Approximate pricing of swaptions in affine and quadratic models

Anna Maria Gambaro*, Ruggero Caldana†, and Gianluca Fusai‡

Abstract

This paper proposes new bounds on the prices of European-style swaptions for affine and quadratic interest rate models. These bounds are computable whenever the joint characteristic function of the state variables is known. In particular, our lower bound involves the computation of a one dimensional Fourier transform independently of the swap length. In addition, we control the error of our method by providing a new upper bound on swaption price applicable to all considered models. We test our bounds on different affine models and on a quadratic Gaussian model. We also apply our procedure to the multiple curve framework. The bounds are found to be accurate and computationally efficient.

JEL classification codes: G12, G13.

KEY WORDS: Pricing, swaptions, affine-quadratic models, Fourier transform, bounds.

1 Introduction

The accurate pricing of swaption contracts is fundamental in interest rate markets; swaptions are among the most liquid over the counter (OTC) derivatives and are largely used for hedging purpose. Many applications require also efficient computation of swaption prices such as calibration, estimation of risk metrics and Credit and Debit Value Adjustment (CVA and DVA) valuation. In the calibration of interest rate models, a large number of swaptions with different maturities, swap lengths and strikes are priced during iterative procedures aimed at fitting market quotations. Similarly in the estimation of risk metrics for a portfolio of swaptions, if a full revaluation setting is used and millions of possible scenarios are considered, a fast pricing

*Dipartimento di Statistica e Metodi Quantitativi, Università Milano Bicocca (DiSMeQ), Via Bicocca degli Arcimboldi 8, 20126 Milano, Italy, Email: a.gambaro@campus.unimib.it (*Corresponding Author*).

†Accenture S.p.A., Via Maurizio Quadrio 17, 20154 Milano, Italy, Email: ruggero.caldana@gmail.com.

‡Faculty of Finance, Cass Business School, City University, 106 Bunhill Row EC1Y 8TZ London, UK and Dipartimento di Studi per l'Economia e l'Impresa, Università del Piemonte Orientale "A. Avogadro", via E. Perrone 18 28100 Novara, Italy, Email: Gianluca.Fusai.1@city.ac.uk.

algorithm is essential to obtain results in a reasonable time. In addition Basel III accords introduced the CVA and DVA charge for OTC contracts; for the simplest and most popular kind of interest rate derivative, i.e. interest rate swap, the two adjustments can be estimated by pricing a portfolio of forward start European swaptions (see Brigo and Masetti (2005)). Hence, the appeal of a fast exact closed form solution for the swaption pricing problem is explained.

The famous Jamshidian (1989) formula is applicable only when the short rate depends on a single stochastic factor; for multi-factor interest rate models, several approximate methods have been developed in the literature. Munk (1999) approximates the price of an option on a coupon bond by a multiple of the price of an option on a zero-coupon bond with time to maturity equal to the stochastic duration of the coupon bond. The method of Schrager and Pelsser (2006) is based on approximating the affine dynamics of the swap rate under the relevant swap measure. These methods are fast but not very accurate for out of the money options. The method of Collin-Dufresne and Goldstein (2002) is based on a Edgeworth expansion of the density of the swap rate and requires a very time consuming calculation of the moments of the coupon bond. An estimation of the error of the Collin-Dufresne and Goldstein (2002) has been provided in Zheng (2013). In addition, it provides reliable estimation only for low volatility level. Singleton and Umantsev (2002) (henceforth S&U) introduce the idea of approximating the exercise region in the space of the state variables. This method has the advantage of producing accurate results for a wide range of strikes, in particular for out of the money swaptions; however, it does not admit a simple extension to general affine interest rate models, because it requires the knowledge in closed form of the joint probability density function of the state variables. Kim (2012) generalizes and simplifies the S&U method. Up to now the Kim's method seems to be the most efficient proposed in the literature. Nevertheless Kim's method requires the calculation of as many Fourier transforms as the number of cash flows in the underlying swap, which implies that the run-time of the algorithm increases with the swap length. Moreover, all these papers do not discuss the direction of the error, i.e. whether the price is overestimated or underestimated. Further, except for Collin-Dufresne and Goldstein (2002), none of the methods proposed in literature is able to estimate or control the approximation error. Only recently a lower and an upper bounds on swaption prices have been proposed in Nunes and Prazeres (2014), but it is applicable only to Gaussian models.

Similarly to S&U and Kim, we propose a lower bound which is based on an approximation of the exercise region via an event set defined through a function of the model factors. Then, our pricing formula consists in the valuation of option on the approximate exercise region and requires a single Fourier transform. Our procedure gives a new perspective with respect to existing method, such as S&U and Kim. Indeed, we prove that their approximations are also lower bounds to the swaption price. To the best of our knowledge, this has been unnoticed up to now. Moreover, we develop a method to control the approximation error by deriving a new upper bound on swaption prices.

Finally, we extend the lower and upper bounds to multiple curve models which reflect the presence of various interest curves in the market after the 2007 crisis. Multiple curve interest rate models are widely discussed in literature (see, among others Ametrano e Bianchetti (2009), Henrard (2009), Morini (2009) and recently Moreni and Pallavicini (2014) and Fanelli (2016)). In particular we concentrate on affine multiple-curve model developed in Moreni and Pallavicini (2014). To the best of our knowledge none of the approximated methods previously described for pricing swaption are developed for a multiple curve interest rate framework.

The paper is organized as follows. Section 2 introduces a general formula for the lower bound on swaption prices based on an approximation of the exercise region. In addition, the popular methods of S&U and Kim are proved to be included in our setting. Then we apply the general lower bound formula to the case of affine models and Gaussian quadratic interest rate models and we find an efficient algorithm to calculate analytically the approximated swaption price. In section 3 the new upper bound is presented for affine-quadratic models. Sections 4 extends the bounds previously described to a multiple curve model. Section 5 shows the results of numerical tests. Conclusive remarks are presented in last section.

2 A Lower Bound on swaption prices

In this section, we discuss the general pricing formula for a receiver European-style swaption and the approximations presented in S&U and Kim. In particular, we prove that these approximations are lower bounds.

A European swaption is a contract that gives the right to its owner to enter into an underlying interest rate swap, i.e. it is an European option on a swap rate. It can be equivalently interpreted as an option on a portfolio of zero coupon bonds (or a coupon bond). Let t be the current date, T the option expiration date, $T_1, ..., T_n$ the underlying swap payment dates (by construction $t < T < T_1 < ... < T_n$) and R the fixed rate of the swap. The payoff of a receiver swaption is

$$\left(\sum_{h=1}^{n} w_h P(T, T_h) - 1\right)^+,$$

where $w_h = R \cdot (T_h - T_{h-1})$ for $h = 1, .., n-1$, $w_n = R \cdot (T_n - T_{n-1}) + 1$, and $P(T, T_h)$ is the price at time T of a zero coupon bond expiring at time T_h. The time t no-arbitrage price is the risk neutral expected value of the discounted payoff,

$$C(t) = \mathbb{E}_t \left[e^{-\int_t^T r(\mathbf{X}(s))ds} \left(\sum_{h=1}^{n} w_h P(T, T_h) - 1 \right)^+ \right] \tag{1}$$

where $r(\mathbf{X}(s))$ is the short rate at time s, and $\mathbf{X}(s)$ denotes the state vector at time s of a multi-factor stochastic model. The price formula (1) after a change of measure to the T-forward measure becomes

$$C(t) \quad = \quad P(t, T) \cdot \mathbb{E}_t^T \left[\left(\sum_{h=1}^{n} w_h P(T, T_h) - 1 \right) I(\mathcal{A}) \right] \tag{2}$$

whit I denoting the indicator function, $\mathcal{A}$ is the exercise region seen as a subset of the space events Ω

$$\mathcal{A} = \{\omega \in \Omega : \sum_{h=1}^{n} w_h P(T, T_h) \geq 1\}.$$

By changing the measure of each expected value from the T-measure to the T_h one, the pricing formula in expression (2) can be written as

$$C(t) = \sum_{h=1}^{n} w_h P(t, T_h) \, \mathbb{P}_t^{T_h}[\mathcal{A}] - P(t, T) \, \mathbb{P}_t^T[\mathcal{A}]$$

where $\mathbb{P}_t^S[\mathcal{A}]$ denotes the time t probability of the exercise set $\mathcal{A}$ under the S-forward measure. S&U and Kim replace the exercise set $\mathcal{A}$ in the above formula by a new one, that makes the computation of the swaption price much simpler, then their approximated pricing formula reads as

$$C_{\mathcal{G}}(t) = \sum_{h=1}^{n} w_h P(t, T_h) \, \mathbb{P}_t^{T_h}[\mathcal{G}] - P(t, T) \, \mathbb{P}_t^T[\mathcal{G}], \tag{3}$$

for $\mathcal{G}$ suitably chosen (see Singleton and Umantsev (2002) and Kim (2012) for further details). The choice of the approximated exercise region is made so that the above probabilities can be computed by performing $n + 1$ Fourier inversion, where n is the number of payments in the underlying swap. We can now show that $C_{\mathcal{G}}(t)$ is a lower bound approximation to the true price. Indeed, we observe that for any event set $\mathcal{G} \subset \Omega$:

$$\mathbb{E}_t^T \left[\left(\sum_{h=1}^{n} w_h P(T, T_h) - 1 \right)^+ \right] \geq \mathbb{E}_t^T \left[\left(\sum_{h=1}^{n} w_h P(T, T_h) - 1 \right)^+ I(\mathcal{G}) \right]$$
$$\geq \mathbb{E}_t^T \left[\left(\sum_{h=1}^{n} w_h P(T, T_h) - 1 \right) I(\mathcal{G}) \right]$$

Then by discounting we obtain:

$$C(t) \geq LB_{\mathcal{G}}(t) := P(t,T) \cdot \mathbb{E}_t^T \left[\left(\sum_{h=1}^{n} w_h P(T,T_h) - 1 \right) I(\mathcal{G}) \right], \tag{4}$$

i.e. $LB_{\mathcal{G}}(t)$ is a lower bound to the swaption price for all possible sets $\mathcal{G}$. Using the same change of measures as in S&U and Kim, it immediately follows that

$$LB_{\mathcal{G}}(t) = C_{\mathcal{G}}(t).$$

Therefore the approximated pricing formula presented in S&U and Kim are indeed lower bounds. This was previously unnoticed and therefore asks for a better discussion of the accuracy of the method. In particular, our new framework allows to control the approximation error by providing an upper bound. In addition, we show how to speed up the computation of the formula (4) by performing a single Fourier transform. This allows a reduction of the computational cost, mainly when we have to price swaptions written on long-maturity swaps.

2.1 Affine and Gaussian quadratic models

In affine and quadratic interest rate models the price at T of a zero coupon bond with expiration T_h can be written as the exponential of a quadratic form of the state variables

$$P(T,T_h) = e^{\mathbf{X}^\top(T) C_h \mathbf{X}(T) + \mathbf{b}_h^\top \mathbf{X}(T) + a_h} \tag{5}$$

for $a_h = A(T - T_h)$, $\mathbf{b}_h = \mathbf{B}(T - T_h)$ and $C_h = C(T - T_h)$ functions of the payment date T_h which are model specific. Fixing a date T_h, C_h is a $d \times d$ symmetric matrix.

From the literature(Ahn, Dittmar and Gallant (2002), Leippold and Wu (2012) and Kim (2012)) we know that, if the risk neutral dynamics of the state variates are Gaussian, then the functions $A(\tau)$, $\mathbf{B}(\tau)$ and $C(\tau)$ are the solution of a system of ordinary differential equations with initial condition $A(0) = 0$, $\mathbf{B}(0) = \underline{0}$, $C(0) = 0_{d\times d}$. Affine models can be obtained forcing C_h to be a null matrix. For affine models, under certain regularity conditions the functions $A(\tau)$ and $\mathbf{B}(\tau)$ are the solution of a system of $d+1$ ordinary differential equations that are completely determined by the specification of the risk-neutral dynamics of the short rate (see Duffie and Kan (1996) and Duffie, Pan and Singleton (2000) for further details). The solutions of these equations are known in closed form for most common affine models.

From Duffie, Pan and Singleton (2000) and Kim (2012), we know that the quadratic

T-forward joint characteristic function of the model factors $\mathbf{X}$ has the form

$$\begin{aligned}\Phi(\boldsymbol{\lambda},\Lambda) &= \mathbb{E}_t^T\left[e^{\boldsymbol{\lambda}^\top \mathbf{X}(T)+\mathbf{X}(T)^\top\Lambda\mathbf{X}(T)}\right] \\ &= e^{\tilde{A}(T-t,\boldsymbol{\lambda},\Lambda)-A(T-t)+(\tilde{\mathbf{B}}(T-t,\boldsymbol{\lambda},\Lambda)-\mathbf{B}(T-t))^\top\mathbf{X}(t)+\mathbf{X}(t)^\top(\tilde{C}(T-t,\boldsymbol{\lambda},\Lambda)-C(T-t))\mathbf{X}(t)}\end{aligned} \tag{6}$$

where $\boldsymbol{\lambda}\in\mathbb{C}^d$ and Λ is a complex $d\times d$ symmetric matrix. If $\mathbf{X}(t)$ is a Gaussian quadratic process (or an affine process, i.e Λ, $\tilde{C}$ and C are null matrices), the functions $\tilde{A}(\tau,\boldsymbol{\lambda},\Lambda)$, $\tilde{\mathbf{B}}(\tau,\boldsymbol{\lambda},\Lambda)$ and $\tilde{C}(\tau,\boldsymbol{\lambda},\Lambda)$ are the solutions of the same ODE system of the zero coupon bond functions, but with initial conditions $\tilde{A}(0,\boldsymbol{\lambda},\Lambda)=0$, $\tilde{\mathbf{B}}(0,\boldsymbol{\lambda},\Lambda)=\boldsymbol{\lambda}$, and $\tilde{C}(0,\boldsymbol{\lambda},\Lambda)=\Lambda$.

In case of a quadratic model, it is convenient to define the approximate exercise region $\mathcal{G}$ using a quadratic form of the state vector

$$\mathcal{G}=\{\omega\in\Omega:\mathbf{X}(T)^\top\Gamma\,\mathbf{X}(T)+\boldsymbol{\beta}^\top\mathbf{X}(T)\geq k\},$$

where Γ is a constant $d\times d$ symmetric matrix, $\boldsymbol{\beta}\in\mathbb{R}^d$ and $k\in\mathbb{R}$.

Proposition 2.1. The lower bound to the European swaption price for quadratic interest rate models is given by the following formula

$$LB(t)=\max_{k\in\mathbb{R},\,\boldsymbol{\beta}\in\mathbb{R}^d,\,\Gamma\in Sym_d(\mathbb{R})} LB_{\boldsymbol{\beta},\Gamma}(k;t), \tag{7}$$

where

$$LB_{\beta,\Gamma}(k;t) = P(t,T)\,\frac{e^{-\delta k}}{\pi}\int_0^{+\infty} Re\left(e^{-i\gamma k}\psi(\delta+i\gamma)\right)d\gamma, \tag{8}$$

and

$$\psi(z)=\left(\sum_{h=1}^n w_h e^{a_h}\,\Phi\left(\mathbf{b}_h+z\boldsymbol{\beta},C_h+z\Gamma\right)-\Phi\left(z\boldsymbol{\beta},z\Gamma\right)\right)\frac{1}{z}, \tag{9}$$

whit $\psi(z)$ defined for $Re(z)>0$. Integrals in formula (8) must be interpreted as Cauchy principal value integral and δ is a positive constant.

Proof: See Appendix D.

For a 2-factor affine interest rate models, Singleton and Umantsev (2002) propose to approximate the exercise boundary of an option on a coupon bond with a straight line that matches closely the exercise boundary where the conditional density of the model factors is concentrated. Kim (2012) improves on the S&U idea and considers three different types of approximation for the exercise region. We choose its approximation "A" because it appears to be the most accurate.[1] It is obtained by a first order Taylor expansion of the coupon bond price, defined as

$$B(\mathbf{X}(T)) = \sum_{h=1}^{n} w_h P(T, T_h), \tag{10}$$

around the point on the true exercise boundary where the density function of the model factors is largest. Moreover Kim (2012) extends his approximation "A" to Gaussian quadratic interest rate models using a second order Taylor expansion of coupon bond. In this way, the optimization of the lower bound (formula (7)) which can be very expensive is not performed. It is instead replaced by a preliminary search of the parameters Γ, $\boldsymbol{\beta}$ and k, which are chosen via the Taylor expansion of the coupon bond.

In particular for affine models, the first order Taylor expansion of the coupon bond is a tangent hyperplane approximation. In fact the approximated exercise boundary is defined as

$$\boldsymbol{\beta}^\top \mathbf{X}(T) + \alpha = 0,$$

with

$$\alpha = -\nabla B(\mathbf{X}^*)^\top \mathbf{X}^*, \quad \boldsymbol{\beta} = \nabla B(\mathbf{X}^*) \quad \text{and} \quad k = -\alpha. \tag{11}$$

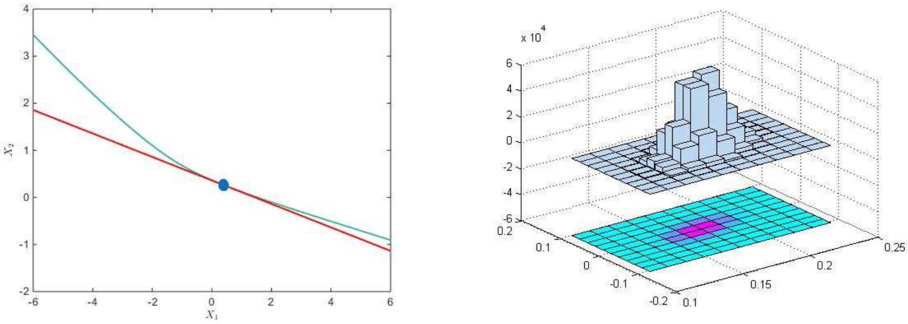
Hence, it is a tangent hyperplane to the true exercise boundary in the point, $\mathbf{X}(T) = \mathbf{X}^*$, where the density function of the model factors is the largest. More details about how to find the point $\mathbf{X}^*$ are in Kim (2012). A two dimensions visualization of the approximate exercise region is in Figure 1.

Once Γ, $\boldsymbol{\beta}$ and k are found, Kim approximation requires the computation of $n+1$ forward probability $\mathbb{P}_t^{T_h}[\mathcal{G}]$, as in formula (3). This is done by performing $n+1$ one-dimensional Fourier inversion. Instead our lower bound is calculated as in formula (8), i.e. performing a single one-dimensional Fourier transform with respect to the parameter k.

3 Upper Bound on Swaption price

In this section we define a new upper bound to swaption prices applicable to all affine and quadratic interest rate models. First of all, it is straightforward to see that for a lower bound

[1] The three approximations presented in Kim (2012) are lower bounds, as proved in section 2. Therefore the most precise is the one that produces the highest price. This was unnoticed in the Kim paper.



(a) Exercise regions (b) Histogram of the density

Figure 1: The first figure shows the true and the approximate exercise boundary for a 2×10 years swaption, with the two factor CIR model. Light blue and red lines are respectively the boundary of the true region and of the approximate set. The region represented by a blue circle is where the joint probability density function of the two factors is highest at the maturity of the option. The second figure is the histogram of the joint probability density function of the two factors at maturity.

defined by a generic approximated exercise set $\mathcal{G}$, the (un-discounted) approximation error is

$$
\begin{aligned}
& \frac{1}{P(t,T)}\left(C(t) - \widehat{LB}(t)\right) \\
= \;& \mathbb{E}_t^T[(B(\mathbf{X}(T)) - 1)^+] - \mathbb{E}_t^T[(B(\mathbf{X}(T)) - 1)I(\mathcal{G})] \\
= \;& \mathbb{E}_t^T[(B(\mathbf{X}(T)) - 1)^+ I(\mathcal{G}^c)] + \mathbb{E}_t^T[(1 - B(\mathbf{X}(T)))^+ I(\mathcal{G})] \\
= \;& \Delta_1 + \Delta_2
\end{aligned}
$$

where $B(\mathbf{X}(T))$ is the coupon bond price defined as in formula (10). In general, Δ_1 and Δ_2 are not explicitly computable. Thus, we can provide upper bounds ϵ_1 and ϵ_2 to them and therefore an upper bound to the swaption price

$$
UB(t) = \widehat{LB}(t) + P(t,T)\,(\epsilon_1 + \epsilon_2), \tag{12}
$$

for $\epsilon_1 \geq \Delta_1$ and $\epsilon_2 \geq \Delta_2$.

For every set of strikes $(K_1, ..., K_n)$ such that $\sum_{h=1}^n K_h = 1$, upper bounds to the errors are

$$
\Delta_1 \leq \epsilon_1 = \sum_{h=1}^{n} \mathbb{E}_t^T[(w_h P(T,T_h) - K_h)^+ \, I(\mathcal{G}^c)], \tag{13}
$$

$$
\Delta_2 \leq \epsilon_2 = \sum_{h=1}^{n} \mathbb{E}_t^T[(K_h - w_h P(T,T_h))^+ \, I(\mathcal{G})], \tag{14}
$$

where $P(T,T_h)$ is the price at time T of the zero coupon bond with maturity T_h. However, without a proper choice of the strikes $(K_1, ..., K_n)$, the approximations can be very rough.

Then we want to find the values of $(K_1, ..., K_n)$ that reduce the error, without performing a time consuming multidimensional numerical minimization. Given that

$$
\begin{aligned}
(B(\mathbf{X}(T)) - 1)^+ &= B(\mathbf{X}(T)) \left(1 - \frac{1}{B(\mathbf{X}(T))}\right)^+ \\
&= \sum_{h=1}^{n} w_h P(T, T_h) \left(1 - \frac{1}{B(\mathbf{X}(T))}\right)^+ \\
&= \sum_{h=1}^{n} \left(w_h P(T, T_h) - \frac{w_h P(T, T_h)}{B(\mathbf{X}(T))}\right)^+
\end{aligned} \tag{15}
$$

as $B(\mathbf{X}(T)) > 0$ and $w_h P(T, T_h) > 0 \ \forall \mathbf{X}(T)$, we note that the following equality holds

$$
\mathbb{E}_t^T[(B(\mathbf{X}(T)) - 1)^+ \, I(\mathcal{G}^c)] = \sum_{h=1}^{n} \mathbb{E}_t^T[(w_h P(T, T_h) - K_h(\mathbf{X}(T)))^+ \ I(\mathcal{G}^c)],
$$

for

$$
K_h(\mathbf{X}(T)) = \frac{w_h P(T, T_h)}{B(\mathbf{X}(T))}.
$$

By similar reasoning we also have

$$
\mathbb{E}_t^T[(1 - B(\mathbf{X}(T)))^+ \, I(\mathcal{G})] = \sum_{h=1}^{n} \mathbb{E}_t^T[(K_h(\mathbf{X}(T)) - w_h P(T, T_h))^+ \ I(\mathcal{G})].
$$

Hence, if in formula (13) and (14) we choose the strikes $(K_1, ..., K_n)$ in the following way

$$
K_h = K_h(\mathbf{X}^*) = w_h \ P(T, T_h)|_{\mathbf{X}(T) = \mathbf{X}^*}, \tag{16}
$$

then the equalities $\epsilon_1 = \Delta_1$ and $\epsilon_2 = \Delta_2$ hold in $\mathbf{X}(T) = \mathbf{X}^*$, the point on the true exercise boundary where the density function of the model factors is largest. More details about how to find the point $\mathbf{X}^*$ are in Kim (2012).

This allows us to avoid a multidimensional optimization with respect to $(K_1, ...K_n)$.

3.1 Affine and Gaussian quadratic models

Proposition 3.1. The upper bound to the European swaption price for quadratic interest rate models is given by the following formula

$$
UB(t) = \widehat{LB}(t) + P(t, T) \left(\epsilon_1(-\alpha) + \epsilon_2(-\alpha)\right) \tag{17}
$$

where

$$\epsilon_1(k) = \frac{1}{2\pi^2}\int_0^{+\infty} d\gamma\, Re\left(\int_{-\infty}^{+\infty} d\omega \sum_{h=1}^{n} w_h e^{a_h}\, e^{-(\delta+i\gamma)k} e^{-(\eta+i\omega)k_h}\, \psi_h(\delta+i\gamma, \eta+i\omega)\right),$$

$$\epsilon_2(k) = -\frac{1}{2\,\pi^2}\int_0^{+\infty} d\gamma\, Re\left(\int_{-\infty}^{+\infty} d\omega \sum_{h=1}^{n} w_h e^{a_h}\, e^{(\delta-i\gamma)k} e^{(\eta-i\omega)k_h}\, \psi_h(-\delta+i\gamma, -\eta+i\omega)\right),$$

and

$$\psi_h(z,y) = -\frac{\Phi\left(z\boldsymbol{\beta} + (y+1)\mathbf{b}_h, z\Gamma + (y+1)C_h\right)}{zy(y+1)}, \tag{18}$$

where $\widehat{LB}(t)$ is given in Proposition 2.1, $k_h = log(K_h) - log(w_h) - a_h$ and K_h are defined in equation (16) and $\Phi(\boldsymbol{\lambda}, \Lambda)$ is defined in equation (6). If $Re(z) < 0$ and $Re(y) > 0$, $\psi_h(z,y)$ is the double Fourier transform of

$$\mathbb{E}_t^T[(e^{\mathbf{b}_h{}^\top\mathbf{X}+\mathbf{X}^\top C_h\mathbf{X}} - e^{k_h})^+\, I(\mathbf{X}^\top\Gamma\mathbf{X} + \boldsymbol{\beta}^\top\mathbf{X} < k)],$$

instead, if $Re(z) > 0$ and $Re(y) < -1$, $\psi_h(z,y)$ is the transform of

$$\mathbb{E}_t^T[(e^{k_h} - e^{\mathbf{b}_h{}^\top\mathbf{X}+\mathbf{X}^\top C_h\mathbf{X}})^+\, I(\mathbf{X}^\top\Gamma\mathbf{X} + \boldsymbol{\beta}^\top\mathbf{X} > k)],$$

with $\delta > 0$, $\eta > 1$ constants.

Proof: See Appendix (E).

We note some important mathematical features of the swaption pricing problem in the affine interest rate model case (C_h and Γ are null matrices), which simplify the upper bound formula. The coupon bond $B(\mathbf{X}(T))$ seen as a function of the model factors $\mathbf{X}(T)$, is convex, as it is a positive linear combination of convex functions, the ZCBs. In fact the zero coupon price seen as a function of state vector ($P(T,T_h) = e^{\mathbf{b}_h^\top\mathbf{X}(T)+a_h}$) is a convex function because it is the composition of a convex monotone functions, the exponential, and a linear function of $\mathbf{X}$. Thus, the convexity of the sub-level $\{B(\mathbf{X}(T)) \leq 1\}$ ensues from the previous argument.

Choosing the tangent hyperplane approximation as lower bound and resorting to the hyperplane separation theorem, it follows immediately that the approximate exercise region is included in the true one, as graphically illustrated in Figure 2 for a two factor case,

$$\mathcal{G} = \{\boldsymbol{\beta}^\top\mathbf{X} + \alpha \geq 0\} \subseteq \{B(\mathbf{X}(T)) \geq 1\},$$

provided that α and $\boldsymbol{\beta}$ are defined as in formula (11).

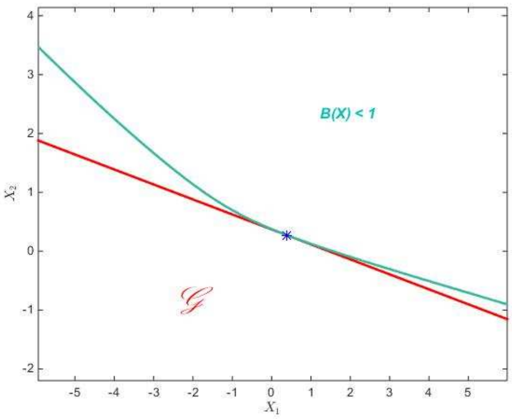


Figure 2: Light Blue line represents the true exercise boundary for a 2 $\times$ 10 years swaption with 2-factors CIR model. The blue star indicates the point $\mathbf{X}^*$. The approximate exercise region $\mathcal{G}$ is the half-space below the red line. Since the sub-level $\{B(\mathbf{X}(T)) \leq 1\}$ is convex, then $\mathcal{G} \cap \{B(\mathbf{X}(T)) \leq 1\} = \emptyset$ by the hyperplane separation theorem.

Hence, the separation theorem guarantees that Δ_2 is zero; this fact allows us to compute only the term ϵ_1 in Proposition 3.1.

It is possible to show that for one-factor affine interest rate models the upper bound coincides with the Jamshidian (1989) formula.

4 Bounds for affine Gaussian specification

For the affine Gaussian model, the lower bound can be calculated analytically

$$LB_{\boldsymbol{\beta}}(k;t) = P(t,T) \left(\sum_{h=1}^{n} w_h \, e^{a_h + \mathbf{b}_h^\top \boldsymbol{\mu} + \frac{1}{2} V_h + \frac{1}{2} d_h^2} \, N(d_h - d) - N(-d) \right)$$

and the upper bound formula can be simplified

$$\epsilon_1(k) = \int_{-\infty}^{d} dz \, \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sum_{h=1}^{n} w_h e^{a_h} \left(e^{M_h + \frac{V_h}{2}} \, N\left(\frac{M_h - \log Y_h + V_h}{\sqrt{V_h}} \right) - Y_h N\left(\frac{M_h - \log Y_h}{\sqrt{V_h}} \right) \right),$$

where $d = \frac{k - \boldsymbol{\beta}^\top \boldsymbol{\mu}}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}}$, $d_h = \mathbf{b}_h^\top \mathbf{v}$, $V_h = \mathbf{b}_h^\top (V - \mathbf{v}\mathbf{v}^\top) \mathbf{b}_h$, $\mathbf{v} = \frac{V \boldsymbol{\beta}}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}}$, $M_h = \mathbf{b}_h^\top \boldsymbol{\mu} + z \mathbf{b}_h^\top \mathbf{v}$, $Y_h = \frac{K_h}{w_h e^{a_h}}$ and $\mu = \mathbb{E}_t^T[\mathbf{X}(T)]$ and $V = Var_t(\mathbf{X}(T))$ are the mean and covariance matrix of the variable $\mathbf{X}(T)$ that is multivariate normal under the T-forward measure. $N(x)$ represents the standard Gaussian cumulative distribution function. Proofs of the simplified bounds are in Appendix (F, G).

5 Approximate pricing of swaption in multiple curve framework

In this section we extend the lower and upper bounds previously described to multiple curve models which better reflect the real behaviour of the interest rate market after the 2007 crisis.

The swaption formula in multi-curve framework becomes

$$C(t) = P(t,T)\,\mathbb{E}_t^T\left[\left(\sum_{j=1}^{n} P(T,T_j)\,x\,(F^x(T,T_j,x) - K)\right)^+\right] \tag{19}$$

where x is the tenor, $T_j - T_{j-1} = x\ \forall j = 1,...,n$ and $T_0 = T$. $F^x(t,T,x)$ is the is the fair rate of a FRA contract written on the Libor rate between $T - x$ and T and tenor x (usually x = 1M, 3M, 6M or 12M). $P(t,T)$ is the price at time t of a risk free zero coupon bond with maturity T.

We test the lower and upper bounds with reference to the Gaussian specification of the multiple curve model presented in Moreni and Pallavicini (2014). In this model the FRA rate and the risk free ZCB price have affine forms. The Markovian-affine representation of the FRA rate is

$$\log\left(\frac{1 + x\,F^x(t,T,x)}{1 + x\,F^x(0,T,x)}\right) = G(t,T,x)^\top \cdot \mathbf{X}(t) + a(t,T,x), \tag{20}$$

where $a(t,T,x)$ is a deterministic coefficient, $G(t,T,x)$ is a deterministic d-dimensional vector and $\mathbf{X}(t)$ is a vector Markovian process that is multivariate normal. A similar Markovian representation can be obtained for the ZCB price

$$\log\left(P(t,T)\frac{P(0,t)}{P(0,T)}\right) = -G(t,T)^\top \cdot \mathbf{X}(t) + a(t,T), \tag{21}$$

where $a(t,T)$ is a deterministic coefficient and $G(t,T)$ is a deterministic d-dimensional vector.

More details about the model and expressions of the coefficients of the Markovian representations are in Appendix B.

5.1 Lower bound formula applied to multi-curve weighted Gaussian model

Using the Markovian representation of the FRA rate and of the risk free ZCBs in the swaption pricing formula (19), we obtain

$$C(t) = P(t,T)\,\mathbb{E}_t^T\left[\left(\sum_{j=1}^{n} w_{1j}\,e^{(G_{1j})^\top \mathbf{X}(T) + a_{1j}} - w_{2j}\,e^{(G_{2j})^\top \mathbf{X}(T) + a_{2j}}\right)\mathrm{I}(\mathcal{A})\right],$$

where

$\mathcal{A}$ is the exercise region and is in form

$$\mathcal{A} = \{\omega \in \Omega : \sum_{j=1}^{n} w_{1j}\, e^{(G_{1j})^\top \mathbf{X}(T)+a_{1j}} - w_{2j}\, e^{(G_{2j})^\top \mathbf{X}(T)+a_{2j}} > 0\},$$

$w_{1j} = \frac{P(t,T_j)}{P(t,T)}(1 + x\, F^x(t,T_j,x))$ and $w_{2j} = \frac{P(t,T_j)}{P(t,T)}(1 + xK)$,

$G_{1j} = G(T,T_j,x) - G(T,T_j)$ and $G_{2j} = -G(T,T_j)$,

$a_{1j} = a(T,T_j,x) + a(T,T_j)$ and $a_{2j} = a(T,T_j)$.

If we substitute the set $\mathcal{A}$ with any other event set $\mathcal{G} \in \Omega$ we obtain a lower bound of the true price. In the affine class models, it is convenient to define the set $\mathcal{G}$ using a linear function of the state variates

$$\mathcal{G} = \{\omega \in \Omega : \boldsymbol{\beta}^\top \mathbf{X}(T) \geq k\},$$

with $\boldsymbol{\beta}$ and α defined in formula (11). The lower bound is provided in the following proposition.

Proposition 5.1. The lower bound to the European swaption price, for the multiple-curve weighted Gaussian model, is given by the following formula

$$\widehat{LB}(t) = \max_{k\in\mathbb{R},\, \beta\in\mathbb{R}^d} LB_\beta(k;t). \tag{22}$$

For fixed parameters k and β the lower bound is

$$\begin{aligned} LB_\beta(k;t) \quad &= \quad P(t,T) \sum_{j=1}^{n} \left(w_{1j}\ \exp\left((G_{1j})^\top \mu + a_{1j} + \frac{1}{2} V_{1j}^G + \frac{1}{2}(d_{1j})^2 \right) \mathcal{N}(d_{1j} - d) \right. \\ &- \quad \left. w_{2j}\ \exp\left((G_{2j})^\top \mu + a_{2j} + \frac{1}{2} V_{2j}^G + \frac{1}{2}(d_{2j})^2 \right) \mathcal{N}(d_{2j} - d) \right), \end{aligned} \tag{23}$$

where $d = \frac{k-\boldsymbol{\beta}^\top \boldsymbol{\mu}}{\sqrt{\boldsymbol{\beta}^\top V \boldsymbol{\beta}}}$, $d_{ij} = (G_{ij})^\top v$ for $i = 1,2$ and $j = 1,...,d$, $v = \frac{V\beta}{\sqrt{\beta^\top V \beta}}$, $V_{ij}^G = (G_{ij})^\top (V - v\, v^\top)\, G_{ij}$ for $i = 1,2$ and $j = 1,...,d$ and $\mu = \mathbb{E}_t^T[\mathbf{X}(T)]$ and $V = Var_t(\mathbf{X}(T))$ are the mean and covariance matrix of the variable $\mathbf{X}(T)$ that is multivariate normal under the T-forward measure.

Proof: The proof is very similar to single curve affine Gaussian case, see Appendix (G).

5.2 Upper bound formula applied to multi-curve weighted Gaussian model

In a multiple curve framework the swaption price can also be written as

$$C(t) = P(t,T)\, \mathbb{E}_t^T \left[(B_1(\mathbf{X}(T)) - B_2(\mathbf{X}(T)))^+ \right] \tag{24}$$

where

$$
\begin{aligned}
B_1(\mathbf{X}(T)) &= \sum_{j=1}^{n} P(T,T_j)\,(1 + x\,F^x(T,T_j,x)) = \sum_{j=1}^{n} w_{1j}\,e^{(G_{1j})^\top \mathbf{X}(T)+a_{1j}},\\
B_2(\mathbf{X}(T)) &= (1 + x\,K)\sum_{j=1}^{n} P(T,T_j) = \sum_{j=1}^{n} w_{2j}\,e^{(G_{2j})^\top \mathbf{X}(T)+a_{2j}}.
\end{aligned}
$$

Hence, the (un-discounted) approximation error of the lower bound defined in Proposition 5.1 is

$$
\begin{aligned}
& \frac{1}{P(t,T)}\left(C(t) - \widehat{LB}(t)\right)\\
= \;& \mathbb{E}_t^T[(B_1(\mathbf{X}(T)) - B_2(\mathbf{X}(T)))^+ I(\mathcal{G}^c)] + \mathbb{E}_t^T[(B_2(\mathbf{X}(T)) - B_1(\mathbf{X}(T)))^+ I(\mathcal{G})]\\
= \;& \Delta_1 + \Delta_2.
\end{aligned}
$$

Applying reasoning as in the single curve case we find that the upper bound is

$$
UB(t) = L\hat{B}(t) + P(t,T)(\epsilon_1 + \epsilon_2), \tag{25}
$$

where ϵ_1 and ϵ_2 are upper bounds for Δ_1 and Δ_2 and their expressions are the following

$$
\begin{aligned}
\epsilon_1 &= \sum_{j=1}^{n} \mathbb{E}_t^T[P(T,T_j)\,(1 + x\,F^x(T,T_j,x) - K_j)^+\,I(\mathcal{G}^c)]\\
&= \sum_{j=1}^{n} \mathbb{E}^T\left[\left(w_{1j}\,e^{G_{1j}^\top \mathbf{X}(T)+a_{1j}} - \tilde{w}_{2j}\,e^{G_{2j}^\top \mathbf{X}(T)+a_{2j}}\right)^+\; I(\mathcal{G}^c)\right], \qquad (26)\\
\epsilon_2 &= \sum_{j=1}^{n} \mathbb{E}_t^T[P(T,T_j)\,(K_j - 1 - x\,F^x(T,T_j,x))^+\,I(\mathcal{G})]\\
&= \sum_{j=1}^{n} \mathbb{E}^T\left[\left(\tilde{w}_{2j}\,e^{G_{2j}^\top \mathbf{X}(T)+a_{2j}} - w_{1j}\,e^{G_{1j}^\top \mathbf{X}(T)+a_{1j}}\right)^+\; I(\mathcal{G})\right], \qquad (27)
\end{aligned}
$$

where $\tilde{w}_{2j} = \frac{P(t,T_j)}{P(t,T)} K_j$ and

$$
K_j = 1 + x\,F(T,T_j,x)|_{\mathbf{X}(T)=\mathbf{X}^*}, \tag{28}
$$

where $\mathbf{X}^*$ is the point on the true exercise boundary (i.e. $B_1(\mathbf{X}(T)) - B_2(\mathbf{X}(T) = 0))$ where the density function of the model factors is largest.

Proposition 5.2. The upper bound to the European swaption price, for the multiple-curve weighted Gaussian model, is given by the following formula

$$UB(t) = \widehat{LB}(t) + P(t,T)\left(\epsilon_1(-\alpha) + \epsilon_2(-\alpha)\right), \tag{29}$$

where

$$\begin{aligned}
\epsilon_1(k) &= \int_{-\infty}^{d} dz\, \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sum_{j=1}^{n} w_{1j}\, e^{M_{1j}+\frac{1}{2}V_{1j}^G}\, N\left(d_{1j}\right) - \tilde{w}_{2j}\, e^{M_{2j}+\frac{1}{2}V_{2j}^G}\, N\left(d_{2j}\right), \\
d_{1j} &= \frac{\log\left(\frac{w_{1j}}{\tilde{w}_{2j}}\right) + M_{1j} + a_{1j} - M_{2j} - a_{2j} + V_{1j}^G - Cov_j}{\sqrt{V_{1j}^G + V_{2j}^G - 2Cov_j}} \\
d_{2j} &= d_{1j} - \sqrt{V_{1j}^G + V_{2j}^G - 2Cov_j} \\
\epsilon_2(k) &= \int_{d}^{+\infty} dz\, \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sum_{j=1}^{n} \tilde{w}_{2j}\, e^{M_{2j}+\frac{1}{2}V_{2j}^G}\, N\left(\delta_{1j}\right) - w_{1j}\, e^{M_{1j}+\frac{1}{2}V_{1j}^G}\, N\left(\delta_{2j}\right), \\
\delta_{1j} &= \frac{-\log\left(\frac{w_{1j}}{\tilde{w}_{2j}}\right) - M_{1j} - a_{1j} + M_{2j} + a_{2j} + V_{2j}^G - Cov_j}{\sqrt{V_{1j}^G + V_{2j}^G - 2Cov_j}} \\
\delta_{2j} &= \delta_{1j} - \sqrt{V_{1j}^G + V_{2j}^G - 2Cov_j}
\end{aligned}$$

and $\widehat{LB}(t)$ is given in Proposition 5.1, $d = \frac{k-\beta^\top\mu}{\sqrt{\beta^\top V\beta}}$, $V_{ij}^G = G_{ij}^\top(V - vv^\top)G_{ij}$ and $Cov_j = G_{1j}^\top(V - vv^\top)G_{2j}$ for $i = 1,2$ and $j = 1, ..., d$, $M_{ij} = G_{ij}^\top\mu + zG_{ij}^\top v$ for $i = 1,2$ and $j = 1, ..., d$ $v = \frac{V\beta}{\sqrt{\beta^\top V\beta}}$, and $\mu = \mathbb{E}_t^T[\mathbf{X}(T)]$ and $V = Var_t(\mathbf{X}(T))$ are the mean and covariance matrix of the variable $\mathbf{X}(T)$ that is multivariate normal under the T-forward measure and $N(x)$ is the standard Gaussian cumulative distribution function.

Proof: The proof is similar to single curve affine Gaussian case, with the difference that instead of the Black formula we apply the Margrabe's formula (Margrabe (1978)) for exchange options, see Appendix (G).

6 Numerical results

For each model we fix a set of parameters and we calculate a matrix of swaption prices with different maturities, swap lengths and three different strikes: ATMF (at the money forward), ITMF (0.85 × ATMF) and OTMF (1.15 × ATMF). This is a common choice in the literature (see for instance Schrager and Pelsser (2006), Singleton and Umantsev (2002) and Kim (2012)). The description and values of the parameters for each model are reported, respectively, in Appendix A and C. The tested models are a 3-factors affine Gaussian model, a 2-factors affine Cox Ingersoll and Ross (CIR) model, a 2-factor Gaussian model with double exponential jumps, a 2-factors Gaussian quadratic model and a 2-factor multiple curve Gaussian model.

Monte Carlo is used as a benchmark for the computation of the true swaption price. The 97.5% mean-centred Monte Carlo confidence interval is used as measure of the accuracy.

For the affine 3-factor Gaussian model and the Gaussian multi-curve model, the lower bounds are obtained via the closed formula described in sections 4 and 5.1. Kim's prices are calculated using the closed price formula for the T-forward probabilities. For the 2-factor CIR model, the Gaussian with jumps and the Gaussian quadratic model, integrals involved in the lower bound and in Kim's method are evaluated by a Gauss-Kronrod quadrature rule, using Matlab's built-in function **quadgk**.

The Matlab function **quadgk** is also used for the integral appearing in the upper bound formula for the 3-factor Gaussian model and for the Gaussian multi-curve model (see section 4 and 5.2). For the 2-factor CIR model, the Gaussian model with jumps, and Gaussian quadratic model the upper bound formula requires the calculus of double integrals that are evaluated using Matlab's function **quad2d**, an iterative algorithm that divides the integration region into quadrants and approximates the integral over each quadrant by a two-dimensional Gauss quadrature rule.

Another important fact is that our lower bound formula is very suitable to be used as a control variate to reduce Monte Carlo simulation error. The approximated formula is easily implemented in a Monte Carlo scheme and turns out to be very effective.In this way the simulation error is considerably reduced.

Relative errors of the bounds with respect to Monte Carlo and the difference between the lower and upper bounds are shown in Figures 3-8. Computational time for each pricing method is also given in Table 2.

6.1 Test with random parameters

We test the robustness of the bounds approximation to parameters change. We use one hundred random extracted parameters for the 2-factor CIR model. The model parameters are extracted uniformly from a reasonable range of possible values specified in Appendix C.

For each set of simulated parameters we calculate a matrix of swaption prices with different maturities and swap lengths and three different strikes ATM, ITMF ($0.85 \times$ ATMF) and OTMF ($1.15 \times$ ATMF). This is a common choice in the literature.

For each swaption we calculate the root mean square deviation (RMSD) of the lower and upper bounds with respect to the Monte Carlo estimation, used as benchmark

$$RMSD = \frac{1}{\sqrt{N}}\sqrt{\frac{\sum_{i=1}^{N}(B_i - MC_i)^2}{(MC_{avg})^2}}, \quad MC_{avg} = \frac{\sum_{i=1}^{N} MC_i}{N},$$

where N is the number of random trials, $B_i = LB_i$ or $B_i = UB_i$ (lower or upper bound) and MC_i is the Monte Carlo estimation of the swaption price with the i^{th} set of random parameters and MC_{avg} is the average of Monte Carlo prices over all random trials. Monte Carlo values are estimated using 10^7 simulations. Numerical results of this test are shown in Table 1.

6.2 Comments on numerical results

Numerical results are presented across a wide class of affine models, for the Gaussian quadratic model and for a multiple curve model. The tangent hyperplane lower bound and the approximation "A" of Kim (2012) produce the same prices, because they are two different implementations of the same approximation. However the new algorithm, that requires the computation of a single Fourier inversion, is much faster across all models for which the characteristic function is known in closed form (see Tables 2). The improvement in computational performance is more evident for swaptions with a large number of cash flows as illustrated in Table 3. Comparing the speed of different method is not simple, because each algorithm should be optimized. However our considerations about the efficiency of an algorithm are also justified by theoretical reasoning and confirmed by our estimations of the computational time.

Our upper bound is applicable to all affine-quadratic models, both in single and multiple curve frameworks and it is particularly efficient for affine ones. In literature upper bounds are available only for Gaussian affine models. The computation of the upper bound is slower than the lower bound calculation, but it is still faster than Monte Carlo simulation. In addition the range between lower and upper bound is always narrow: so in practice the combined use of the two bounds provides an accurate estimate of the true price.

Moreover, for the multiple curve model, we compare our bounds with an approximate method widely used in the market, the freezing drift approximation (see Moreni and Pallavicini (2014)) and we find that lower bound and upper bounds perform better with comparable computational times.

The RMSD computation performed for the 2-factor CIR model and reported in Table 1 is an important validation for the stability of the accuracy of the bounds to changes in the parameter set. The RMSD of the lower bound for at the money and in the money options is less than 0.1% of the Monte Carlo average price, which is a good result. The relative error is larger for out of the money options in particular for the swaptions with long swap length. Indeed the maximum error is around 0.3% of the Monte Carlo price. The RMSDs of the upper bound are greater than the RMSDs of the lower bound, in particular for swaptions with longer swap lengths. However the maximum RMSD of the upper bound is about 0.8% of the Monte Carlo price, which is also a confirmation of the good performance of the upper bound.

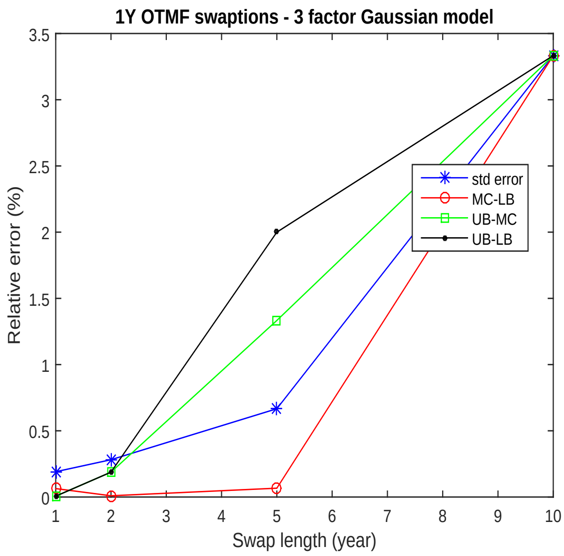
7 Tables and Figures

2-factor CIR model: RMSD calculation

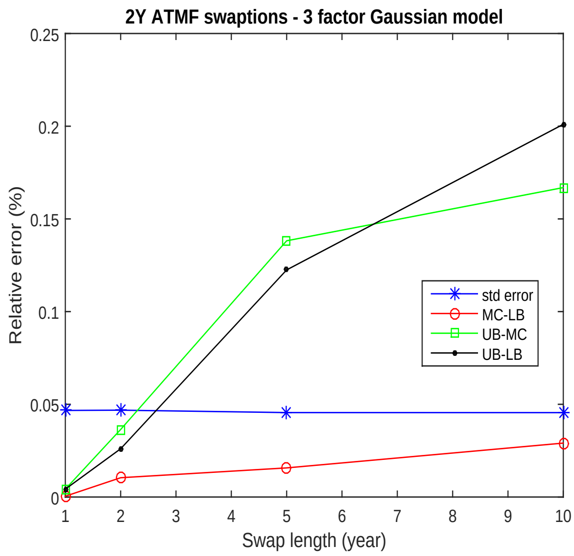
RMSD - LB											
ATM	1	2	5	ITM	1	2	5	OTM	1	2	5
1	0.05%	0.07%	0.08%	1	0.01%	0.02%	0.02%	1	0.2%	0.2%	0.2%
2	0.05%	0.07%	0.08%	2	0.01%	0.01%	0.01%	2	0.2%	0.2%	0.2%
5	0.05%	0.07%	0.09%	5	0.004%	0.01%	0.01%	5	0.2%	0.3%	0.3%
10	0.05%	0.07%	0.09%	10	0.004%	0.01%	0.01%	10	0.2%	0.3%	0.3%

RMSD - UB											
ATM	1	2	5	ITM	1	2	5	OTM	1	2	5
1	0.05%	0.07%	0.08%	1	0.01%	0.02%	0.01%	1	0.2%	0.2%	0.2%
2	0.06%	0.09%	0.11%	2	0.01%	0.02%	0.02%	2	0.2%	0.3%	0.3%
5	0.14%	0.20%	0.28%	5	0.03%	0.05%	0.08%	5	0.3%	0.5%	0.6%
10	0.15%	0.22%	0.32%	10	0.04%	0.06%	0.09%	10	0.3%	0.5%	0.7%

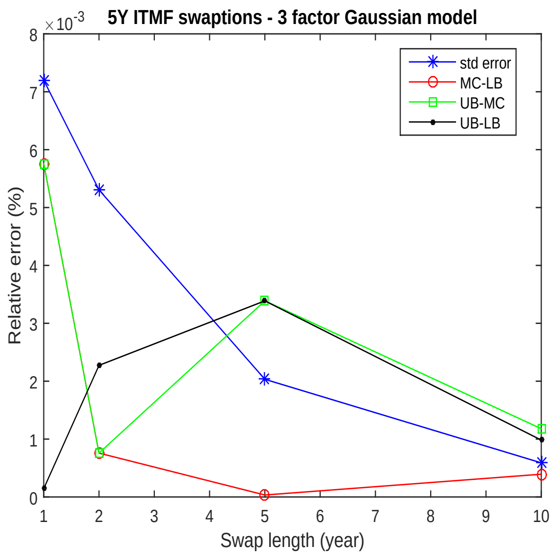
Table 1: These tables report for each swaption the RMSD value of the bounds with respect to the Monte Carlo Value obtained by randomly sampling one hundred parameters sets.



(a)

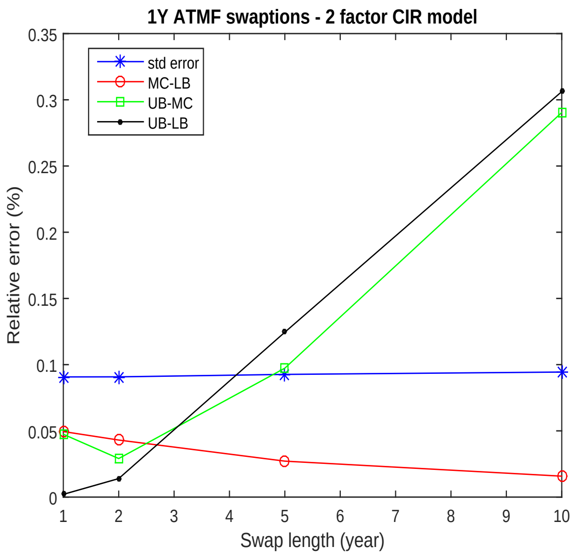


(b)

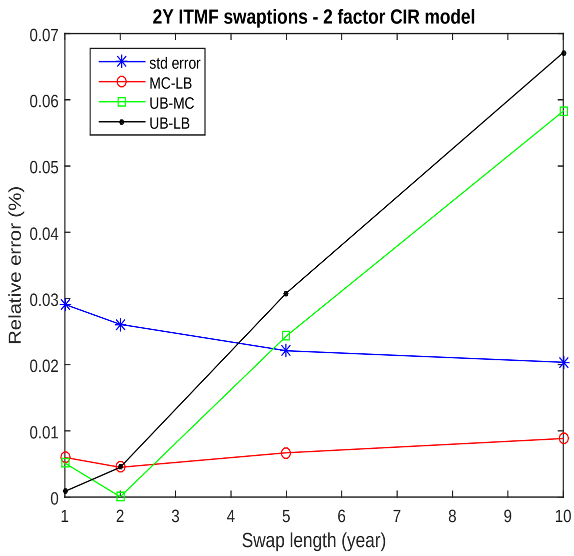


(c)

Figure 3: The figures show three examples of results for the 3 factors Gaussian model. The graphs report relative errors in percentage of the swaption prices for different maturities, swap lengths and strikes. For each swaption we estimate the price with Monte Carlo method (MC), the hyperplane approximation lower bound (LB) and the upper bound (UB). The relative error is calculated as the ratio between the error and Monte Carlo price. Monte Carlo values are estimated using 10^7 simulations, antithetic variates method and the exact probability distribution. The ratio between the confidence interval at 97.5% and the Monte Carlo price is reported as standard error (std error).



(a)



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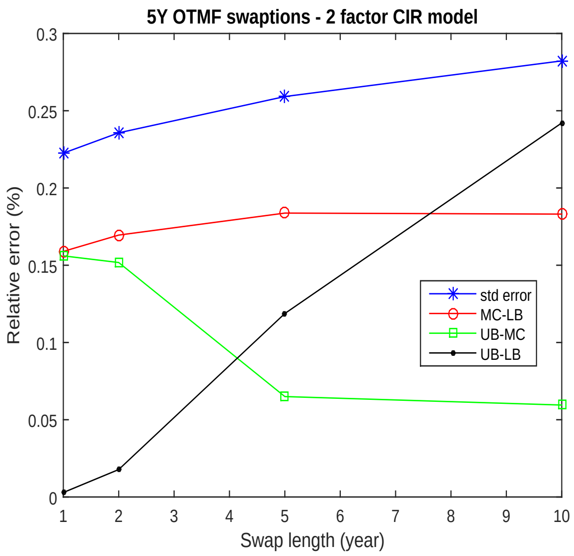

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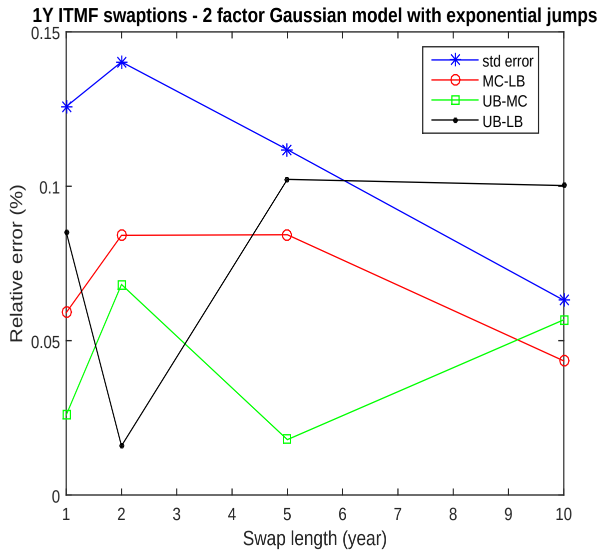
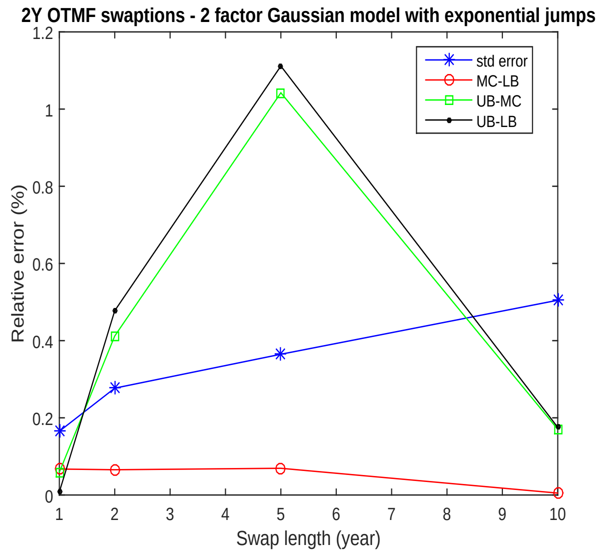
Figure 4: The figures show three examples of results for the 2 factors CIR model. The graphs report relative errors in percentage of the swaption prices for different maturities, swap lengths and strikes. For each swaption we estimate the price with Monte Carlo method (MC), the hyperplane approximation lower bound (LB) and the upper bound (UB). The relative error is calculated as the ratio between the error and Monte Carlo price. Monte Carlo prices are estimated using 10^7 simulations and the exact probability distribution. The ratio between the confidence interval at 97.5% and the Monte Carlo price is reported as standard error (std error).



(a)



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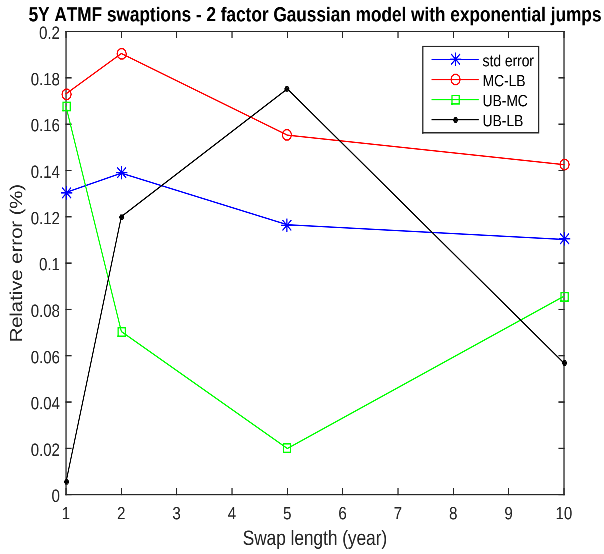
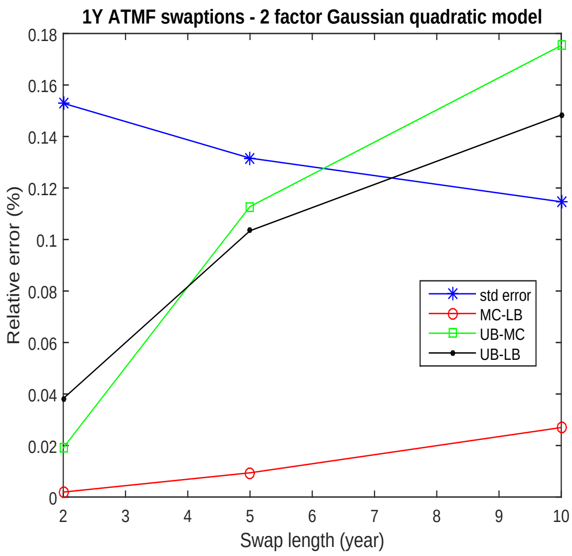


(c)

Figure 5: The figures show three examples of results for the 2 factors Gaussian model with double exponential jump size. Parameters are calibrated to the Euribor 6M curve from January 4th 2015. The graphs report relative errors in percentage of the swaption prices for different maturities, swap lengths and strikes. For each swaption we estimate the price with Monte Carlo method (MC), the hyperplane approximation lower bound (LB) and the upper bound (UB). The relative error is calculated as the ratio between the error and Monte Carlo price. Monte Carlo prices are estimated using 4×10^6 simulations, an Euler scheme with time step equal to 0.0005 and the antithetic variates technique. The ratio between the confidence interval at 97.5% and the Monte Carlo price is reported as standard error (std error).



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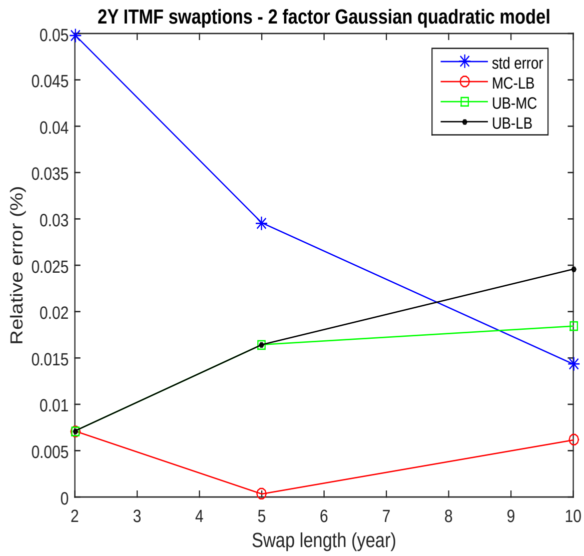

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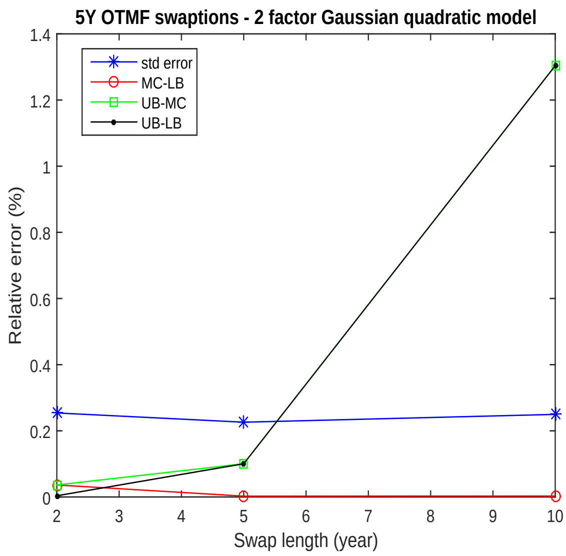


(c)

Figure 6: The figures show three examples of results for the 2 factor Gaussian quadratic model. The graphs report relative errors in percentage of the swaption prices for different maturities, swap lengths and strikes. For each swaption we estimate the price with Monte Carlo method (MC), the hyperplane approximation lower bound (LB) and the upper bound (UB). The relative error is calculated as the ratio between the error and Monte Carlo price. Monte Carlo prices are estimated using 4×10^6 simulations, an Euler scheme with time step equal to 0.0005 and the antithetic variates technique. The ratio between the confidence interval at 97.5% and the Monte Carlo price is reported as standard error (std error).

3 factor Gaussian model

Overall time (sec)	MC	LB (HP)	UB	Kim
ATMF	32.045	0.084	0.140	0.141
ITMF	32.023	0.170	0.223	0.219
OTMF	32.024	0.169	0.223	0.219

2 factor CIR model

Overall time (sec)	MC	LB (HP)	UB	Kim
ATMF	23.118	0.146	17.054	0.391
ITMF	23.121	0.150	17.015	0.341
OTMF	23.121	0.152	17.018	0.395

2 factor Gaussian model with exponential jumps (A)

Overall time (sec)	MC	LB (HP)	UB	Kim
ATMF	35×10^3	1.957	132.229	1.968
ITMF	35×10^3	0.868	129.218	0.977
OTMF	35×10^3	0.845	149.071	0.966

2 factor Gaussian model with exponential jumps (B)

Overall time (sec)	MC	LB (HP)	UB	Kim
ATMF	30×10^3	1.014	151.070	1.106
ITMF	30×10^3	0.878	151.260	1.018
OTMF	30×10^3	1.023	152.690	1.172

2 factor Gaussian quadratic model

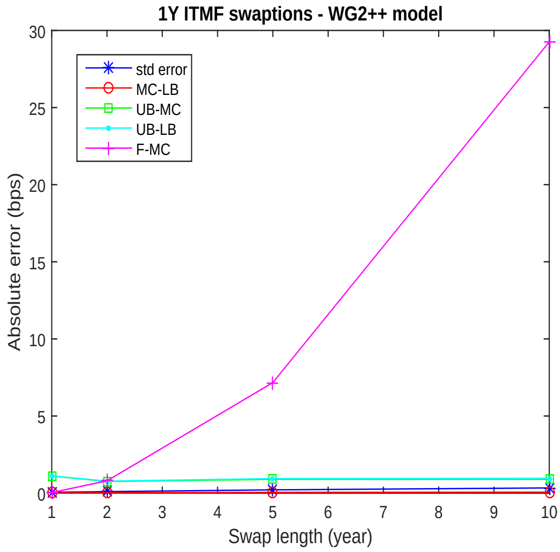
Overall time (sec)	MC	LB (HP)	UB	Kim
ATMF	1.472×10^3	0.861	587.403	0.665
ITMF	1.472×10^3	1.124	635.807	0.717
OTMF	1.472×10^3	1.019	509.202	0.633

Table 2: Computational times reported in tables are the overall time needed for calculating a matrix of swaption prices with four different tenors, 1Y, 2Y, 5Y, 10Y and three different maturities 1Y, 2Y, 5Y.

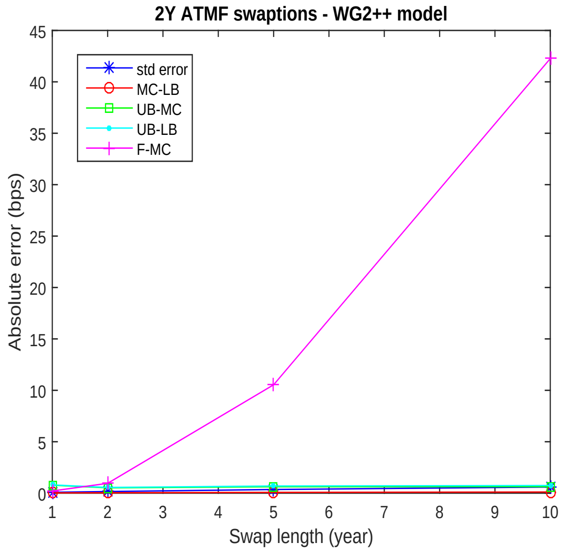
2-factor CIR model: comparison of the algorithms performance

Swap length (y)	LB (HP) (sec)	Kim (sec)	LB (HP) (%)	Kim (%)
1	0.024	0.022	-	-
2	0.023	0.026	0%	20%
5	0.023	0.034	0%	55%
10	0.032	0.051	34%	132%
15	0.040	0.071	69%	225%
20	0.048	0.089	102%	305%

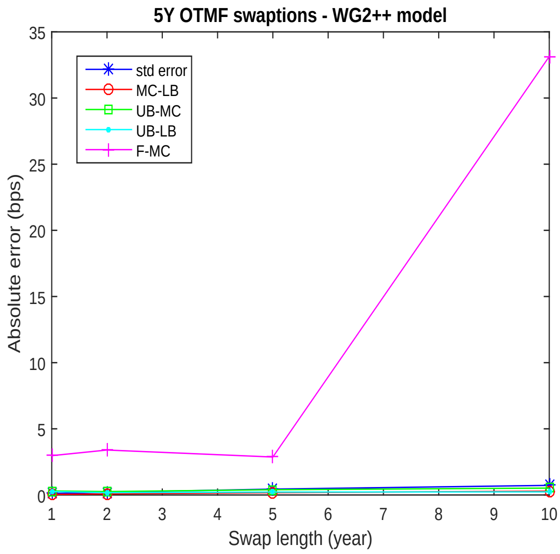
Table 3: For each swaption we report in the first two columns the run time in seconds and in the last two columns the percentage variation of the two run times with respect to the first row. The maturity of the swaptions is 2 years and the frequency of payments is six months.



(a)

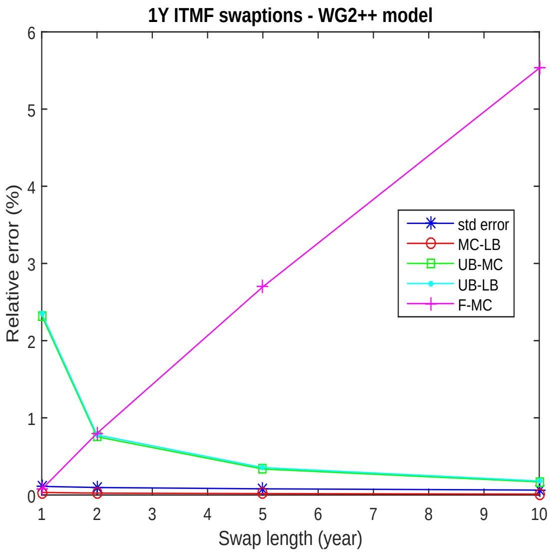


(b)

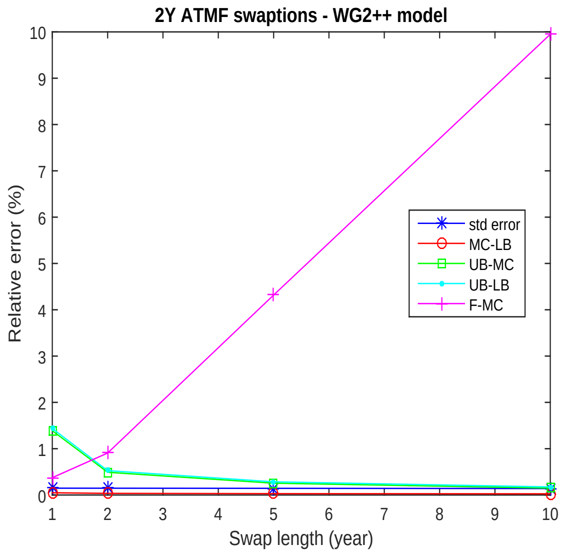


(c)

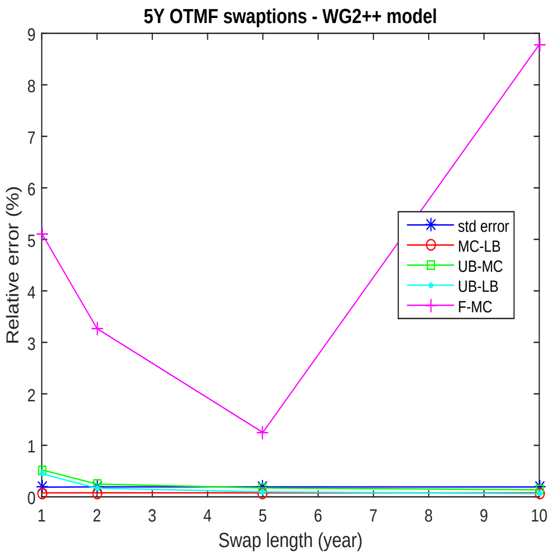
Figure 7: The figures show three examples of results for the multi-curve 2 factors weighted Gaussian model. The graphs report relative errors in percentage of the swaption prices for different maturities, swap lengths and strikes. For each swaption we estimate the price with Monte Carlo method (MC), the hyperplane approximation lower bound (LB), the upper bound (UB) and the freezing technique (F). The absolute error of the price is calculated in basis point. Monte Carlo values are estimated using 10^7 simulations, antithetic variates method and the exact probability distribution. The confidence interval at 97.5% of the Monte Carlo price is reported as standard error (std error).



(a)



(b)



(c)

Figure 8: The figures show three examples of results for the multi-curve 2 factors weighted Gaussian model. The graphs report relative errors in percentage of the swaption prices for different maturities, swap lengths and strikes. For each swaption we estimate the price with Monte Carlo method (MC), the hyperplane approximation lower bound (LB), the upper bound (UB) and the freezing technique (F). The relative error of the price is calculated as the ratio between the error and Monte Carlo price. Monte Carlo values are estimated using 10^7 simulations, antithetic variates method and the exact probability distribution. The ratio between the confidence interval at 97.5% and the Monte Carlo price is reported as standard error (std error).

Conclusions

In this paper we propose a general lower bound formula of the swaption price, based on an approximation of the exercise region. We note that previous approximations, such as Kim (2012) and Singleton and Umantsev (2002) methods, represent a particular case of our general formula and so they can be interpreted as lower bounds too. Moreover, we provide a new algorithm to implement the lower bound that is found to be more efficient for interest rate models in which the joint characteristic function of state variables is known in analytical form. Further, this work provides a new upper bound to swaption prices that is applicable to all affine-quadratic models and that is accurate and computable in a reasonable time. So the lower bound approximation error is controlled. Finally, we extend lower and upper bounds to multiple curve models. Numerical results confirm our hypothesis about the performance of the new algorithm, in terms of computational times for the calculus of lower bound, except for quadratic models in which the characteristic function is not analytic. Moreover, numerical tests show a very good accuracy of the new upper bound for different models, across tenors, maturities and strikes.

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A Models description

This section presents the considered affine and quadratic models.

A.1 Affine Gaussian models

Affine Gaussian models assign the following stochastic differential equation (SDE) to the state variable $\mathbf{X}$

$$d\mathbf{X}(t) = K(\boldsymbol{\theta} - \mathbf{X}(t))\,dt + \Sigma\,d\mathbf{W}(t) \quad \text{and} \quad \mathbf{X}(0) = \mathbf{x}_0$$

where $\mathbf{W}_t$ is a standard d-dimensional Brownian motion, K is a $d \times d$ diagonal matrix and Σ is a $d \times d$ triangular matrix. The short rate is obtained as a linear combination of the state vector $\mathbf{X}$; it is always possible to rescale the components $X_i(t)$ and assume that $r(t) = \phi + \sum_{i=1}^{d} X_i(t)$, $\phi \in \mathbb{R}$ without loss of generality.

The ZCB formula (33) and T-forward characteristic function (6) of $\mathbf{X}$ can be obtained in closed form using the moment generating function of a multivariate normal variable or solving the ODE system in Duffie, Pan and Singleton (2000), the solution is given, for example, in Collin-Dufresne and Goldstein (2002).

A.2 Multi-factor Cox-Ingersoll-Ross (CIR) model

In this model, the risk neutral dynamics of the state variates is

$$dX_i(t) \quad = \quad a_i(\,\theta_i - X_i(t))dt + \sigma_i\sqrt{X_i(t)}dW^i(t) \quad \text{and} \quad \mathbf{X}(0) = \mathbf{x}_0,$$

where $i = 1, ..., d$, $W^i(t)$ are independent standard Brownian motions, a_i, θ_i and σ_i are positive constants. The short rate is obtained by $r(t) = \phi + \sum_{i=1}^{d} X_i(t)$, where $\phi \in \mathbb{R}$.

In multi-factor CIR models the bond price (33) and the characteristic function (6) have closed-form expressions, given, for example, in Collin-Dufresne and Goldstein (2002).

A.3 Gaussian model with double exponential jumps

In this model, the risk neutral dynamics of the state variates is

$$d\mathbf{X}(t) = K(\boldsymbol{\theta} - \mathbf{X}(t))\,dt + \Sigma\,d\mathbf{W}(t) + d\mathbf{Z}^{+}(t) - d\mathbf{Z}^{-}(t) \quad \text{and} \quad \mathbf{X}(0) = \mathbf{x}_0,$$

where $\mathbf{W}_t$ is a standard d-dimensional Brownian motion, K is a $d \times d$ diagonal matrix, Σ is a $d \times d$ triangular matrix and $\mathbf{Z}^{\pm}$ are pure jumps processes whose jumps have fixed probability distribution ν on $\mathbb{R}^d$ and constant intensity $\mu^{\pm}$. The short rate is obtained as a linear combination of the state vector $\mathbf{X}$. In particular $\mathbf{Z}^{\pm}$ are compounded Poisson processes with jump size exponentially distributed, i.e.

$$Z_l^{\pm} = \sum_{j=1}^{N^{\pm}(t)} Y_{j,l}^{\pm}$$

where $l = 1, ..., d$ is the factor index, $N^{\pm}(t)$ are Poisson processes with intensity $\frac{\mu^{\pm}}{d}$ and $Y_{j,l}^{\pm}$, for a fixed l, are independent identically distributed exponential random variables of mean parameters $m_l^{\pm}$.

Since $\mu^{\pm}$ do not depend on $\mathbf{X}$, we know that

$$\Phi(\boldsymbol{\lambda}) = \mathbb{E}_t^T\left[e^{\boldsymbol{\lambda}^\top \mathbf{X}(T)}\right] = \Phi^D(\boldsymbol{\lambda})\, e^{\tilde{A}^J(T-t,\boldsymbol{\lambda})-A^J(T-t)} \tag{30}$$

where $\Phi^D(\boldsymbol{\lambda})$ is the T-forward characteristic function of affine Gaussian model and the function $\tilde{A}^J(\tau, \boldsymbol{\lambda})$ is available in closed form (see Duffie, Pan and Singleton (2000) for further details).

A.4 Gaussian quadratic model

In this model, the risk neutral dynamics of the state variates is

$$d\mathbf{X}(t) = K(\boldsymbol{\theta} - \mathbf{X}(t))\, dt + \Sigma\, d\mathbf{W}_t \quad \text{and} \mathbf{X}(0) = \mathbf{x}_0,$$

where $\mathbf{W}_t$ is a standard d-dimensional Brownian motion, $\boldsymbol{\theta}$ is a d-dimensional constant vector, K and Σ are $d \times d$ matrix. The short rate is a quadratic function of the state variates, $r(t) = a_r + b_r^\top \mathbf{X}(t) + \mathbf{X}(t)^\top C_r \mathbf{X}(t)$, $a_r \in \mathbb{R}$, $b_r \in \mathbb{R}^d$ and C_r is a $d \times d$ symmetric matrix.

We solve the system fo ordinary differential equation for the functions $\tilde{A}(\tau, \boldsymbol{\lambda}, \Lambda)$, $\tilde{\mathbf{B}}(\tau, \boldsymbol{\lambda}, \Lambda)$, $\tilde{C}(\tau, \boldsymbol{\lambda}, \Lambda)$ in formula (6), using the method proposed in Cheng and Scaillet (2007). Closed form evaluation of functions requires the calculus of a matrix exponentiation and also a numerical integration. However, numerical tests show that the method of Cheng and Scaillet (2007) is much faster than solving numerically the ODE system using Runge-Kutta or Dormand-Prince schemes.

B Multiple curve model

We test the lower and upper bounds to the multiple curve weighted Gaussian model presented in Moreni and Pallavicini (2014). They model the Libor FRA rate $F^x(t, T, x)$, which is the fair rate of a FRA contract with underling the Libor rate with tenor x (usually x = 1M, 3M, 6M or 12M). Under the risk neutral $\mathbb{P}$ measure, the FRA rate is in the form

$$F^x(t,T,x) = \frac{1}{x}\left[(1 + x\,F^x(0,T,x))e^{\int_0^t \Sigma^x(s,T,x)^\top \cdot dW(s) + \int_0^t A^x(s,T,x)\,ds} - 1\right], \tag{31}$$

where

- $\Sigma^x(s,T,x) = \int_{T-x}^{T} \sigma(s,u;T,x)\,du$ is a d-dimensional volatility function,

- in order to satisfy condition (ii) $\sigma(s,T;T,0) = \sigma(s,T)$,

- to satisfy condition (i)

$$A^x(s,T,x) = -\frac{1}{2}\Sigma^x(s,T,x)^\top \cdot \Sigma^x(s,T,x) + \Sigma^x(s,T,x)^\top \cdot \Sigma(s,T) \tag{32}$$

To model the FRA rate these constraints are respected: $F^x(t,T,x)$ has to be a martingale under the T-forward measure and $\lim_{x\to 0} F^x(t,T,x) = \lim_{x\to 0} F^0(t,T,x)$ and $F^x(t,T,x) \sim F^0(t,T,x)$ if $x \sim 0$, where $F^0(t,T,x)$ is the simply compounding forward rate at time t for the interval $[T-x,\, T]$ in a classical single curve framework.

Hence, the zero coupon bond price process has the following dynamic

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\, e^{\int_0^t (\Sigma(s,t)-\Sigma(s,T))^\top \cdot dW(s) + \int_0^t (A(s,t)-A(s,T))\,ds}, \tag{33}$$

where

$\Sigma(t,T) = \int_t^T \sigma(t,u)\,du$ is a d-dimensional vector volatility function,

$W(t)$ is a d-dimensional standard Brownian motion,

$A(t,T) = \frac{1}{2}\Sigma(t,T)^\top\Sigma(t,T)$.

B.1 Volatility specification

The weighted Gaussian specification of the multiple curve model assumes a deterministic volatility in form

$$\begin{aligned}
\sigma(t,u;T,x) &= h(t)\,q(u;T,x)\,g(t,u)\\
g(t,u) &= \exp\left(-\lambda(u-t)\right)\\
h(t) &= \epsilon(t)\,h\,R
\end{aligned}$$

where λ is a deterministic array function, h is a diagonal matrix, R is an upper triangular matrix such that $\rho = R^\top R$ is a correlation matrix and the model allows for a time varying common volatility shape $\epsilon(t)$ of the form

$$\epsilon(t) = 1 + (\beta_0 - 1 + \beta_1\, t)e^{\beta_2\, t},$$

where β_0, β_1 and β_2 are three positive constants. Furthermore, the matrix q is given by

$$q_{i,j}(u;T,x) \quad = \quad e^{-\eta_i\, x}\, \mathrm{I}(i=j) \quad \text{for } \; i,j = 1,...,d$$

where η is a deterministic constant vectors.

B.2 Markovian specification for the weighted Gaussian model

By plugging the expression for the volatility into formula (31), it is possible to work out the expression ending up with the following Markovian representation of the FRA rate

$$\log\left(\frac{1 + x\, F^x(t,T,x)}{1 + x\, F^x(0,T,x)}\right) = G(t,T,x)^\top \cdot \mathbf{X}(t) + a(t,T,x), \tag{34}$$

where $a(t,T,x)$ is a deterministic coefficient and it has the following form

$$\begin{aligned} a(t,T,x) &= G(t,T,x)^\top \cdot Y(t)\left(G(t,T) - \frac{1}{2}G(t,T,x)\right) \\ (Y(t))_{ik} &= \int_0^t g_i(s,t)(h^\top(s)\cdot h(s))_{ik} g_k(s,t) ds \quad i,k = 1,...,d, \end{aligned}$$

$G(t,T,x)$ is a deterministic vector with components

$$G_i(t,T,x) = \int_{T-x}^{T} q_{ii}(u;T,x)\, g_i(t,u)\, du,$$

$G(t,T)$ is a deterministic vector with components

$$G_i(t,T) = \int_t^T g_i(t,u)\, du,$$

and $\mathbf{X}(t)$ is a vector Markovian process with components, under the risk neutral measure, in form

$$X_i(t) = \sum_{j=1}^{d} \int_0^t g_i(s,t)\left(h_{i,j}^\top(s) dW_j(s) + (h^\top(s)\cdot h(s))_{i,j}\left(\int_s^t g_i(s,y)dy\right)\, ds\right).$$

A similar Markovian representation can be obtained for the ZCB price

$$\log \left(P(t,T) \frac{P(0,t)}{P(0,T)} \right) = -G(t,T)^\top \cdot \mathbf{X}(t) + a(t,T), \tag{35}$$

where $a(t,T)$ is a deterministic coefficient and it has the following form

$$a(t,T) \quad = \quad -\frac{1}{2} G(t,T)^\top Y(t) G(t,T).$$

C Parameters values

C.1 Three-factors Gaussian model and Cox-Ingersoll and Ross model

We verify the accuracy of our bounds using models and parameter values already examined in literature[2]

- 3-factors Gaussian model: $K = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$, $\theta = [0,\, 0,\, 0]^\top$, $\boldsymbol{\sigma} = [0.01,\, 0.005,\, 0.002]^\top$, $\rho = \begin{bmatrix} 1 & -0.2 & -0.1 \\ -0.2 & 1 & 0.3 \\ -0.1 & 0.3 & 1 \end{bmatrix}$, $\Sigma = \mathrm{diag}(\boldsymbol{\sigma}) \cdot \mathrm{chol}(\rho)$[3], $x_0 = [0.01,\, 0.005,\, -0.02]^\top$ and $\phi =$ 0.06;

- 2-factors Cox-Ingersoll and Ross model $\mathbf{a} = [0.5080,\, -0.0010]^\top$, $\boldsymbol{\theta} = [0.4005,\, -0.7740]^\top$, $\boldsymbol{\sigma} = [0.023,\, 0.019]^\top$, $x_0 = [0.374,\, 0.258]^\top$ and $\phi =$ -0.58.

Examples of numerical results for this model are shown in Figures 3 and 4.

Moreover we specify the interval of parameters of the 2-factors CIR model from which we extract the one hundred parameters sets for the RMSD calculation: $x_0 \in [0.001, 0.5] \times [0.001, 0.5]$, $\phi \in [0.001, 1]$, $\mathbf{a} \in [0.001, 1] \times [0.001, 1]$, $\boldsymbol{\theta} \in [0.001, 1] \times [0.001, 1]$, $\boldsymbol{\sigma} \in [0.001, \sqrt{2a(1)\theta(1)}] \times [0.001, \sqrt{2a(2)\theta(2)}]$.

C.2 Two-factor Gaussian model with double exponential jumps

We test the affine Gaussian model with exponentially distributed jumps, using parameter values obtained by minimization of the least square distance between the model and the market

[2] Schrager and Pelsser (2006) and Duffie and Singleton (1997) for the 2-factors CIR model.

[3] $\mathrm{diag}(\boldsymbol{\sigma})$ means the diagonalization of the vector $\boldsymbol{\sigma}$ and $\mathrm{chol}(\rho)$ means the Cholesky decomposition of the correlation matrix ρ, where $\boldsymbol{\sigma}$ and ρ are the volatility vector and the correlation matrix, respectively, of the original paper.

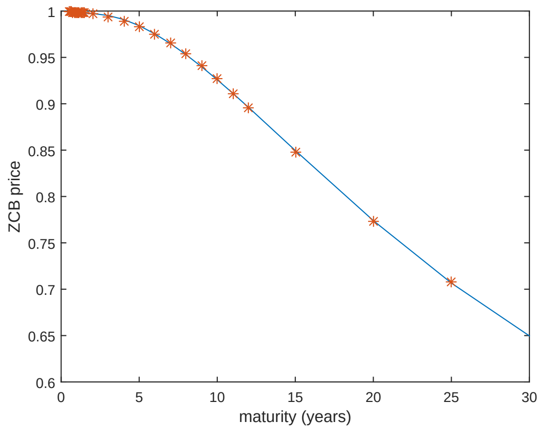

Figure 9: The red dots are the market price of ZCBs, the blue line is the calibrated curve. The quadratic error obtained from the least square minimization is 2.069×10^{-5}.

discount curve implied by bootstrapping the Euribor six months swap curve up to thirty years. The calibration is performed on at January 4^{th}, 2015 to obtain parameters set reported below.

Parameters:

- Gaussian parameters: $K = \begin{bmatrix} 0.050926 & 0 \\ 0 & 1.3687 \end{bmatrix}$, $\theta = [0,\ 0]^\top$, $\boldsymbol{\sigma} = [0.0048887,\ 0.24025]^\top$,
  $\rho = \begin{bmatrix} 1 & -0.1482 \\ -0.1482 & 1 \end{bmatrix}$, $\Sigma = \text{diag}(\boldsymbol{\sigma}) \cdot \text{chol}(\rho)$,
  $x_0 = [0.00035256,\ 0.00035497]^\top$ and $\phi = 4.332\ \times 10^{-5}$;

- Jumps parameters: $\boldsymbol{\mu}^+ = 0.4372$, $\mathbf{m}^+ = [0.027372, 0.045667]^\top$,
  $\boldsymbol{\mu}^- = 0.1101$, $\mathbf{m}^- = [0.027043, 0.012339]^\top$.

Figure 9 shows fitting of the calibration. Examples of numerical results for this model are shown in Figure 5.

C.3 Two-factor quadratic Gaussian model

The last considered model is the 2-factor quadratic Gaussian model. We use the following parameter values, proposed by Kim (2007) $K = \begin{bmatrix} -0.0541 & 0.0361 \\ -1.2113 & 0.4376 \end{bmatrix}$,

$\theta = [0.1932,\ 0.1421]^\top$, $\Sigma = \begin{bmatrix} 0.0145 & 0 \\ 0 & 0.0236 \end{bmatrix}$, $x_0 = [0.1690,\ -0.0501]^\top$,

$a_r = 0.0444$, $b_r = [0,\ 0]^\top$ and $C_r = \begin{bmatrix} 1 & 0.4412 \\ 0.4412 & 1 \end{bmatrix}$;

Examples of numerical results for this model are shown in Figure 6.

C.4 Multiple curve two-factors weighted Gaussian model

We verify the accuracy of our bounds using following fixed parameters:

$\lambda = [0.0073, 4.7344]$, $\eta = [0.1581, 0.8894]$, $h = [0.0059, 0.0411]$, $\rho_{12} = -0.8577$, $\beta_0 = 1.3160$, $\beta_1 = 1.3327$ and $\beta_2 = 0.5900$.

Examples of numerical results for this model are shown in Figures 7 and 8.

D Proof Proposition 2.1

We consider the lower bound to the swaption price as in formula (4) for affine models:

$$LB_{\boldsymbol{\beta}}(k;t) = P(t,T)\,\mathbb{E}_t^T\left[\left(\sum_{h=1}^{n} w_h\, e^{\mathbf{X}(T)^\top C_h \mathbf{X}(T) + \mathbf{b}_h^\top \mathbf{X}(T) + a_h} - 1\right) I(\mathcal{G})\right]$$

where the set $\mathcal{G} = \{\omega \in \Omega : \mathbf{X}(T)^\top \Gamma\, \mathbf{X}(T) + \boldsymbol{\beta}^\top \mathbf{X}(T) \geq k\}$.

We apply the extended Fourier transform (refer to Titchmarsh (1975) for a comprehensive treatment and to Hubalek, Kallsen and Krawczyk (2006) for examples of financial applications) with respect to the variable k to the T-forward expected value

$$\psi(z) \;=\; \int_{-\infty}^{+\infty} e^{zk}\mathbb{E}_t^T\left[\left(\sum_{h=1}^{n} w_h\, e^{\mathbf{b}_h^\top \mathbf{X}(T) + a_h} - 1\right) I(\boldsymbol{\beta}^\top \mathbf{X}(T) \geq k)\right] dk.$$

Assuming that we can apply Fubini's Theorem, which is verified in concrete cases, we have

$$\begin{aligned}\psi(z) \;=\; \mathbb{E}_t^T\Bigg[&\left(\sum_{h=1}^{n} w_h\, e^{\mathbf{X}(T)^\top C_h \mathbf{X}(T) + \mathbf{b}_h^\top \mathbf{X}(T) + a_h} - 1\right)\\ &\int_{-\infty}^{+\infty} e^{zk} I(\mathbf{X}(T)^\top \Gamma\, \mathbf{X}(T) + \boldsymbol{\beta}^\top \mathbf{X}(T) \geq k)\, dk\Bigg]\end{aligned}$$

The function $\psi(z)$ is defined for $k \to -\infty$ if $Re(z) > 0$ and

$$\psi(z) \;=\; \mathbb{E}_t^T\left[\left(\sum_{h=1}^{n} w_h e^{\mathbf{X}(T)^\top C_h \mathbf{X}(T) + \mathbf{b}_h^\top \mathbf{X}(T) + a_h} - 1\right) e^{z(\mathbf{X}(T)^\top \Gamma\, \mathbf{X}(T) + \boldsymbol{\beta}^\top \mathbf{X}(T))}\right]\frac{1}{z}$$

Using the (quadratic) characteristic function of $\mathbf{X}$, calculated under the T-forward measure ϕ, the function $\psi(z)$ can be written as

$$\psi(z) = \left(\sum_{h=1}^{n} w_h e^{a_h}\,\Phi\left(\mathbf{b}_h + z\boldsymbol{\beta}, C_h + z\Gamma\right) - \Phi\left(z\boldsymbol{\beta}, z\Gamma\right)\right)\frac{1}{z}, \tag{36}$$

Finally the lower bound is the inverse transform of $\psi(z)$ in the sense of Chauchy principal value integral

$$LB_{\boldsymbol{\beta}}(k;t) \quad = \quad P(t,T)\,\frac{1}{i2\pi}\lim_{\xi\to\infty}\int_{\delta-i\xi}^{\delta+i\xi} e^{-kz}\psi(z)dz,$$

where δ is a positive constant. the function $\psi(\delta+i\gamma)$ is the Fourier transform of the real function $e^{-\delta k}LB_{\boldsymbol{\beta}}(k;t)$, then $\psi(\delta+i\gamma)$ has a even real part and a odd imaginary part. This is useful to simplify the expression above.

$$LB_{\boldsymbol{\beta}}(k;t) \quad = \quad P(t,T)\,\frac{e^{-\delta k}}{\pi}\int_{0}^{+\infty} Re(e^{-i\gamma k}\psi(\delta+i\gamma)d\gamma$$

E Proof of Proposition 3.1

We consider the function

$$f(k,k_h) = \mathbb{E}_t^T[(e^{\mathbf{X}(T)^\top C_h\mathbf{X}(T)+\mathbf{b}_h^\top\mathbf{X}(T)} - e^{k_h})^+\, I(\mathbf{X}(T)^\top\Gamma\,\mathbf{X}(T)+\boldsymbol{\beta}^\top\mathbf{X}(T)<k)],$$

we apply the extended Fourier transform with respect to the variable k and the Fubini's theorem and we obtain

$$\int_{-\infty}^{+\infty} e^{zk}\, f(k,k_h)\, dk = -\mathbb{E}_t^T\left[(e^{\mathbf{X}(T)^\top C_h\mathbf{X}(T)+\mathbf{b}_h^\top\mathbf{X}(T)} - e^{k_h})^+\,\frac{e^{z\,(\mathbf{X}(T)^\top\Gamma\,\mathbf{X}(T)+\boldsymbol{\beta}^\top\mathbf{X}(T))}}{z}\right]$$

The integral converges for $k\to+\infty$ if $Re(z)<0$, then we apply a second extended Fourier transform with respect to the variable k_h

$$\begin{aligned}
&- \int_{-\infty}^{+\infty} e^{y\,k_h}\,\frac{1}{z}\,\mathbb{E}_t^T\left[(e^{\mathbf{X}(T)^\top C_h\mathbf{X}(T)+\mathbf{b}_h^\top\mathbf{X}(T)} - e^{k_h})^+\, e^{z\,(\mathbf{X}(T)^\top\Gamma\,\mathbf{X}(T)+\boldsymbol{\beta}^\top\mathbf{X}(T))}\right] dk_h\\
&= -\frac{1}{z}\,\mathbb{E}_t^T\Big[\Big(\int_{-\infty}^{+\infty} e^{y\,k_h}\,\left(e^{\mathbf{X}(T)^\top C_h\mathbf{X}(T)+\mathbf{b}_h^\top\mathbf{X}(T)} - e^{k_h}\right)\\
&\qquad I(\mathbf{X}(T)^\top C_h\mathbf{X}(T)+\mathbf{b}_h^\top\mathbf{X}(T)>k_h)\,dk_h\Big)\, e^{z\,(\mathbf{X}(T)^\top\Gamma\,\mathbf{X}(T)+\boldsymbol{\beta}^\top\mathbf{X}(T))}\Big]
\end{aligned}$$

The integral converges for $k_h\to-\infty$ if $Re(y)>0$. Then the function $\psi(z,y)$ is in form

$$\begin{aligned}
\psi(z,y) &= \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk_h\, e^{zk}e^{yk_h}\, f(k,k_h)\\
&= -\frac{\Phi(z\boldsymbol{\beta}+(y+1)\mathbf{b}_h, z\,\Gamma+(y+1)C_h)}{zy(y+1)}
\end{aligned}$$

and it is defined for $Re(z)<0$ and $Re(y)>0$.

Finally $f(k,k_h)$ is the inverse transform of $\psi(z,y)$ in the sense of Cauchy principal value integral

$$f(k,k_h) \quad = \quad \frac{1}{(i2\pi)^2}\lim_{\xi\to\infty}\lim_{\varsigma\to\infty}\int_{\delta-i\xi}^{\delta+i\xi} dz\, e^{-zk}\int_{\eta-i\varsigma}^{\eta+i\varsigma} dy\, e^{-yk_h}\psi(z,y),$$

where $\delta < 0$ and $\eta > 0$ are constants. Recognising that $\psi(\delta+i\gamma,\eta+i\omega)$ is the double Fourier transform of the function $e^{\delta k}e^{\eta k_h}f(k,k_h)$ we obtain

$$f(k,k_h) \quad = \quad \frac{e^{-\delta k}e^{-\eta k_h}}{4\pi^2}\lim_{\xi\to\infty}\lim_{\varsigma\to\infty}\int_{-\xi}^{+\xi} d\gamma\, e^{-i\gamma k}\int_{-\varsigma}^{+\varsigma} d\omega\, e^{-i\omega k_h}\psi(\delta+i\gamma,\eta+i\omega),$$

where $\delta < 0$ and $\eta > 0$ are constants. The inner integral of the above formula is the Fourier transform of a real function, then we can use the same symmetry properties explained in Appendix D and we obtain

$$f(k,k_h) = \frac{e^{-\delta k}e^{-\eta k_h}}{2\pi^2}\lim_{\xi\to\infty}\int_{0}^{+\xi} d\gamma\, Re\left(e^{-i\gamma k}\lim_{\varsigma\to\infty}\int_{-\varsigma}^{+\varsigma} d\omega\, e^{-i\omega k_h}\psi(\delta+i\gamma,\eta+i\omega)\right).$$

F Proof of the analytical lower bound for Gaussian affine models

Since $\mathbf{X}(T) \sim N(\boldsymbol{\mu}, V)$ in T-forward measure, then the approximate exercise region $\mathcal{G}$ becomes

$$\mathcal{G} = \{\omega \in \Omega : \boldsymbol{\beta}^\top \mathbf{X}(T) > k\} = \{\omega \in \Omega : z > d\},$$

where z is a standard normal random variable and $d = \frac{k-\boldsymbol{\beta}^\top\boldsymbol{\mu}}{\sqrt{\boldsymbol{\beta}^\top V\boldsymbol{\beta}}}$.

The lower bound expression can be written using the law of iterative expectation

$$LB_{\boldsymbol{\beta}}(k;t) = P(t,T)\,\mathbb{E}_t^T\left[\mathbb{E}_t^T\left[\left(\sum_{h=1}^{n} w_h e^{\mathbf{b}_h{}^\top\mathbf{X}(T)+a_h} - 1\right)|z\right] I(z>d)\right].$$

Conditionally to the random variable z, the variable $\mathbf{X}$ is distributed as a multivariate normal with mean and variance

$$\mathbb{E}_t^T[\mathbf{X}|z] = \boldsymbol{\mu} + z\cdot\mathbf{v} \text{ and } Var(\mathbf{X}|z) = V - \mathbf{v}\mathbf{v}^\top, \text{ with } \mathbf{v} = \frac{V\boldsymbol{\beta}}{\sqrt{\boldsymbol{\beta}^\top V\boldsymbol{\beta}}}.$$

We can now compute the inner expectation

$$
\begin{aligned}
LB_{\boldsymbol{\beta}}(k;t) &= P(t,T)\left(\sum_{h=1}^{n} w_h\,\mathbb{E}_t^T\left[e^{\mathbf{b}_h^\top\boldsymbol{\mu}+z\mathbf{b}_h^\top\mathbf{v}+\frac{1}{2}V_h}I(z>d)\right]-\mathbb{E}_t^T\left[I(z>d)\right]\right)\\
&= \sum_{h=1}^{n} w_h e^{a_h+\mathbf{b}_h^\top\boldsymbol{\mu}+\frac{1}{2}V_h+\frac{1}{2}d_h^2}N(d_h-d)-N(-d).
\end{aligned}
$$

where $V_h=\mathbf{b}_h^\top(V-\mathbf{v}\mathbf{v}^\top)\mathbf{b}_h$, $d_h=\mathbf{b}_h^\top\mathbf{v}$ and $N(x)$ is the cumulative distribution function of standard normal variable.

G Proof of the upper bound formula for Gaussian affine models

Since $\mathbf{X}\sim\mathcal{N}(\boldsymbol{\mu},V)$ in T-forward measure and using the law of iterative expectations, then

$$
\begin{aligned}
&\mathbb{E}_t^T[(w_h e^{a_h+b_h^\top\mathbf{X}(T)}-K_h)^+ I(\beta^\top X<k)]\\
=\ &\mathbb{E}_t^T[\mathbb{E}_t^T[(w_h e^{a_h+b_h^\top\mathbf{X}(T)}-K_h)^+|Z]I(Z<d)],\\
=\ &\int_{-\infty}^{d}dz\,\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,\mathbb{E}_t^T[(w_h e^{a_h+\mathbf{b}_h^\top\mathbf{X}(T)}-K_h)^+|Z=z].
\end{aligned}
$$

where $Z\sim\mathcal{N}(0,1)$ and $d=\frac{k-\beta^\top\mu}{\sqrt{\beta^\top V\beta}}$.

Since $\mathbf{b}_h^\top\mathbf{X}$ conditioned to the variable Z, is a normal random variable with mean and variance

$$
\begin{aligned}
&M_h=\mathbb{E}_t^T[\mathbf{b}_h^\top\mathbf{X}|Z=z]=\mathbf{b}_h^\top\boldsymbol{\mu}+z\mathbf{b}_h^\top\mathbf{v},\\
&V_h=Var_t[\mathbf{b}_h^\top\mathbf{X}|Z=z]=\mathbf{b}_h^\top(V-\mathbf{v}\mathbf{v}^\top)\mathbf{b}_h\\
&\mathbf{v}=\frac{V\boldsymbol{\beta}}{\sqrt{\boldsymbol{\beta}^\top V\boldsymbol{\beta}}},
\end{aligned}
$$

then the conditioned expectation can be evaluated with a Black formula

$$
\begin{aligned}
&\mathbb{E}_t^T[(w_h e^{a_h+b_h^\top\mathbf{X}(T)}-K_h)^+|Z=z]\\
&=w_h e^{a_h}\left(e^{M_h+\frac{V_h}{2}}N\left(\frac{M_h-\log Y_h+V_h}{\sqrt{V_h}}\right)-Y_h N\left(\frac{M_h-\log Y_h}{\sqrt{V_h}}\right)\right),
\end{aligned}
$$

where $Y_h=\frac{K_h}{w_h e^{a_h}}$ and $N(x)$ is the cumulative distribution function of a standard normal variable.