Capital Allocation for Set-Valued Risk Measures

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Abstract

We introduce the notion of set-valued Capital Allocation rule, and study Capital allocation principles for multivariate set-valued coherent and convex risk measures. We compare these rules with some of those mostly used for univariate (single-valued) risk measures.

Key words: Risk management, Capital Allocation rules, Set-Valued Risk Measures, Coherent and Convex Risk Measures.

1 Introduction

In the risk management literature, the topic known as *Capital Allocation* problem consists in choosing a fair way for sharing a risk capital held by firms, insurances or in general portfolio managers in order to hedge the uncertain future net worth of their positions, among the different business lines (see, for example [9], [10], [24]): Capital Allocation problems and the theory of risk measures are hence naturally linked through the role played by risk measures in determining risk capital ([3]).

A large part of the literature on Capital Allocation is based on the idea that fairness amounts to allocating to a sub-portfolio a cost corresponding to its marginal contribution to the riskiness of the whole portfolio (see, for example [10], [26]) and, in this respect, many Capital Allocation rules are

based on the assumption of differentiability of the underlying risk measure. In a previous paper we have extended some known allocation rules to cover the case of non-differentiability, introducing a family of Capital Allocation rules based on sub-differentials, which thus give rise to a set-valued map ([7]).
Indeed, quite recently, risk measures have been studied and extended to a more general setting where they can be set-valued ([23], [20]): one among the different financial motivations for studying set-valued risk measures is given by the existence of portfolios of financial positions in different currencies that can not be aggregated for reasons such as liquidity constraints and/or transaction costs ([23], [20]). In this case, in fact, it seems more reasonable to consider risk measures that associate to any financial portfolio in different currencies a set of hedging deterministic positions. Dual representations for set-valued risk measures can be found, among many others, in [19], [20] while extensions to the dynamic framework can be found in [12], [13], [4]. Recently, many well known risk measures have been extended to the set-valued case (see, for example [1], [15], [25]) and, furthermore, set-valued risk measures have also been applied to the study of systemic risk ([2],[14]).

In this paper, we define and extend the concept of Capital Allocation rule to set-valued risk measures, and provide some Capital Allocation rule once more based on the idea of marginality and so, linked to the concept of subdifferential for set-valued functions.
The paper is organized as follows: in Section 2 we introduce the mathematical setup and recall the main results about representation of set-valued risk measures and their subdifferentials, in Section 3 we introduce the Capital Allocation problem in the set-valued context, define some set-valued Capital Allocation rules based on the directional derivative and on sub-differentials of a set-valued function, and study their properties. We also briefly investigate the case when we reduce to a scalar-valued risk measure. Moreover, by adopting Kalkbrener's view ([24]), we also derive set-valued risk measures with suitable properties, by starting with a general Capital Allocation rule. Section 4 is devoted to some examples. Section 5 sums up and provides some conclusions.

2 Mathematical Preliminaries

In the following, $(\Omega, \mathcal{F}_T, P)$ will denote a probability space where T is an a priori fixed time horizon.
Let L^∞_d be the usual linear space of P-equivalence classes of $\mathcal{F}_T$-measurable

functions $X : \Omega \longrightarrow \mathbb{R}^d$, such that ess $\sup_{\omega\in\Omega} |X(\omega)| < +\infty$. We will assume L^∞_d endowed with the $\sigma(L^\infty_d, L^1_d)$ topology. Given a vector $X = (X_1, \dots, X_d)$, $E[X]$ will denote the componentwise expectation of X under P while, if $Q = (Q_1, \dots, Q_d)$, $E_Q[X]$ will denote $(E_{Q_1}[X_1], \dots, E_{Q_d}[X_d])$. Every element $X \in L^\infty_d$ has components in L^∞, and $\mathbf{1}$ denotes the random variable in L^∞_d assuming, P-a.s, the constant value 1. The value of each component X_i, $i = 1, \dots d$ of the random vector X is interpreted as the profit and loss of asset i in some market, at maturity T.

Let M be a linear subspace of $\mathbb{R}^d$ having dimension $1 \le m \le d$. The financial meaning of M is that a regulator accepts compensation for risk only in a given subset of the d markets. Moreover, it will be assumed that $M \cap \mathbb{R}^d_+ \neq \emptyset$, meaning that there exists at least a position with non-negative components which is accepted to hedge risk. Furthermore, a closed convex cone $K \subset \mathbb{R}^d$, termed as solvency cone, will be considered to induce a partial order $\le_K$ on $\mathbb{R}^d$, namely $x \le_K y$ if and only if $y - x \in K$ for every $x, y \in \mathbb{R}^d$. If a deterministic position belongs to the solvency cone, it means that it can be converted, by payment of a transaction cost, into an acceptable position with nonnegative components. Hence, we will be interested in the positions belonging to the closed convex cone $K_M = K \cap M$.

The partial order induced by K extends from $\mathbb{R}^d$ to $\mathcal{P}(\mathbb{R}^d)$, the set of all subsets of $\mathbb{R}^d$, in the following way: $A \le_K B$ if and only if $B \subseteq A + K$, for $A, B \in \mathcal{P}(\mathbb{R}^d)$ (where $+$ is the usual elementwise Minkowski sum). If we set $\mathcal{P}(\mathbb{R}^d, K) \triangleq \{A \in \mathcal{P}(\mathbb{R}^d) : A = A + K\}$, then for every $A, B \in \mathcal{P}(\mathbb{R}^d, K)$, we have that $A \le_K B \iff A \supseteq B$. It is well known (see [21] for more details on this topic), that $((\mathbb{R}^d, K), \supseteq)$ is a complete lattice where, for $\mathcal{A} \subseteq (\mathbb{R}^d, K)$, $\inf \mathcal{A} = \bigcup_{A\in\mathcal{A}} A$, and $\sup \mathcal{A} = \bigcap_{A\in\mathcal{A}} A$. For every $A, B \in \mathcal{P}(\mathbb{R}^d)$, we define the operation of *inf-residuation*: $A \mathbin{-\!\!\cdot} B \triangleq \inf\{D \in \mathcal{P}(\mathbb{R}^d, K) : A \supseteq B + D\}$.

Let K^+ also denote the dual positive cone of K in $\mathbb{R}^d$. Set $L^\infty_d(K) \triangleq \{X \in L^\infty_d : X \in K,\ P-\text{a.s.}\}$ and let $L^1_d(K^+)$ denote its positive dual cone. This cone generates a partial order on the set of random variables vectors which extends the one generated by K on $\mathbb{R}^d$.

We point out that, for the sake of simplicity, we work in L^∞_d even if the results also hold in the case of L^p_d, with $p \ge 1$. Let now $\mathbb{F}_M = \{D \subseteq M : D = cl(D + K_M)\}$ be the family of upper closed subsets of M.

Given a function $R : L^\infty_d \to \mathbb{F}_M$, its effective domain is the set $domR = \{X \in L^\infty_d : R(X) \neq \emptyset\}$. A function $R : L^\infty_d \to \mathbb{F}_M$ is said to be proper if $domR \neq \emptyset$ and $R(X) \neq M$, for every $X \in L^\infty_d$.

We are now ready to recall the definition and the dual representation of set-valued risk measures (see [20]).

Definition 1 (see [20]) A set-valued convex risk measure is a function:

$$R : L_d^\infty \to \mathbb{F}_M$$

satisfying the following properties:

- *normalization*: $R(0) \supseteq K_M$ and $R(0) \cap (-int\, K_M) = \emptyset$;
- *M-translation*: $R(X + u\mathbf{1}) = R(X) - u$ for any $X \in L_d^\infty$ and $u \in M$;
- *monotonicity*: if $Y - X \in K$, then $R(Y) \supseteq R(X)$;
- *convexity*: $R(\alpha X + (1-\alpha)Y) \supseteq \alpha R(X) + (1-\alpha)R(Y)$ for any $X, Y \in L_d^\infty$, $\alpha \in (0,1)$.

A *set-valued coherent risk measure* is a set-valued convex risk measure satisfying also:

- *positive homogeneity*: $R(\alpha X) \supseteq \alpha R(X)$ for any $\alpha > 0$ and $X \in L_d^\infty$.

We can now recall the dual representations for coherent and convex set-valued risk measures (see [20]). Set $\mathbb{G}_M = \{D \subseteq M : D = cl\, co(D + K_M)\}$, and $M^\perp = \{v \in \mathbb{R}^d : v^T u = 0, \forall u \in M\}$. Denote by $\mathcal{M}_{1,d}^P(\Omega, \mathcal{F}_T)$ the set of all vectors $Q = (Q_1, \ldots, Q_d)$ of probability measures, such that $Q_i \ll P$, $i = 1, \ldots, d$. Set $\mathcal{W} = \{(Q, w) \in \mathcal{M}_{1,d}^P \times K^+ \setminus M^\perp : diag(w)\frac{dQ}{dP} \in L_d^1(K^+)\}$, where $diag(w)$ denotes the square diagonal matrix having the components of w on the diagonal, and $\frac{dQ}{dP} = (\frac{dQ_1}{dP}, \ldots \frac{dQ_n}{dP})$. Then it holds:

- any proper, convex and $\sigma(L_d^\infty, L_d^1)$ - closed set-valued risk measure $R : L_d^\infty \to \mathbb{G}_M$ can be represented as

$$R(X) = \bigcap_{(Q,w)\in\mathcal{W}} \{-\alpha_R(Q, w) + (E_Q[-X] + G(w)) \cap M\} \qquad (1)$$

  where $G(w) \triangleq \{x \in \mathbb{R}^d : w^T x \geq 0\}$, and $\alpha_R(Q, w)$ is a $\mathbb{G}_M$-valued function defined on $\mathcal{M}_{1,d}^P \times K^+ \setminus M^\perp$. In particular, α_R can be given by

$$-\alpha(Q, w) = cl\left(\bigcup_X [R(X) + (E_Q[-X] + G(w)) \cap M]\right) \qquad (2)$$

- any proper, coherent and $\sigma(L_d^\infty, L_d^1)$ - closed set-valued risk measure $R : L_d^\infty \to \mathbb{G}_M$ can be represented as

$$R(X) = \bigcap_{(Q,w)\in\mathcal{W}_R} (E_Q[-X] + G(w)) \cap M \qquad (3)$$

  where $\mathcal{W}_R$ is a suitable subset of $\mathcal{W}$.

We briefly recall the interpretation of the dual variables (Q, w) (see also [20]): the vector Q represents probabilistic scenarios which can differ in the various markets, while the vector w is connected to the preferences of the investors among the reference instruments. Roughly speaking, while Q has the same interpretation as in the scalar case, the component w is new and is "responsible" of giving rise to a set by means of $G(w)$. Remember also that the operation of intersection in the representation is a supremum w.r.t. the order relation on the set $\mathbb{G}_M$, in analogy to the scalar single-valued case.

In [21], Prop. 5.19, the authors, taken $w \in K^+$ and $\bar{X} \in L^\infty_d$, also define the w-subdifferential of a $\mathbb{G}_M$-valued set function at $\bar{X}$ and characterize it as the following (nonempty) set:

$$\begin{aligned}&\partial_w R(\bar{X}) \\ &= \{Q_w : \big(E_{Q_w}[X] + G(w)\big) \cap M \mathbin{-^{\cdot}} R(X) \supseteq \big(E_{Q_w}[\bar{X}] + G(w)\big) \cap M \mathbin{-^{\cdot}} R(\bar{X}) \text{ for any } X\}.\end{aligned} \tag{4}$$

3 Capital Allocation for Set-Valued Risk Measures

We now extend the classical notion of Capital Allocation rule ([24]) to the set-valued setting.

Definition 2 Given a set-valued risk measure R, a *set-valued Capital Allocation rule* (associated to R) is a set-valued map

$$\Lambda : L^\infty_d \times L^\infty_d \to \mathbb{F}_M$$

satisfying $\Lambda(X, X) = R(X)$ for any X. We will refer to *weak set-valued Capital Allocation rule* when the previous property is replaced by the weaker condition $\Lambda(X, X) \supseteq R(X)$ for any X.
Also in this setting we can define some desirable properties:

- *no undercut*: $\Lambda(Y; X) \supseteq R(Y)$ for any $X, Y \in L^\infty_d$;
- *sub-allocation*: $\Lambda(X_1 + X_2 + ... + X_n, X) \supseteq \sum_{i=1}^n \Lambda(X_i, X)$ once $X_1 + X_2 + ... + X_n = X$.

We interpret $\Lambda(Y; X)$ as the set of all deterministic positions that can be allocated to Y in order to compensate for its risk as a sub-portfolio of X. With this in mind, also the interpretation of the properties becomes clear: indeed the no undercut means that all the deterministic positions that hedge the risk of Y as a sub-portfolio of itself, also cover the risk of Y as a sub-portfolio of X, while the sub-allocation property means that, if we split a portfolio X into n sub-portfolios X_i, then its risk can be covered in more ways than only by using deterministic vectors obtained by summing positions that cover the risk of the X_i's as sub-portfolios of X. As in the scalar case, therefore, under no undercut there is no incentive to split the sub-portfolio Y from the whole X since the riskiness $R(Y)$ of Y as a stand-alone portfolio is contained in $\Lambda(Y;X)$.

The condition $\Lambda(X,X) \supseteq R(X)$ required for a weak capital allocation rule is automatically fulfilled for R satisfying the no undercut and can be interpreted similarly to that property. That is, all the deterministic positions covering the risk of X can also be allocated to X in order to compensate for its risk as a sub-portfolio of itself. For capital allocation rules, the same holds for all and only deterministic positions as before.

Proposition 3 *Assume that $R : L_d^\infty \to \mathbb{G}_M$ is a proper, convex and closed set-valued risk measure, and suppose that, for every $X \in L_d^\infty$, there existed $(Q_X, w_X) \in \mathcal{W}$, such that $R(X) = -\alpha_R(Q_X, w_X) + (E_{Q_X}[-X] + G(w_X)) \cap M$.*
Then, the set-valued map defined by:

$$\Lambda(Y;X) = -\alpha_R(Q_X, w_X) + (E_{Q_X}[-Y] + G(w_X)) \cap M,$$

for every $Y, X \in L_d^\infty$, is a set-valued capital allocation rule satisfying no undercut. Moreover, if the following condition holds true for any $X_1, \dots, X_n, X \in L_d^\infty$ with $X_1 + \dots + X_n = X$:

$$(*) \qquad -\alpha_R(Q_{w,X}, w_X) + \sum_i (E_{Q_{X,w_X}}[-X_i] + G(w_X)) \cap M$$
$$\supseteq \sum_i [-\alpha_R(Q_X, w_X) + (E_{Q_{X,w_X}}[-X_i] + G(w_X)) \cap M],$$

then Λ satisfies also sub-allocation.

Proof. $\Lambda(X;X) = R(X)$ holds trivially.
For $Y, X \in L_d^\infty$, by the assumption that there exists $(Q_X, w_X) \in \mathcal{W}$ s.t. $R(X) = -\alpha_R(Q_X, w_X) + (E_{Q_X}[-X] + G(w_X)) \cap M$, it holds that $\Lambda(Y;X) \supseteq \bigcap_{(Q,w)\in\mathcal{W}} \{-\alpha_R(Q,w) + (E_Q[-Y] + G(w)) \cap M\} = R(Y)$.

Take now X and $X_1, \ldots, X_n$ s.t. $X = X_1 + \cdots + X_n$. Then

$$\begin{aligned} \Lambda(X;X) &= -\alpha_R(Q_X, w_X) + \left(\sum_{i=1}^{n} E_{Q_X}[-X_i] + G(w_X) \right) \cap M \\ &\supseteq \sum_{i=1}^{n} (-\alpha_R(Q_X, w_X) + (E_{Q_X}[-X_i] + G(w_X)) \cap M), \end{aligned}$$

where the last step follows directly from Lemma 4.1 and Proposition 4.2 in [20]. ■

Some necessary and sufficient conditions for the existence of a global $(Q, w) \in \mathcal{W}$ realizing the intersection in (1) for any X can be found in Hamel et al. [21]. Such conditions include lattice-lower semicontinuity of ρ at 0 and some other quite strong assumption on ρ, such as additivity. Concerning the existence of $(Q_X, w_X) \in \mathcal{W}$ as in Proposition 3, we provide two examples (see Examples 12 and 13) showing that for general R existence may occur or not.

Remark 4 *Notice that, if the risk measure R is coherent, condition $(*)$ is automatically fulfilled.*

Remark 5 *The previous result can be regarded as a set-valued version of the classical capital allocation rules based on marginal contribution. Indeed, if there exists (Q_X, w_X) in $\mathcal{W}$ such that the intersection in the representation is realized at this point, then it is easy to show that $-Q_X \in \partial_{w_X} R(X)$. Indeed,*

$$E_{Q_X}[-X_i] + G(w_X)) \cap M = R(X) + \alpha_R(Q_X, w_X)$$

and hence, by the definition of inf-residuation:

$$\alpha_R(Q_X, w_X) = E_{Q_X}[-X_i] + G(w_X)) \cap M -^{\cdot} R(X).$$

Therefore $-Q_X \in \partial_{w_X} R(X)$.
Notice that the result above resembles those true for single-valued scalar risk measures where the existence of a generalized scenario Q_X realizing the supremum in the dual representation of $\rho(X)$ implies that $-Q_X \in \partial \rho(X)$. The interested reader can see Kalkbrener [24] and Delbaen [8] (coherent case), Centrone and Rosazza Gianin [7] (convex case) for details. Differently from the classical (single-valued scalar) case, however, in the set-valued one the dependence on w appears as usual.

Since the hypothesis of Proposition 3 are quite strong, we investigate now what happens when the condition of the existence of $(Q_X, w_X) \in \mathcal{W}$ is weakened and replaced by the existence (for any fixed w) of some $Q_{X,w}$ such that $R(X) = \bigcap_w \left\{-\alpha_R(Q_{X,w}, w) + \left(E_{Q_{X,w}}[-X] + G(w)\right) \cap M\right\}$.

Proposition 6 *Assume that $R : L^\infty_d \to \mathbb{G}_M$ is a proper, convex and closed set-valued risk measure, and suppose that, for every $X \in L^\infty_d$ and any fixed w, there exists $Q_{X,w}$ such that $(Q_{X,w}, w) \in \mathcal{W}$ and*

$$R(X) = \bigcap_w \left\{-\alpha_R(Q_{X,w}, w) + \left(E_{Q_{X,w}}[-X] + G(w)\right) \cap M\right\}.$$

Then, the set-valued map defined by

$$\overline{\Lambda}(Y; X) = \bigcap_w \left\{-\alpha_R(Q_{X,w}, w) + \left(E_{Q_{X,w}}[-Y] + G(w)\right) \cap M\right\}, \qquad (5)$$

for every $Y, X \in L^\infty_d$, is a set-valued capital allocation rule satisfying no undercut. Moreover, if the following condition holds true for any w and any $X_1, \dots, X_n, X \in L^\infty_d$ with $X_1 + \dots + X_n = X$:

$$(**) \qquad -\alpha_R(Q_{w,X}, w) + \sum_i \left(E_{Q_X,w}[-X_i] + G(w)\right) \cap M$$
$$\supseteq \sum_{i=1}^n [-\alpha_R(Q_X, w) + (E_{Q_X,w}[-X_i] + G(w)) \cap M],$$

then $\overline{\Lambda}$ satisfies also sub-allocation.

Proof. It follows immediately that $\overline{\Lambda}(X; X) = R(X)$ and that $\overline{\Lambda}$ is a set-valued capital allocation rule.

No undercut. By (5),

$$\overline{\Lambda}(Y; X) \supseteq \bigcap_{(Q,w)\in\mathcal{W}} \{-\alpha_R(Q, w) + (E_Q[-Y] + G(w)) \cap M\} = R(Y),$$

hence no undercut is verified.

Sub-allocation. Let $X_1, \dots, X_n$ satisfy $X_1 + \cdots + X_n = X$ and assume $(**)$. Then

$$
\begin{aligned}
\overline{\Lambda}(X_1 + \cdots + X_n; X) &= \bigcap_w \{-\alpha_R(Q_{X,w}, w) + (E_{Q_{X,w}}[-X_1 - \cdots - X_n] + G(w)) \cap M\} \\
&\supseteq \bigcap_w \left\{-\alpha_R(Q_{X,w}, w) + \sum_{i=1}^n (E_{Q_{X,w}}[-X_i] + G(w)) \cap M\right\} \\
&\supseteq \bigcap_w \left\{\sum_{i=1}^n \left[-\alpha_R(Q_{X,w}, w) + (E_{Q_{X,w}}[-X_i] + G(w)) \cap M\right]\right\} \\
&\supseteq \sum_{i=1}^n \left\{\bigcap_w \left[-\alpha_R(Q_{X,w}, w) + (E_{Q_{X,w}}[-X_i] + G(w)) \cap M\right]\right\} \\
&= \sum_{i=1}^n \overline{\Lambda}(X_i; X),
\end{aligned}
$$

where the first inclusion is due to the coherence of $(E_Q[-Y] + G(w)) \cap M$ (see Prop. 4.2 and Lemma 4.1 in Hamel and Heyde [20]), while the second to condition $(**)$. ■

Notice that condition $(**)$ implies $(*)$ of Proposition 3 and that, as for condition $(*)$, also $(**)$ is automatically satisfied when R is coherent.

It is worth to emphasize that in the scalar case the two previous approaches collapse to one in line with the gradient approach for single-valued scalar risk measures.

Remark 7 ***(Scalar case)*** *Assume to be in the scalar case, that is $d = 1$, $M = \mathbb{R}$ and $K = \mathbb{R}^+$.*

Firstly, let R be a coherent risk measure. By (3), it follows that

$$
\begin{aligned}
R(X) &= \bigcap_{(Q,w) \in \mathcal{W}_R} \left(E_Q[-X] + \mathbb{R}^+\right) \cap M \\
&= \bigcap_Q [E_Q[-X]; +\infty) \qquad (6) \\
&= \left[\sup_Q E_Q[-X]; +\infty\right) \qquad (7)
\end{aligned}
$$

since $K^+ = \mathbb{R}^+$ and, consequently, $G(w) = \mathbb{R}^+$ for any $w \in K^+$. Hence $R(X) = [\rho(X); +\infty)$ where $\rho(X) \triangleq \sup_Q E_Q[-X]$ is a single-valued scalar coherent risk measure.

Since the dependence on w disappears in (6) *and* (7)*, the approaches in Propositions 3 and 6 reduce to requiring that for any X there exist some* $Q_X \in argmax_Q E_Q[-X]$*. In that case,*

$$R(X) = [E_{Q_X}[-X]; +\infty) \qquad \text{and} \qquad \Lambda(Y;X) = [E_{Q_X}[-Y]; +\infty) \tag{8}$$

Since $Q_X \in argmax_Q E_Q[-X]$ *implies that* $-Q_X \in \partial\rho(X)$*, the approach above generalizes the gradient approach for scalar single-valued risk measures where* $\Lambda_\rho(Y;X) = E_{Q_X}[-Y]$ *with* $-Q_X \in \partial\rho(X)$*. See [24], [8] and [7].*

Similar arguments hold also for the convex case. Let, indeed, R be a convex risk measure. By (1) *it follows that*

$$\begin{aligned} R(X) &= \bigcap_{(Q,w)\in\mathcal{W}} [-\alpha_R(Q,w) + [E_Q[-X]; +\infty)] \\ &= \bigcap_{Q} [E_Q[-X] - \alpha_R(Q); +\infty) \qquad (9) \\ &= \left[\sup_Q \{E_Q[-X] - \alpha_R(Q)\}; +\infty\right) \qquad (10) \end{aligned}$$

since, by (2)*,*

$$\begin{aligned} -\alpha_R(Q,w) &= cl\left(\bigcup_X [R(X) + (E_Q[-X] + G(w)) \cap M]\right) \\ &= \left[-\sup_X \{E_Q[X] - \inf R(X)\}; +\infty\right) \\ &= [-\alpha_R(Q); +\infty) \end{aligned}$$

where $\alpha_R(Q) \triangleq \sup_X\{E_Q[X] - \inf R(X)\}$*. Hence* $R(X) = [\rho(X); +\infty)$ *where* $\rho(X) \triangleq \sup_Q\{E_Q[-X] - \alpha_R(Q)\}$ *is a scalar single-valued convex risk measure. Notice also that* α_R *is the minimal penalty term associated to* ρ *because* $\rho(X) = \inf R(X)$*.*

As for coherent risk measures, also in this case the dependence on w disappears in (9) *and* (10)*. The approaches in Propositions 3 and 6 reduce therefore to requiring that for any X there exist some* $Q_X \in argmax_Q\{E_Q[-X] - \alpha_R(Q)\}$*. As before, then, the approach above generalizes the gradient approach for scalar single-valued risk measures where* $\Lambda_\rho(Y;X) = E_{Q_X}[-Y] - \alpha_R(Q_X)$ *with* $-Q_X \in \partial\rho(X)$*. See [7].*

We recall that, in the framework of scalar single-valued risk measures, Kalkbrener [24] showed that the directional derivative has a special role as

a capital allocation rule. The use of the subdifferential, instead, has been investigated by Delbaen [8] and Centrone and Rosazza Gianin [7], among others. Inspired by the results above, we investigate whether the directional derivative could still be a good capital allocation rule also for set-valued risk measures or not.
To this aim, we recall from Hamel et al. [21] that the (lower Dini) directional derivative of R with respect to w at Y in the direction X is defined as

$$D_w R(Y;X) \triangleq \liminf_{t\downarrow 0} \frac{(R(Y+tX)+G(w))\cap M -^{\cdot} R(Y)}{t}, \qquad (11)$$

while the subdifferential of R with respect to w at X can be equivalently be defined as in (4). Moreover, for a convex set-valued risk measure R satisfying the additional hypothesis of Theorem 5.18 in [21] it holds that, given w, for any X there exists $-\bar{Q}_{w,X} \in \partial_w R(X)$ such that

$$D_w R(Y;X) = \left(E_{\bar{Q}_{w,X}}[-Y] + G(w)\right) \cap M. \qquad (12)$$

See [21] for details.
In the spirit of Kalkbrener [24], we define now

$$\Lambda_w(Y;X) \triangleq D_w R(Y;X) \qquad (13)$$

once w is given.

Proposition 8 *Let w be given and let R be a set-valued risk measure satisfying the hypothesis of Theorem 5.18 in [21].*
(a) If R is coherent, then Λ_w is a weak capital allocation satisfying no undercut and sub-allocation.
(b) If R is convex, then Λ_w satisfies the following generalized no undercut

$$R(Y) \subseteq -\alpha_R(\bar{Q}_{w,X}, w) + \Lambda_w(Y;X), \text{ for any } X, Y \in L^\infty_d. \qquad (14)$$

Proof. (a) By the dual representation of R, (12) and (13), the no undercut follows immediately.

Consider now any $X_1, ..., X_n, X$ with $X_1 + X_2 + ... + X_n = X$. Then

$$\begin{aligned} \Lambda_w(X_1+X_2+...+X_n;X) &= \left(E_{\bar{Q}_{w,X}}[-(X_1+X_2+...+X_n)] + G(w)\right) \cap M \\ &\supseteq \sum_{i=1}^n \left(E_{\bar{Q}_{w,X}}[-X_i] + G(w)\right) \cap M \\ &= \sum_{i=1}^n \Lambda_w(X_i;X), \end{aligned}$$

where the inclusion is due to coherence of $(E_Q[-Z]+G(w))\cap M$ (see Prop. 4.2. and Lemma 4.1 in Hamel and Heyde [20]). Sub-allocation is then proved.

(b) The generalized no undercut can be deduced immediately from the dual representation of R and (12). ■

Notice that the previous result is in line with those of Kalkbrener [24] and Centrone and Rosazza Gianin [7] for coherent and convex scalar single-valued risk measures. While, indeed, for coherent risk measures both no undercut and sub-allocation hold true, in the convex case sub-allocation in general fails and no undercut should be replaced by a generalized one taking into account the penalty term. This generalized no undercut (14) can be interpreted and motivated as in the classical case (see [7]). The main differences between coherent and convex risk measures consist, indeed, in positive homogeneity that can be seen as a scaling invariance property and, in terms of dual representation, in a penalty term. Differently from coherent risk measures, convex ones are able to take into account liquidity aspects, hence to distinguish between portfolios formed by different sizes of the same assets. It seems therefore to be financially reasonable that for convex risk measures the generalized no undercut in terms of $\Lambda_w(Y;X)$ would depend not only on the sub-portfolio Y but also on the whole portfolio X (for instance, by means of its size) and on the penalty term.
Once w is fixed, $\Lambda_w(Y;X)$ as defined in (13) provides a possible rule of capital allocation depending on w. Anyway, there is freedom on the choice of w and, depending on w, we will have different capital allocation rules. This fact again reflects the preferences of the investors among the reference instruments. An alternative way to define Λ independently on the choice of w is, however, to consider

$$\widetilde{\Lambda}(Y;X) \triangleq \bigcap_{w} \Lambda_w(Y;X). \tag{15}$$

So far, we have examined the properties of a capital allocation stemming from a coherent/convex set-valued risk measure. We now assume the converse standpoint: starting from a set-valued capital allocation, we define a set-valued risk function and study its properties, in the spirit of Kalkbrener's paper ([24]).

Proposition 9 *Let $\Lambda : L_d^\infty \times L_d^\infty \longrightarrow \mathbb{F}_M$ be a set-valued map, and define $R(X)=\Lambda(X,X)$, for every $X \in L_d^\infty$.*

If Λ satisfies: (a) $\Lambda(Y;X) \supseteq \Lambda(Y;Y)$, for every $X, Y \in L^\infty_d$, (b) $\Lambda(X_1 + \cdots + X_n; X) \supseteq \sum_{i=1}^n \Lambda(X_i, X)$, (c) $\Lambda(X; \alpha Z) = \Lambda(\alpha X; Z) = \alpha\Lambda(X; Z)$, for every $\alpha > 0, X, Z \in L^\infty_d$, then R is a positively homogeneous and subadditive. If Λ satisfies (a), (b) and the second equality in (c), then R is convex.

Proof. We have $R(\alpha X) = \Lambda(\alpha X; \alpha X) \supseteq \alpha\Lambda(\alpha X; X) = \alpha\Lambda(X; \alpha X) \supseteq \alpha\Lambda(X; X) = \alpha R(X)$. This proves positive homogeneity. Moreover note that it also holds $R(\alpha X) = \Lambda(\alpha X; \alpha X) \subseteq \Lambda(\alpha X; \alpha X) = \alpha\Lambda(X; X) = \alpha R(X)$.
$R(X+Y) = \Lambda(X+Y; X+Y) = \Lambda(X; X+Y) + \Lambda(Y; X+Y) \supseteq \Lambda(X; X) + \Lambda(Y; Y) = R(X) + R(Y)$. This proves subadditivity.
As for the second part of the proposition, we have: $R(\alpha X + (1-\alpha)Y) = \Lambda(\alpha X + (1-\alpha)Y; \alpha X + (1-\alpha)Y) = \Lambda(\alpha X; \alpha X + (1-\alpha)Y) + \Lambda((1-\alpha)Y; \alpha X + (1-\alpha)Y) \supseteq \Lambda(\alpha X; \alpha X) + \Lambda((1-\alpha)Y; (1-\alpha)Y) = \alpha\Lambda(X; \alpha X) + (1-\alpha)\Lambda(Y; (1-\alpha)Y) \supseteq \alpha\Lambda(X; X) + (1-\alpha)\Lambda(Y; Y) = \alpha R(X) + (1-\alpha)R(Y)$.
■

4 Examples

In this section we study some examples of set-valued capital allocation rules, also in connection with the rule we introduced in Proposition 3.
The first two examples deal with the scalar case that is particularly relevant for its interpretation and its relation with the case of single-valued risk measures.

Example 10 (Scalar case) *Consider $M = \mathbb{R}$ and $K = \mathbb{R}_+$. Let R be the 1-dimensional set-valued risk measure defined as $R(X) \triangleq [\rho(X); +\infty)$, where ρ is a (single-valued) coherent risk measure that is continuous from below. Hence $\rho(X) = \max_{Q \in \mathcal{P}} E_Q[-X]$ for any $X \in L^\infty$.*

It can be easily checked that R is a coherent set-valued risk measure and that

$$R(X) = [E_{Q^*_X}[-X]; +\infty), \tag{16}$$

where $Q^*_X \in$ $argmax_{Q \in \mathcal{P}} E_Q[-X]$. For any fixed w, it follows that $-\partial_w R(X) = -\partial\rho(X)$, hence $R(X) = \bigcap_Q [E_Q[-X]; +\infty) = [E_{Q^*_X}[-X]; +\infty)$ with $-Q^*_X$ belonging to $\partial_w R(X)$ for any w. By the dual representation of R and the definition of $\partial_w R(X)$, indeed, for any fixed w it holds that

$$-\partial_w R(X) = \{Q : [E_Q[-Y]; +\infty) -^{\cdot} R(Y) \supseteq [E_Q[-X]; +\infty) -^{\cdot} R(X) \text{ for any } Y\}$$

since $G(w) = \{v \in \mathbb{R} : w \cdot v \geq 0\} = \mathbb{R}^+$ because $w \geq 0$. Furthermore, by (16), the definition of Q^*_X, Q^*_Y, and the definition of inf-residuation, it *follows that*

$$\begin{aligned}
& -\partial_w R(X) \\
= \;& \{Q : [E_Q[-Y] - E_{Q^*_Y}[-Y]; +\infty) \supseteq [E_Q[-X] - E_{Q^*_X}[-X]; +\infty) \text{ for any } Y\} \\
= \;& \{Q : [E_Q[-Y] - \rho(Y); +\infty) \supseteq [E_Q[-X] - \rho(X); +\infty) \text{ for any } Y\} \\
= \;& \{Q : \rho(Y) - \rho(X) \geq E_Q[-(Y - X)] \text{ for any } Y\} \\
= \;& -\partial\rho(X).
\end{aligned}$$

Assume now that R be defined as $R(X) \triangleq [\rho(X); +\infty)$, where ρ is a (single-valued) convex risk measure that is continuous from below. Hence $\rho(X) = \max_{Q\in\mathcal{P}}\{E_Q[-X] - \alpha(Q)\}$ for any $X \in L^\infty$. It can be easily checked that R is a convex set-valued risk measure and that

$$R(X) = [E_{Q^*_X}[-X] - \alpha(Q^*_X); +\infty), \tag{17}$$

where $Q^*_X \in \; argmax_{Q\in\mathcal{P}}\{E_Q[-X] - \alpha(Q)\}$.

Similarly to above, one can check that for any fixed w it holds that $-\partial_w R(X) = -\partial\rho(X)$. Hence $R(X) = \bigcap_Q [E_Q[-X] - \alpha(Q); +\infty) = [E_{Q^*_X}[-X] - \alpha(Q^*_X); +\infty)$ with $-Q^*_X$ belonging to $\partial_w R(X)$ for any w.

Example 11 (Aumann and Shapley set-valued capital allocation) *Set $M = \mathbb{R}$ and $K = \mathbb{R}^+$ and let ρ be a scalar-valued lower semicontinuous coherent risk measure. Take $Q_X \in argmax_{Q\in\mathcal{P}} E_Q[-X]$. Since ρ is coherent it holds: $argmax_Q E_Q[-X] \equiv argmax_Q E_Q[-\gamma X]$ for any $\gamma \in [0, 1]$ and $X \in L^\infty$. Then $\rho(X) = \int_0^1 E_{Q_{\gamma X}}[-X] d\gamma$.*
Define the set-valued risk measure $R(X) \triangleq [\rho(X), +\infty)$ and the set-valued map $\Lambda(Y; X) = [\int_0^1 E_{Q_{\gamma X}}[-Y] d\gamma, +\infty)$. Then Λ is a set-valued capital allocation rule.
Indeed, by construction $\Lambda(Y; X) = R(X)$ is trivially fulfilled while no undercut follows from $\int_0^1 E_{Q_{\gamma Y}}[-Y] d\gamma \geq \int_0^1 E_{Q_{\gamma X}}[-Y] d\gamma$.

The following two examples show that it may happen or not that there exist some (Q, w) realizing the intersection in the dual representation of R.

Example 12 *Let $d = 2$, $\Omega = \{\omega_1, \omega_2\}$. Set $M = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Take the set $\mathcal{Q} = \{Q_1 = (Q^1_1, Q^2_1), Q_2 = (Q^1_2, Q^2_2)\}$ where $Q^1_1(\omega_1) = Q^1_1(\omega_2) = 1/2, Q^2_1(\omega_1) = 1/4, Q^2_1(\omega_2) = 3/4$, and $Q^1_2(\omega_1) = 1/3, Q^1_2(\omega_2) = 2/3, Q^2_2(\omega_1) = 3/4, Q^2_2(\omega_2) = 1/4$. Consider now*

$$\mathcal{W}_R = \mathcal{Q} \times \{\widetilde{w}\}$$

with $\widetilde{w} = (1,1)$. *Hence* $G(\widetilde{w}) = \{(v_1, v_2) : v_1 + v_2 \geq 0\}$. *Let* $X = (X_1, X_2)$, *with*

$$X_1 = \begin{cases} x_1^1; & \omega_1, \\ x_1^2; & \omega_2. \end{cases}, \quad X_2 = \begin{cases} x_2^1; & \omega_1, \\ x_2^2; & \omega_2. \end{cases}$$

Hence

$$E_{Q_1}[-X] = (E_{Q_1^1}[-X_1], E_{Q_1^2}[-X_2]) = \left(\frac{x_1^1 + x_1^2}{2}, \frac{x_2^1 + 3x_2^2}{4}\right)$$

and

$$E_{Q_2}[-X] = (E_{Q_2^1}[-X_1], E_{Q_2^2}[-X_2]) = \left(\frac{x_1^1 + 2x_1^2}{3}, \frac{3x_2^1 + x_2^2}{4}\right).$$

Therefore, by (3) *and* $\mathcal{W}_R = \mathcal{Q} \times \{\widetilde{w}\}$,

$$R(X) = \left\{\left(\frac{x_1^1 + x_1^2}{2}, \frac{x_2^1 + 3x_2^2}{4}\right) + G(\widetilde{w})\right\} \cap \left\{\left(\frac{x_1^1 + 2x_1^2}{3}, \frac{3x_2^1 + x_2^2}{4}\right) + G(\widetilde{w})\right\}$$

is a coherent set-valued risk measure. So, depending on the various configurations of the realizations of X_1 *and* X_2, $R(X)$ *is equal to either* $E_{Q_1}[-X] \cap G(\widetilde{w})$ *or to* $E_{Q_2}[-X] \cap G(\widetilde{w})$. *In other words, for any* X *there exists* $(Q_X, w_X) \in \mathcal{W}_R$ *realizing the intersection in the dual representation of* R. *In particular,* $(Q_X, w_X) = (Q_X, \widetilde{w})$ *with* $Q_X \in \{Q_1, Q_2\}$ *depending on* X. *Indeed, the existence of a maximizer as above is not always guaranteed, as the next example shows.*

Example 13 *In the setting of the previous example, let*

$$X_1 = \begin{cases} 0; & \omega_1, \\ 3; & \omega_2. \end{cases}, \quad X_2 = \begin{cases} -2; & \omega_1, \\ 1; & \omega_2. \end{cases}$$

Then, $E_{Q_1}[-X] = (3/2, 1/4)$ *and* $E_{Q_2}[-X] = (2, -5/4)$. *In this case, taken* $\widetilde{w}$ *as before, it is immediate to see that* $R(X) = E_{Q_1}[-X] + G(\widetilde{w})$.
Take now also $w^* = (2,3)$. *An easy computation shows that in this case it does not exist a couple* (Q_X, w_X) *in the set* $\mathcal{Q} \times \{\widetilde{w}, w^*\}$ *such that* $R(X)$ *can be expressed as* $R(X) = E_{Q_X}[X] + G(w_X)$.

5 Conclusions

In this paper we have extended the well known definition of Capital Allocation rule, as well as its properties, to the setting of set-valued risk measures.

This is motivated by the growing interest in set-valued risk measures, supported by financial arguments. We have defined some Capital Allocation rules based on representation theorems for coherent and convex set-valued risk measures and have shown that, under suitable assumptions, they are linked to the subdifferential of set-valued functions, in analogy to what happens in the scalar case. Also, again inspired by the single-valued case, we have defined a Capital allocation rule through the set-valued directional derivative. We have finally provided some examples, some of which are related to the scalar case.

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