Optimal Filtering of the Local Polynomial Trend in Seasonal Time Series

Filtro Ottimale del Trend Polinomiale nelle Serie Storiche Stagionali

Paolo Chirico

Facoltà di Scienze Politiche, Università degli Studi di Torino
Via Verdi 25, 10124 Torino, chiricop@katamail.com

Riassunto: Il presente lavoro propone una metodologia per determinare medie mobili centrate in grado di filtrare correttamente, da dati storici stagionali, il ciclo-trend, quando questo sia localmente approssimabile con una funzione polinomiale del tempo.

Keywords: MA (Moving Average, Filter), Local Polynomial Trend.

1. Introduction

The aim of this paper is to provide a method to filter the polynomial trend in seasonal time series by moving average (MA). If we suppose the classical additive decomposition:

$$y_t = f_t + s_t + a_t$$

(1)

a MA "m" ([1]) has to satisfy the following conditions:

$$\text{m} f_t = f_t \tag{2}$$
$$\text{m} s_t = 0 \tag{3}$$
$$\text{Var}(\text{m} a_t) \leq \text{Var}(a_t) \tag{4}$$

It is known that a simple MA satisfy the condition 2) (cancelling seasonal fluctuation) and 3) (smoothing noise), but no 1) (conserving trend) if f_t is a polynomial trend of order p≥2. Nevertheless there are weighted MA, that conserve local polynomial trend and smooth efficiently the noise (classical optimal MA). They smooth seasonal fluctuation too, but don't cancel it. Neither a double filtering, resolve the problem.
Neither Spencer nor Henderson's MAs prove an unbiased filter for seasonal time series. They have powerful smoothing features, but they don't exactly cancel seasonal component. There are methods like X-11 that prove acceptable estimates, but by an iterative method.
In the following page it is shown a method to get efficient and unbiased filters of local polynomial trend in seasonal time series.

2. Optimal filters

---

[1]() In this case we consider a moving average like a operator.

A centred MA m of order $d = 2k+1$ ([2]) can be represented like a $d \times 1$ vector :

$$\mathbf{m'} = \left[\theta_{-k}, .., \theta_0, .., \theta_k\right] \tag{7}$$

Hence:

$$m\, y_t = \mathbf{m'}\, \mathbf{y_t} \tag{8}$$

and

$$\mathrm{Var}(\, m\, a_t\,) = \sigma_a^2\, \mathbf{m'm} \tag{9}$$

where $\mathbf{y_t} = \left[y_{t-k}, .., y_t, .., y_{t+k}\right]$ and $\mathrm{Var}(a_t) = \sigma_a^2 \;\; \forall\; t.$

It' s clear that m is an efficient filter if its degree of smoothness $\mathbf{m'm} < 1$

Now, let be f_t a pth order polynomial trend in time interval $t \pm k$, where $k \geq p/2$.
A classic optimal MA is proved by solution $\mathbf{m^*}$ of the following problem:

$$\begin{cases} min\; \mathbf{m'}\, \mathbf{m} \\ \mathbf{T'}\, \mathbf{m} = \mathbf{e_1} \end{cases} \tag{10}$$

where $\mathbf{e_1}$ is an elementary vector of $p+1$ elements (the first is 1 and the others 0); $\mathbf{T}$ is a $d \times (p+1)$ matrix:

$$\mathbf{T} = \begin{bmatrix} 1 & k & k^2 & ... & k^p \\ 1 & k-1 & ... & ... & (k-1)^p \\ \vdots & & & & \\ 1 & 0 & ... & ... & 0 \\ \vdots & & & & \\ 1 & -k & (-k)^2 & ... & (-k)^p \end{bmatrix}$$

(11)

In this case, $\mathbf{m^*}$ ([3]) is the most efficient among polynomial trend conserving MAs, but don't cancel seasonal fluctuation so it is not an unbiased filter.

2.1 A Seasonal Unbiased Moving Average (SUMA)

Consider a seasonal time series and let $\mathbf{m_s}$ a simple MA of order s, that cancel seasonal fluctuation. For example:

$$\mathbf{m_5'} = \left[0.125,\; 0.25,\; 0.25,\; 0.25,\; 0.125\right] \tag{12}$$

cancels seasonal fluctuation in quarterly data.
Let $\mathbf{D_s}$ the following matrix:

---

[2]() k is a positive integer; d is obvious odd.

[3]() Now we consider a moving average indifferently like a operator or a vector.

$$\mathbf{D_s} = \begin{bmatrix} \mathbf{m_s} & 0 & .. & 0 \\ 0 & \mathbf{m_s} & & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & .. & \mathbf{m_s} \end{bmatrix} \qquad (13)$$

$\mathbf{D_s}$ has order $d \times (d-s+1)$.

Now we can change the optimisation problem (10) into:

$$\begin{cases} min \ \mathbf{g_s' D_s' D_s g_s} \\ \mathbf{T' D_s g_s = e_1} \end{cases} \qquad (14)$$

with $\mathbf{g_s}$ vector of $d-s+1$ symmetric coefficient. The solution $\mathbf{g_s^*}$ proves an unbiased filter, which is the most efficient of the seasonal unbiased moving average, SUMA:

$$\mathbf{m_s^* = D_s g_s^*} \qquad (15)$$

Of course the efficiency of $\mathbf{m_s^*}$ is measured by $\mathbf{m_s^{*'} m_s^*}$ and is obvious that major is d greater efficiency. We remember $d=2k+1$ and $k \geq p/2$. Moreover, if $\mathbf{g_s^*}$ is not banal, $d \geq s+2$, also:

$$\begin{cases} k \geq max[p/2, (s+1)/2] \\ d = 2k+1 \end{cases} \qquad (16)$$

3. Validation and Conclusion

In order to validation the suggested filter, we have considered a SUMA of order 9 for quarterly data:

$$\mathbf{m_s^*} = [-0.057, 0.020, 0.125, 0.230, 0.364, 0.230, 0.125, 0.020, -0.057] \qquad (17)$$

and the Handerson's MA, HMA, of order 9.

We have tested the two MA on 100 different quarterly time series, simulated by the pattern (1), where:

$$(^4) \qquad i = 1, 2, ...100 \qquad (18)$$

and s_t is a quarterly fluctuation with standard deviation σ_s; a_t is a random number with normal distribution and standard deviation σ_a.

We show the estimated Squared Error, S.E., of f_t for the two MA in the in following table:

---

[4] () A_0, A_{ij}, B_{ij}, α_{ij}, β_{ij} are parameters, different for every series.

Table 1: *The squared error in the SUMA and HMA ([5])*

SUMA	$m^{*\prime}m^*$ =	27.7%	HMA	$m^{*\prime}m^*$ =	28.3%
S. E.	$\sigma_s = 4.5\%$	$\sigma_s = 18\%$	**S. E.**	$\sigma_s = 4.5\%$	$\sigma_s = 18\%$
$\sigma_a = 1\%$	1.3%	1.3%	$\sigma_a = 1\%$	1.3%	1.4%
$\sigma_a = 10\%$	13.1%	13.1%	$\sigma_a = 10\%$	13.2%	13.3%

We can note that the SUMA is, obviously, independent by the width of seasonal fluctuation and its degree of smoothness is a little smaller than HMA. Because to its structural features, HMA is more efficient only if the width of seasonal and random fluctuations are little respect to error of approximation to polynomial trend. Of course the filtering of trend is only a formal abstraction based on some hypotheses: it doesn't exist a true trend of a time series and is questionable to assume constant effects of seasonal fluctuation. Nevertheless structural decomposition can be useful for many analyses and the proved method, we think, assure some good features.

References

Dagum E. B. (1978) Modelling, Forecasting and Seasonal Adjustment Economic Time Series with Arima Method, *Cambridge Conference on Time Series Analysis.*
Di Fonzo T., Lisi F. (2001) *Complementi di Statistica Economica*: *Analisi delle Serie Storiche Univariate,*Cleup Editrice, Padova.
Faliva M. (1978) Optimal filtering for seasonal Adjustment of Quarterly Data, *Rivista Internazionale di scienze sociali*, LXXXVI, 1.
Passamani G. (1986) *Analisi Classica delle componenti di Trend e Periodiche di una Serie Storica,* Mimeo, Padova.
Zani S. (1982) *Indicatori statistici della congiuntura*, Loescher, Torino
Zavanella B. (2001) *Analisi Classica delle Serie Storiche*, CUESP, Milano.

---

[5] The squared error and standard deviation are shown in percentage respect the medium level of series.