_1^{\#,S}$ for all $c \in \mathbb{R}$, with $c \neq 0$. This can be easily deduced by (3.4).

Lemma 3.18. *Let Ω be a bounded domain in \mathbb{R}^N of class C^2 . Let $\rho_\varepsilon \in \mathcal{R}^N$ be as in (3.11). Then the following statements hold.*

- i) *For all $\varphi \in H^2(\Omega)/\mathbb{R}$, $\pi_{\rho_\varepsilon}^{\#,N}[\varphi] \rightarrow \pi_1^{\#,S}[\varphi]$ in $L^2(\Omega)$ (hence also in $H^2(\Omega)$) as $\varepsilon \rightarrow 0$;*
- ii) *If $u_\varepsilon \rightarrow u$ in $H^2(\Omega)/\mathbb{R}$, then (possibly passing to a subsequence) $\pi_{\rho_\varepsilon}^{\#,N}[u_\varepsilon] \rightarrow \pi_1^{\#,S}[u]$ in $L^2(\Omega)$ as $\varepsilon \rightarrow 0$;*
- iii) *Assume that $u_\varepsilon, u, w_\varepsilon, w \in H^2(\Omega)$ are such that $u_\varepsilon \rightarrow u$, $w_\varepsilon \rightarrow w$ in $L^2(\Omega)$, $\text{Tr}[u_\varepsilon] \rightarrow \text{Tr}[u]$, $\text{Tr}[w_\varepsilon] \rightarrow \text{Tr}[w]$ in $L^2(\partial\Omega)$ as $\varepsilon \rightarrow 0$. Moreover assume that there exists a constant $C > 0$ such that $\|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq C$, $\|\nabla w_\varepsilon\|_{L^2(\Omega)} \leq C$ for all $\varepsilon \in]0, \varepsilon_0[$. Then*

$$\int_{\Omega} \rho_\varepsilon (u_\varepsilon - u) w_\varepsilon dx \rightarrow 0$$

and

$$\int_{\Omega} \rho_\varepsilon (w_\varepsilon - w) u dx \rightarrow 0$$

as $\varepsilon \rightarrow 0$

Proof. The proof is standard but long and we omit it. We refer to [34] and references therein for details. We remark that in order to prove this lemma we need to assume that Ω is of class C^2 , since the Tubular Neighborhood Theorem is used to perform computations on the strip $\Omega \setminus \overline{\Omega}_\varepsilon$. \square

Proof of Theorem 3.17. It is sufficient to prove that the family $\{T_{\rho_\varepsilon}^N\}_{\varepsilon \in]0, \varepsilon_0[}$ of compact operators, compactly converges to the compact operator $T_{\frac{M}{|\partial\Omega|}}^S$. This implies, in fact, that

$$\lim_{\varepsilon \rightarrow 0} \left\| \left(T_{\rho_\varepsilon}^N - T_{\frac{M}{|\partial\Omega|}}^S \right)^2 \right\|_{\mathcal{L}(H^2(\Omega)/\mathbb{R}, H^2(\Omega)/\mathbb{R})} = 0. \quad (3.19)$$

Then, since the operators $\{T_{\rho_\varepsilon}^N\}_{\varepsilon \in]0, \varepsilon_0[}$ and $T_{\frac{M}{|\partial\Omega|}}^S$ are selfadjoint, property (3.19) is equivalent to convergence in norm. We refer to [4, 41] for a more detailed discussion on compact convergence of compact operators on Hilbert spaces. We recall that, by definition, $T_{\rho_\varepsilon}^N$ compactly converges to $T_{\frac{M}{|\partial\Omega|}}^S$ if the following requirements are fulfilled:

- i) if $\|u_\varepsilon\|_{H^2(\Omega)/\mathbb{R}} \leq C$ for all $\varepsilon \in]0, \varepsilon_0[$, then the family $\{T_{\rho_\varepsilon}^N u_\varepsilon\}_{\varepsilon \in]0, \varepsilon_0[}$ is relatively compact in $H^2(\Omega)/\mathbb{R}$;
- ii) if $u_\varepsilon \rightarrow u$ in $H^2(\Omega)/\mathbb{R}$, then $T_{\rho_\varepsilon}^N u_\varepsilon \rightarrow T_{\frac{M}{|\partial\Omega|}}^S u$ in $H^2(\Omega)/\mathbb{R}$.

We prove i) first. For a fixed $u \in H^2(\Omega)/\mathbb{R}$ we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_{\varepsilon} \pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[u] dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_{\varepsilon} \left(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[u] - \pi_1^{\sharp, \mathcal{S}}[u] \right) dx \\ &+ \left(\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_{\varepsilon} \pi_1^{\sharp, \mathcal{S}}[u] dx - \frac{M}{|\partial\Omega|} \int_{\partial\Omega} \pi_1^{\sharp, \mathcal{S}}[u] d\sigma \right) \\ &+ \frac{M}{|\partial\Omega|} \int_{\partial\Omega} \pi_1^{\sharp, \mathcal{S}}[u] d\sigma. \end{aligned} \quad (3.20)$$

By Lemma 3.18 we have that the first summand in the right-hand side of (3.20) goes to zero as $\varepsilon \rightarrow 0$ and by standard calculus we have that the second term goes to zero as $\varepsilon \rightarrow 0$. Moreover, the equality $(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}})^{-1} \circ (\mathcal{P}_{\rho_{\varepsilon}}^{\mathcal{N}})^{-1} = (\pi_1^{\sharp, \mathcal{S}})^{-1} \circ (\mathcal{P}_1^{\mathcal{S}})^{-1}$ holds. Therefore, it follows that $T_{\rho_{\varepsilon}}^{\mathcal{N}} u$ is bounded for each $u \in H^2(\Omega)/\mathbb{R}$. Thus, by Banach-Steinhaus Theorem, there exists C' such that $\|T_{\rho_{\varepsilon}}^{\mathcal{N}}\|_{\mathcal{L}(H^2(\Omega)/\mathbb{R}, H^2(\Omega)/\mathbb{R})} \leq C'$ for all $\varepsilon \in]0, \varepsilon_0[$. Moreover, since $\|u_{\varepsilon}\|_{H^2(\Omega)/\mathbb{R}} \leq C$ for all $\varepsilon \in]0, \varepsilon_0[$, possibly passing to a subsequence, we have that $u_{\varepsilon} \rightharpoonup u$ in $H^2(\Omega)/\mathbb{R}$, for some $u \in H^2(\Omega)/\mathbb{R}$. This implies that, possibly passing to a subsequence, $T_{\rho_{\varepsilon}}^{\mathcal{N}} u_{\varepsilon} \rightharpoonup w$ in $H^2(\Omega)/\mathbb{R}$ as $\varepsilon \rightarrow 0$, for some $w \in H^2(\Omega)/\mathbb{R}$. We show that $w = T_{\frac{M}{|\partial\Omega|}}^{\mathcal{S}} u$. To shorten our notation we set $w_{\varepsilon} := T_{\rho_{\varepsilon}}^{\mathcal{N}} u_{\varepsilon}$. By Lemma 3.18 we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} D^2(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[w_{\varepsilon}]) : D^2(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[\varphi]) + \tau \nabla(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[w_{\varepsilon}]) \cdot \nabla(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[\varphi]) dx \\ = \int_{\Omega} D^2(\pi_1^{\sharp, \mathcal{S}}[w]) : D^2(\pi_1^{\sharp, \mathcal{S}}[\varphi]) + \tau \nabla(\pi_1^{\sharp, \mathcal{S}}[w]) \cdot \nabla(\pi_1^{\sharp, \mathcal{S}}[\varphi]) dx, \end{aligned}$$

for all $\varphi \in H^2(\Omega)/\mathbb{R}$. On the other hand, since $(\mathcal{P}_{\rho_{\varepsilon}}^{\mathcal{N}} \circ \pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}) w_{\varepsilon} = (\mathcal{J}_{\rho_{\varepsilon}}^{\mathcal{N}} \circ i \circ \pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}) u_{\varepsilon}$, we have that

$$\int_{\Omega} D^2(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[w_{\varepsilon}]) : D^2(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[\varphi]) + \tau \nabla(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[w_{\varepsilon}]) \cdot \nabla(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[\varphi]) dx = \int_{\Omega} \rho_{\varepsilon} \pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[u_{\varepsilon}] \pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[\varphi] dx$$

Then, by Lemma 3.18, *iii*) we have

$$\begin{aligned} \langle w, \varphi \rangle_{H^2(\Omega)/\mathbb{R}} &= \lim_{\varepsilon \rightarrow 0} \langle w_{\varepsilon}, \varphi \rangle_{H^2(\Omega)/\mathbb{R}} = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_{\varepsilon} \pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[u_{\varepsilon}] \pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[\varphi] dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_{\varepsilon} \left(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[u_{\varepsilon}] - \pi_1^{\sharp, \mathcal{S}}[u] \right) \pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[\varphi] dx + \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_{\varepsilon} \pi_1^{\sharp, \mathcal{S}}[u] \left(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[\varphi] - \pi_1^{\sharp, \mathcal{S}}[\varphi] \right) dx \\ &+ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_{\varepsilon} \pi_1^{\sharp, \mathcal{S}}[u] \pi_1^{\sharp, \mathcal{S}}[\varphi] dx = \frac{M}{|\partial\Omega|} \int_{\partial\Omega} \pi_1^{\sharp, \mathcal{S}}[u] \pi_1^{\sharp, \mathcal{S}}[\varphi] d\sigma = \langle T_{\frac{M}{|\partial\Omega|}}^{\mathcal{S}} u, \varphi \rangle_{H^2(\Omega)/\mathbb{R}}, \end{aligned}$$

hence $w = T_{\frac{M}{|\partial\Omega|}}^{\mathcal{S}} u$. In a similar way one can prove that $\|w_{\varepsilon}\|_{H^2(\Omega)/\mathbb{R}} \rightarrow \|w\|_{H^2(\Omega)/\mathbb{R}}$. In fact

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \|w_{\varepsilon}\|_{H^2(\Omega)/\mathbb{R}}^2 &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_{\varepsilon} \left(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[u_{\varepsilon}] - \pi_1^{\sharp, \mathcal{S}}[u] \right) \pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[w_{\varepsilon}] dx \\ &+ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_{\varepsilon} \pi_1^{\sharp, \mathcal{S}}[u] \left(\pi_{\rho_{\varepsilon}}^{\sharp, \mathcal{N}}[w_{\varepsilon}] - \pi_1^{\sharp, \mathcal{S}}[w] \right) dx \\ &+ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_{\varepsilon} \pi_1^{\sharp, \mathcal{S}}[u] \left(\pi_1^{\sharp, \mathcal{S}}[w_{\varepsilon}] - \pi_1^{\sharp, \mathcal{S}}[w] \right) dx \\ &+ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_{\varepsilon} \pi_1^{\sharp, \mathcal{S}}[u] \pi_1^{\sharp, \mathcal{S}}[w] dx \\ &= \frac{M}{|\partial\Omega|} \int_{\partial\Omega} \pi_1^{\sharp, \mathcal{S}}[u] \pi_1^{\sharp, \mathcal{S}}[w] d\sigma = \|w\|_{H^2(\Omega)/\mathbb{R}}^2. \end{aligned}$$

This proves *i*). As for point *ii*), let $u_{\varepsilon} \rightarrow u$ in $H^2(\Omega)/\mathbb{R}$. Then there exists C'' such that $\|u_{\varepsilon}\|_{H^2(\Omega)/\mathbb{R}} \leq C''$ for all ε . Then, by the same argument used for point *i*), for each sequence $\varepsilon_j \rightarrow 0$, possibly passing to a subsequence, we have $T_{\rho_{\varepsilon_j}}^{\mathcal{N}} u_{\varepsilon_j} \rightarrow T_{\frac{M}{|\partial\Omega|}}^{\mathcal{S}} u$. Since this is true for each $\{\varepsilon_j\}_{j \in \mathbb{N}}$, we have the convergence for the whole family, i.e., $T_{\rho_{\varepsilon}}^{\mathcal{N}} u_{\varepsilon} \rightarrow T_{\frac{M}{|\partial\Omega|}}^{\mathcal{S}} u$. This concludes the proof. \square

Now we recall the following well-known result.

Theorem 3.21. *Let H be a real Hilbert space and $\{A_\varepsilon\}_{\varepsilon \in]0, \varepsilon_0[}$ a family of bounded selfadjoint operators converging in norm to the bounded selfadjoint operator A , i.e., $\lim_{\varepsilon \rightarrow 0} \|A_\varepsilon - A\|_{\mathcal{L}(H, H)} = 0$. Then isolated eigenvalues λ of A of finite multiplicity are exactly the limits of eigenvalues of A_ε , counting multiplicity; moreover, the corresponding eigenprojections converge in norm.*

Thanks to Theorem 3.21, as an immediate corollary of Theorem 3.17 we have

Corollary 3.22. *Let Ω be a bounded domain in \mathbb{R}^N of class C^2 . Let $\lambda_j(\rho_\varepsilon)$ be the eigenvalues of problem (3.12) on Ω for all $j \in \mathbb{N}$. Let λ_j , $j \in \mathbb{N}$ denote the eigenvalues of problem (3.1) corresponding to the constant surface density $\frac{M}{|\partial\Omega|}$. Then $\lim_{\varepsilon \rightarrow 0} \lambda_j(\rho_\varepsilon) = \lambda_j$ for all $j \in \mathbb{N}$.*

4 Symmetric functions of the eigenvalues. Isovolumetric perturbations

In this section we compute Hadamard-type formulas for both the Steklov and the Neumann problems, which will be used to investigate the behavior of the eigenvalues subject to isovolumetric perturbations. To do so, we use the so called transplantation method, see [27] for a general introduction to this approach. We will study problems (1.1) and (3.10) in $\phi(\Omega)$, for a suitable homeomorphism ϕ , where Ω has to be thought as a fixed bounded domain of class C^1 . Therefore, we introduce the following class of functions

$$\Phi(\Omega) = \left\{ \phi \in (C^2(\overline{\Omega}))^N : \phi \text{ injective and } \inf_{\Omega} |\det D\phi| > 0 \right\}.$$

We observe that if Ω is of class C^1 and $\phi \in \Phi(\Omega)$, then also $\phi(\Omega)$ is of class C^1 and $\phi^{(-1)} \in \Phi(\phi(\Omega))$. Therefore, it makes sense to study both problem (1.1) and problem (3.10) on $\phi(\Omega)$. Moreover, we endow the space $C^2(\overline{\Omega})$ with the standard norm

$$\|f\|_{C^2(\overline{\Omega})} = \sup_{|\alpha| \leq 2, x \in \overline{\Omega}} |D^\alpha f(x)|.$$

Note that $\Phi(\Omega)$ is open in $(C^2(\overline{\Omega}))^N$ (see [31, Lemma 3.11]).

We recall that it has been pointed out that balls play a relevant role in the study of isovolumetric perturbations of the domain Ω for all the eigenvalues of the Dirichlet and Neumann Laplacian. We refer to [31, 33], where the authors prove that the elementary symmetric functions of the eigenvalues depend real analytically on the domain, providing also Hadamard-type formulas for the corresponding derivatives. Then, in [32] they show that balls are critical points for such functions under volume constraint.

From now on we will consider problems (1.1) and (3.10) with constant mass density $\rho \equiv 1$.

4.1 The Steklov problem

We plan to study the Steklov problem in the domain $\phi(\Omega)$ for $\phi \in \Phi(\Omega)$, i.e.,

$$\begin{cases} \Delta^2 u - \tau \Delta u = 0, & \text{in } \phi(\Omega), \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\phi(\Omega), \\ \tau \frac{\partial u}{\partial \nu} - \operatorname{div}_{\partial\phi(\Omega)}(D^2 u \cdot \nu) - \frac{\partial \Delta u}{\partial \nu} = \lambda u, & \text{on } \partial\phi(\Omega). \end{cases} \quad (4.1)$$

To do so, we pull it back to Ω . Therefore, we are interested in the operator \mathcal{P}_ϕ^S from $H_\phi^{2,S}(\Omega)$ to $F(\Omega)$, defined by

$$\begin{aligned} \mathcal{P}_\phi^S[u][\varphi] &:= \int_{\Omega} (D^2(u \circ \phi^{-1}) \circ \phi) : (D^2(\varphi \circ \phi^{-1}) \circ \phi) |\det D\phi| dx \\ &+ \tau \int_{\Omega} (\nabla(u \circ \phi^{-1}) \circ \phi) \cdot (\nabla(\varphi \circ \phi^{-1}) \circ \phi) |\det D\phi| dx, \quad \forall u \in H_\phi^{2,S}(\Omega), \varphi \in H^2(\Omega), \end{aligned} \quad (4.2)$$

where

$$H_\phi^{2,S}(\Omega) := \left\{ u \in H^2(\Omega) : \int_{\partial\Omega} u |\nu(\nabla\phi)^{-1}| |\det D\phi| d\sigma = 0 \right\}.$$

Moreover, for every $\phi \in \Phi(\Omega)$, we consider the map \mathcal{J}_ϕ^S from $L^2(\partial\Omega)$ to $H^2(\Omega)'$ defined by

$$\mathcal{J}_\phi^S[u][\varphi] := \int_{\partial\Omega} u\varphi|\nu(\nabla\phi)^{-1}||\det D\phi|d\sigma, \quad \forall u \in L^2(\partial\Omega), \varphi \in H^2(\Omega).$$

It is easily seen that the form (4.2) is a scalar product on $H_\phi^{2,S}(\Omega)$. We will think of the space $H_\phi^{2,S}(\Omega)$ as endowed with the scalar product (4.2). We denote by π_ϕ^S the map from $H^2(\Omega)$ to $H_\phi^{2,S}(\Omega)$ defined by

$$\pi_\phi^S(u) =: u - \frac{\int_{\partial\Omega} u|\nu(\nabla\phi)^{-1}||\det D\phi|dx}{\int_{\partial\Omega} |\nu(\nabla\phi)^{-1}||\det D\phi|dx},$$

and by $\pi_\phi^{\sharp,S}$ the map from $H^2(\Omega)/\mathbb{R}$ onto $H_\phi^{2,S}(\Omega)$ defined by the equality $\pi_\phi^S = \pi_\phi^{\sharp,S} \circ p$. Clearly, $\pi_\phi^{\sharp,S}$ is a homeomorphism, and we can recast problem (4.1) as

$$\lambda^{-1}u = W_\phi^S u,$$

where

$$W_\phi^S := (\pi_\phi^{\sharp,S})^{-1} \circ (\mathcal{P}_\phi^S)^{-1} \circ \mathcal{J}_\phi^S \circ \text{Tr} \circ \pi_\phi^{\sharp,S}.$$

The operator W_ϕ^S can be shown to be compact and selfadjoint, as we have done for the operator T_ρ^S defined by (3.5) in Theorem 3.7 (see also [33, Theorem 2.1]).

In order to avoid bifurcation phenomena, which usually occur when dealing with multiple eigenvalues, we turn our attention to the elementary symmetric functions of the eigenvalues. This is the aim of the following

Theorem 4.3. *Let Ω be a bounded domain of \mathbb{R}^N of class C^1 . Let F be a finite non-empty subset of \mathbb{N} . Let*

$$\mathcal{A}_\Omega[F] := \{\phi \in \Phi(\Omega) : \lambda_l[\phi] \notin \{\lambda_j[\phi] : j \in F\} \quad \forall l \in \mathbb{N} \setminus F\}.$$

Then the following statements hold.

- i) *The set $\mathcal{A}_\Omega[F]$ is open in $\Phi(\Omega)$. The map P_F of $\mathcal{A}_\Omega[F]$ to the space $\mathcal{L}(H^2(\Omega), H^2(\Omega))$ which takes $\phi \in \mathcal{A}_\Omega[F]$ to the orthogonal projection of $H_\phi^{2,S}(\Omega)$ onto its (finite dimensional) subspace generated by*

$$\left\{ u \in H_\phi^{2,S}(\Omega) : \mathcal{P}_{\text{Id}}^S[u \circ \phi^{-1}] = \lambda_j[\phi] \mathcal{J}_{\text{Id}}^S \circ \text{Tr}[u \circ \phi^{-1}] \text{ for some } j \in F \right\}$$

is real analytic.

- ii) *Let $s \in \{1, \dots, |F|\}$. The function $\Lambda_{F,s}$ from $\mathcal{A}_\Omega[F]$ to \mathbb{R}^N defined by*

$$\Lambda_{F,s}[\phi] := \sum_{j_1 < \dots < j_s \in F} \lambda_{j_1}[\phi] \cdots \lambda_{j_s}[\phi]$$

is real analytic.

- iii) *Let*

$$\Theta_\Omega[F] := \{\phi \in \mathcal{A}_\Omega[F] : \lambda_j[\phi] \text{ have a common value } \lambda_F[\phi] \quad \forall j \in F\}.$$

Then the real analytic functions

$$\left(\binom{|F|}{1}^{-1} \Lambda_{F,1} \right)^{\frac{1}{|F|}}, \dots, \left(\binom{|F|}{|F|}^{-1} \Lambda_{F,|F|} \right)^{\frac{1}{|F|}},$$

of $\mathcal{A}_\Omega[F]$ to \mathbb{R}^N coincide on $\Theta_\Omega[F]$ with the function which takes ϕ to $\lambda_F[\phi]$.

Proof. The proof can be done adapting that of [33, Theorem 2.2 and Corollary 2.3] (see also [31]). \square

In order to compute explicit formulas for the differentials of the functions $\Lambda_{F,s}$, we need the following technical lemma.

Lemma 4.4. *Let Ω be a bounded domain in \mathbb{R}^N of class C^1 , and let $\tilde{\phi} \in \Phi(\Omega)$ be such that $\tilde{\phi}(\Omega)$ is of class C^2 . Let $u_1, u_2 \in H^2(\Omega)$ be such that $v_i = u_i \circ \tilde{\phi}^{(-1)} \in H^4(\tilde{\phi}(\Omega))$ for $i = 1, 2$ and*

$$\frac{\partial^2 v_1}{\partial \nu^2} = \frac{\partial^2 v_2}{\partial \nu^2} = 0 \quad \text{on} \quad \partial \tilde{\phi}(\Omega).$$

Then we have

$$\begin{aligned} d|_{\phi=\tilde{\phi}} \mathcal{P}_\phi^S[\psi][u_1][u_2] &= \int_{\partial \tilde{\phi}(\Omega)} (D^2 v_1 : D^2 v_2 + \tau \nabla v_1 \cdot \nabla v_2) \mu \cdot \nu d\sigma \\ &\quad + \int_{\partial \tilde{\phi}(\Omega)} \left(\operatorname{div}_{\partial \tilde{\phi}(\Omega)}(D^2 v_1 \cdot \nu) \nabla v_2 + \operatorname{div}_{\partial \tilde{\phi}(\Omega)}(D^2 v_2 \cdot \nu) \nabla v_1 \right) \cdot \mu d\sigma \\ &\quad + \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial \Delta v_1}{\partial \nu} \nabla v_2 + \frac{\partial \Delta v_2}{\partial \nu} \nabla v_1 \right) \cdot \mu d\sigma - \tau \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \nabla v_2 + \frac{\partial v_2}{\partial \nu} \nabla v_1 \right) \cdot \mu d\sigma \\ &\quad - \int_{\tilde{\phi}(\Omega)} \left((\Delta^2 v_1 - \tau \Delta v_1) \nabla v_2 + (\Delta^2 v_2 - \tau \Delta v_2) \nabla v_1 \right) \cdot \mu d\sigma, \end{aligned} \quad (4.5)$$

for all $\psi \in (C^2(\bar{\Omega}))^N$, where $\mu = \psi \circ \tilde{\phi}^{-1}$.

Proof. We have

$$\begin{aligned} d|_{\phi=\tilde{\phi}} \mathcal{P}_\phi^S[\psi][u_1][u_2] &= \int_{\Omega} (d|_{\phi=\tilde{\phi}} D^2(u_1 \circ \phi^{-1}) \circ \phi)[\psi] : (D^2(u_2 \circ \tilde{\phi}^{-1}) \circ \tilde{\phi}) | \det D\tilde{\phi} | dx \\ &\quad + \tau \int_{\Omega} (d|_{\phi=\tilde{\phi}} \nabla(u_1 \circ \phi^{-1}) \circ \phi)[\psi] \cdot (\nabla(u_2 \circ \tilde{\phi}^{-1}) \circ \tilde{\phi}) | \det D\tilde{\phi} | dx \\ &\quad + \int_{\Omega} (D^2(u_1 \circ \tilde{\phi}^{-1}) \circ \tilde{\phi}) : (d|_{\phi=\tilde{\phi}} D^2(u_2 \circ \phi^{-1}) \circ \phi)[\psi] | \det D\tilde{\phi} | dx \\ &\quad + \tau \int_{\Omega} (\nabla(u_1 \circ \tilde{\phi}^{-1}) \circ \tilde{\phi}) \cdot (d|_{\phi=\tilde{\phi}} \nabla(u_2 \circ \phi^{-1}) \circ \phi)[\psi] | \det D\tilde{\phi} | dx \\ &\quad + \int_{\Omega} (D^2(u_1 \circ \tilde{\phi}^{-1}) \circ \tilde{\phi}) : (D^2(u_2 \circ \tilde{\phi}^{-1}) \circ \tilde{\phi}) d|_{\phi=\tilde{\phi}} | \det D\phi | [\psi] dx \\ &\quad + \tau \int_{\Omega} (\nabla(u_1 \circ \tilde{\phi}^{-1}) \circ \tilde{\phi}) \cdot (\nabla(u_2 \circ \tilde{\phi}^{-1}) \circ \tilde{\phi}) d|_{\phi=\tilde{\phi}} | \det D\phi | [\psi] dx, \end{aligned} \quad (4.6)$$

and we note that the last two summands in (4.6) equals

$$\int_{\tilde{\phi}(\Omega)} (D^2 v_1 : D^2 v_2 + \tau \nabla v_1 \cdot \nabla v_2) \operatorname{div} \mu dy.$$

(See also Proposition 4.18). By standard calculus we have (see [13, formula (2.15)])

$$D^2(u \circ \phi^{-1}) \circ \phi = (\nabla \phi)^{-t} D^2 u (\nabla \phi)^{-1} + \left(\sum_{k,l=1}^N \frac{\partial u}{\partial x_k} \frac{\partial \sigma_{k,i}}{\partial x_l} \sigma_{l,j} \right)_{i,j},$$

where $\sigma = (\nabla \phi)^{-1}$. This yields the following formula

$$d|_{\phi=\tilde{\phi}} (D^2(u \circ \phi^{-1}) \circ \phi)[\psi] \circ \tilde{\phi}^{-1} = -D^2 v \nabla \mu - \nabla \mu^t D^2 v - \sum_{r=1}^N \frac{\partial v}{\partial y_r} D^2 \mu_r, \quad (4.7)$$

where $\mu = \psi \circ \tilde{\phi}^{-1}$ and $v = u \circ \tilde{\phi}^{-1}$. We rewrite formula (4.7) componentwise getting

$$\begin{aligned} &\left(d|_{\phi=\tilde{\phi}} (D^2(u \circ \phi^{-1}) \circ \phi)[\psi] \circ \tilde{\phi}^{-1} \right)_{i,j} \\ &= - \sum_{r=1}^N \left(\frac{\partial^2 v}{\partial y_i \partial y_r} \frac{\partial \mu_r}{\partial y_j} + \frac{\partial^2 v}{\partial y_j \partial y_r} \frac{\partial \mu_r}{\partial y_i} + \frac{\partial^2 \mu_r}{\partial y_i \partial y_j} \frac{\partial v}{\partial y_r} \right). \end{aligned}$$

Moreover (see [31, Lemma 3.26])

$$\left(d|_{\phi=\tilde{\phi}}(\nabla(u \circ \phi^{-1}) \circ \phi)[\psi] \circ \tilde{\phi}^{-1}\right)_i = -\sum_{r=1}^N \frac{\partial v}{\partial y_r} \frac{\partial \mu_r}{\partial y_i}.$$

Now we use Einstein notation, dropping all the summation symbols. The first summand of the right hand side of (4.6) equals

$$-\int_{\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_1}{\partial y_i \partial y_r} \frac{\partial \mu_r}{\partial y_j} + \frac{\partial^2 v_1}{\partial y_j \partial y_r} \frac{\partial \mu_r}{\partial y_i} + \frac{\partial^2 \mu_r}{\partial y_i \partial y_j} \frac{\partial v_1}{\partial y_r} \right) \frac{\partial^2 v_2}{\partial y_i \partial y_j} dy. \quad (4.8)$$

In order to compute (4.8), integrating by parts, we have

$$\begin{aligned} \int_{\tilde{\phi}(\Omega)} \frac{\partial^2 v_1}{\partial y_i \partial y_r} \frac{\partial \mu_r}{\partial y_j} \frac{\partial^2 v_2}{\partial y_i \partial y_j} dy &= \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \mu_r}{\partial y_j} \nu_r \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma \\ &\quad - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \operatorname{div} \mu}{\partial y_j} \frac{\partial^2 v_2}{\partial y_i \partial y_j} dy - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \mu_r}{\partial y_j} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} dy \\ &= \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \mu_r}{\partial y_j} \nu_r \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \mu_r}{\partial y_j} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} dy \\ &\quad - \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \operatorname{div} \mu \frac{\partial^2 v_2}{\partial y_i \partial y_j} \nu_j d\sigma + \int_{\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \operatorname{div} \mu dy \\ &\quad + \int_{\tilde{\phi}(\Omega)} \operatorname{div} \mu \nabla v_1 \cdot \nabla \Delta v_2 dy, \end{aligned}$$

and

$$\begin{aligned} \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial^2 \mu_r}{\partial y_i \partial y_j} \frac{\partial^2 v_2}{\partial y_i \partial y_j} dy &= \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \nu_j \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma \\ &\quad - \int_{\tilde{\phi}(\Omega)} \frac{\partial^2 v_1}{\partial y_r \partial y_j} \frac{\partial \mu_r}{\partial y_i} \frac{\partial^2 v_2}{\partial y_i \partial y_j} dy - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \frac{\partial \Delta v_2}{\partial y_i} dy \\ &= \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \nu_j \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \frac{\partial \Delta v_2}{\partial y_i} dy \\ &\quad - \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \nu_r \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j} \frac{\partial \operatorname{div} \mu}{\partial y_i} \frac{\partial^2 v_2}{\partial y_i \partial y_j} dy \\ &\quad + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} dy \\ &= \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \nu_j \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \frac{\partial \Delta v_2}{\partial y_i} dy \\ &\quad - \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \nu_r \frac{\partial^2 v_2}{\partial y_i \partial y_j} d\sigma + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j} \frac{\partial \mu_r}{\partial y_i} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} dy \\ &\quad + \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_j} \operatorname{div} \mu \frac{\partial^2 v_2}{\partial y_i \partial y_j} \nu_i d\sigma - \int_{\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \operatorname{div} \mu dy \\ &\quad - \int_{\tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla \Delta v_2 \operatorname{div} \mu dy. \end{aligned}$$

We also have

$$\begin{aligned} \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r} \frac{\partial \mu_r}{\partial y_i} \frac{\partial v_2}{\partial y_i} dy &= \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_2}{\partial \nu} \nabla v_1 \cdot \mu d\sigma - \int_{\tilde{\phi}(\Omega)} \Delta v_2 \nabla v_1 \cdot \mu dy \\ &\quad - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_2}{\partial y_i} \frac{\partial^2 v_1}{\partial y_i \partial y_r} \mu_r dy = \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial v_2}{\partial \nu} \nabla v_1 \cdot \mu d\sigma - \int_{\tilde{\phi}(\Omega)} \Delta v_2 \nabla v_1 \cdot \mu dy \\ &\quad - \int_{\partial \tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla v_2 \mu \cdot \nu d\sigma + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial^2 v_2}{\partial y_i \partial y_r} \mu_r dy + \int_{\tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla v_2 \operatorname{div} \mu dy. \end{aligned}$$

It follows that

$$\begin{aligned}
& d|_{\phi=\tilde{\phi}} \mathcal{P}_\phi^S[\psi][u_1][u_2] \\
&= - \int_{\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \operatorname{div} \mu dy - \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i} \frac{\partial^2 v_2}{\partial y_i \partial y_j} + \frac{\partial v_2}{\partial y_i} \frac{\partial^2 v_1}{\partial y_i \partial y_j} \right) \frac{\partial \mu_r}{\partial y_j} \nu_r d\sigma \\
&\quad + \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} + \frac{\partial v_2}{\partial y_i} \frac{\partial^3 v_1}{\partial y_i \partial y_j \partial y_r} \right) \frac{\partial \mu_r}{\partial y_j} dy \\
&\quad + \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i} \frac{\partial^2 v_2}{\partial y_i \partial y_j} + \frac{\partial v_2}{\partial y_i} \frac{\partial^2 v_1}{\partial y_i \partial y_j} \right) \nu_j \operatorname{div} \mu d\sigma \\
&\quad - \int_{\tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1) \operatorname{div} \mu dy \\
&\quad - \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_r} \frac{\partial^2 v_2}{\partial y_i \partial y_j} + \frac{\partial v_2}{\partial y_r} \frac{\partial^2 v_1}{\partial y_i \partial y_j} \right) \nu_j \frac{\partial \mu_r}{\partial y_i} d\sigma \\
&\quad + \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_r} \frac{\partial \Delta v_2}{\partial y_i} + \frac{\partial v_2}{\partial y_r} \frac{\partial \Delta v_1}{\partial y_i} \right) \frac{\partial \mu_r}{\partial y_i} dy \\
&\quad - \tau \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \nabla v_2 + \frac{\partial v_2}{\partial \nu} \nabla v_1 \right) \cdot \mu d\sigma \\
&\quad + \tau \int_{\tilde{\phi}(\Omega)} (\Delta v_1 \nabla v_2 + \Delta v_2 \nabla v_1) \cdot \mu dy + \tau \int_{\partial \tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla v_2 \mu \cdot \nu d\sigma. \quad (4.9)
\end{aligned}$$

Now we recall that

$$\operatorname{div} \mu = \operatorname{div}_{\partial \tilde{\phi}(\Omega)} \mu + \frac{\partial \mu}{\partial \nu} \cdot \nu \quad \text{on } \partial \tilde{\phi}(\Omega),$$

(see also [16, §8.5]) and that, since $\nu = \nabla b$, where b is the distance from the boundary defined in an appropriate tubular neighborhood of the boundary, then $\nabla \nu = (\nabla \nu)^t$ and $\frac{\partial \nu}{\partial \nu} = 0$, from which it follows that

$$\nabla_{\partial \tilde{\phi}(\Omega)} \nu = (\nabla_{\partial \tilde{\phi}(\Omega)} \nu)^t \quad \text{on } \partial \tilde{\phi}(\Omega).$$

We will use these identities throughout all the following computations.

Using the fact that

$$\frac{\partial^2 v_1}{\partial \nu^2} = \frac{\partial^2 v_2}{\partial \nu^2} = 0 \quad \text{on } \partial \tilde{\phi}(\Omega),$$

we get that the sixth summand in (4.9) equals

$$\begin{aligned}
& - \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_r} (D^2 v_2 \cdot \nu)_{\partial \tilde{\phi}(\Omega)} + \frac{\partial v_2}{\partial y_r} (D^2 v_1 \cdot \nu)_{\partial \tilde{\phi}(\Omega)} \right) \cdot \nabla_{\partial \tilde{\phi}(\Omega)} \mu_r d\sigma \\
&\quad - \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_r} \frac{\partial^2 v_2}{\partial \nu^2} + \frac{\partial v_2}{\partial y_r} \frac{\partial^2 v_1}{\partial \nu^2} \right) \frac{\partial \mu_r}{\partial \nu} d\sigma \\
&= \int_{\partial \tilde{\phi}(\Omega)} \left(\nabla_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_r} \right) (D^2 v_2 \cdot \nu)_{\partial \tilde{\phi}(\Omega)} + \nabla_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_2}{\partial y_r} \right) (D^2 v_1 \cdot \nu)_{\partial \tilde{\phi}(\Omega)} \right) \mu_r d\sigma \\
&\quad + \int_{\partial \tilde{\phi}(\Omega)} \left(\operatorname{div}_{\partial \tilde{\phi}(\Omega)} (D^2 v_1 \cdot \nu)_{\partial \tilde{\phi}(\Omega)} \nabla v_2 + \operatorname{div}_{\partial \tilde{\phi}(\Omega)} (D^2 v_2 \cdot \nu)_{\partial \tilde{\phi}(\Omega)} \nabla v_1 \right) \cdot \mu d\sigma \\
&\quad = \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_1}{\partial y_i \partial y_r} \frac{\partial^2 v_2}{\partial y_i \partial y_j} + \frac{\partial^2 v_2}{\partial y_i \partial y_r} \frac{\partial^2 v_1}{\partial y_i \partial y_j} \right) \nu_j \mu_r d\sigma \\
&\quad + \int_{\partial \tilde{\phi}(\Omega)} \left(\operatorname{div}_{\partial \tilde{\phi}(\Omega)} (D^2 v_1 \cdot \nu)_{\partial \tilde{\phi}(\Omega)} \nabla v_2 + \operatorname{div}_{\partial \tilde{\phi}(\Omega)} (D^2 v_2 \cdot \nu)_{\partial \tilde{\phi}(\Omega)} \nabla v_1 \right) \cdot \mu d\sigma. \quad (4.10)
\end{aligned}$$

The seventh summand in (4.9) equals

$$\begin{aligned}
& \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial\Delta v_1}{\partial\nu} \nabla v_2 + \frac{\partial\Delta v_2}{\partial\nu} \nabla v_1 \right) \cdot \mu d\sigma - \int_{\tilde{\phi}(\Omega)} (\Delta^2 v_1 \nabla v_2 + \Delta^2 v_2 \nabla v_1) \cdot \mu d\sigma \\
& \quad - \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_1}{\partial y_i \partial y_r} \frac{\partial\Delta v_2}{\partial y_i} + \frac{\partial^2 v_2}{\partial y_i \partial y_r} \frac{\partial\Delta v_1}{\partial y_i} \right) \mu_r dy \\
& = \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial\Delta v_1}{\partial\nu} \nabla v_2 + \frac{\partial\Delta v_2}{\partial\nu} \nabla v_1 \right) \cdot \mu d\sigma - \int_{\tilde{\phi}(\Omega)} (\Delta^2 v_1 \nabla v_2 + \Delta^2 v_2 \nabla v_1) \cdot \mu d\sigma \\
& \quad - \int_{\partial\tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1) \mu \cdot \nu d\sigma + \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i} \frac{\partial^2 \Delta v_2}{\partial y_i \partial y_r} + \frac{\partial v_2}{\partial y_i} \frac{\partial^2 \Delta v_1}{\partial y_i \partial y_r} \right) \mu_r dy \\
& \quad + \int_{\tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1) \operatorname{div} \mu dy. \quad (4.11)
\end{aligned}$$

The second summand in (4.9) equals

$$\begin{aligned}
& - \int_{\partial\tilde{\phi}(\Omega)} \nabla(\nabla v_1 \cdot \nabla v_2) \nabla(\mu_r) \nu_r d\sigma \\
& = - \int_{\partial\tilde{\phi}(\Omega)} \nabla_{\partial\tilde{\phi}(\Omega)}(\nabla v_1 \cdot \nabla v_2) \nabla_{\partial\tilde{\phi}(\Omega)}(\mu_r) \nu_r d\sigma - \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial}{\partial\nu}(\nabla v_1 \cdot \nabla v_2) \frac{\partial\mu_r}{\partial\nu} \nu_r d\sigma. \quad (4.12)
\end{aligned}$$

The third summand in (4.9) equals

$$\begin{aligned}
& \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} + \frac{\partial v_2}{\partial y_i} \frac{\partial^3 v_1}{\partial y_i \partial y_j \partial y_r} \right) \nu_j \mu_r d\sigma \\
& \quad - \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i} \frac{\partial^2 \Delta v_2}{\partial y_i \partial y_j \partial y_r} + \frac{\partial v_2}{\partial y_i} \frac{\partial^2 \Delta v_1}{\partial y_i \partial y_j \partial y_r} \right) \mu_r dy \\
& \quad - \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_1}{\partial y_i \partial y_j} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} + \frac{\partial^2 v_2}{\partial y_i \partial y_j} \frac{\partial^3 v_1}{\partial y_i \partial y_j \partial y_r} \right) \mu_r dy \\
& = \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} + \frac{\partial v_2}{\partial y_i} \frac{\partial^3 v_1}{\partial y_i \partial y_j \partial y_r} \right) \nu_j \mu_r d\sigma \\
& \quad - \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i} \frac{\partial^2 \Delta v_2}{\partial y_i \partial y_j \partial y_r} + \frac{\partial v_2}{\partial y_i} \frac{\partial^2 \Delta v_1}{\partial y_i \partial y_j \partial y_r} \right) \mu_r dy \\
& \quad - \int_{\partial\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \mu \cdot \nu d\sigma + \int_{\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \operatorname{div} \mu dy. \quad (4.13)
\end{aligned}$$

From (4.9)-(4.13), it follows that

$$\begin{aligned}
d|_{\phi=\tilde{\phi}} \mathcal{P}_\phi^S[\psi][u_1][u_2] & = - \int_{\partial\tilde{\phi}(\Omega)} \nabla_{\partial\tilde{\phi}(\Omega)}(\nabla v_1 \cdot \nabla v_2) \nabla_{\partial\tilde{\phi}(\Omega)}(\mu_r) \nu_r d\sigma \\
& \quad - \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial}{\partial\nu}(\nabla v_1 \cdot \nabla v_2) \frac{\partial\mu_r}{\partial\nu} \nu_r d\sigma + \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial}{\partial\nu}(\nabla v_1 \cdot \nabla v_2) \operatorname{div} \mu d\sigma \\
& \quad + \int_{\partial\tilde{\phi}(\Omega)} \left(\operatorname{div}_{\partial\tilde{\phi}(\Omega)}(D^2 v_1 \cdot \nu)_{\partial\tilde{\phi}(\Omega)} \nabla v_2 + \operatorname{div}_{\partial\tilde{\phi}(\Omega)}(D^2 v_2 \cdot \nu)_{\partial\tilde{\phi}(\Omega)} \nabla v_1 \right) \cdot \mu d\sigma \\
& \quad + \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_1}{\partial y_i \partial y_r} \frac{\partial^2 v_2}{\partial y_i \partial y_j} + \frac{\partial^2 v_2}{\partial y_i \partial y_r} \frac{\partial^2 v_1}{\partial y_i \partial y_j} \right) \nu_j \mu_r d\sigma \\
& \quad + \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i} \frac{\partial^3 v_2}{\partial y_i \partial y_j \partial y_r} + \frac{\partial v_2}{\partial y_i} \frac{\partial^3 v_1}{\partial y_i \partial y_j \partial y_r} \right) \nu_j \mu_r d\sigma \\
& \quad - \int_{\partial\tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \mu \cdot \nu d\sigma - \int_{\partial\tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1) \mu \cdot \nu d\sigma \\
& \quad + \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial\Delta v_1}{\partial\nu} \nabla v_2 + \frac{\partial\Delta v_2}{\partial\nu} \nabla v_1 \right) \cdot \mu d\sigma - \int_{\tilde{\phi}(\Omega)} (\Delta^2 v_1 \nabla v_2 + \Delta^2 v_2 \nabla v_1) \cdot \mu d\sigma \\
& \quad - \tau \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial\nu} \nabla v_2 + \frac{\partial v_2}{\partial\nu} \nabla v_1 \right) \cdot \mu d\sigma
\end{aligned}$$

$$\begin{aligned}
& + \tau \int_{\tilde{\phi}(\Omega)} (\Delta v_1 \nabla v_2 + \Delta v_2 \nabla v_1) \cdot \mu dy + \tau \int_{\partial \tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla v_2 \mu \cdot \nu d\sigma \\
= & - \int_{\partial \tilde{\phi}(\Omega)} \nabla_{\partial \tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla v_2) \nabla_{\partial \tilde{\phi}(\Omega)} (\mu_r) \nu_r d\sigma + \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial}{\partial \nu} (\nabla v_1 \cdot \nabla v_2) \operatorname{div}_{\partial \tilde{\phi}(\Omega)} \mu d\sigma \\
& + \int_{\partial \tilde{\phi}(\Omega)} \left(\operatorname{div}_{\partial \tilde{\phi}(\Omega)} (D^2 v_1 \cdot \nu)_{\partial \tilde{\phi}(\Omega)} \nabla v_2 + \operatorname{div}_{\partial \tilde{\phi}(\Omega)} (D^2 v_2 \cdot \nu)_{\partial \tilde{\phi}(\Omega)} \nabla v_1 \right) \cdot \mu d\sigma \\
& + \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial \Delta v_1}{\partial \nu} \nabla v_2 + \frac{\partial \Delta v_2}{\partial \nu} \nabla v_1 \right) \cdot \mu d\sigma - \tau \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \nabla v_2 + \frac{\partial v_2}{\partial \nu} \nabla v_1 \right) \cdot \mu d\sigma \\
& \quad + \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial}{\partial \nu} \left(\frac{\partial}{\partial y_r} (\nabla v_1 \cdot \nabla v_2) \right) \mu_r d\sigma \\
& - \int_{\partial \tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \mu \cdot \nu d\sigma - \int_{\partial \tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1) \mu \cdot \nu d\sigma \\
& \quad - \int_{\tilde{\phi}(\Omega)} (\Delta^2 v_1 \nabla v_2 + \Delta^2 v_2 \nabla v_1) \cdot \mu d\sigma \\
& \quad + \tau \int_{\tilde{\phi}(\Omega)} (\Delta v_1 \nabla v_2 + \Delta v_2 \nabla v_1) \cdot \mu dy + \tau \int_{\partial \tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla v_2 \mu \cdot \nu d\sigma. \quad (4.14)
\end{aligned}$$

The first summand on the right hand side of (4.14) equals

$$\int_{\partial \tilde{\phi}(\Omega)} \Delta_{\partial \tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla v_2) \mu \cdot \nu d\sigma + \int_{\partial \tilde{\phi}(\Omega)} \nabla_{\partial \tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla v_2) \cdot (\nabla_{\partial \tilde{\phi}(\Omega)} \nu_r) \mu_r d\sigma,$$

while the sixth one equals

$$\begin{aligned}
& \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial^2}{\partial \nu^2} (\nabla v_1 \cdot \nabla v_2) \mu \cdot \nu d\sigma + \int_{\partial \tilde{\phi}(\Omega)} \nabla_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial}{\partial \nu} (\nabla v_1 \cdot \nabla v_2) \right) \cdot \mu d\sigma \\
& \quad - \int_{\partial \tilde{\phi}(\Omega)} \nabla_{\partial \tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla v_2) \cdot (\nabla_{\partial \tilde{\phi}(\Omega)} \nu_r) \mu_r d\sigma.
\end{aligned}$$

Using the fact that

$$\int_{\partial \tilde{\phi}(\Omega)} \operatorname{div}_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial}{\partial \nu} (\nabla v_1 \cdot \nabla v_2) \cdot \mu \right) d\sigma = \int_{\partial \tilde{\phi}(\Omega)} K \frac{\partial}{\partial \nu} (\nabla v_1 \cdot \nabla v_2) \mu \cdot \nu d\sigma,$$

where K denotes the mean curvature of $\partial \tilde{\phi}(\Omega)$ (see [16, §8.5]), we obtain

$$\begin{aligned}
d|_{\phi=\tilde{\phi}} \mathcal{P}_\phi^S[\psi][u_1][u_2] & = \int_{\partial \tilde{\phi}(\Omega)} \Delta_{\partial \tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla v_2) \mu \cdot \nu d\sigma \\
& \quad + \int_{\partial \tilde{\phi}(\Omega)} K \frac{\partial}{\partial \nu} (\nabla v_1 \cdot \nabla v_2) \mu \cdot \nu d\sigma + \int_{\partial \tilde{\phi}(\Omega)} \frac{\partial^2}{\partial \nu^2} (\nabla v_1 \cdot \nabla v_2) \mu \cdot \nu d\sigma \\
& \quad - \int_{\partial \tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \mu \cdot \nu d\sigma - \int_{\partial \tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1) \mu \cdot \nu d\sigma \\
& \quad + \int_{\partial \tilde{\phi}(\Omega)} \left(\operatorname{div}_{\partial \tilde{\phi}(\Omega)} (D^2 v_1 \cdot \nu)_{\partial \tilde{\phi}(\Omega)} \nabla v_2 + \operatorname{div}_{\partial \tilde{\phi}(\Omega)} (D^2 v_2 \cdot \nu)_{\partial \tilde{\phi}(\Omega)} \nabla v_1 \right) \cdot \mu d\sigma \\
& \quad + \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial \Delta v_1}{\partial \nu} \nabla v_2 + \frac{\partial \Delta v_2}{\partial \nu} \nabla v_1 \right) \cdot \mu d\sigma - \tau \int_{\partial \tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \nabla v_2 + \frac{\partial v_2}{\partial \nu} \nabla v_1 \right) \cdot \mu d\sigma \\
& \quad \quad - \int_{\tilde{\phi}(\Omega)} (\Delta^2 v_1 \nabla v_2 + \Delta^2 v_2 \nabla v_1) \cdot \mu d\sigma \\
& \quad + \tau \int_{\tilde{\phi}(\Omega)} (\Delta v_1 \nabla v_2 + \Delta v_2 \nabla v_1) \cdot \mu dy + \tau \int_{\partial \tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla v_2 \mu \cdot \nu d\sigma \\
& = \int_{\partial \tilde{\phi}(\Omega)} \Delta (\nabla v_1 \cdot \nabla v_2) \mu \cdot \nu d\sigma - \int_{\partial \tilde{\phi}(\Omega)} D^2 v_1 : D^2 v_2 \mu \cdot \nu d\sigma \\
& \quad - \int_{\partial \tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1) \mu \cdot \nu d\sigma
\end{aligned}$$

$$\begin{aligned}
& + \int_{\partial\tilde{\phi}(\Omega)} \left(\operatorname{div}_{\partial\tilde{\phi}(\Omega)}(D^2v_1 \cdot \nu)_{\partial\tilde{\phi}(\Omega)} \nabla v_2 + \operatorname{div}_{\partial\tilde{\phi}(\Omega)}(D^2v_2 \cdot \nu)_{\partial\tilde{\phi}(\Omega)} \nabla v_1 \right) \cdot \mu d\sigma \\
& + \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial\Delta v_1}{\partial\nu} \nabla v_2 + \frac{\partial\Delta v_2}{\partial\nu} \nabla v_1 \right) \cdot \mu d\sigma - \tau \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial\nu} \nabla v_2 + \frac{\partial v_2}{\partial\nu} \nabla v_1 \right) \cdot \mu d\sigma \\
& - \int_{\tilde{\phi}(\Omega)} \left((\Delta^2 v_1 - \tau\Delta v_1) \nabla v_2 + (\Delta^2 v_2 - \tau\Delta v_2) \nabla v_1 \right) \cdot \mu d\sigma \\
& + \tau \int_{\partial\tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla v_2 \mu \cdot \nu d\sigma.
\end{aligned}$$

Using the equality

$$\Delta(\nabla v_1 \cdot \nabla v_2) = \nabla\Delta v_1 \cdot \nabla v_2 + \nabla v_1 \cdot \nabla\Delta v_2 + 2D^2v_1 : D^2v_2$$

we finally get formula (4.5). \square

Now we can compute Hadamard-type formulas for the eigenvalues of problem (4.1).

Theorem 4.15. *Let Ω be a bounded domain in \mathbb{R}^N of class C^1 . Let F be a finite non-empty subset of \mathbb{N} . Let $\tilde{\phi} \in \Theta_\Omega[F]$ be such that $\partial\tilde{\phi}(\Omega) \in C^4$. Let $v_1, \dots, v_{|F|}$ be an orthonormal basis of the eigenspace associated with the eigenvalue $\lambda_F[\tilde{\phi}]$ of problem (4.1) in $L^2(\partial\tilde{\phi}(\Omega))$. Then*

$$\begin{aligned}
d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] &= -\lambda_F^{s-1}[\tilde{\phi}] \binom{|F|-1}{s-1} \sum_{l=1}^{|F|} \int_{\partial\tilde{\phi}(\Omega)} \left(\lambda_F K v_l^2 \right. \\
&\quad \left. + \lambda_F \frac{\partial(v_l^2)}{\partial\nu} - \tau|\nabla v_l|^2 - |D^2v_l|^2 \right) \mu \cdot \nu d\sigma,
\end{aligned}$$

for all $\psi \in (C^2(\overline{\Omega}))^N$, where $\mu = \psi \circ \tilde{\phi}^{(-1)}$, and K denotes the mean curvature of $\partial\tilde{\phi}(\Omega)$.

Proof. First of all we note that $v_1, \dots, v_{|F|} \in H^4(\tilde{\phi}(\Omega))$ (see e.g., [19, §2.5]). We set $u_l = v_l \circ \tilde{\phi}$ for $l = 1, \dots, |F|$. For $|F| > 1$ (case $|F| = 1$ is similar), $s \leq |F|$, we have

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = -\lambda_F^{s+2}[\tilde{\phi}] \binom{|F|-1}{s-1} \sum_{l=1}^{|F|} \mathcal{P}_\tilde{\phi}^S \left[d|_{\phi=\tilde{\phi}} W_\phi^S[\psi][p(u_l)] \right] [p(u_l)]. \quad (4.16)$$

We refer to [31, Theorem 3.38] for a proof of formula (4.16).

By standard calculus in normed spaces we have:

$$\begin{aligned}
\mathcal{P}_\tilde{\phi}^S \left[d|_{\phi=\tilde{\phi}} \left((\pi_\tilde{\phi}^{\#,S})^{-1} \circ (\mathcal{P}_\tilde{\phi}^S)^{-1} \circ \mathcal{J}_\tilde{\phi}^S \circ \operatorname{Tr} \circ \pi_\tilde{\phi}^{\#,S} \right) [\psi][p(u_l)] \right] [p(u_l)] \\
= \mathcal{P}_\tilde{\phi}^S \left[(\pi_\tilde{\phi}^{\#,S})^{-1} \circ (\mathcal{P}_\tilde{\phi}^S)^{-1} \circ d|_{\phi=\tilde{\phi}} \left(\mathcal{J}_\tilde{\phi}^S \circ \operatorname{Tr} \circ \pi_\tilde{\phi}^{\#,S} \right) [\psi][p(u_l)] \right] [p(u_l)] \\
+ \mathcal{P}_\tilde{\phi}^S \left[d|_{\phi=\tilde{\phi}} \left((\pi_\tilde{\phi}^{\#,S})^{-1} \circ (\mathcal{P}_\tilde{\phi}^S)^{-1} \right) [\psi] \circ \mathcal{J}_\tilde{\phi}^S \circ \operatorname{Tr} \circ \pi_\tilde{\phi}^{\#,S} [p(u_l)] \right] [p(u_l)].
\end{aligned}$$

Now note that:

$$\begin{aligned}
\mathcal{P}_\tilde{\phi}^S \left[(\pi_\tilde{\phi}^{\#,S})^{-1} \circ (\mathcal{P}_\tilde{\phi}^S)^{-1} \circ d|_{\phi=\tilde{\phi}} \left(\mathcal{J}_\tilde{\phi}^S \circ \operatorname{Tr} \circ \pi_\tilde{\phi}^{\#,S} \right) [\psi][p(u_l)] \right] [p(u_l)] \\
= \int_{\partial\tilde{\phi}(\Omega)} \left(K v_l^2 + \frac{\partial(v_l^2)}{\partial\nu} \right) \mu \cdot \nu d\sigma - \int_{\partial\tilde{\phi}(\Omega)} \nabla(v_l^2) \cdot \mu d\sigma,
\end{aligned}$$

(see also [30, Lemma 3.3]) and

$$\begin{aligned}
\mathcal{P}_\tilde{\phi}^S \left[d|_{\phi=\tilde{\phi}} \left((\pi_\tilde{\phi}^{\#,S})^{-1} \circ (\mathcal{P}_\tilde{\phi}^S)^{-1} \right) [\psi] \circ \mathcal{J}_\tilde{\phi}^S \circ \operatorname{Tr} \circ \pi_\tilde{\phi}^{\#,S} [p(u_l)] \right] [p(u_l)] \\
= -\lambda_F^{-1} d|_{\phi=\tilde{\phi}} \left(\mathcal{P}_\tilde{\phi}^S \circ \pi_\tilde{\phi}^S \right) [\psi][u_l][\pi_\tilde{\phi}^S(u_l)].
\end{aligned}$$

(We refer to [33, Lemma 2.4] for more explicit computations). Using formula (4.5) we obtain

$$\begin{aligned}
\mathcal{P}_\tilde{\phi}^S \left[d|_{\phi=\tilde{\phi}} \left((\pi_\tilde{\phi}^{\#,S})^{-1} \circ (\mathcal{P}_\tilde{\phi}^S)^{-1} \right) [\psi] \circ \mathcal{J}_\tilde{\phi}^S \circ \operatorname{Tr} \circ \pi_\tilde{\phi}^{\#,S} [p(u_l)] \right] [p(u_l)] \\
= -\lambda_F^{-1} \int_{\partial\tilde{\phi}(\Omega)} (|D^2v_l|^2 + \tau|\nabla v_l|^2) \mu \cdot \nu d\sigma + \int_{\partial\tilde{\phi}(\Omega)} \nabla(v_l^2) \cdot \mu d\sigma.
\end{aligned}$$

This concludes the proof. \square

Now we turn our attention to extremum problems of the type

$$\min_{\mathcal{V}(\phi)=\text{const.}} \Lambda_{F,s}[\phi] \text{ or } \max_{\mathcal{V}(\phi)=\text{const.}} \Lambda_{F,s}[\phi],$$

where $\mathcal{V}(\phi)$ denotes the measure of $\phi(\Omega)$, i.e.,

$$\mathcal{V}(\phi) := \int_{\phi(\Omega)} dx = \int_{\Omega} |\det D\phi| dx. \quad (4.17)$$

In particular, all ϕ 's realizing the extremum are critical points under measure constraint, i.e., $\text{Ker } d\mathcal{V}(\phi) \subseteq \text{Ker } d\Lambda_{F,s}[\phi]$. We have the following result (see [32, Proposition 2.10]).

Proposition 4.18. *Let Ω be a bounded domain in \mathbb{R}^N of class C^1 . Then the following statements hold.*

i) *The map \mathcal{V} from $\Phi(\Omega)$ to \mathbb{R} defined in (4.17) is real analytic. Moreover, the differential of \mathcal{V} at $\tilde{\phi} \in \Phi(\Omega)$ is given by the formula*

$$d|_{\phi=\tilde{\phi}} \mathcal{V}(\phi)[\psi] = \int_{\tilde{\phi}(\Omega)} \text{div}(\psi \circ \tilde{\phi}^{-1}) dy = \int_{\partial\tilde{\phi}(\Omega)} (\psi \circ \tilde{\phi}^{-1}) \cdot \nu d\sigma.$$

ii) *For $\mathcal{V}_0 \in]0, +\infty[$, let*

$$V(\mathcal{V}_0) := \{\phi \in \Phi(\Omega) : \mathcal{V}(\phi) = \mathcal{V}_0\}.$$

If $V(\mathcal{V}_0) \neq \emptyset$, then $V(\mathcal{V}_0)$ is a real analytic manifold of $(C^2(\overline{\Omega}))^N$ of codimension 1.

Using Lagrange Multipliers Theorem, it is easy to prove the following

Theorem 4.19. *Let Ω be a bounded domain in \mathbb{R}^N of class C^1 . Let F be a non-empty finite subset of \mathbb{N} . Let $\mathcal{V}_0 \in]0, +\infty[$. Let $\tilde{\phi} \in V(\mathcal{V}_0)$ be such that $\partial\tilde{\phi}(\Omega) \in C^4$ and $\lambda_j[\tilde{\phi}]$ have a common value $\lambda_F[\tilde{\phi}]$ for all $j \in F$ and $\lambda_l[\tilde{\phi}] \neq \lambda_F[\tilde{\phi}]$ for all $l \in \mathbb{N} \setminus F$. For $s = 1, \dots, |F|$, the function $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ on $V(\mathcal{V}_0)$ if and only if there exists an orthonormal basis $v_1, \dots, v_{|F|}$ of the eigenspace corresponding to the eigenvalue $\lambda_F[\tilde{\phi}]$ of problem (4.1) in $L^2(\partial\tilde{\phi}(\Omega))$, and a constant $c \in \mathbb{R}$ such that*

$$\sum_{l=1}^{|F|} \left(\lambda_F[\tilde{\phi}] \left(K v_l^2 + \frac{\partial(v_l^2)}{\partial\nu} \right) - \tau |\nabla v_l|^2 - |D^2 v_l|^2 \right) = c, \text{ a.e. on } \partial\tilde{\phi}(\Omega). \quad (4.20)$$

Now that we have a characterization for the criticality of $\tilde{\phi}$, we may wonder whether balls are critical domains. This is the aim of the following

Theorem 4.21. *Let Ω be a bounded domain of \mathbb{R}^N of class C^1 . Let $\tilde{\phi} \in \Phi(\Omega)$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of problem (4.1) in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ on $V(\mathcal{V}(\tilde{\phi}))$, for all $s = 1, \dots, |F|$.*

Proof. Using Lemma 4.22 below and the fact that the mean curvature is constant for a ball, condition (4.20) is immediately seen to be satisfied. \square

Lemma 4.22. *Let B be the unit ball in \mathbb{R}^N centered at zero, and let λ be an eigenvalue of problem (4.1) in B . Let F be the subset of \mathbb{N} of all indices j such that the j -th eigenvalue of problem (4.1) in B coincides with λ . Let $v_1, \dots, v_{|F|}$ be an orthonormal basis of the eigenspace associated with the eigenvalue λ , where the orthonormality is taken with respect to the scalar product in $L^2(\partial B)$. Then*

$$\sum_{j=1}^{|F|} v_j^2, \sum_{j=1}^{|F|} |\nabla v_j|^2, \sum_{j=1}^{|F|} |D^2 v_j|^2$$

are radial functions.

Proof. Let $O_N(\mathbb{R})$ denote the group of orthogonal linear transformations in \mathbb{R}^N . Since the Laplace operator is invariant under rotations, then $v_k \circ A$, where $A \in O_N(\mathbb{R})$, is still an eigenfunction with eigenvalue λ ; moreover, $\{v_j \circ A : j = 1, \dots, |F|\}$ is another orthonormal basis for

the eigenspace associated with λ . Since both $\{v_j : j = 1, \dots, |F|\}$ and $\{v_j \circ A : j = 1, \dots, |F|\}$ are orthonormal bases, then there exists $R[A] \in O_N(\mathbb{R})$ with matrix $(R_{ij}[A])_{i,j=1,\dots,|F|}$ such that

$$v_j = \sum_{l=1}^{|F|} R_{jl}[A] v_l \circ A.$$

This implies that

$$\sum_{j=1}^{|F|} v_j^2 = \sum_{j=1}^{|F|} (v_j \circ A)^2,$$

from which we get that $\sum_{j=1}^{|F|} v_j^2$ is radial. Moreover, using standard calculus, we get

$$\sum_{j=1}^{|F|} |\nabla v_j|^2 = \sum_{l_1, l_2=1}^{|F|} R_{jl_1}[A] R_{jl_2}[A] (\nabla v_{l_1} \circ A) \cdot (\nabla v_{l_2} \circ A) = \sum_{l=1}^{|F|} |\nabla v_l \circ A|^2,$$

and

$$\begin{aligned} D^2 v_j \cdot D^2 v_j &= \sum_{l_1, l_2=1}^{|F|} R_{jl_1}[A] R_{jl_2}[A] A^t \cdot (D^2 v_{l_1} \circ A) \cdot A \cdot A^t \cdot (D^2 v_{l_2} \circ A) \cdot A \\ &= \sum_{l_1, l_2=1}^{|F|} R_{jl_1}[A] R_{jl_2}[A] A^t \cdot (D^2 v_{l_1} \circ A) \cdot (D^2 v_{l_2} \circ A) \cdot A, \end{aligned}$$

hence

$$|D^2 v_j|^2 = \text{tr}(D^2 v_j \cdot D^2 v_j) = \sum_{l_1, l_2=1}^{|F|} R_{jl_1}[A] R_{jl_2}[A] (D^2 v_{l_1} \circ A) : (D^2 v_{l_2} \circ A),$$

from which we get

$$\sum_{j=1}^{|F|} |D^2 v_j|^2 = \sum_{j=1}^{|F|} |D^2 v_j \circ A|^2.$$

□

4.2 The Neumann problem

As we have done for the Steklov problem, we study the Neumann problem in $\phi(\Omega)$, i.e.,

$$\begin{cases} \Delta^2 u - \tau \Delta u = \lambda u, & \text{in } \phi(\Omega), \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial \phi(\Omega), \\ \tau \frac{\partial u}{\partial \nu} - \text{div}_{\partial \phi(\Omega)}(D^2 u \cdot \nu) - \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial \phi(\Omega). \end{cases} \quad (4.23)$$

We consider the operator $\mathcal{P}_\phi^{\mathcal{N}}$ from $H_\phi^{2,\mathcal{N}}(\Omega)$ to $F(\Omega)$, defined by

$$\begin{aligned} \mathcal{P}_\phi^{\mathcal{N}}[u][\varphi] &:= \int_{\Omega} (D^2(u \circ \phi^{-1}) \circ \phi) : (D^2(\varphi \circ \phi^{-1}) \circ \phi) |\det D\phi| dx \\ &+ \tau \int_{\Omega} (\nabla(u \circ \phi^{-1}) \circ \phi) \cdot (\nabla(\varphi \circ \phi^{-1}) \circ \phi) |\det D\phi| dx, \quad \forall u \in H_\phi^{2,\mathcal{N}}(\Omega), \varphi \in H^2(\Omega), \end{aligned} \quad (4.24)$$

where

$$H_\phi^{2,\mathcal{N}}(\Omega) := \left\{ u \in H^2(\Omega) : \int_{\Omega} u |\det D\phi| dx = 0 \right\},$$

Moreover, for every $\phi \in \Phi(\Omega)$, we consider the map $\mathcal{J}_\phi^{\mathcal{N}}$ from $L^2(\Omega)$ to $H^2(\Omega)'$ defined by

$$\mathcal{J}_\phi^{\mathcal{N}}[u][\varphi] := \int_{\Omega} u \varphi |\det D\phi| d\sigma, \quad \forall u \in L^2(\Omega), \varphi \in H^2(\Omega).$$

We will think of the space $H_\phi^{2,\mathcal{N}}(\Omega)$ as endowed with the scalar product induced by (4.24). We denote by $\pi_\phi^\mathcal{N}$ the map from $H^2(\Omega)$ to $H_\phi^{2,\mathcal{N}}(\Omega)$ defined by

$$\pi_\phi^\mathcal{N}(u) =: u - \frac{\int_\Omega u |\det D\phi| dx}{\int_\Omega |\det D\phi| dx},$$

and by $\pi_\phi^{\sharp,\mathcal{N}}$ the map from $H^2(\Omega)/\mathbb{R}$ onto $H_\phi^{2,\mathcal{N}}(\Omega)$ defined by the equality $\pi_\phi^\mathcal{N} = \pi_\phi^{\sharp,\mathcal{N}} \circ p$. Clearly, $\pi_\phi^{\sharp,\mathcal{N}}$ is a homeomorphism, and we can recast problem (4.23) as

$$\lambda^{-1}u = W_\phi^\mathcal{N}u,$$

where $W_\phi^\mathcal{N} := (\pi_\phi^{\sharp,\mathcal{N}})^{-1} \circ (\mathcal{P}_\phi^\mathcal{N})^{-1} \circ \mathcal{J}_\phi^\mathcal{N} \circ i \circ \pi_\phi^{\sharp,\mathcal{N}}$ and i is the canonical embedding of $H^2(\Omega)$ into $L^2(\Omega)$. An analogue of Theorem 4.3 can be stated also in this case. Therefore, we can compute Hadamard-type formulas for the Neumann eigenvalues. This is contained in the following

Theorem 4.25. *Let Ω be a bounded domain in \mathbb{R}^N of class C^1 . Let F be a finite non-empty subset of \mathbb{N} . Let $\tilde{\phi} \in \Theta_\Omega[F]$ be such that $\partial\tilde{\phi}(\Omega) \in C^4$. Let $v_1, \dots, v_{|F|}$ be an orthonormal basis of the eigenspace associated with the eigenvalue $\lambda_F[\tilde{\phi}]$ of problem (4.23) in $L^2(\tilde{\phi}(\Omega))$. Then*

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = -\lambda_F^{s-1}[\tilde{\phi}] \binom{|F|-1}{s-1} \sum_{l=1}^{|F|} \int_{\partial\tilde{\phi}(\Omega)} (\lambda_F v_l^2 - \tau |\nabla v_l|^2 - |D^2 v_l|^2) \mu \cdot \nu d\sigma,$$

for all $\psi \in (C^2(\overline{\Omega}))^N$, where $\mu = \psi \circ \tilde{\phi}^{-1}$.

Proof. The proof is similar to that of Theorem 4.15.

First of all we note that, by elliptic regularity theory, $v_1, \dots, v_{|F|} \in H^4(\tilde{\phi}(\Omega))$ (see [19, §2.5]). We set $u_l = v_l \circ \tilde{\phi}$ for $l = 1, \dots, |F|$. For $|F| > 1$ (case $|F| = 1$ is similar), $s \leq |F|$, we have

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = -\lambda_F^{s+2}[\tilde{\phi}] \binom{|F|-1}{s-1} \sum_{l=1}^{|F|} \mathcal{P}_\phi^\mathcal{N} \left[d|_{\phi=\tilde{\phi}} W_\phi^\mathcal{N}[\psi][p(u_l)] \right] [p(u_l)].$$

By standard calculus in normed spaces we have:

$$\begin{aligned} & \mathcal{P}_\phi^\mathcal{N} \left[d|_{\phi=\tilde{\phi}} \left((\pi_\phi^{\sharp,\mathcal{N}})^{-1} \circ (\mathcal{P}_\phi^\mathcal{N})^{-1} \circ \mathcal{J}_\phi^\mathcal{N} \circ i \circ \pi_\phi^{\sharp,\mathcal{N}} \right) [\psi][p(u_l)] \right] [p(u_l)] \\ &= \mathcal{P}_\phi^\mathcal{N} \left[(\pi_\phi^{\sharp,\mathcal{N}})^{-1} \circ (\mathcal{P}_\phi^\mathcal{N})^{-1} \circ d|_{\phi=\tilde{\phi}} \left(\mathcal{J}_\phi^\mathcal{N} \circ i \circ \pi_\phi^{\sharp,\mathcal{N}} \right) [\psi][p(u_l)] \right] [p(u_l)] \\ & \quad + \mathcal{P}_\phi^\mathcal{N} \left[d|_{\phi=\tilde{\phi}} \left((\pi_\phi^{\sharp,\mathcal{N}})^{-1} \circ (\mathcal{P}_\phi^\mathcal{N})^{-1} \right) [\psi] \circ \mathcal{J}_\phi^\mathcal{N} \circ i \circ \pi_\phi^{\sharp,\mathcal{N}} [p(u_l)] \right] [p(u_l)]. \end{aligned}$$

Now note that

$$\begin{aligned} & \mathcal{P}_\phi^\mathcal{N} \left[(\pi_\phi^{\sharp,\mathcal{N}})^{-1} \circ (\mathcal{P}_\phi^\mathcal{N})^{-1} \circ d|_{\phi=\tilde{\phi}} \left(\mathcal{J}_\phi^\mathcal{N} \circ i \circ \pi_\phi^{\sharp,\mathcal{N}} \right) [\psi][p(u_l)] \right] [p(u_l)] \\ &= \int_{\tilde{\phi}(\Omega)} v_l^2 \operatorname{div} \mu dy, \end{aligned}$$

(see also Proposition 4.18) and

$$\begin{aligned} & \mathcal{P}_\phi^\mathcal{N} \left[d|_{\phi=\tilde{\phi}} \left((\pi_\phi^{\sharp,\mathcal{N}})^{-1} \circ (\mathcal{P}_\phi^\mathcal{N})^{-1} \right) [\psi] \circ \mathcal{J}_\phi^\mathcal{N} \circ i \circ \pi_\phi^{\sharp,\mathcal{N}} [p(u_l)] \right] [p(u_l)] \\ &= -\lambda_F^{-1} d|_{\phi=\tilde{\phi}} (\mathcal{P}_\phi^\mathcal{N} \circ \pi_\phi^\mathcal{N}) [\psi][u_l][\pi_\phi^\mathcal{N}(u_l)]. \end{aligned}$$

Using formula (4.5) we obtain

$$\begin{aligned} & \mathcal{P}_\phi^\mathcal{N} \left[d|_{\phi=\tilde{\phi}} \left((\pi_\phi^{\sharp,\mathcal{N}})^{-1} \circ (\mathcal{P}_\phi^\mathcal{N})^{-1} \right) [\psi] \circ \mathcal{J}_\phi^\mathcal{N} \circ i \circ \pi_\phi^{\sharp,\mathcal{N}} [p(u_l)] \right] [p(u_l)] \\ &= -\lambda_F^{-1} \int_{\partial\tilde{\phi}(\Omega)} (|D^2 v_l|^2 + \tau |\nabla v_l|^2) \mu \cdot \nu d\sigma + \int_{\tilde{\phi}(\Omega)} \nabla(v_l^2) \cdot \mu dy. \end{aligned}$$

To conclude, just observe that

$$\int_{\tilde{\phi}(\Omega)} \nabla(v_l^2) \cdot \mu dy = \int_{\partial\tilde{\phi}(\Omega)} v_l^2 \mu \cdot \nu d\sigma - \int_{\tilde{\phi}(\Omega)} (v_l^2) \operatorname{div} \mu dy.$$

□

Now we can state the analogue of Theorem 4.19 for problem (4.23).

Theorem 4.26. *Let Ω be a bounded domain in \mathbb{R}^N of class C^1 . Let F be a non-empty finite subset of \mathbb{N} . Let $\mathcal{V}_0 \in]0, +\infty[$. Let $\tilde{\phi} \in V(\mathcal{V}_0)$ be such that $\partial\tilde{\phi}(\Omega) \in C^4$ and $\lambda_j[\tilde{\phi}]$ have a common value $\lambda_F[\tilde{\phi}]$ for all $j \in F$ and $\lambda_l[\tilde{\phi}] \neq \lambda_F[\tilde{\phi}]$ for all $l \in \mathbb{N} \setminus F$. For $s = 1, \dots, |F|$, the function $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ on $V(\mathcal{V}_0)$ if and only if there exists an orthonormal basis $v_1, \dots, v_{|F|}$ of the eigenspace corresponding to the eigenvalue $\lambda_F[\tilde{\phi}]$ of problem (4.23) in $L^2(\tilde{\phi}(\Omega))$, and a constant $c \in \mathbb{R}$ such that*

$$\sum_{l=1}^{|F|} (\lambda_F v_l^2 - \tau |\nabla v_l|^2 - |D^2 v_l|^2) = c, \text{ a.e. on } \partial\tilde{\phi}(\Omega).$$

We observe that Lemma 4.22 holds for problem (4.23) as well, since in the proof we have only used the rotation invariance of the Laplace operator. Then, we are led to the following

Theorem 4.27. *Let Ω be a bounded domain in \mathbb{R}^N of class C^1 . Let $\tilde{\phi} \in \Phi(\Omega)$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of problem (4.23) in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ on $V(\mathcal{V}(\tilde{\phi}))$, for all $s = 1, \dots, |F|$.*

5 The fundamental tone of the ball. The isoperimetric inequality

In the previous section we have shown that the ball is a critical point for all the elementary symmetric functions of the eigenvalues of problem (1.1). In this section we prove that the ball is actually a maximizer for the fundamental tone, that is

$$\lambda_2(\Omega) \leq \lambda_2(\Omega^*), \tag{5.1}$$

where Ω^* is a ball such that $|\Omega| = |\Omega^*|$.

5.1 Eigenvalues and eigenfunctions on the ball

We characterize the eigenvalues and the eigenfunctions of (1.1) when $\Omega = B$ is the unit ball in \mathbb{R}^N centered at the origin. It is convenient to use spherical coordinates (r, θ) , where $\theta = (\theta_1, \dots, \theta_{N-1})$. The corresponding transformation of coordinates is

$$\begin{aligned} x_1 &= r \cos(\theta_1), \\ x_2 &= r \sin(\theta_1) \cos(\theta_2), \\ &\vdots \\ x_{N-1} &= r \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{N-2}) \cos(\theta_{N-1}), \\ x_N &= r \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{N-2}) \sin(\theta_{N-1}), \end{aligned}$$

with $\theta_1, \dots, \theta_{N-2} \in [0, \pi]$, $\theta_{N-1} \in [0, 2\pi[$ (here it is understood that $\theta_1 \in [0, 2\pi[$ if $N = 2$).

The boundary conditions of (1.1) in this case are written as

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} \Big|_{r=1} = 0, \\ \tau \frac{\partial u}{\partial r} - \frac{1}{r^2} \Delta_S \left(\frac{\partial u}{\partial r} - \frac{u}{r} \right) - \frac{\partial \Delta u}{\partial r} \Big|_{r=1} = \lambda u \Big|_{r=1}, \end{cases}$$

where Δ_S is the angular part of the Laplacian. It is well known that the eigenfunctions can be written as a product of a radial part and an angular part (see [14] for details). The radial

part is given in terms of ultraspherical modified Bessel functions and powertype functions. The ultraspherical modified Bessel functions $i_l(z)$ and $k_l(z)$ are defined as follows

$$\begin{aligned} i_l(z) &:= z^{1-\frac{N}{2}} I_{\frac{N}{2}-1+l}(z), \\ k_l(z) &:= z^{1-\frac{N}{2}} K_{\frac{N}{2}-1+l}(z), \end{aligned}$$

for $l \in \mathbb{N}$, where $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions of first and second kind respectively. We recall that $i_l(z)$ and all its derivatives are positive on $]0, +\infty[$ (see [1, §9.6]). We recall that the Bessel functions J_ν and N_ν solve the Bessel equation

$$z^2 y''(z) + zy'(z) + (z^2 - \nu^2)y(z) = 0,$$

while the modified Bessel functions I_ν and K_ν solve the modified Bessel equation

$$z^2 y''(z) + zy'(z) + (z^2 + \nu^2)y(z) = 0.$$

We have the following

Theorem 5.2. *Let Ω be the unit ball in \mathbb{R}^N centered at the origin. Any eigenfunction u_l of problem (1.1) is of the form $u_l(r, \theta) = R_l(r)Y_l(\theta)$ where $Y_l(\theta)$ is a spherical harmonic of some order $l \in \mathbb{N}$ and*

$$R_l(r) = A_l r^l + B_l i_l(\sqrt{\tau}r),$$

where A_l and B_l are suitable constants such that

$$B_l = \frac{l(1-l)}{\tau i_l''(\sqrt{\tau})} A_l.$$

Moreover, the eigenvalue $\lambda_{(l)}$ associated with the eigenfunction u_l is delivered by formula

$$\begin{aligned} \lambda_{(l)} = l \left((1-l)li_l(\sqrt{\tau}) + \tau i_l''(\sqrt{\tau}) \right)^{-1} & \left[3(l-1)l(l+N-2)i_l(\sqrt{\tau}) \right. \\ & - (l-1)\sqrt{\tau}(N-1+2Nl+2l(l-2)l+\tau)i_l'(\sqrt{\tau}) \\ & + \tau((l-1)(l+2N-3)+\tau)i_l''(\sqrt{\tau}) \\ & \left. + (l-1)\tau\sqrt{\tau}i_l'''(\sqrt{\tau}) \right], \quad (5.3) \end{aligned}$$

for any $l \in \mathbb{N}$.

Proof. Solutions of problem (1.1) in the unit ball are smooth (see e.g., [19, Theorem 2.20]). We consider two cases: $\Delta u = 0$ and $\Delta u \neq 0$.

Let u be such that $\Delta u = 0$. The Laplacian can be written in spherical coordinates as

$$\Delta = \partial_{rr} + \frac{N-1}{r} \partial_r + \frac{1}{r^2} \Delta_S.$$

Separating variables so that $u = R(r)Y(\theta)$ we obtain the equations

$$R'' + \frac{N-1}{r} R' - \frac{l(l+N-2)}{r^2} R = 0 \quad (5.4)$$

and

$$\Delta_S Y = -l(l+N-2)Y. \quad (5.5)$$

The solutions of equation (5.4) are given by $R(r) = ar^l + br^{2-N-l}$ if $l > 0, N \geq 2$, and by $R(r) = a + b \log(r)$ if $l = 0, N = 2$. Since the solutions cannot blow up at $r = 0$, we must impose $b = 0$. The solutions of the second equation are the spherical harmonics of order l . Then u can be written as

$$u(r, \theta) = a_l r^l Y_l(\theta)$$

for some $l \in \mathbb{N}$.

Let us consider now the case $\Delta u \neq 0$. We set $v = \Delta u$ and solve the equation

$$\Delta v = \tau v.$$

By writing $v = R(r)Y(\theta)$ we obtain that R solves the equation

$$R'' + \frac{N-1}{r}R' - \frac{l(l+N-2)}{r^2}R = \tau R, \quad (5.6)$$

while Y solves equation (5.5). Equation (5.6) is the modified ultraspherical Bessel equation that is solved by the modified ultraspherical Bessel functions of first and second kind $i_l(\sqrt{\tau}r)$ and $k_l(\sqrt{\tau}r)$. Since the solutions cannot blow up at $r = 0$, we must choose only $i_l(z)$ since $k_l(z)$ has a singularity at $z = 0$. Then

$$v(r, \theta) = b_{l_1} i_{l_1}(\sqrt{\tau}r) Y_{l_1}(\theta)$$

for some $l_1 \in \mathbb{N}$. Now $v = \frac{\Delta v}{\tau} = \Delta u$, that is $\Delta(v/\tau - u) = 0$. This means that

$$u(r, \theta) = \frac{b_{l_1}}{\tau} i_{l_1}(\sqrt{\tau}r) Y_{l_1}(\theta) - c_{l_2} r^{l_2} Y_{l_2}(\theta) \quad (5.7)$$

for some $l_2 \in \mathbb{N}$.

Now we prove that the indexes l_1 and l_2 in (5.7) must coincide. This can be shown by imposing the boundary condition $\frac{\partial^2 u}{\partial r^2}|_{r=1} = 0$, which can be written as

$$b_{l_1} i_{l_1}''(\sqrt{\tau}) Y_{l_1}(\theta) - c_{l_2} l_2(l_2 - 1) Y_{l_2}(\theta) = 0. \quad (5.8)$$

If the two indexes do not agree, the coefficients of Y_{l_i} , $i = 1, 2$ must vanish since spherical harmonics with different indexes are linearly independent on $\partial\Omega$. Since $i_{l_1}''(\sqrt{\tau}) > 0$, this implies $b_{l_1} = 0$ and therefore $l_2 = 0$ or $l_2 = 1$. Then we have

$$u_l(r, \theta) = \left(A_l r^l + B_l i_l(\sqrt{\tau}r) \right) Y_l(\theta), \quad (5.9)$$

with suitable constants A_l, B_l . In the case $l \neq 0, 1$, again from the boundary condition (5.8) we have

$$l(l-1)A_l + \tau i_l''(\sqrt{\tau}) B_l = 0, \quad (5.10)$$

then $B_l = \frac{l(1-l)}{\tau i_l''(\sqrt{\tau})} A_l$. Note that the formula holds also in the case $l = 0, 1$ since these indexes correspond to $B_l = 0$.

Finally, let us consider the boundary condition

$$\tau \frac{\partial u}{\partial r} - \frac{1}{r^2} \Delta_S \left(\frac{\partial u}{\partial r} - \frac{u}{r} \right) - \frac{\partial \Delta u}{\partial r} \Big|_{r=1} = \lambda u|_{r=1}. \quad (5.11)$$

Using in (5.11) the representation of u_l provided by formula (5.9), we get

$$\begin{aligned} & \left[\left(-\lambda + l((l-1)(l+N-2) + \tau) \right) A_l + \left(- (3l(l+N-2) + \lambda) i_l(\sqrt{\tau}) \right. \right. \\ & \quad \left. \left. - \sqrt{\tau}((N-1-2Nl-2(l-2)l - \tau) i_l'(\sqrt{\tau}) + (N-1)\sqrt{\tau} i_l''(\sqrt{\tau}) \right. \right. \\ & \quad \left. \left. + \tau i_l'''(\sqrt{\tau})) \right) B_l \right] Y_l(\theta) = \lambda (A_l + B_l i_l(\sqrt{\tau})) Y_l(\theta). \end{aligned}$$

Using equality (5.10) we get that u_l given by (5.9) is an eigenfunction of (1.1) on the unit ball. Moreover, as a consequence, we also get formula (5.3) for the associated eigenvalue. This concludes the proof. \square

We are ready to state and prove the following theorem concerning the first positive eigenvalue.

Theorem 5.12. *Let Ω be the unit ball in \mathbb{R}^N centered at the origin. The first positive eigenvalue of (1.1) is $\lambda_2 = \lambda_{(1)} = \tau$. The corresponding eigenspace is generated by $\{x_1, x_2, \dots, x_N\}$.*

Proof. By Theorem 5.2, $0 = \lambda_{(0)} < \tau = \lambda_{(1)}$. We consider formula (5.3) with $l = 2$. We have

$$\begin{aligned} \lambda_{(2)} = 2 \left(\tau i_2''(\sqrt{\tau}) - 2i_2(\sqrt{\tau}) \right)^{-1} & \left[6N i_2(\sqrt{\tau}) - \sqrt{\tau}(5N-1+\tau) i_2'(\sqrt{\tau}) \right. \\ & \left. + \tau(2N-1+\tau) i_2''(\sqrt{\tau}) + \tau \sqrt{\tau} i_2'''(\sqrt{\tau}) \right]. \quad (5.13) \end{aligned}$$

In order to prove that $\lambda_{(2)} > \tau$, we use some well-known recurrence relations between ultraspherical Bessel functions (see [1, p. 376]),

$$\begin{aligned} i'_l(\sqrt{\tau}) &= \frac{l}{\sqrt{\tau}}i_l(\sqrt{\tau}) + i_{l+1}(\sqrt{\tau}), \\ i''_l(\sqrt{\tau}) &= \frac{l(l-1)}{\tau}i_l(\sqrt{\tau}) + \frac{l+2}{\tau}i_{l+1}(\sqrt{\tau}) + i_{l+2}(\sqrt{\tau}), \\ i'''_l(\sqrt{\tau}) &= \frac{l(l-1)(l-2)}{\tau\sqrt{\tau}}i_l(\sqrt{\tau}) + \frac{l(2l+1)}{\tau}i_{l+1}(\sqrt{\tau}) + \frac{2(l+2)}{\sqrt{\tau}}i_{l+2}(\sqrt{\tau}) + i_{l+3}(\sqrt{\tau}). \end{aligned}$$

Using these relations in (5.13), we obtain an equivalent formula for $\lambda_{(2)}$,

$$\begin{aligned} \lambda_{(2)} &= 2\left(5\sqrt{\tau}i_3(\sqrt{\tau}) + \tau i_4(\sqrt{\tau})\right)^{-1} \left[(10N - 2 + 2\tau)i_2(\sqrt{\tau}) \right. \\ &\quad \left. + (2 - 10N + (7 + 10N)\sqrt{\tau} - 2\tau + 5\tau\sqrt{\tau})i_3(\sqrt{\tau}) \right. \\ &\quad \left. + \tau(8 + 2N + \tau)i_4(\sqrt{\tau}) + \tau\sqrt{\tau}i_5(\sqrt{\tau}) \right]. \end{aligned}$$

By well-known properties of the functions I_ν (see [1, §9]), it follows that $i_l \geq i_{l+1}$ for all $l \in \mathbb{N}$. This implies

$$\begin{aligned} (10N - 2 + 2\tau)i_2(\sqrt{\tau}) + (2 - 10N + (7 + 10N)\sqrt{\tau} - 2\tau + 5\tau\sqrt{\tau})i_3(\sqrt{\tau}) \\ + \tau(8 + 2N + \tau)i_4(\sqrt{\tau}) + \tau\sqrt{\tau}i_5(\sqrt{\tau}) \geq \left(5\tau\sqrt{\tau}i_3(\sqrt{\tau}) + \tau^2i_4(\sqrt{\tau})\right), \end{aligned}$$

then

$$\lambda_{(2)} \geq 2\tau > \tau = \lambda_{(1)}.$$

Now it remains to prove that $\lambda_{(l)}$ is an increasing function of l for $l \geq 2$. We adapt the method used in [14, Theorem 3]. We claim that for any smooth radial function $R(r)$ the Rayleigh quotient

$$\mathcal{Q}(R(r)Y_l(\theta)) = \frac{\int_B |D^2(R(r)Y_l(\theta))|^2 + \tau|\nabla(R(r)Y_l(\theta))|^2 dx}{\int_{\partial B} R(r)^2 Y_l(\theta)^2 d\sigma}$$

is an increasing function of l for $l \geq 2$. We consider the spherical harmonics to be normalized with respect to the $L^2(\partial B)$ scalar product. In particular, we have that the denominator $D[R(r)Y_l(\theta)]$ of $\mathcal{Q}(R(r)Y_l(\theta))$ is $R^2(1)$. For the numerator $N[R(r)Y_l(\theta)]$ of the Rayleigh quotient we have

$$\begin{aligned} N[R(r)Y_l(\theta)] &= \int_0^1 \left(\frac{2k}{r^4} \left(rR' - \frac{3}{2}R \right)^2 + \frac{k(k-N-1/2)}{r^4} R^2 + \tau \frac{kR^2}{r^2} \right) r^{N-1} dr \\ &\quad + \int_0^1 \left((R''^2) + \frac{N-1}{r^2} (R')^2 + \tau(R')^2 \right) r^{N-1} dr, \end{aligned}$$

where $k = l(l + N - 2)$. The above expression is increasing in k for $k \geq N + 1/2$ and since k is an increasing function of l , we easily get that each term involving l is an increasing function of l for $l \geq 2$. Thus the claim above is proved.

For each $l \in \mathbb{N}$,

$$\lambda_{(l)} = \inf \mathcal{Q}(u) = \inf \frac{\int_B |D^2 u|^2 + \tau|\nabla u|^2 dx}{\int_{\partial B} u^2 d\sigma}, \quad (5.14)$$

where the infimum is taken among all functions u that are $L^2(\partial B)$ -orthogonal to the first $m-1$ eigenfunctions u_i and $m \in \mathbb{N}$ is such that $\lambda_{(l)} = \lambda_m$ is the m -th eigenvalue of problem (1.1). The eigenfunctions u_l are of the form $u_l = R_l(r)Y_l(\theta)$, and u_l realizes the infimum in (5.14). Then

$$\lambda_{(l)} = \mathcal{Q}(R_l(r)Y_l(\theta)) \leq \mathcal{Q}(R_{l+1}(r)Y_l(\theta)) \leq \mathcal{Q}(R_{l+1}(r)Y_{l+1}(\theta)) = \lambda_{(l+1)},$$

where the first inequality follows from the fact that $R_{l+1}(r)Y_l(\theta)$ is also orthogonal with respect to the $L^2(\partial B)$ scalar product to the first $m-1$ eigenfunctions $R_i(r)Y_i(\theta)$ for $i = 1, \dots, m-1$, and then it is a suitable trial function in (5.14). The second inequality follows from the fact that the quotient $\mathcal{Q}(R(r)Y_l(\theta))$ is an increasing function of l , for $l \geq 2$. This concludes the proof. \square

5.2 The isoperimetric inequality

In this section we prove the isoperimetric inequality (5.1). Actually, we prove a stronger result, that is a quantitative version of (5.1). We adapt to our case a result of [8], where the authors prove a quantitative version of the Brock-Weinstock inequality for the Steklov Laplacian. We also refer to [21, 22] where these kind of questions have been considered for the first time (see also [9, 18]).

Throughout this section Ω is a bounded domain of class C^1 . We recall the following lemma from [8].

Lemma 5.15. *Let Ω be an open set with Lipschitz boundary and $p > 1$. Then*

$$\int_{\partial\Omega} |x|^p d\sigma \geq \int_{\partial\Omega^*} |x|^p d\sigma \left(1 + c_{N,p} \left(\frac{|\Omega \Delta \Omega^*|}{|\Omega|} \right)^2 \right),$$

where Ω^* is the ball centered at zero with the same measure as Ω , $\Omega \Delta \Omega^*$ is the symmetric difference of Ω and Ω^* , and $c_{N,p}$ is a constant depending only on N and p given by

$$c_{N,p} := \frac{(N+p-1)(p-1)}{4} \frac{\sqrt[N]{2}-1}{N} \left(\min_{t \in [1, \sqrt[N]{2}]} t^{p-1} \right).$$

We also recall the following characterization of the inverses of the eigenvalues of (1.1) from [28] (see also [7]).

Lemma 5.16. *Let Ω be a bounded domain of class C^1 in \mathbb{R}^N . Then the eigenvalues of problem (1.1) on Ω satisfy,*

$$\sum_{l=k+1}^{k+N} \frac{1}{\lambda_l(\Omega)} = \max \left\{ \sum_{l=k+1}^{k+N} \int_{\partial\Omega} v_l^2 d\sigma \right\}, \quad (5.17)$$

where the maximum is taken over the families $\{v_l\}_{l=k+1}^{k+N}$ in $H^2(\Omega)$ satisfying $\int_{\Omega} D^2 v_i : D^2 v_j + \tau \nabla v_i \cdot \nabla v_j dx = \delta_{ij}$, and $\int_{\partial\Omega} v_i u_j d\sigma = 0$ for all $i = k+1, \dots, k+N$ and $j = 1, 2, \dots, k$, where u_1, u_2, \dots, u_k are the first k eigenfunctions of problem (1.1).

For every open set $\Omega \in \mathbb{R}^N$ with finite measure, we recall the definition of Fraenkel asymmetry

$$\mathcal{A}(\Omega) := \inf \left\{ \frac{\|\chi_{\Omega} - \chi_B\|_{L^1(\mathbb{R}^N)}}{|\Omega|} : B \text{ ball with } |B| = |\Omega| \right\}.$$

The quantity $\mathcal{A}(\Omega)$ is the distance in the $L^1(\mathbb{R}^N)$ norm of a set Ω from the set of all balls of the same measure as Ω . This quantity turns out to be a suitable distance between sets for the purposes of stability estimates of eigenvalues. Note that $\mathcal{A}(\Omega)$ is scaling invariant and $0 \leq \mathcal{A}(\Omega) < 2$.

We are ready to prove the following

Theorem 5.18. *For every domain Ω in \mathbb{R}^N of class C^1 the following estimate holds*

$$\lambda_2(\Omega) \leq \lambda_2(\Omega^*) (1 - \delta_N \mathcal{A}(\Omega)^2), \quad (5.19)$$

where δ_N is given by

$$\delta_N := \frac{N+1}{8N} \left(\sqrt[N]{2} - 1 \right),$$

and Ω^* is a ball with the same measure as Ω .

Proof. Let Ω be a bounded domain of class C^1 in \mathbb{R}^N with the same measure as the unit ball B . We consider in (5.17) $l = 2, \dots, N+1$ and $v_l = (\tau|\Omega|)^{-1/2} x_l$ as trial functions. The trial functions must have zero integral mean over $\partial\Omega$. This can be obtained by a change of coordinates $x = y - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} y d\sigma$. Moreover, the functions v_l satisfy the normalization condition of Lemma 5.16. Then v_l are suitable trial functions to test in formula (5.17). We get

$$\sum_{l=2}^{N+1} \frac{1}{\lambda_l(\Omega)} \geq \frac{1}{\tau|\Omega|} \int_{\partial\Omega} |x|^2 d\sigma.$$

We use Lemma 5.15 with $p = 2$. This yields

$$\begin{aligned} \sum_{l=2}^{N+1} \frac{1}{\lambda_l(\Omega)} &\geq \frac{1}{\tau|\Omega|} \int_{\partial B} |x|^2 d\sigma \left(1 + c_{N,2} \left(\frac{|\Omega \Delta B|}{|\Omega|} \right)^2 \right) \\ &= \frac{N|B|}{\tau|B|} \left(1 + c_{N,2} \left(\frac{|\Omega \Delta B|}{|\Omega|} \right)^2 \right) = \frac{N}{\tau} \left(1 + c_{N,2} \left(\frac{|\Omega \Delta B|}{|\Omega|} \right)^2 \right) \\ &= \sum_{l=2}^{N+1} \frac{1}{\lambda_l(B)} \left(1 + c_{N,2} \left(\frac{|\Omega \Delta B|}{|\Omega|} \right)^2 \right). \end{aligned}$$

Suppose now that $\lambda_2(\Omega) \geq \frac{\tau}{2}$, otherwise estimate (5.19) is trivially true, since $0 \leq \mathcal{A}(\Omega) < 2$. Since $\lambda_2(\Omega) \leq \lambda_l(\Omega)$ for all $l \geq 3$, the previous inequality and the definition of $\mathcal{A}(\Omega)$ yield

$$\lambda_2(\Omega) (1 + c_{N,2} \mathcal{A}(\Omega)^2) \leq \lambda_2(B).$$

This implies (5.19) with $\delta_N = \frac{1}{8} \min\{1, \frac{N+1}{N}(\sqrt[3]{2} - 1)\}$. Note that $\min\{1, \frac{N+1}{N}(\sqrt[3]{2} - 1)\} = \frac{N+1}{N}(\sqrt[3]{2} - 1)$. This concludes the proof in the case Ω has the same measure as the unit ball.

The proof for general finite values of $|\Omega|$ relies on the well-known scaling properties of the eigenvalues. Namely, for all $\alpha > 0$, if we write an eigenvalue of problem (1.1) as $\lambda(\tau, \Omega)$, we have

$$\lambda(\tau, \Omega) = \alpha^3 \lambda(\alpha^{-2}\tau, \alpha\Omega).$$

This is easy to prove by looking at the variational characterization of $\lambda(\tau, \Omega)$ and $\lambda(\alpha^{-2}\tau, \alpha\Omega)$ and performing a change of variable $x \mapsto x/\alpha$ in the Rayleigh quotient (3.9). This last observation concludes the proof of the theorem. \square

The isoperimetric inequality 5.1 is an immediate consequence of Theorem 5.18.

Corollary 5.20. *Among all bounded domains of class C^1 with fixed measure, the ball maximizes the first non-negative eigenvalue of problem (1.1), that is $\lambda_2(\Omega) \leq \lambda_2(\Omega^*)$, where $\lambda_2(\Omega)$ has been defined in (3.9) and Ω^* is a ball with the same measure as Ω .*

Remark 5.21. *In [8] the authors prove that the quantitative version of the Brock-Weinstock inequality that they find is sharp. We think that it would be of interest to consider the problem of the sharpness of inequality 5.19 as well. Unfortunately, the results of [8] do not apply immediately to our case. Also, we do not discuss the sharpness here since we think it is out of the purposes of the present paper. Such a discussion will be part of a future work.*

6 Concluding remarks

Throughout this paper we have only considered problems (1.1) and (3.10) with $\tau > 0$. If we set $\tau = 0$, problem (1.1) reads

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ -\operatorname{div}_{\partial\Omega}(D^2 u \cdot \nu) - \frac{\partial \Delta u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega, \end{cases} \quad (6.1)$$

while problem (3.10) reads

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \operatorname{div}_{\partial\Omega}(D^2 u \cdot \nu) + \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \quad (6.2)$$

Problems (6.1) and (6.2) model free vibrating plates which are not subject to lateral tension. These problems have a sequence of non-negative eigenvalues of finite multiplicity and the corresponding eigenfunctions form an orthonormal basis of $H^2(\Omega)$. The coordinate functions x_1, \dots, x_N and the constants are eigenfunctions of both problems (6.1) and (6.2) corresponding to the eigenvalue $\lambda = 0$, which has multiplicity $N + 1$. Therefore, the first non-zero eigenvalue is the $(N + 2)$ -th eigenvalue.

As we did in Theorem 3.17, we can define the family of problems

$$\begin{cases} \Delta^2 u = \lambda \rho_\varepsilon u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \operatorname{div}_{\partial\Omega}(D^2 u \cdot \nu) + \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \quad (6.3)$$

where ρ_ε is defined as in (3.11). We have the following theorem, whose proof can be easily done adapting that of Theorem 3.17.

Theorem 6.4. *Let Ω be a bounded domain in \mathbb{R}^N of class C^2 . Let ρ_ε be defined as in (3.11). Let $\lambda_j(\rho_\varepsilon)$ be the eigenvalues of problem (6.3) on Ω for all $j \in \mathbb{N}$. Let λ_j , $j \in \mathbb{N}$ denote the eigenvalues of problem (6.1) corresponding to the constant surface density $\frac{M}{|\partial\Omega|}$. Then we have $\lim_{\varepsilon \rightarrow 0} \lambda_j(\rho_\varepsilon) = \lambda_j$ for all $j \in \mathbb{N}$.*

It is clear that a discussion similar to that of Section 4 can be carried out for problems (6.1) and (6.2) as well, by means of a change of the projections π_ϕ^S, π_ϕ^N according to the kernel. In particular, all the formulas in Section 4 remain true, by setting $\tau = 0$. Then we have the following

Theorem 6.5. *Let Ω be a domain in \mathbb{R}^N . Let $\tilde{\phi} \in \Phi(\Omega)$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of problem (6.1) (problem (6.2) respectively) in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ on $V(\mathcal{V}(\tilde{\phi}))$, for all $s = 1, \dots, |F|$.*

Moreover, for problem (6.1), it is possible to identify the fundamental modes and the fundamental tone on the ball. We have the following

Theorem 6.6. *Let $\Omega = B$ be the unit ball in \mathbb{R}^N . The eigenfunctions of problem (6.1) are of the form*

$$u_l(r, \theta) = (A_l r^l + B_l r^{2+l}) Y_l(\theta),$$

for $l \in \mathbb{N}$, where A_l and B_l are suitable constants such that

$$B_l = -\frac{l(l-1)}{(l+2)(l+1)} A_l.$$

The eigenvalues $\lambda_{(l)}$ of problem (6.1) corresponding to the eigenfunctions $u_l(r, \theta)$ are delivered by the formula

$$\lambda_{(l)} = \frac{l(l-1)(N+2Nl+(l-1)(2+3l))}{1+2l}.$$

The first positive eigenvalue is

$$\lambda_{N+2} = \lambda_{(2)} = 2 \left(N + \frac{8}{5} \right),$$

and the corresponding eigenfunctions are

$$u_2(r, \theta) = (6r^2 - r^4) Y_2(\theta).$$

Proof. The proof is similar to that of Theorem 5.2, from which it differs only for the use of biharmonic functions on the ball as solutions of the differential equation $\Delta^2 u = 0$. For a characterization of biharmonic functions on the ball we refer to [2, 3, 38]. \square

We have an explicit form for the fundamental tone and for the corresponding eigenfunctions in the case of the unit ball which suggests how to construct trial functions for the Rayleigh quotient of λ_{N+2} . Unfortunately, if we want to use a function of the form $R(r)Y_2(\theta)$ as a test function as we did in Theorem 5.18 we must impose that $R(r)Y_2(\theta)$ is orthogonal to the constants and to the coordinate functions with respect to the $L^2(\partial\Omega)$ scalar product and we can no more obtain this just by translating the domain Ω .

We remark that functions of the form $R(r)Y_2(\theta)$ where $R(r) = 6r^2 - r^4$ for $r \in [0, 1]$ and $R(r) = 8r - 3$ for $r > 1$ are suitable trial functions for the annuli. Explicit computations show that, for example, in dimension 2 or 3 (where the formulas are less involved), the ball is a maximizer among radial domains with a fixed measure.

The results contained in this section suggest that the ball should be a maximizer also for problems (6.1) and (6.2). For what concerns problem (6.2), a characterization of the fundamental tone is still unavaiable. A deeper analysis of problems (6.1) and (6.2) will be part of a future work.

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References

- [1] M. Abramowitz, I.A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, National Bureau of Standards Applied Mathematics Series, U.S. Government Printing Office, Washington D.C., 1964.
- [2] E. Almansi, Sull’integrazione dell’equazione differenziale $\Delta^2 \Delta^2 = 0$, Atti di Torino 31 (1886) 881–888.
- [3] E. Almansi, Sull’integrazione dell’equazione differenziale $\Delta^{2n} = 0$, Annali di matematica, serie III, t. II (1899) 1–59.
- [4] P.M. Anselone, T.W. Palmer, Spectral analysis of collectively compact, strongly convergent operator sequences, Pacific J. Math. 25 (1968) 423–431.
- [5] J.M. Arrieta, Á. Jimnez-Casas, A. Rodríguez-Bernal, Flux terms and Robin boundary conditions as limit of reactions and potentials concentrating at the boundary, Rev. Mat. Iberoam. 24 no.1 (2008) 183–211.
- [6] M.S. Ashbaugh, R.D. Benguria, On Rayleigh’s conjecture for the clamped plate and its generalization to three dimensions, Duke Math. J. 78 no.1 (1995) 1–7.
- [7] C. Bandle, Isoperimetric inequalities and applications, Pitman advanced publishing program, monographs and studies in mathematics, vol. 7, 1980.
- [8] L. Brasco, G. De Philippis, B. Ruffini, Spectral optimization for the Stekloff-Laplacian: the stability issue, J. Funct. Anal. 262 no.11 (2012) 4675–4710.
- [9] L. Brasco, A. Pratelli, Sharp stability of some spectral inequalities, Geom. Funct. Anal. 22 no. 1 (2012) 107–135.
- [10] F. Brock, An isoperimetric inequality for eigenvalues of the Stekloff problem, Z. Angew. Math. Mech. 81 no.1 (2001) 69–71.
- [11] D. Bucur, A. Ferrero, F. Gazzola, On the first eigenvalue of a fourth order Steklov problem, Calculus of Variations and Partial Differential Equations 35 (2009) 103–131.
- [12] D. Buoso, P.D. Lamberti, Eigenvalues of polyharmonic operators on variable domains, ESAIM: COCV 19 (2013) 1225–1235.
- [13] D. Buoso, P.D. Lamberti, Shape deformation for vibrating hinged plates, Mathematical Methods in the Applied Sciences 37 (2014) 237–244.
- [14] L.M. Chasman, An isoperimetric inequality for fundamental tones of free plates, Comm. Math. Phys. 303 no.2 (2011) 421–449.
- [15] M. Dalla Riva, P. Musolino, Real analytic families of harmonic functions in a domain with a small hole, J. Differential Equations 252 no.12 (2012) 6337–6355.
- [16] M.C. Delfour, J.P. Zolésio, Shapes and geometries. Analysis, differential calculus, and optimization, Advances in Design and Control, 4, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001.

- [17] G. Faber, Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt, Sitz. Ber. Bayer. Akad. Wiss. (1923) 169–172.
- [18] N. Fusco, F. Maggi, A. Pratelli, Stability estimates for certain Faber-Krahn, isocapacitary and Cheeger inequalities, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 8 (2009) no. 1 51–71.
- [19] F. Gazzola, H-C. Grunau, G. Sweers, Polyharmonic boundary value problems. Positivity preserving and nonlinear higher order elliptic equations in bounded domains, *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 2010.
- [20] A. Girouard, I. Polterovich, Spectral geometry of the Steklov problem. arXiv:1411.6567 (2014).
- [21] W. Hansen, N. Nadirashvili, Isoperimetric inequalities in potential theory, *Potential Anal.* 3 (1994) 1–14.
- [22] A.D. Melas, The stability of some eigenvalue estimates, *J. Differential Geom.* 36 no. 1 (1992) 19–33.
- [23] D. Gómez, M. Lobo, S.A. Nazarov, E. Pérez, Spectral stiff problems in domains surrounded by thin bands: asymptotic and uniform estimates for eigenvalues, *J. Math. Pures Appl.* 85 no.4 (2006) 598–632.
- [24] D. Gómez, M. Lobo, S.A. Nazarov, E. Pérez, Asymptotics for the spectrum of the Wentzell problem with a small parameter and other related stiff problems, *J. Math. Pures Appl.* 86 no.5 (2006) 369–402.
- [25] J.K. Hale, *Eigenvalues and perturbed domains. Ten mathematical essays on approximation in analysis and topology*, Elsevier B.V., Amsterdam, 2005.
- [26] A. Henrot, *Extremum problems for eigenvalues of elliptic operators*, *Frontiers in Mathematics*, Birkhäuser Verlag, Basel, 2006.
- [27] D. Henry, *Perturbation of the boundary in boundary-value problems of partial differential equations. With editorial assistance from Jack Hale and Antnio Luiz Pereira*, *London Mathematical Society Lecture Note Series*, 318, Cambridge University Press, Cambridge, 2005.
- [28] G.N. Hile, Z.Y. Xu, Inequalities for sums of reciprocals of eigenvalues, *J. Math. Anal. Appl.* 180 no.2 (1993) 412–430.
- [29] E. Krahn, Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises, *Math. Ann.* 94 (1924) 97–100.
- [30] P.D. Lamberti, Steklov-type eigenvalues associated with best Sobolev trace constants: domain perturbation and overdetermined systems, *Complex Var. Elliptic Equ.* 59 no.3 (2014) 309–323.
- [31] P.D. Lamberti, M. Lanza de Cristoforis, A real analyticity result for symmetric functions of the eigenvalues of a domain dependent Dirichlet problem for the Laplace operator, *J. Nonlinear Convex Anal.* 5 no.1 (2004) 19–42.
- [32] P.D. Lamberti, M. Lanza de Cristoforis, Critical points of the symmetric functions of the eigenvalues of the Laplace operator and overdetermined problems, *J. Math. Soc. Japan* 58 no.1 (2006) 231–245.
- [33] P.D. Lamberti, M. Lanza de Cristoforis, A real analyticity result for symmetric functions of the eigenvalues of a domain-dependent Neumann problem for the Laplace operator, *Mediterr. J. Math.* 4 no.4 (2007) 435–449.
- [34] P.D. Lamberti, L. Provenzano, Viewing the Steklov eigenvalues of the Laplace operator as critical Neumann eigenvalues, *Current Trends in Analysis and Its Applications, Proceedings of the 9th ISAAC Congress, Kraków 2013 (to appear)*, Birkhäuser, Basel (2015).

- [35] M. Lanza de Cristoforis, Simple Neumann eigenvalues for the Laplace operator in a domain with a small hole. A functional analytic approach, *Rev. Mat. Complut.* 25 no.2 (2012) 369–412.
- [36] E. Mohr, Über die Rayleighsche Vermutung: unter allen Platten von gegebener Fläche und konstanter Dichte und Elastizität hat die kreisförmige den tiefsten, Grundton. *Ann. Mat. Pura Appl.* 104 (1975) 85–122.
- [37] N.S. Nadirashvili, Rayleigh’s conjecture on the principal frequency of the clamped plate, *Arch. Rational Mech. Anal.* 129 no.1 (1995) 1–10.
- [38] M. Nicolesco, *Les fonctions polyharmoniques. Actualités scientifiques et industrielles*, 331, Herman & C, éditeurs, Paris, 1936.
- [39] S. Stekloff, Sur les problèmes fondamentaux de la physique mathématique (suite et fin), *Ann. Sci. École Norm. Sup.* 3 no.19 (1902) 455–490.
- [40] G. Szegő, On membranes and plates, *Proc. Nat. Acad. Sci.* 36 (1950) 210–216.
- [41] J. Weidmann, Strong operator convergence and spectral theory of ordinary differential operators, *Universitatis Iagellonicae acta mathematica* (1997) Fasciculus XXXIV.
- [42] H.F. Weinberger, An isoperimetric inequality for the N -dimensional free membrane problem, *J. Rat. Mech. Anal.* no.5 (1956) 633–636.
- [43] R. Weinstock, *Calculus of variations with applications to physics and engineering*, McGraw-Hill Book Company Inc., New York-Toronto-London, 1952.
- [44] R. Weinstock, Inequalities for a classical eigenvalue problem, *J. Rational Mech. Anal.* 3 (1954) 745–753.

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