Published for SISSA by Springer

Received: *April 5, 2017*
Revised: *June 27, 2017*
Accepted: *October 10, 2017*
Published: *October 23, 2017*

Observables and dispersion relations in κ-Minkowski spacetime

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Abstract: We revisit the notion of quantum Lie algebra of symmetries of a noncommutative spacetime, its elements are shown to be the generators of infinitesimal transformations and are naturally identified with physical observables. Wave equations on noncommutative spaces are derived from a quantum Hodge star operator.

This general noncommutative geometry construction is then exemplified in the case of κ-Minkowski spacetime. The corresponding quantum Poincaré-Weyl Lie algebra of infinitesimal translations, rotations and dilatations is obtained. The d'Alembert wave operator coincides with the quadratic Casimir of quantum translations and it is deformed as in Deformed Special Relativity theories. Also momenta (infinitesimal quantum translations) are deformed, and correspondingly the Einstein-Planck relation and the de Broglie one. The energy-momentum relations (dispersion relations) are consequently deduced. These results complement those of the phenomenological literature on the subject.

Keywords: Non-Commutative Geometry, Quantum Groups, Space-Time Symmetries

ArXiv ePrint: 1703.08726

https://doi.org/10.1007/JHEP10(2017)152

Contents

1 Introduction

The open challenge of a quantum description of the gravitational interaction and the corresponding notion of quantum spacetime is particularly demanding because of the very limited experimental hints at our disposal. Indeed the quantum structure of spacetime, due to quantum gravity effects, is expected to become relevant at Planck scales ($l_p = \sqrt{\frac{G}{\hbar c^3}} \sim 10^{-33}$ cm). Since these energy regimes are not directly accessible we have to study indirect signatures, for example related to cosmological data near inflationary epoch (a few orders of magnitude away from regimes characterised by Planck scale energies) or related to modified dispersion relations of light.

Indeed it is conceivable that light travelling in a quantum spacetime (a dynamical spacetime that interacts with photons) has a velocity dependent on the photons energy.

Even a tiny modification of the usual dispersion relations could then be detected due to the cumulative effect of light travelling long distances. A natural setting for this study is that of Gamma Ray Bursts (GRB) from distant galaxies [1]. Another possibility is that of high precision (quantum) optics experiments based on interferometry techniques like the Holometer one in Fermilab [2] or the table top experiment devised in [3].

These possibilities motivate the study of phenomenological models that lead to modified dispersion relations. Lorentz invariance violating (LIV) theories generically provide modified dispersion relations, see for example [4]. A whole class where the Lorentz group (or its realization) is modified, so that new relativistic symmetries replace the classical one, goes under the name of Deformed (or Doubly) special relativity theories (DSR theories) [5–8]. Many of these phenomenological models describe spacetime features and wave equations that are typical of noncommutative spacetimes, the prototypical example being κ-Minkowski spacetime, where coordinates satisfy the relations $x^0 \star x^j - x^j \star x^0 = \frac{i}{\kappa} x^j$, $x^i \star x^j - x^j \star x^i = 0$; here we consider $\frac{1}{\kappa}$ of the order of the Planck length. See [9] for an early relation between DSR and κ-deformed symmetries. The relation between DSR theories and noncommutative spacetimes is very interesting because noncommutative spacetimes, and their quantum symmetry groups, independently from DSR studies, naturally arise as models of quantum spacetime where discretization and indetermination relations are realized in spacetime itself rather than phase-space. This is indeed expected by gedanken experiments probing spacetime structure at Planck scale; furthermore noncommutative spacetimes features, like generalized uncertainty relations (involving space and time coordinates) or minimal area and volume elements arise also in String Theory and Loop Quantum Gravity, see e.g. [10].

In this paper we complement the phenomenological or bottom up approach of constructing (realizations of) spacetime symmetries, observables and field equations that model expected and possibly detectable quantum gravity effects with a top-down approach where we just assume a general noncommutative structure of spacetime and then use noncommutative differential geometry to derive the physics of fields propagating in this noncommutative spacetime.

We focus on spacetimes that are obtained via a Drinfeld twist procedure, these form a very large class of noncommutative spacetimes (including the most studied Moyal-Weyl one $x^i \star x^j - x^j \star x^i = i\theta^{ij}$ [11, 12] and the κ-Minkowski one, $x^0 \star x^i - x^i \star x^0 = \frac{i}{\kappa} x^i$ [13–15]). We present a general method on how to study observables, wave equations and the corresponding dispersion relations, thus providing a wide framework for physical models of fields in noncommutative spaces. The construction is quite constrained by the mathematical consistency of the noncommutative differential geometry and the notion of infinitesimal symmetry generators (e.g. translations-momenta). We thus contribute filling the gap between theory and phenomenology, providing sound and mathematically consistent tools for phenomenological studies as well as a deeper physical understanding of the mathematical structures associated with noncommutative spaces.

We then apply the theory thus developed to the κ-Minkowski spacetime and derive the same equations of motion studied in the phenomenological (bottom up) approach considered in [7].

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We show that the usual Einstein-Planck relation $E = \hbar\omega$ is modified to $\hbar\omega/(1-\omega/c\kappa)$ leading to a maximum Planck energy or frequency per elementary particle. Similarly, the de Broglie momentum-wavelength relation is modified. This implies that massless fields in κ-Minkowski spacetime have no modified dispersion relations, in agreement with [7] and disagreement with [16, 17].

As we show in the sequel to this paper [18] the methods here developed do in general imply modified dispersion relations for massless fields once we leave flat noncommutative spacetime by turning on a nontrivial metric background.

Our primary interest in the present paper is the equation of motion of massless fields in flat spacetime. In the commutative case these propagate in Minkowski spacetime, and the relevant symmetry group is the conformal group. Hence we consider the noncommutative differential geometry on κ-Minkowski spacetime that is covariant under the action of a quantum conformal group (in particular the Minkowski metric won't be invariant but covariant under quantum dilatations). For simplicity and brevity rather than deforming the conformal group we focus on the most relevant part, the Poincaré-Weyl (also simply called Weyl) group of Poincaré transformations and dilatations, the inclusion of the special conformal transformations being straightforward. The choice to focus on Poincaré-Weyl symmetry rather than Poincaré symmetry is motivated by the study of massless fields and the interest in applying the general noncommutative differential geometry construction via twist deformation that we develop in section 2 to the case of the well known κ-Minkowski noncommutative spacetime.

We further observe that the only term breaking conformal symmetry in the standard model action is the Higgs mass term and that there are interesting scenarios where there is no such term in the classical action, so that the conformal group is a fundamental symmetry of the tree level action. There, according to the seminal work [19], the standard model masses are generated via radiative corrections that break the classical scale invariance of the Higgs potential thus inducing the spontaneous symmetry breaking of the electroweak interaction. These models, see e.g. [20–22], provide a solution to the hierarchy problem up to Planck scale thanks to classical scale invariance. It is then intriguing to speculate that due to quantum gravity effects the conformal symmetry at Planck scale is twist deformed to the quantum conformal symmetry associated with the quantum Poincaré-Weyl Lie algebra that we study in section 3.2. The deformation parameter $\frac{1}{\kappa}$ is indeed dimensionful and considered of the order of the Planck length.

A key point in order to have a physical interpretation of the wave equation (e.g. of the corresponding dispersion relations) is that of identifying the physical generators of Poincaré-Weyl transformations and in particular the four-momentum operators. In this paper we first attack the problem mathematically and show that given a twist (hence a given quantum group, deformation of a usual group) there is a unique notion of quantum Lie algebra (with quantum Lie bracket and quantum Jacobi identity). Quantum Lie algebra elements act on fields on noncommutative spacetime as quantum infinitesimal transformations because they satisfy a specific deformation of the Leibniz rule. They are obtained via a quantization map $\mathcal{D}$ applied to classical Lie algebra elements. Then we confirm their interpretation as generators of infinitesimal transformations by recalling their associated differential geometry; in particular on κ-Minkowski space the quantum Lie algebra of momentum generators is shown to give the differential calculus. Because of these properties these quantum Lie algebra generators are naturally identified with the physical observables. Of course it would be interesting to confirm this identification with a Noether theorem analysis. Noether theorem for theories on noncommutative spacetimes has to be further understood. For preliminary work we refer for example to [23, 24]. The methods we advocate, possibly combined with Drinfeld twist deformed Hamiltonian mechanics, see e.g. [25], should provide powerful tools in the study of conserved charges in noncommutative spacetimes.

The same quantization map $\mathcal{D}$ used to single out the quantum Lie algebra of infinitesimal generators from the undeformed Lie algebra is then used to obtain the quantum Hodge star operator from the usual Hodge star operator. This procedure is very general and does not need to rely on a flat metric. The associated Laplace-Beltrami operator and the corresponding wave equation is then presented. In the case of κ-Minkowski space the Laplace-Beltrami operator becomes the d'Alembert operator and we show that it equals the quadratic Casimir of quantum translations. This implies covariance of the wave equation under the quantum Poincaré-Weyl group. Hence, in particular, besides the parameter c (velocity of light) also the dimensionful deformation parameter $\frac{1}{\kappa}$ (the Planck length) is constant under quantum Poincaré-Weyl transformations. In other words, as usual in noncommutative geometry, the quantum group generalization of the notion of symmetry allows to extend the principle of relativity of frames to that of relativity of frames related by quantum symmetry transformations,[1] (cf. also [26, section 8]).

Finally, we compare our results with the pioneering approach in [7, 8]. We show that the nonlinear realization of the Lorentz group considered in [7] is implemented by the quantization map $\mathcal{D}$ that gives the Poincaré-Weyl quantum Lie algebra. Since the quantization map $\mathcal{D}$ does not depend on the representation considered, this allows to extend the construction in [7], based on momentum space, to arbitrary representations; in particular to position space. Furthermore, in our scheme the wave equation turns out to be the same as that in [7], however the identification of the physical momenta differs from that in [7].

2 Twist noncommutative geometry

The noncommutative deformations of spacetime we consider arise from the action of a symmetry group (group of transformations) on spacetime, e.g. Poincaré, Poincaré-Weyl, or conformal groups on Minkowski spacetime. Physical observables on these noncommutative spacetimes are related to their symmetries. Like in the commutative case, where energy and momentum are the generator of spacetime translation, in the noncommutative case we identify energy and momentum as the generators of noncommutative spacetime translations. Since infinitesimal transformations are obtained as actions of the Lie algebra of the symmetry group on the spacetime and on the matter fields, we first have to recall the notion of quantum group and quantum Lie algebra of a quantum group. Then we consider their

[1] All frames related by quantum Poincaré-Weyl transformations detect the same values of c and $\frac{1}{\kappa}$. In this paper for simplicity we do not consider finite transformations, we focus on quantum Lie algebras and infinitesimal transformations, with c and $\frac{1}{\kappa}$ that are constant under these transformations.

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symmetry transformations on the corresponding noncommutative spaces. The construction of the differential geometry on these spaces, covariant under these symmetry transformations, leads to wave equations describing fields propagating in noncommutative spacetimes.

2.1 Quantum symmetries and quantum Lie algebras

Let G be a Lie group of transformations on a manifold M; infinitesimal transformations are governed by its Lie algebra g, or equivalently by the universal enveloping algebra Ug. We recall that Ug is the sum of products of elements of g (modulo the Lie algebra relations) and that it is a Hopf algebra with coproduct $\Delta : Ug \to Ug \otimes Ug$, counit $\epsilon : Ug \to \mathbb{C}$ and antipode S given on the generators $u \in g$ as: $\Delta(u) = u \otimes 1 + 1 \otimes u$, $\varepsilon(u) = 0$, $S(u) = -u$ and extended to all Ug by requiring Δ and ε to be linear and multiplicative while S is linear and antimultiplicative. In the following we use Sweedler notation for the coproduct

$$\Delta(\xi) = \xi_1 \otimes \xi_2 \tag{2.1}$$

where $\xi \in Ug$, $\xi_1 \otimes \xi_2 \in Ug \otimes Ug$ and a sum over ξ_1 and ξ_2 is understood ($\xi_1 \otimes \xi_2 = \sum_i \xi_{1i} \otimes \xi_{2i}$). There is a canonical action of Ug on itself obtained from the coproduct and the antipode, it is given by the adjoint action, defined by, for all $\zeta, \xi \in Ug$,

$$\zeta(\xi) \equiv ad_\zeta(\xi) = \zeta_1 \xi S(\zeta_2)\,. \tag{2.2}$$

When restricted to Lie algebra elements this becomes the Lie bracket, indeed we have, for all $u, u' \in g$,

$$[u, u'] = u_1 u' S(u_2) = uu' - u'u \tag{2.3}$$

where we used that $\Delta(u) = u \otimes 1 + 1 \otimes u$ and that $S(u) = -u$. Notice that the Ug-adjoint action restricts to and action on g, i.e., for all $\xi \in Ug$, $v \in g$, $\zeta(v) \in g$. This is easily seen by writing ζ as sums of products $uu' \dots u''$ of Lie algebra elements, and then by iteratively using (2.3): $(uu' \dots u'')(v) = [u[u' \dots [u'', v]]]$.

We deform spacetime by first deforming the Hopf algebra Ug and then functions and matter fields on spacetime itself. Since we work in the deformation quantization context we extend the notion of enveloping algebra to formal power series in λ and we correspondingly consider the Hopf algebra $Ug[[\lambda]]$ over the ring $\mathbb{C}[[\lambda]]$ of formal power series with complex coefficients. In the following, for the sake of brevity we will often denote $Ug[[\lambda]]$ by Ug.[2]

Following Drinfeld, a twist element $\mathcal{F}$ on the universal enveloping algebra $Ug[[\lambda]]$ of a Lie algebra g is an invertible element of $Ug[[\lambda]] \otimes Ug[[\lambda]]$ satisfying the cocycle and

[2]Physically we are interested in the lowest order corrections in λ to dispersion relations and related expressions. In this case λ is linked to the Planck energy E_p and the expansion will be in $\hbar\omega/E_p$ with ω the frequency of the wave. The lowest order corrections are then well defined also nonformally, i.e. with $\hbar\omega/E_p$ a real number. Specific expression can be well defined nonformally also at all powers in $\hbar\omega/E_p$, see e.g. the Einstein-Planck relations (4.3). The treatment of λ as a formal parameter is a mathematical requirement for the $\star$-product between arbitrary smooth functions to be well defined. However for the exponential (hence analytic) functions we consider as solutions of the wave equation it is consistent to consider $\hbar\omega/E_p$ a real number.

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normalisation conditions:

$$(\mathcal{F}\otimes 1)(\Delta\otimes id)\mathcal{F} = (1\otimes\mathcal{F})(id\otimes\Delta)\mathcal{F}, \tag{2.4}$$

$$(id\otimes\epsilon)\mathcal{F} = (\epsilon\otimes id)\mathcal{F} = 1\otimes 1\,. \tag{2.5}$$

The first relation implies associativity of the $\star$-products induced by the twist; consistently with the second relation we also require that $\mathcal{F} = 1\otimes 1 + \mathcal{O}(\lambda)$ so that for $\lambda\to 0$ we recover the classical (undeformed) products.

Given a twist $\mathcal{F}$ on the Hopf algebra Ug we have a new Hopf algebra structure $Ug^{\mathcal{F}} = \big(Ug, m, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}}\big)$; $Ug^{\mathcal{F}}$ equals Ug as vector space and also as an algebra (the product m is undeformed, and also the counit ε) the deformed coproduct and antipode are defined by, for all $\xi\in Ug$,

$$\Delta^{\mathcal{F}}(\xi) = \mathcal{F}\Delta(\xi)\mathcal{F}^{-1}\,, \qquad S^{\mathcal{F}}(\xi) = \chi S(\xi)\chi^{-1} \tag{2.6}$$

where $\chi = \mathrm{f}^\alpha S(\mathrm{f}_\alpha)$ and we used the notation: $\mathcal{F} = \mathrm{f}^\alpha\otimes\mathrm{f}_\alpha$, $\quad\mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha\otimes\bar{\mathrm{f}}_\alpha$ (sum over α understood). We denote by $ad^{\mathcal{F}} : Ug^{\mathcal{F}}\otimes Ug^{\mathcal{F}}\to Ug^{\mathcal{F}}$ the $Ug^{\mathcal{F}}$-adjoint action, it is given by, for all $\xi,\zeta\in Ug^{\mathcal{F}}$, $ad^{\mathcal{F}}_\xi(\zeta) = \xi_{1^{\mathcal{F}}}\zeta S^{\mathcal{F}}(\xi_{2^{\mathcal{F}}})$, where we used the Sweedler notation $\Delta^{\mathcal{F}}(\xi) = \xi_{1^{\mathcal{F}}}\otimes\xi_{2^{\mathcal{F}}}$.

We now study the quantum Lie algebra $g^{\mathcal{F}}$ of the Hopf algebra $Ug^{\mathcal{F}}$. To this aim let us recall that there is a one-to-one correspondence between Lie algebras g and universal enveloping algebras Ug: given g then Ug is canonically constructed, vice versa $g\subset Ug$ is the subspace of primitive elements of Ug, i.e., of those elements that satisfy $\Delta(u) = u\otimes 1 + 1\otimes u$. From this expression it is not difficult to show that $\varepsilon(u) = 0$ and $S(u) = -u$ so that as in (2.3) the adjoint action of two primitive elements u and v equals their commutator: $[u,v] := ad_u(v) = uv - vu$. Now it is easy to show that $uv - vu$ is again a primitive element thus proving that g with the bracket $[\ ,\ ]$ is a Lie algebra: the Lie algebra of the universal enveloping algebra Ug. Similarly, in the quantum case, following [27], we have a quantum Lie algebra $g^{\mathcal{F}}$ of a quantum universal enveloping algebra $Ug^{\mathcal{F}}$ when

i) $g^{\mathcal{F}}$ generates $Ug^{\mathcal{F}}$,

ii) $\Delta^{\mathcal{F}}(g^{\mathcal{F}})\subset g^{\mathcal{F}}\otimes 1 + Ug^{\mathcal{F}}\otimes g^{\mathcal{F}}$

iii) $[g^{\mathcal{F}}, g^{\mathcal{F}}]_{\mathcal{F}}\subset g^{\mathcal{F}}$

where the quantum Lie bracket $[\ ,\ ]_{\mathcal{F}}$ is the restriction of the $Ug^{\mathcal{F}}$-adjoint action to $g^{\mathcal{F}}$, i.e., for all $u^{\mathcal{F}}, v^{\mathcal{F}}\in g^{\mathcal{F}}$ we have

$$[u^{\mathcal{F}}, v^{\mathcal{F}}]_{\mathcal{F}} := u^{\mathcal{F}}_{1^{\mathcal{F}}}\, v^{\mathcal{F}} S^{\mathcal{F}}(u^{\mathcal{F}}_{2^{\mathcal{F}}})\,. \tag{2.7}$$

Property *iii)* states that the restriction to $g^{\mathcal{F}}$ of the $Ug^{\mathcal{F}}$-adjoint action is well defined on $g^{\mathcal{F}}$. Since from Property *i)* we have that $g^{\mathcal{F}}$ generates $Ug^{\mathcal{F}}$ we conclude that $g^{\mathcal{F}}\subset Ug^{\mathcal{F}}$ is invariant under the $Ug^{\mathcal{F}}$-adjoint action, i.e. for all $\xi\in Ug^{\mathcal{F}}$, $v^{\mathcal{F}}\in g^{\mathcal{F}}$, $ad^{\mathcal{F}}_\xi(v^{\mathcal{F}})\in g^{\mathcal{F}}$. For example $ad^{\mathcal{F}}_{u^{\mathcal{F}}u'^{\mathcal{F}}}(v^{\mathcal{F}}) = ad^{\mathcal{F}}_{u^{\mathcal{F}}}(ad^{\mathcal{F}}_{u'^{\mathcal{F}}}(v)) = [u^{\mathcal{F}}, [u'^{\mathcal{F}}, v]_{\mathcal{F}}]_{\mathcal{F}}\in g^{\mathcal{F}}$.

Property *ii)* states that the elements in $g^{\mathcal{F}}$ are quasi-primitive, i.e., we require a minimal deviation from the usual coproduct property $\Delta(u) = u\otimes 1 + 1\otimes u$ of elements $u\in g$.

Combining *ii)* with *iii)* it can be shown [28] [29, section 2.3] that the bracket $[\,,\,]_{\mathcal{F}}$ is a deformed commutator which is quadratic in the generators of the Lie algebra $g^{\mathcal{F}}$.

In the twist deformation case we are considering there is a canonical construction in order to obtain the quantum Lie algebra $g^{\mathcal{F}}$ of the Hopf algebra $Ug^{\mathcal{F}}$ [30] [26, section 7.7]. We simply have $g^{\mathcal{F}} = \mathcal{D}(g)$, i.e.,

$$g^{\mathcal{F}} := \{u^{\mathcal{F}} \in Ug^{\mathcal{F}};\ u^{\mathcal{F}} = \mathcal{D}(u),\ \text{with } u \in g\} \tag{2.8}$$

where, for all $\xi \in Ug$,

$$\mathcal{D}(\xi) = \bar{\mathrm{f}}^{\alpha}(\xi)\bar{\mathrm{f}}_{\alpha} = \bar{\mathrm{f}}^{\alpha}_{1}\xi S(\bar{\mathrm{f}}^{\alpha}_{2})\bar{\mathrm{f}}_{\alpha}\,, \tag{2.9}$$

the action $\bar{\mathrm{f}}^{\alpha}(\xi)$ being the adjoint action (2.2). Property *i)* follows form the twist property $\mathcal{F} = 1 \otimes 1 + \mathcal{O}(\lambda)$ that implies $\mathcal{D} = \mathrm{id} + \mathcal{O}(\lambda)$.[3] Property *ii)* holds because, cf. (3.12) and (3.17) in [30], for all $u \in g$,

$$\Delta^{\mathcal{F}}(u^{\mathcal{F}}) = u^{\mathcal{F}} \otimes 1 + \bar{R}^{\alpha} \otimes (\bar{R}_{\alpha}(u))^{\mathcal{F}} \tag{2.10}$$

where $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$ is the so-called universal R-matrix, and we used the notation $\mathcal{R}^{-1} = \bar{R}^{\alpha} \otimes \bar{R}_{\alpha}$ (the action $\bar{R}^{\alpha}(u)$ of $\bar{R}^{\alpha}$ on $u \in g$ is the adjoint action (2.2), and $\bar{R}^{\alpha}(u) \in g$, cf. sentence after (2.3)). Property *iii)* holds because it is equivalent to equation (4.6) in [30] (just apply $\mathcal{D}$ to (4.6)). Similarly to the undeformed case we then have that the adjoint action (2.7) equals the braided commutator, for all $u^{\mathcal{F}}, v^{\mathcal{F}} \in g^{\mathcal{F}}$,

$$[u^{\mathcal{F}}, v^{\mathcal{F}}]_{\mathcal{F}} = u^{\mathcal{F}} v^{\mathcal{F}} - (\bar{R}^{\alpha}(v))^{\mathcal{F}}(\bar{R}_{\alpha}(u))^{\mathcal{F}}\,. \tag{2.11}$$

Furthermore the deformed Lie bracket satisfies the braided-antisymmetry property and the braided-Jacobi identity

$$[u^{\mathcal{F}}, v^{\mathcal{F}}]_{\mathcal{F}} = -[(\bar{R}^{\alpha}(v))^{\mathcal{F}}, (\bar{R}_{\alpha}(u))^{\mathcal{F}}]_{\mathcal{F}} \tag{2.12}$$

$$[u^{\mathcal{F}}, [v^{\mathcal{F}}, z^{\mathcal{F}}]_{\mathcal{F}}]_{\mathcal{F}} = [[u^{\mathcal{F}}, v^{\mathcal{F}}]_{\mathcal{F}}, z^{\mathcal{F}}]_{\mathcal{F}} + [(\bar{R}^{\alpha}(v))^{\mathcal{F}}, [(\bar{R}_{\alpha}(u))^{\mathcal{F}}, z^{\mathcal{F}}]_{\mathcal{F}}]_{\mathcal{F}}\,. \tag{2.13}$$

A proof of (2.11)–(2.13) follows immediately by applying $\mathcal{D}$ to eq. (3.7) (3.9), (3.10) in [30]; indeed the quantum Lie algebras presented here and there are isomorphic via $\mathcal{D}^{-1}$ (cf. also [26, section 7 and section 8]).

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[3] *Proof.* Since as vector spaces $Ug^{\mathcal{F}} = Ug[[\lambda]]$, any element $\xi \in Ug^{\mathcal{F}} = Ug[[\lambda]]$ is a sum $\xi = \sum_{j\geq 0} \lambda^j \xi_j$ with $\xi_j \in Ug$ (and no λ dependence in ξ_j). Since g generates Ug we have $\xi_j = \sum_{m_1\ldots m_n} c_{j,m_1,\ldots m_n} u_1^{m_1} u_2^{m_2} \ldots u_n^{m_n}$ where $u_1, u_2, \ldots u_n$ are a basis of g. Then we have

$$\xi = \sum_j \lambda^j \xi_j = \sum_{j,m_1,\ldots m_n} \lambda^j c_{j,m_1,\ldots m_n} u_1^{m_1} \ldots u_n^{m_n} = \sum_{j,m_1,\ldots m_n} \lambda^j c_{j,m_1,\ldots m_n} \mathcal{D}(u_1)^{m_1} \ldots \mathcal{D}(u_n)^{m_n} + \lambda \xi^{(1)}$$

where $\xi^{(1)} \in Ug[[\lambda]]$. We can therefore expand $\xi^{(1)}$ in the same way as done with ξ and obtain $\lambda\xi^{(1)} = \sum_{j,m_1,\ldots m_n} \lambda^{j+1} c^{(1)}_{j,m_1,\ldots m_n} \mathcal{D}(u_1)^{m_1} \ldots \mathcal{D}(u_n)^{m_n} + \lambda^2 \xi^{(2)}$. Iterating this procedure we thus arrive at the expression $\xi = \sum_{j,m_1,\ldots m_n} (\lambda^j c_{j,m_1,\ldots m_n} + \lambda^{j+1} c^{(1)}_{j,m_1,\ldots m_n} + \ldots \lambda^{j+k} c^{(k)}_{j,m_1,\ldots m_n}) \mathcal{D}(u_1)^{m_1} \ldots \mathcal{D}(u_n)^{m_n} + \lambda^{(k+1)} \xi^{(k+1)}$ that shows that ξ is generated by the elements $\mathcal{D}(u_1), \mathcal{D}(u_2), \ldots \mathcal{D}(u_n)$ because this is so up to order λ^k with k an arbitrarily big integer $\Box$. An explicit proof for the algebra we will consider follows immediately from (3.9) and (3.28).

In the classical case the physical operators associated with a symmetry Hopf algebra Ug are the elements of the Lie algebra g (e.g. momenta boosts and rotations in the universal enveloping algebra of the Poincaré group). It is then natural that the physical operators associated with a symmetry Hopf algebra $Ug^{\mathcal{F}}$ are the elements of the quantum Lie algebra $g^{\mathcal{F}}$. To clarify this point we have to study representations of $Ug^{\mathcal{F}}$.

2.2 Noncommutative algebras

Let G be a Lie group of transformations on a manifold M (e.g. the spacetime manifold) and let A be the algebra of smooth functions on M (extended to formal power series in λ). If the action of G on M is via diffeomorphisms, then the Lie algebra g acts on A via the Lie derivative, which is extended as an action to all Ug, i.e., for all $\xi, \zeta \in Ug$ $\mathcal{L}_\xi \mathcal{L}_\zeta = \mathcal{L}_{\xi\zeta}$. The compatibility between the Ug action and the product of A is encoded in the property that the Lie algebra elements $u \in g$ act as derivations, i.e., for all $u \in g$ and $f, h, \in A$, $\mathcal{L}_u(fh) = \mathcal{L}_u(f)h + f\mathcal{L}_u(h)$, or simply $u(fh) = u(f)h + fu(h)$. Since $u(fh) = u(f)h + fu(h) = u_1(f)u_2(h)$ and the u's generate Ug this implies that for all $\xi \in Ug$, $f, h \in A$,

$$\xi(fh) = \xi_1(f)\, \xi_2(h)\,. \tag{2.14}$$

We say that A is a Ug-module algebra because it carries a representation of Ug and the Ug-action is compatible with the product of A as in (2.14).

Corresponding to the deformation $Ug \to Ug^{\mathcal{F}}$ we have the algebra deformation $A \to A_\star$. As vector spaces A and $A_\star$ coincide. The action of $Ug^{\mathcal{F}}$ on $A^{\mathcal{F}}$ is also the same as that of Ug on A. The space $A_\star$ is therefore still the space of smooth functions on M; it is the product that is deformed in the $\star$-product,

$$f \star h = \mu\{\mathcal{F}^{-1}(f \otimes h)\} = \bar{\mathrm{f}}^\alpha(f)\, \bar{\mathrm{f}}_\alpha(h)\,, \tag{2.15}$$

for all functions $f, h \in A_\star$, where $\bar{\mathrm{f}}^\alpha(f) = \mathcal{L}_{\bar{\mathrm{f}}^\alpha}(f)$. Associativity of the product follows from the twist cocycle property (2.4), moreover the product has been defined so that it is compatible with the $Ug^{\mathcal{F}}$ action: for all $\xi \in Ug^{\mathcal{F}}$, $\xi(f \star h) = \xi_{1^{\mathcal{F}}}(f)\, \xi_{2^{\mathcal{F}}}(h)$. This proves that $A_\star$ is a $Ug^{\mathcal{F}}$-module algebra. Equivalently we say that it is a $Ug^{\mathcal{F}}$-module algebra because the quantum Lie algebra elements $\mathcal{D}(u) \in g^{\mathcal{F}}$ act as twisted derivations, i.e., they obey the deformed Leibniz rule (cf. (2.10)),

$$\mathcal{D}(u)(f \star h) = \mathcal{D}(u)(f) \star h + \bar{R}^\alpha(f) \star \mathcal{D}(\bar{R}_\alpha(u))(h)\,. \tag{2.16}$$

A special case is when the manifold M is the group manifold G itself, then the Lie algebra g is identified with that of left invariant vector fields on G. The differential geometry on A (smooth functions on G) is described via these vector fields and the corresponding left invariant one-forms. Similarly, the differential geometry on $A_\star$ (the quantum group) is described in terms of the quantum Lie algebra $g^{\mathcal{F}}$: the quantum Lie algebra of left invariant vector fields on $A_\star$ [27, 31].

We can now sharpen our claim: among the many generators of the universal enveloping algebra $Ug^{\mathcal{F}}$ the physically relevant ones are the elements of the quantum Lie algebra $g^{\mathcal{F}}$ (and not for example of g). This is so because of their geometric properties: they generate $Ug^{\mathcal{F}}$, they are closed under the $Ug^{\mathcal{F}}$-adjoint action and their action on representations is via twisted derivations, so that they are *quantum infinitesimal transformations.*

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2.3 Differential calculus

Consider the algebra A of smooth functions on spacetime M and the action of the Lie algebra g on A via the Lie derivative. The Lie derivative lifts to the algebra $\Omega^\bullet = A \oplus \Omega^1 \oplus \Omega^2 \oplus \ldots$ of exterior forms on M. We then twist deform this algebra to the algebra $\Omega^\bullet_\star$, which as a vector space is the same as $\Omega^\bullet$ (more precisely we should write $\Omega^\bullet[[\lambda]]$, i.e. power series in the deformation parameter λ of classical exterior forms) but has the new product

$$\omega \wedge_\star \omega' = \bar{\mathrm{f}}^\alpha(\omega) \wedge \bar{\mathrm{f}}_\alpha(\omega') \,. \tag{2.17}$$

In particular when ω is a zero form f, then the wedge product is usually omitted and correspondingly the wedge $\star$-product reads $f \star \omega' = \bar{\mathrm{f}}^\alpha(f)\bar{\mathrm{f}}_\alpha(\omega')$.

Since the action of the Lie algebra g on forms is via the Lie derivative and the Lie derivative commutes with the exterior derivative the usual (undeformed) exterior derivative satisfies the Leibniz rule $\mathrm{d}(f \star g) = \mathrm{d}f \star g + f \star \mathrm{d}g$, and more in general, for forms of homogeneous degree $\omega \in \Omega^r$,

$$\mathrm{d}(\omega \wedge_\star \omega') = \mathrm{d}\omega \wedge_\star \omega' + (-1)^r \omega \wedge_\star \omega' \,. \tag{2.18}$$

We have constructed a differential calculus on the deformed algebra of exterior forms $\Omega^\bullet_\star$.

2.4 Hodge star operator

Another key ingredient in order to formulate field theories on a spacetime M is a metric. It acts on exterior forms via the Hodge star operator. Let us first recall few facts related with the Hodge star in the classical (undeformed) case. For an $n-$dimensional manifold M with metric g the Hodge $*$-operation is a linear map $* : \Omega^r(M) \to \Omega^{n-r}(M)$. In local coordinates an r-form is given by $\omega = \frac{1}{r!}\omega_{\mu_1 \ldots \mu_r} \mathrm{d}x^{\mu_1} \wedge \ldots \mathrm{d}x^{\mu_r}$ and the Hodge $*$-operator reads

$$* \, \omega = \frac{\sqrt{g}}{r!\,(n-r)!} \omega_{\mu_1 \ldots \mu_r} \epsilon^{\mu_1 \ldots \mu_r}{}_{\nu_{r+1} \ldots \ldots \nu_n} \mathrm{d}x^{\nu_{r+1}} \wedge \ldots \mathrm{d}x^{\nu_n} \tag{2.19}$$

where $\sqrt{g}$ is the square root of the absolute value of the determinant of the metric, the completely antisymmetric tensor $\epsilon_{\nu_1 \ldots \nu_n}$ is normalized to $\epsilon_{1 \ldots n} = 1$ and indices are lowered and raised with the metric g and its inverse. There is a one to one correspondence between metrics and Hodge star operators.

We define metrics on noncommutative spaces by defining the corresponding Hodge star operators on the $\star$-algebra of exterior forms $\Omega^\bullet_\star$. We first observe that the undeformed Hodge $*$-operator is A-linear: $*(\omega f) = *(\omega) f$, for any form ω and function f (of course, since A is commutative we equivalently have $*(f\omega) = f(*\omega)$). We then require the Hodge $*$-operator $*^{\mathcal{F}}$ on $\Omega^\bullet_\star$ to map r-forms into $n-r$-forms, and to be right $A_\star$-linear

$$*^{\mathcal{F}} (\omega \star f) = *^{\mathcal{F}}(\omega) \star f \tag{2.20}$$

for any form ω and function f. There is a canonical way to deform A-linear maps to right $A_\star$-linear maps, this is given by the "quantization map" $\mathcal{D}$ studied in [32–34]. Let V be a right A-module (i.e., for all $v \in V$, $f, h \in A$ we have $vf \in V$ and $(vf)h = v(fh)$), then V

is a right $A_\star$-module by defining $v \star f = \bar{\mathrm{f}}^\alpha(v)\bar{\mathrm{f}}_\alpha(f)$ (the property $(v \star f) \star h = v \star (f \star h)$ follows from the cocycle condition (2.4) for the twist). Now let V, W be right A-modules and $P : V \to W$ be a right A-linear map, i.e., $P(vf) = P(v)f$, for all $v \in V$, $f \in A$. Then, similarly to the way we obtained the quantum Lie algebra generators we have the quantization $P^{\mathcal{F}} = \mathcal{D}(P)$ of the right A-linear map P,

$$P^{\mathcal{F}} = \mathcal{D}(P) := \mathcal{L}_{\bar{\mathrm{f}}_1^\alpha} \circ P \circ \mathcal{L}_{S(\bar{\mathrm{f}}_2^\alpha)\bar{\mathrm{f}}_\alpha} \tag{2.21}$$

that is a right $A_\star$-linear map: $P^{\mathcal{F}}(v \star f) = P^{\mathcal{F}}(v) \star f$. In particular the deformed or quantum Hodge $*$-operator explicitly reads

$$\begin{aligned} *^{\mathcal{F}} = \mathcal{D}(*) : \Omega_\star^\bullet &\longrightarrow \Omega_\star^\bullet \\ \omega &\longmapsto *^{\mathcal{F}}(\omega) = \bar{\mathrm{f}}_1^\alpha\Big(*\left(S(\bar{\mathrm{f}}_2^\alpha)\bar{\mathrm{f}}_\alpha(\omega)\right)\Big) \end{aligned} \tag{2.22}$$

where $\bar{\mathrm{f}}_1^\alpha\Big(*\left(S(\bar{\mathrm{f}}_2^\alpha)\bar{\mathrm{f}}_\alpha(\omega)\right)\Big)$ is a shorthand notation for $\mathcal{L}_{\bar{\mathrm{f}}_1^\alpha}\Big(*\left(\mathcal{L}_{S(\bar{\mathrm{f}}_2^\alpha)\bar{\mathrm{f}}_\alpha}(\omega)\right)\Big)$. For any exterior form ω and function f we have the right $A_\star$-linearity property $*^{\mathcal{F}}(\omega \star f) = *^{\mathcal{F}}(\omega) \star f$.

Finally we notice that the metric structure we have introduced via the Hodge star operator is a priori independent from the twist $\mathcal{F}$ determining the noncommutativity of spacetime.

2.5 Wave equations

The wave equation in (curved) spacetime is governed by the Laplace-Beltrami operator $\Box = \delta d + d\delta$. In the case of even dimensional Lorenzian manifolds (like Minkowski spacetime) the adjoint of the exterior derivative is defined by $\delta = *d*$. For a scalar field (i.e., a function or 0-form) we have

$$\Box\varphi = *\mathrm{d} * \mathrm{d}\varphi = \frac{1}{\sqrt{g}}\partial_\nu\left[\sqrt{g}g^{\nu\mu}\partial_\mu\varphi\right] \tag{2.23}$$

(cf. e.g. [35] for a proof). We can now immediately consider wave equations in noncommutative spacetime. We just define the Laplace-Beltrami operator by replacing the Hodge $*$-operator with the $*^{\mathcal{F}}$-operator, for even dimensional noncommutative spaces with Lorenzian metric: $\Box^{\mathcal{F}} = *^{\mathcal{F}}\mathrm{d} *^{\mathcal{F}} \mathrm{d} + \mathrm{d} *^{\mathcal{F}} \mathrm{d}*^{\mathcal{F}}$. In particular on a scalar field we have

$$\Box^{\mathcal{F}}\varphi = *^{\mathcal{F}}\mathrm{d} *^{\mathcal{F}} \mathrm{d}\varphi = 0\,, \tag{2.24}$$

and more in general for a massive scalar field $(\Box^{\mathcal{F}} + m^2)\varphi = 0$.

We can similarly consider Maxwell equations on noncommutative spacetime (without sources)

$$\begin{aligned} \mathrm{d}F &= 0 \\ \mathrm{d} *^{\mathcal{F}} F &= 0 \end{aligned} \tag{2.25}$$

The first equation implies (locally) the existence of a gauge potential 1-form A such that $F = dA$, the second one then becomes the noncommutative Maxwell equation for the gauge potential A. These Maxwell equations differ from the usual equations for a $\mathcal{U}(1)$-gauge field in noncommutative space, where $F = dA + A \wedge_\star A$.

3 Differential geometry on κ-Minkowski spacetime

The Weyl algebra or Poincaré-Weyl algebra $\mathfrak{pw} = span\{M_{\mu\nu}, P_\mu, D\}$ is the extension of the Poincaré algebra with the dilatation generator D. It is described by the following set of commutators and structure constants:

$$[M_{\mu\nu}, M_{\rho\lambda}] = i(\eta_{\mu\lambda}M_{\nu\rho} - \eta_{\nu\lambda}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\rho}M_{\nu\lambda}), \tag{3.1}$$

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu)\,, \qquad [P_\mu, P_\lambda] = 0, \tag{3.2}$$

$$[D, P_\mu] = iP_\mu\,, \qquad [D, M_{\mu\nu}] = 0\,, \tag{3.3}$$

where $\eta_{\mu\nu}$ is the flat Minkowski space metric. The representation of the Poincaré-Weyl generators as infinitesimal transformations on Minkowski spacetime is given by the differential operators

$$M_{\mu\nu} = i\,(x_\mu\partial_\nu - x_\nu\partial_\mu)\;; \qquad P_\mu = -i\partial_\mu\,; \qquad D = -ix^\mu\partial_\mu\,. \tag{3.4}$$

The universal enveloping algebra of the Poincaré-Weyl algebra is $U\mathfrak{pw}$. This is a Hopf algebra with coproduct counit and antipode as recalled at the beginning of section 2.1 (in particular $\Delta(u) = u \otimes 1 + 1 \otimes u, \varepsilon(u) = 0, S(u) = -u$ for all $u \in \mathfrak{pw}$).

We study a twist deformation of the Poincaré-Weyl algebra that leads to the noncommutative algebra of κ-Minkowski spacetime. We recall that the coordinates of κ-Minkowski spacetime satisfy the commutation relations

$$x^0 \star x^j - x^j \star x^0 = \frac{i}{\kappa}x^j\,, \qquad x^i \star x^j - x^j \star x^i = 0 \tag{3.5}$$

where i and j run over the space indices $1, \ldots n-1$. For this twist deformation we list the corresponding quantum Lie algebra $\mathfrak{pw}^\mathcal{F}$, the quadratic Casimir operator $\eta^{\mu\nu}P^\mathcal{F}_\mu P^\mathcal{F}_\nu$ of the Poincaré subalgebra, the commutation relations of the algebra of exterior forms, the differential calculus and the (massless) fields wave equations, including the Dirac equation. We show that the d'Alembert opertor obtained from the quadratic Casimir coincides with the Laplace-Beltrami operator obtained from the differential and the Hodge $*^\mathcal{F}$-operators.

3.1 The Jordanian twist

The Jordanian twist of the Poincaré-Weyl algebra is defined by [15]:

$$\mathcal{F} = \exp\left(-iD \otimes \sigma\right)\;; \qquad \sigma = \ln\left(1 + \frac{1}{\kappa}P_0\right). \tag{3.6}$$

Its inverse is $\mathcal{F}^{-1} = \exp\left(iD \otimes \sigma\right)$. Equivalent expressions are

$$\mathcal{F}^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!}(iD)^n \otimes \sigma^n \quad \text{and} \quad \mathcal{F}^{-1} = \sum_{n=0}^{\infty} \frac{(iD)^{\underline{n}}}{n!} \otimes \left(\frac{1}{\kappa}P_0\right)^n \tag{3.7}$$

where $X^{\underline{n}} = X(X-1)(X-2)\ldots(X-(n-1))$ is the so-called lower factorial. The last expression is the power series expansion in $\frac{1}{\kappa}$ of $\mathcal{F}^{-1}$ and follows observing that $\mathcal{F}^{-1}$ is analytic in D and P_0 and that $D\otimes 1$ commutes with $1\otimes P_0$:

$$\begin{aligned}\mathcal{F}^{-1} &= e^{(iD\otimes 1)(1\otimes\sigma)} = (e^{1\otimes\sigma})^{iD\otimes 1} = (e^{1\otimes\ln(1+\frac{1}{\kappa}P_0)})^{iD\otimes 1} = \left(1\otimes\left(1+\frac{1}{\kappa}P_0\right)\right)^{iD\otimes 1} \\ &= \sum_{n=0}^{\infty}\binom{iD\otimes 1}{n}\left(1\otimes\frac{1}{\kappa}P_0\right)^n = \sum_{n=0}^{\infty}\binom{iD}{n}\otimes\left(\frac{1}{\kappa}P_0\right)^n = \sum_{n=0}^{\infty}\frac{(iD)^{\underline{n}}}{n!}\otimes\left(\frac{1}{\kappa}P_0\right)^n,\end{aligned} \tag{3.8}$$

where $\binom{X}{n} = \frac{X^{\underline{n}}}{n!}$ denotes the generalized binomial coefficient.

3.2 Quantum Poincaré-Weyl Lie algebra

The quantum Lie algebra $\mathfrak{pw}^{\mathcal{F}}$ has twisted generators (cf. (2.8), (2.9)):

$$P_\mu^{\mathcal{F}} = P_\mu\frac{1}{1+\frac{1}{\kappa}P_0} = P_\mu e^{-\sigma}\,, \qquad M_{\mu\nu}^{\mathcal{F}} = M_{\mu\nu}\,, \qquad D^{\mathcal{F}} = D\,. \tag{3.9}$$

In order to obtain the first expression use that $\mathcal{F}^{-1} = \exp(-iD\otimes -\sigma)$ and that $-iD$ on momenta acts as the identity operator: $-iD(P_\mu) = [-iD, P_\mu] = P_\mu$, hence $\mathcal{F}^{-1} = \exp(-iD\otimes -\sigma) = \exp(1\otimes -\sigma)$ when we consider the adjoint action of its first leg on P_μ. Similarly, $M_{\mu\nu}^{\mathcal{F}} = M_{\mu\nu}$ and $D^{\mathcal{F}} = D$, because D acts trivially on $M_{\mu\nu}$ and D. The inverse of the universal R-matrix is

$$\mathcal{R}^{-1} = \mathcal{F}\mathcal{F}_{21}^{-1} = e^{-iD\otimes\sigma}e^{\sigma\otimes iD}\,. \tag{3.10}$$

In order to calculate the twisted commutators we first compute

$$\begin{aligned}\mathcal{R}^{-1}(P_\rho\otimes M_{\mu\nu}) &= P_\rho\otimes e^{\sigma}(M_{\mu\nu}) = P_\rho\otimes M_{\mu\nu} + P_\rho\otimes\frac{1}{\kappa}[P_0, M_{\mu\nu}] \\ \mathcal{R}^{-1}(P_\mu\otimes D) &= P_\mu\otimes e^{\sigma}(D) = P_\mu\otimes D - P_\mu\otimes\frac{i}{\kappa}P_0\end{aligned} \tag{3.11}$$

and $\mathcal{R}^{-1}(M_{\rho\sigma}\otimes M_{\mu\nu}) = M_{\rho\sigma}\otimes M_{\mu\nu}$, $\mathcal{R}^{-1}(P_\nu\otimes P_\mu) = P_\nu\otimes P_\mu$, $\mathcal{R}^{-1}(M_{\mu\nu}\otimes D) = M_{\mu\nu}\otimes D$. From (3.11) it immediately follows the nontriviality of the twisted commutators (cf. (2.11))

$$\begin{aligned}[M_{\mu\nu}^{\mathcal{F}}, P_\rho^{\mathcal{F}}]_{\mathcal{F}} &= M_{\mu\nu}^{\mathcal{F}}P^{\mathcal{F}} - P_\rho^{\mathcal{F}}M_{\mu\nu}^{\mathcal{F}} - P_\rho^{\mathcal{F}}\frac{1}{\kappa}([P_0, M_{\mu\nu}])^{\mathcal{F}}\,, \\ [D^{\mathcal{F}}, P_\mu^{\mathcal{F}}]_{\mathcal{F}} &= D^{\mathcal{F}}P_\mu^{\mathcal{F}} - P_\mu^{\mathcal{F}}D^{\mathcal{F}} + P^{\mathcal{F}}\frac{i}{\kappa}P_0\,,\end{aligned} \tag{3.12}$$

while the remaining twisted commutators are usual commutators. Use of (3.9), of the identities $[M_{\mu\nu}, e^{-\sigma}] = -e^{-2\sigma}[M_{\mu\nu}, e^{\sigma}] = -e^{-2\sigma}[M_{\mu\nu}, \frac{1}{\kappa}P_0]$, $[D, e^{-\sigma}] = -e^{-2\sigma}[D, e^{\sigma}] = -e^{-2\sigma}\frac{i}{\kappa}P_0$ and of the undeformed Poincaré-Weyl Lie algebra relations then leads to the $\mathfrak{pw}^{\mathcal{F}}$ quantum Lie algebra

$$\left[M_{\mu\nu}^{\mathcal{F}}, M_{\rho\lambda}^{\mathcal{F}}\right]_{\mathcal{F}} = i(\eta_{\mu\lambda}M_{\nu\rho}^{\mathcal{F}} - \eta_{\nu\lambda}M_{\mu\rho}^{\mathcal{F}} + \eta_{\nu\rho}M_{\mu\lambda}^{\mathcal{F}} - \eta_{\mu\rho}M_{\nu\lambda}^{\mathcal{F}}), \tag{3.13}$$

$$\left[M_{\mu\nu}^{\mathcal{F}}, P_\rho^{\mathcal{F}}\right]_{\mathcal{F}} = i(\eta_{\nu\rho}P_\mu^{\mathcal{F}} - \eta_{\mu\rho}P_\nu^{\mathcal{F}})\,, \qquad \left[P_\mu^{\mathcal{F}}, P_\lambda^{\mathcal{F}}\right]_{\mathcal{F}} = 0 \tag{3.14}$$

$$\left[D^{\mathcal{F}}, P_\lambda^{\mathcal{F}}\right]_{\mathcal{F}} = iP_\lambda^{\mathcal{F}}\,, \qquad \left[D^{\mathcal{F}}, M_{\mu\nu}^{\mathcal{F}}\right]_{\mathcal{F}} = 0\,. \tag{3.15}$$

Notice that the structure constants are undeformed; this is also the case for the Moyal-Weyl noncommutative Minkowski space [26, section 7.7], in general however this does not hold, as the example of the twisted quantum Lorentz Lie algebra in [36, eq. (6.65)] shows.

We complete the description of the quantum Lie algebra $\mathfrak{pw}^{\mathcal{F}}$ by listing the twisted coproduct and the antipode on the generators[4]

$$\Delta^{\mathcal{F}}\left(M^{\mathcal{F}}_{\mu\nu}\right) = M^{\mathcal{F}}_{\mu\nu}\otimes 1 + 1\otimes M^{\mathcal{F}}_{\mu\nu} + \frac{1}{\kappa}D^{\mathcal{F}}\otimes\left(\tau_\mu P^{\mathcal{F}}_\nu - \tau_\nu P^{\mathcal{F}}_\mu\right) \tag{3.22}$$
$$\Delta^{\mathcal{F}}\left(P^{\mathcal{F}}_{\mu}\right) = P^{\mathcal{F}}_\mu\otimes 1 + e^{-\sigma}\otimes P^{\mathcal{F}}_\mu \tag{3.23}$$
$$\Delta^{\mathcal{F}}\left(D^{\mathcal{F}}\right) = 1\otimes D^{\mathcal{F}} + D^{\mathcal{F}}\otimes e^{-\sigma} \tag{3.24}$$
$$S^{\mathcal{F}}\left(M^{\mathcal{F}}_{\mu\nu}\right) = -M^{\mathcal{F}}_{\mu\nu} + \frac{1}{\kappa}D^{\mathcal{F}}\left(\tau_\mu P^{\mathcal{F}}_\nu - \tau_\nu P^{\mathcal{F}}_\mu\right)e^{\sigma} \tag{3.25}$$
$$S^{\mathcal{F}}\left(P^{\mathcal{F}}_{\mu}\right) = -P^{\mathcal{F}}_\mu e^{\sigma} \tag{3.26}$$
$$S^{\mathcal{F}}\left(D^{\mathcal{F}}\right) = -D^{\mathcal{F}}e^{\sigma} \tag{3.27}$$

Where we have set $\tau^\mu = (1,0,\ldots 0)$, $\tau_\mu = \eta_{\mu\nu}\tau^\nu$, and e^σ is here seen as dependent on $P^{\mathcal{F}}_0$ via

$$e^\sigma = 1 + \frac{1}{\kappa}P_0 = \frac{1}{1-\frac{1}{\kappa}P^{\mathcal{F}}_0}\,. \tag{3.28}$$

An easy way to proceed in order to prove the twisted coproduct expressions (3.22)–(3.24) is to recall the general one (2.10) and use the properties

$$\begin{aligned}
\bar{R}^\alpha\otimes\bar{R}_\alpha(M_{\mu\nu}) &= \mathrm{f}^\alpha\otimes\mathrm{f}_\alpha(M_{\mu\nu}) = 1\otimes M_{\mu\nu} - iD\otimes\frac{1}{\kappa}[P_0, M_{\mu\nu}]\\
\bar{R}^\alpha\otimes\bar{R}_\alpha(P_\mu) &= \bar{\mathrm{f}}_\alpha\otimes\bar{\mathrm{f}}^\alpha(P_\mu) = e^{-\sigma}\otimes P_\mu\\
\bar{R}^\alpha\otimes\bar{R}_\alpha(D) &= \mathrm{f}^\alpha\otimes\mathrm{f}_\alpha(D) = 1\otimes D - D\otimes\frac{1}{\kappa}P_0\,.
\end{aligned} \tag{3.29}$$

The twisted antipode on the generators can then be easily obtained by applying $S^{\mathcal{F}}\otimes\mathrm{id}$ to (3.24), (3.22) and $\mathrm{id}\otimes S^{\mathcal{F}}$ to (3.23) and then recalling the Hopf algebra properties $S^{\mathcal{F}}(\xi_{1\mathcal{F}})\xi_{2\mathcal{F}} = \varepsilon(\xi)$ and $\xi_{1\mathcal{F}}S^{\mathcal{F}}(\xi_{2\mathcal{F}}) = \varepsilon(\xi)$.

If we focus only on the algebra part of the quantum Lie algebra $\mathfrak{pw}^{\mathcal{F}}$ we notice that the bracket $[\,,\,]_{\mathcal{F}}$ closes on the twisted Poincaré generators. However, from the coproduct

[4]For sake of comparison we list also the twisted coproduct and antipode on the untwisted generators (3.1)–(3.3) because this is the usual set of generators considered for the Hopf algebra $U\mathfrak{pw}^{\mathcal{F}}$, even though they do not span a quantum Lie algebra as defined in *i), ii), iii)* before equation (2.7).

$$\Delta^{\mathcal{F}}(P_\mu) = P_\mu\otimes e^{\sigma} + 1\otimes P_\mu, \tag{3.16}$$
$$\Delta^{\mathcal{F}}(M_{\mu\nu}) = M_{\mu\nu}\otimes 1 + 1\otimes M_{\mu\nu} + \frac{1}{\kappa}D\otimes(\tau_\mu P_\nu - \tau_\nu P_\mu)\,e^{-\sigma}, \tag{3.17}$$
$$\Delta^{\mathcal{F}}(D) = D\otimes 1 + 1\otimes D - \frac{1}{\kappa}D\otimes P_\tau e^{-\sigma} = 1\otimes D + D\otimes e^{-\sigma}, \tag{3.18}$$
$$S^{\mathcal{F}}(P_\mu) = -P_\mu e^{-\sigma}, \tag{3.19}$$
$$S^{\mathcal{F}}(M_{\mu\nu}) = -M_{\mu\nu} + \frac{1}{\kappa}D\,(\tau_\mu P_\nu - \tau_\nu P_\mu)\,, \tag{3.20}$$
$$S^{\mathcal{F}}(D) = -D\left(1+\frac{1}{\kappa}P_\tau\right) = -De^{\sigma}\,. \tag{3.21}$$

and the antipode, because of the appearance of the dilatation generator, we see that the Poincaré generators do not form a quantum Lie subalgebra of $\mathfrak{pw}^{\mathcal{F}}$. Notice however that the Lie algebra $so(3)$ of spatial rotations is an undeformed Lie subalgebra of the quantum Lie algebra $\mathfrak{pw}^{\mathcal{F}}$ (indeed $M_{ij}^{\mathcal{F}} = M_{ij}, \Delta^{\mathcal{F}}(M_{ij}^{\mathcal{F}}) = \Delta(M_{ij}), S^{\mathcal{F}}(M_{ij}^{\mathcal{F}}) = S(M_{ij})$). Moreover the twisted translations $P^{\mathcal{F}}$ span a quantum Lie algebra that is a quantum Lie subalgebra of $\mathfrak{pw}^{\mathcal{F}}$ because the twisted bracket $[\,,\,]_{\mathcal{F}}$, the deformed coproduct $\Delta^{\mathcal{F}}$ and antipode $S^{\mathcal{F}}$ on translations are just expressed in terms of translations. Explicitly, from (3.14), (3.23) and (3.26), the quantum Lie algebra of translations reads

$$[P_\mu^{\mathcal{F}}, P_\lambda^{\mathcal{F}}]_{\mathcal{F}} = 0\,, \quad \Delta^{\mathcal{F}}\left(P_\mu^{\mathcal{F}}\right) = P_\mu^{\mathcal{F}} \otimes 1 + e^{-\sigma} \otimes P_\mu^{\mathcal{F}}\,, \quad S^{\mathcal{F}}\left(P_\mu^{\mathcal{F}}\right) = -P_\mu^{\mathcal{F}} e^{\sigma} \tag{3.30}$$

with $P_\mu^{\mathcal{F}} = \frac{P_\mu}{1+\frac{1}{\kappa}P_0}$ and $e^\sigma = 1+\frac{1}{\kappa}P_0 = \frac{1}{1-\frac{1}{\kappa}P_0^{\mathcal{F}}}$. Thus the twisted momenta $\{P_\mu^{\mathcal{F}}\}$ generate a Hopf algebra: the quantum universal enveloping algebra of momenta. All relevant formulae that in the following we derive for the differential geometry on κ-Minkowski spacetime and the dispersion relations depend only the quantum Lie algebra of translations $P_\mu^{\mathcal{F}}$ summarized in (3.30).

3.2.1 Addition of momenta

From the quantum Lie algebra of momenta we can immediately extract the addition law of energy and momentum for multiparticle states. It is dictated by the coproduct (3.24) on the quantum enveloping algebra. In particular the total energy-momentum $p_\mu^{\mathcal{F}\,\mathrm{tot}}$ of two free particles respectively of momenta $p_\mu^{\mathcal{F}}$ and $p_\mu'^{\mathcal{F}}$, eigenvalues of the energy-momentum operators $P_\mu^{\mathcal{F}}$, is derived from (3.23) to be

$$p_\mu^{\mathcal{F}\,\mathrm{tot}} = p_\mu^{\mathcal{F}} + p_\mu'^{\mathcal{F}} - \frac{1}{\kappa} p_0^{\mathcal{F}} p_\mu'^{\mathcal{F}}\,. \tag{3.31}$$

We read from this expression a typical feature of addition of momenta derived from quantum groups, cf. [37], the total momentum is not invariant under the exchange of the two particles. This however does not mean that there is no symmetry between the states s and s' of the unprimed and primed particle (momentum eigenstates with eigenvalues $p^{\mathcal{F}}$ and $p'^{\mathcal{F}}$). The two particle states is the tensor product $s \otimes s'$. Instead of realizing the permutation as $s \otimes s' \to s' \otimes s$, it is realized via the appropriate action of the R-matrix as $s \otimes s' \to \bar{R}^\alpha(s') \otimes \bar{R}_\alpha(s)$; hence the exchange of the two particles is implemented via the R-matrix. We remark that the present R-matrix $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$ is obtained from a twist 2-cocycle $\mathcal{F}$ and hence it is triangular and provides a representation of the permutation group.

More importantly we notice that the deviation in (3.31) from the usual addition law is quadratic in the momenta (at all orders in $\frac{1}{\kappa}$), and not exponential like in κ-Poincaré in the standard basis [38] of generators as well as in the bicrossproduct basis [39]. Furthermore the total energy is invariant under the usual permutation of the two particles. These characteristics should impose less stringent constraints on multiparticle applications of the model.

3.2.2 Quadratic Casimir operator

Associated with the quantum Lie algebra of momenta $\{P_\mu^{\mathcal{F}}\}$ we have the quadratic Casimir operator

$$\Box^{\mathcal{F}} = P_\mu^{\mathcal{F}} P^{\mu\mathcal{F}} = P_\mu P^\mu \frac{1}{\left(1+\frac{1}{\kappa}P_0\right)^2} = \Box e^{-2\sigma} . \tag{3.32}$$

The adjoint action of the twisted Poincaré generators on the quadratic Casimir vanishes, and that of the dilatation generator is as in the classical case:

$$\left[P_\mu^{\mathcal{F}}, \Box^{\mathcal{F}}\right]_{\mathcal{F}} = 0 \tag{3.33}$$

$$\left[M_{\mu\nu}^{\mathcal{F}}, \Box^{\mathcal{F}},\right]_{\mathcal{F}} = 0 \tag{3.34}$$

$$\left[D^{\mathcal{F}}, \Box^{\mathcal{F}}\right]_{\mathcal{F}} = -2i\Box^{\mathcal{F}} \tag{3.35}$$

where we have defined $[u^{\mathcal{F}}, \xi^{\mathcal{F}}]_{\mathcal{F}} := ad^{\mathcal{F}}_{u^{\mathcal{F}}}(\xi^{\mathcal{F}}) = u^{\mathcal{F}}_{1^{\mathcal{F}}}\xi^{\mathcal{F}} S^{\mathcal{F}}(u^{\mathcal{F}}_{2^{\mathcal{F}}})$ for all $u \in g$ and $\xi \in Ug$. It is easy to prove the first relation: $\left[P_\mu^{\mathcal{F}}, \Box^{\mathcal{F}}\right]_{\mathcal{F}} = P_\mu^{\mathcal{F}}\Box^{\mathcal{F}} + e^{-\sigma}\Box^{\mathcal{F}} S^{\mathcal{F}}(P_\mu^{\mathcal{F}}) = P_\mu^{\mathcal{F}}\Box^{\mathcal{F}} - e^{-\sigma}\Box^{\mathcal{F}} P_\mu^{\mathcal{F}} e^{\sigma} = 0$. The other relations can be similarly proven, although the required algebra is longer. A quicker way is to use that the quantization map $\mathcal{D}$ intertwines the $U\mathfrak{pw}$ and the $U\mathfrak{pw}^{\mathcal{F}}$ adjoint actions, see [33, Theorem 3.10] (with $\mathbb{A} = U\mathfrak{pw}$), so that $\mathcal{D}([M_{\mu\nu}^{\mathcal{F}}, \Box]) = [M_{\mu\nu}^{\mathcal{F}}, \mathcal{D}(\Box)]_{\mathcal{F}}$ and $\mathcal{D}([D_{\mu\nu}^{\mathcal{F}}, \Box]) = [D_{\mu\nu}^{\mathcal{F}}, \mathcal{D}(\Box)]_{\mathcal{F}} = -2i\Box^{\mathcal{F}}$. Then from $[M_{\mu\nu}^{\mathcal{F}}, \Box] = [M_{\mu\nu}, \Box] = 0$ and $[D_{\mu\nu}^{\mathcal{F}}, \Box] = [D_{\mu\nu}, \Box] = -2i\Box$ we immediately obtain (3.34) and (3.35).

Instead of defining the quadratic Casimir $\Box^{\mathcal{F}} = P_\mu^{\mathcal{F}} P^{\mu\mathcal{F}}$ as the square (with the usual flat Minkowski metric) of the twisted momenta, one could deform the usual Casimir operator using the quantization map (2.9). This deformation procedure leads to the same quadratic Casimir $\Box^{\mathcal{F}}$,

$$\mathcal{D}(\Box) := \bar{\mathrm{f}}^\alpha(\Box)\,\bar{\mathrm{f}}_\alpha = \Box e^{-2\sigma} = \Box^{\mathcal{F}} \tag{3.36}$$

indeed $-iD(P_\mu) = P_\mu$ implies $-iD(\Box) = 2\Box$ so that $\mathcal{F}^{-1} = \exp\left(-iD \otimes -\sigma\right) = \exp\left(2 \otimes -\sigma\right)$ when we consider the adjoint action of its first leg on $\Box$. In section 3.5 we further show that $\Box^{\mathcal{F}} = *^{\mathcal{F}} \mathrm{d} *^{\mathcal{F}} \mathrm{d}$ when the quadratic Casimir is represented as a differential operator on functions.

Finally we observe that the quadratic Casimir $\Box^{\mathcal{F}}$ coincides with the quadratic invariant considered by Magueijo and Smolin in [7]. However the viewpoint on this invariant is different: there momenta are undeformed while boosts are deformed (and act nonlinearly in momentum space), here all Poincaré-Weyl generators are twist-deformed: momenta turn out to be nontrivially deformed $P_\mu^{\mathcal{F}} \neq P_\mu$ while boosts are trivially deformed $M_{0j}^{\mathcal{F}} = M_{0j}$. These twisted momenta (rather than the undeformed ones) are given physical relevance because they close a quantum Lie algebra: that of translations in κ-Minkowski noncommutative space. As we discuss in section 4.3 the physics of the dispersion relations associated with the Casimir operator $\Box^{\mathcal{F}}$ will then turn out to be different from that considered in [7].

3.3 κ-Minkowski spacetime and differential calculus

Using the representation (3.4) of the Poincaré-Weyl generators as differential operators, the $\star$-product of two coordinate functions is easily seen to be $x^\mu \star x^\nu = \mu\{\mathcal{F}^{-1}(x^\mu \otimes x^\nu)\} =$

$x^\mu x^\nu - \frac{i}{\kappa}x^\mu\partial_0 x^\nu$, henceforth the $\star$-commutator of the coordinates satisfies the κ-Minkowski relations (3.5).

The differential calculus on κ-Minkowski spacetime induced by the Jordanian twist $\mathcal{F}$ is easily described using the basis of 1-forms $\mathrm{d}x^\mu$. The action of the dilatation and translation generators is given by the Lie derivative and on these forms it explicitly reads

$$D(\mathrm{d}x^\mu) = \mathcal{L}_{-ix^\mu\partial_\mu}(\mathrm{d}x^\mu) = -i\mathrm{d}x^\mu\,, \quad P_\mu(\mathrm{d}x^\mu) = \mathcal{L}_{-i\partial_\mu}(\mathrm{d}x^\nu) = 0\,. \tag{3.37}$$

Since in the Jordanian twist each term $\bar{\mathrm{f}}_\alpha$ in the second leg of the tensor product $\mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha$ is a power of translation operators it is immediate to see that

$$f \star \mathrm{d}x^\mu = f\mathrm{d}x^\mu\,, \quad f \star \mathrm{d}x^\mu \wedge_\star \mathrm{d}x^\nu = (f \star \mathrm{d}x^\mu) \wedge \mathrm{d}x^\nu = f\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

and iterating

$$f \star \mathrm{d}x^{\mu_1} \wedge_\star \mathrm{d}x^{\mu_2} \wedge_\star \dots \mathrm{d}x^{\mu_r} = f\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \dots \mathrm{d}x^{\mu_r}\,. \tag{3.38}$$

The star product with functions on the right is however nontrivial

$$\mathrm{d}x^\mu \star f = \mathrm{d}x^\mu f - \frac{i}{\kappa}\mathrm{d}x^\mu\partial_0 f = \mathrm{d}x^\mu\left(1 - \frac{i}{\kappa}\partial_0\right)f \tag{3.39}$$

so that

$$f \star \mathrm{d}x^\mu - \mathrm{d}x^\mu \star f = \frac{i}{\kappa}\partial_0 f\mathrm{d}x^\mu = \frac{i}{\kappa}\partial_0 f \star \mathrm{d}x^\mu\,. \tag{3.40}$$

One can prove the first relation using the form of the twist (3.6) written as a sum (recall that the twist acts via the Lie derivative on forms):

$$\begin{aligned}\mathrm{d}x^\mu \star f &= \sum_n \frac{1}{n!}\left(\mathcal{L}_{x^\rho\partial_\rho}\right)^{\underline{n}}(\mathrm{d}x^\mu)\left(-\frac{i}{\kappa}\partial_0\right)^n(f)\\ &= \mathrm{d}x^\mu f - \frac{i}{\kappa}\mathcal{L}_{x^\rho\partial_\rho}(\mathrm{d}x^\mu)\,\partial_0 f + \frac{1}{2!}\left(\frac{i}{\kappa}\right)^2\left(\mathcal{L}_{x^\rho\partial_\rho}\right)^{\underline{2}}(\mathrm{d}x^\mu)\,\partial_0^2 f + \dots\\ &= \mathrm{d}x^\mu f - \frac{i}{\kappa}\mathrm{d}x^\mu\partial_0 f\end{aligned} \tag{3.41}$$

where we used the definition of the lower factorial, i.e. $X^{\underline{n}} = X(X-1)\dots(X-n+1)$, and the fact that $\left(\mathcal{L}_{x^\rho\partial_\rho}\right)^{\underline{2}}(\mathrm{d}x^\mu) = \left(\mathcal{L}_{x^\sigma\partial_\sigma} - 1\right)\mathcal{L}_{x^\rho\partial_\rho}(\mathrm{d}x^\mu) = \left(\mathcal{L}_{x^\sigma\partial_\sigma} - 1\right)\mathrm{d}\big(\mathcal{L}_{x^\rho\partial_\rho}x^\mu\big) = \left(\mathcal{L}_{x^\sigma\partial_\sigma} - 1\right)\mathrm{d}x^\mu = 0$, so that also all the higher order terms vanish. Expression (3.40) for $f = x^\nu$ has been considered also in [40] (as special case $\mathcal{S}_1$ in their equation (18)).

From (3.40) it follows that

$$f \star \mathrm{d}x^\mu = \mathrm{d}x^\mu \star \frac{1}{(1 - \frac{i}{\kappa}\partial_0)}f = \mathrm{d}x^\mu \star e^{-\sigma}(f)\,. \tag{3.42}$$

Iterating this expression and using associativity of the $\wedge_\star$-product we immediately obtain:

$$f \star \mathrm{d}x^{\mu_1} \wedge_\star \mathrm{d}x^{\mu_2} \dots \wedge_\star \mathrm{d}x^{\mu_r} = \mathrm{d}x^{\mu_1} \wedge_\star \mathrm{d}x^{\mu_2} \dots \wedge_\star \mathrm{d}x^{\mu_r} \star \frac{1}{\left(1 - \frac{i}{\kappa}\partial_0\right)^r}f\,. \tag{3.43}$$

We now express the exterior derivative on κ-Minkowski spacetime in terms of the twisted momenta $P^{\mathcal{F}}_\mu$, thus confirming that they have the interpretation of quantum infinitesimal transformations. Corresponding to the representation $P_\mu = -i\partial_\mu$ and the relation $P^{\mathcal{F}}_\mu = P_\mu \frac{1}{1+\frac{1}{\kappa}P_0}$ we have the representation $P^{\mathcal{F}}_\mu = -i\partial^{\mathcal{F}}_\mu$, where

$$\partial^{\mathcal{F}}_\mu := \frac{1}{1-\frac{i}{\kappa}\partial_0}\partial_\mu \,. \tag{3.44}$$

The twisted momenta act as quantum infinitesimal translations because for any function f on κ-Minkowski spacetime we have

$$\mathrm{d}f = \mathrm{d}x^\mu \star iP^{\mathcal{F}}_\mu(f)\,, \tag{3.45}$$

i.e., $\mathrm{d}f = \mathrm{d}x^\mu \star \partial^{\mathcal{F}}_\mu f$. Indeed, $\mathrm{d}x^\mu \star \partial^{\mathcal{F}}_\mu f = \mathrm{d}x^\mu \star \frac{1}{1-\frac{i}{\kappa}\partial_0}\partial_\mu f = \mathrm{d}x^\mu\,\partial_\mu f = \mathrm{d}f$, where we used (3.39) and that the exterior derivative is the usual undeformed one (cf. section 2.3). It is also instructive to see that the deformed Leibniz rule (2.16), that in the present representation reads, cf. (3.23),

$$\partial^{\mathcal{F}}_\mu(f\star h) = \partial^{\mathcal{F}}_\mu(f)\star h + e^{-\sigma}(f)\star\partial^{\mathcal{F}}_\mu(h)\,, \tag{3.46}$$

combines with the commutation property (3.42) to give the undeformed Leibniz rule for the exterior derivative $\mathrm{d}(f\star h) = \mathrm{d}f\star h + f\star \mathrm{d}h$.

3.4 Hodge star operator

The Hodge $*^{\mathcal{F}}$-operator is defined in (2.22) as the quantization $*^{\mathcal{F}} = \mathcal{D}(*)$ of the Hodge $*$-operator. On an s-form $\mathrm{d}x^{\mu_1}\wedge_\star \mathrm{d}x^{\mu_2}\ldots\wedge_\star \mathrm{d}x^{\mu_s} = \mathrm{d}x^{\mu_1}\wedge \mathrm{d}x^{\mu_2}\ldots\wedge \mathrm{d}x^{\mu_s}$, it reads (in the last line recall that each term $\bar{\mathrm{f}}_\alpha$ in the second leg of the tensor product $\mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha\otimes\bar{\mathrm{f}}_\alpha$ is a power of translation operators, so that it acts trivially on $\mathrm{d}x^{\mu_1}\wedge_\star \mathrm{d}x^{\mu_2}\ldots\wedge_\star \mathrm{d}x^{\mu_s}$)

$$\begin{aligned} *^{\mathcal{F}}\left(\mathrm{d}x^{\mu_1}\wedge_\star \mathrm{d}x^{\mu_2}\ldots\wedge_\star \mathrm{d}x^{\mu_s}\right) &= \bar{\mathrm{f}}^\alpha_1\Big(*\left(S\left(\bar{\mathrm{f}}^\alpha_2\right)\bar{\mathrm{f}}_\alpha\left(\mathrm{d}x^{\mu_1}\wedge_\star \mathrm{d}x^{\mu_2}\ldots\wedge_\star \mathrm{d}x^{\mu_s}\right)\right)\Big) \\ &= *\left(\mathrm{d}x^{\mu_1}\wedge_\star \mathrm{d}x^{\mu_2}\ldots\wedge_\star \mathrm{d}x^{\mu_s}\right)\,. \end{aligned} \tag{3.47}$$

Hence the Hodge $*^{\mathcal{F}}$-operator equals the commutative Hodge $*$-operator on the exterior forms $\mathrm{d}x^{\mu_1}\wedge_\star \mathrm{d}x^{\mu_2}\ldots\wedge_\star \mathrm{d}x^{\mu_s}$. Furthermore, recalling that the Hodge star $\star^{\mathcal{F}}$ is right $A_\star$-linear we have the general expression:

$$*^{\mathcal{F}}\left(\mathrm{d}x^{\mu_1}\wedge_\star \mathrm{d}x^{\mu_2}\ldots\wedge_\star \mathrm{d}x^{\mu_s}\star f\right) = *\left(\mathrm{d}x^{\mu_1}\wedge_\star \mathrm{d}x^{\mu_2}\ldots.\wedge_\star \mathrm{d}x^{\mu_s}\right)\star f; \tag{3.48}$$

explicitly, $*^{\mathcal{F}}\left(\mathrm{d}x^{\mu_1}\wedge_\star \mathrm{d}x^{\mu_2}\ldots\wedge_\star \mathrm{d}x^{\mu_s}\star f\right) = \frac{1}{(n-s)!}\varepsilon^{\mu_1\mu_2\ldots\mu_s}{}_{\nu_{s+1}\nu_{s+2}\ldots\nu_n}\mathrm{d}x^{\nu_{s+1}}\wedge_\star \mathrm{d}x^{\nu_{s+2}}\ldots\wedge_\star \mathrm{d}x^{\nu_n}\star f$. In particular we see that $*^{\mathcal{F}}$ squares to $\pm$ the identity.

It is now easy to show that for an arbitrary form $\omega\in\Omega^s_\star$ of homogeneous degree s in n-dimensional κ-Minkowski space we have $*^{\mathcal{F}}(\omega) = (1-\frac{i}{\kappa}\partial_0)^{n-2s} * (\omega)$. Indeed, use of (3.43) and (3.48) gives

$$*^{\mathcal{F}}(f \star \mathrm{d}x^{\mu_1} \wedge_\star \mathrm{d}x^{\mu_2} \ldots \wedge_\star \mathrm{d}x^{\mu_s}) = *(\mathrm{d}x^{\mu_1} \wedge_\star \mathrm{d}x^{\mu_2} \ldots \wedge_\star \mathrm{d}x^{\mu_s}) \star \left(1 - \frac{i}{\kappa}\partial_0\right)^{-s} f \tag{3.49}$$
$$= \left(1 - \frac{i}{\kappa}\partial_0\right)^{n-2s} f \star \left(*\left(\mathrm{d}x^{\mu_1} \wedge_\star \mathrm{d}x^{\mu_2} \ldots \wedge_\star \mathrm{d}x^{\mu_s}\right)\right)$$
$$= \left(1 - \frac{i}{\kappa}\partial_0\right)^{n-2s} *\left(f \star \mathrm{d}x^{\mu_1} \wedge_\star \mathrm{d}x^{\mu_2} \ldots \wedge_\star \mathrm{d}x^{\mu_s}\right).$$

3.5 Field equations

Scalar fields. The d'Alembert operator $\square^{\mathcal{F}}$ on κ-Minkowski spacetime can be defined:

(1) as the quadratic Casimir $P_\mu^{\mathcal{F}} P^{\mu\mathcal{F}}$ (once the generators of translations are represented as differential operators);

(2) as the quantization of the d'Alembert operator on commutative Minkowski spacetime $\mathcal{D}(\square) := \bar{\mathrm{f}}^\alpha(\square)\bar{\mathrm{f}}_\alpha$, cf. equation (3.36);

(3) via the Hodge $*^{\mathcal{F}}$-operator, as the Laplace-Beltrami operator $*^{\mathcal{F}}\mathrm{d} *^{\mathcal{F}} \mathrm{d}$.

These definitions coincide: we already saw in (3.36) the equivalence of the first two definitions, for the third definition, use of (3.49) gives $*^{\mathcal{F}}\mathrm{d} *^{\mathcal{F}} \mathrm{d}\varphi = \frac{1}{(1-\frac{i}{\kappa}\partial_0)^2}\square\varphi$ showing the equality with the other two. In conclusion for a (massless) scalar field φ we have the wave equation

$$\square^{\mathcal{F}}\varphi = *^{\mathcal{F}}\mathrm{d} *^{\mathcal{F}} \mathrm{d}\varphi = \frac{1}{(1 - \frac{i}{\kappa}\partial_0)^2}\square\varphi = 0\,, \tag{3.50}$$

where $\square = \partial_\mu\partial^\mu$.

Notice that this wave equation is equivalent to the undeformed one $\square\varphi = 0$, indeed the differential operator $\frac{1}{(1-\frac{i}{\kappa}\partial_0)^2}$ is invertible. This result agrees with the well known one for free fields in noncommutative Moyal-Weyl space. There the result is trivial because the $\star$-product is simpler and translation invariant (so that ${P_\mu}^{\mathcal{F}_{MW}} = P_\mu$ where $\mathcal{F}_{MW} = e^{-i\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}$). The Moyal-Weyl $\star$-product is also compatible with the usual integral on commutative Minkowski space, thus already the free field actions coincide $\int \partial_\mu\varphi \star \partial^\mu\varphi\, \mathrm{d}^n x = \int \partial_\mu\varphi\partial^\mu\varphi\, \mathrm{d}^n x$. Notice however that while in Minkowski spacetime with Moyal-Weyl noncommutativity the equations of motion and the action remain undeformed also when considering massive scalar fields, $\int \partial_\mu\varphi \star \partial^\mu\varphi - m^2\varphi\star\varphi\, \mathrm{d}^n x = \int \partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2\, \mathrm{d}^n x$, this is no more the case in k-Minkowski spacetime, indeed

$$(\square^{\mathcal{F}} + m^2)\varphi = 0 \tag{3.51}$$

is not equivalent to $(\square + m^2)\varphi = 0$. It is easy to check covariance of (3.51) under the quantum Lie algebra of momenta $P_\mu^{\mathcal{F}}$, indeed

$$P_\mu^{\mathcal{F}}\left((\square^{\mathcal{F}} + m^2)\varphi\right) = (\square^{\mathcal{F}} + m^2)P_\mu^{\mathcal{F}}\varphi \tag{3.52}$$

as follows from considering the coproduct (3.23) and recalling the triviality of the quantum adjoint action of $P_\mu^{\mathcal{F}}$ on $\square^{\mathcal{F}}$, cf. equation (3.33).

Among the three considered formulations of the operator $\square^{\mathcal{F}}$, the advantage of the Laplace-Beltrami operator formulation is that it immediately applies to the case of curved and noncommutative spacetime. As shown in [18], when a gravitational background is turned on then the wave equation in noncommutative spacetime in general differs from that in commutative spacetime.

Spin 1 fields and twisted Maxwell equations. Maxwell equations in κ-Minkowski spacetime read $\mathrm{d}F = 0$, $\mathrm{d} *^{\mathcal{F}} F = 0$. The first equation is undeformed because the exterior derivative is undeformed. Also the second equation is equivalent to the undeformed one $\mathrm{d} * F = 0$, indeed $\mathrm{d} *^{\mathcal{F}} F = \mathrm{d}(1 - \frac{i}{\kappa}\partial_0)^{n-4} * F = (1 - \frac{i}{\kappa}\partial_0)^{n-4}\mathrm{d} * F$, and since $(1 - \frac{i}{\kappa}\partial_0)^{n-4}$ is invertible $\mathrm{d} *^{\mathcal{F}} F = 0$ is equivalent to $\mathrm{d} * F = 0$. This is no more the case if sources are present or if we are in curved noncommutative spacetime.

Spin 1/2 fields and twisted Dirac equation. The classical Dirac operator in Minkowski spacetime, $\not{\partial}$, can be written as $-\gamma^\mu P_\mu$ and is twisted according to the quantization map $\mathcal{D}$ by considering the twisted momenta $\mathcal{D}(P_\mu) = P_\mu^{\mathcal{F}}$, so that we have $-\gamma^\mu P_\mu^{\mathcal{F}}$, i.e.,

$$\not{\partial}^{\mathcal{F}} = i\gamma^\mu \partial_\mu^{\mathcal{F}} = i\gamma^\mu \frac{1}{1 - \frac{i}{\kappa}\partial_0}\partial_\mu = \frac{1}{1 - \frac{i}{\kappa}\partial_0}\not{\partial}\,. \tag{3.53}$$

It is immediate to check that the twisted Dirac operator squares to the twisted d'Alembert opertor $\left(\not{\partial}^{\mathcal{F}}\right)^2 = \square^{\mathcal{F}}$. The Dirac equation for a massless spinor field ψ reads

$$\not{\partial}^{\mathcal{F}}\psi = 0; \tag{3.54}$$

it is equivalent to the undeformed one because $\frac{1}{1-\frac{i}{\kappa}\partial_0}$ is invertible. As in the case of scalar fields, the massive Dirac field equation on κ-Minkowski however differs from the undeformed one.

4 Dispersion relations in κ-Minowski spacetime

Given the Jordanian twist $\mathcal{F}$ in (3.6), we derived the wave operator $\square^{\mathcal{F}}$ for scalar fields in flat noncommutative spacetime using three different techniques that lead to the same result: $\square^{\mathcal{F}}$ as the quadratic Casimir $P_\mu^{\mathcal{F}} P^{\mu\mathcal{F}}$, as the deformation $\mathcal{D}(\square)$ of the commutative wave operator $\square$, and as the Laplace-Beltrami operator $\square^{\mathcal{F}} = *^{\mathcal{F}}\mathrm{d} *^{\mathcal{F}} \mathrm{d}$.

We are now in a position to study the relation between energy, momentum, frequency and wave vector and the corresponding dispersion relations. We first study the massless case considering both the momentum space and the position space perspectives. We similarly study the massive case. There, as discussed in the introduction, we can consider masses as emerging from a conformal anomaly and the κ-Poincaré-Weyl symmetry as a broken symmetry.

4.1 Massless fields: undeformed dispersion relations and modified Einstein-Planck relations

The massless wave equation is undeformed, indeed $\square^{\mathcal{F}}\varphi = 0$ is equivalent to $\square\varphi = 0$. The *energy - momentum dispersion relations* are $P_\mu^{\mathcal{F}} P^{\mu\mathcal{F}} = 0$, and are undeformed as well. We immediately have (inserting c and setting $P^{\mathcal{F}\,2} = P_i^{\mathcal{F}} P^{i\,\mathcal{F}}$) $\frac{\mathrm{d}E^{\mathcal{F}}}{\mathrm{d}P^{\mathcal{F}}} = c$. More elegantly, let $\varphi(\boldsymbol{x}, t)$ be an eigenvector of the energy - momentum operators (cf. (3.4) and (3.9))

$$P_\mu^{\mathcal{F}} = \frac{P_\mu}{1 + \frac{1}{\kappa} P_0} = \frac{-i\partial_\mu}{1 - \frac{i}{\kappa}\partial_0} \tag{4.1}$$

with eigenvalues $p_\mu^{\mathcal{F}} = (-E^{\mathcal{F}}, \boldsymbol{p}^{\mathcal{F}})$, i.e., $p^{\mu\mathcal{F}} = (E^{\mathcal{F}}, \boldsymbol{p}^{\mathcal{F}})$, then $\varphi(\boldsymbol{x}, t)$ is a solution of the wave equation $\square^{\mathcal{F}}\varphi = 0$ if $p_\mu^{\mathcal{F}} p^{\mu\mathcal{F}} = 0$, hence $\frac{\mathrm{d}E^{\mathcal{F}}}{\mathrm{d}p^{\mathcal{F}}} = c$ (with $p^{\mathcal{F}}$ the modulus of $\boldsymbol{p}^{\mathcal{F}}$).

The same result follows if we consider the *frequency-wave vector dispersion relations*. To this aim we observe that the wave functions $\varphi(\boldsymbol{x}, t)$ of definite momentum are proportional to $e^{i(\boldsymbol{k}\boldsymbol{x} - \omega t)}$, and that they are a solution of the wave equation $\square^{\mathcal{F}}\varphi = 0$ if the usual undeformed dispersion relation holds, $\omega^2 = c^2 k^2$. The group velocity is therefore undeformed as well, $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = c$, and $v_g = \frac{\mathrm{d}E^{\mathcal{F}}}{\mathrm{d}p^{\mathcal{F}}}$.

Finally, evaluation of the energy momentum operator (4.1) on the monochromatic wave $e^{i(\boldsymbol{k}\boldsymbol{x} - \omega t)}$ leads to the modified Einstein-Planck relations

$$E^{\mathcal{F}} = \frac{\hbar\omega}{1 - \frac{\omega}{c\kappa}}\,, \qquad \boldsymbol{p}^{\mathcal{F}} = \frac{\hbar\boldsymbol{k}}{1 - \frac{\omega}{c\kappa}}\,. \tag{4.2}$$

If we assume a negative value for the deformation parameter κ then we can define $E_p = -c\hbar\kappa$ that we identify with Planck energy. In this case we have

$$E^{\mathcal{F}} = \frac{\hbar\omega}{1 + \frac{\hbar\omega}{E_p}}\,, \qquad \boldsymbol{p}^{\mathcal{F}} = \frac{\hbar\boldsymbol{k}}{1 + \frac{\hbar\omega}{E_p}}\,; \tag{4.3}$$

these deformed Einstein-Planck relations imply a maximum energy reachable by an elementary particle: Planck energy. The other choice $E_p = c\hbar\kappa$, corresponding to a positive value of the deformation parameter κ is also possible, in this case we have a maximum frequency $c\kappa$ that is associated with infinite energy.

In conclusion, for massless elementary particles we have the usual dispersion relations, however the Einstein-Planck relations are deformed so that there is a maximum energy reachable by an elementary particle: Planck energy, or, depending on the sign of the deformation parameter, a maximum frequency.

4.2 Massive fields

We proceed similarly with the equation for massive fields. An eigenvector $\varphi(\boldsymbol{x}, t)$ of the energy-momentum operators $P_\mu^{\mathcal{F}}$ with eigenvalues $p_\mu^{\mathcal{F}} = (-E^{\mathcal{F}}, \boldsymbol{p}^{\mathcal{F}})$ solves the wave equation $(\square^{\mathcal{F}} + m^2)\varphi = 0$ if the usual energy-momentum dispersion relations $(E^{\mathcal{F}})^2 = (p^{\mathcal{F}})^2 + m^2$ hold. Hence, inserting c, $\frac{\mathrm{d}E^{\mathcal{F}}}{\mathrm{d}p^{\mathcal{F}}} = c^2 \frac{p^{\mathcal{F}}}{E^{\mathcal{F}}}$. The Einstein-Planck and de Broglie relations are as in (4.2). Therefore for negative deformation parameter we can set $E_p = -c\hbar\kappa$ and we see that also massive elementary particles have Planck energy as maximum energy, and correspondingly a maximum momentum. On the other hand, for positive valued deformation parameter we can set $E_p = c\hbar\kappa$ and we have the maximum frequency $c\kappa$.

The frequency-wave vector dispersion relations are modified to

$$\omega^2 - c^2 k^2 = \frac{m^2 c^4}{\hbar^2}\left(1 - \frac{\omega}{c\kappa}\right)^2 ; \tag{4.4}$$

and therefore they differ from the energy-momentum dispersion relations. In particular the group velocity at first order in $\mathcal{O}\left(\frac{1}{\kappa^2}\right)$ reads

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = c^2 \frac{k}{\omega}\left(1 + \frac{m^2c^3}{\hbar^2\omega\kappa}\right) + \mathcal{O}\left(\frac{1}{\kappa^2}\right)$$

and differs from $\frac{\mathrm{d}E^{\mathcal{F}}}{\mathrm{d}p^{\mathcal{F}}} = c^2\frac{p^{\mathcal{F}}}{E^{\mathcal{F}}} = c^2\frac{k}{\omega}$ (cf. (4.2)). It is easy to see from $(E^{\mathcal{F}})^2 = (p^{\mathcal{F}})^2 + m^2c^4$ and (4.2) that at first order in the deformation this difference is limited by the ratio mc^2/E_p. The discrepancy between v_g and $\frac{\mathrm{d}E^{\mathcal{F}}}{\mathrm{d}p^{\mathcal{F}}}$ is thus proportional to the particle mass and inversely proportional to E_p or κ. These are indeed the two scales arising in the breaking of conformal symmetry scenario discussed in the introduction.

4.3 Relation to Deformed Special Relativity (DSR) theories

It is very interesting to compare the energy-momentum undeformed dispersion relations result, that we have derived from a general noncommutative differential geometry construction, with previous results in the literature, in particular with the Deformed Special Relativity (DSR) proposed in [7, 8], also known as DSR2. There, Magueijo and Smolin consider a nonlinear modification of the action of the Lorentz generators on momentum space. It is induced by the differential operator $U = e^{\lambda p_0 p_\mu \frac{\partial}{\partial p_\mu}}$ on momentum space itself. Since it is the exponential of a vector field it is an element, say φ^U, of the group of diffeomorphism on momentum space. For f, h functions on momentum space we then have $U(f)(p) = f(\varphi^U(p))$, $U(h)(p) = h(\varphi^U(p))$, and from $(fh)(\varphi^U(p)) = f(\varphi^U(p))\, h(\varphi^U(p))$ we immediately obtain $U(fh) = U(f)U(h)$. Hence the action of U is determined by that on the coordinate functions, that explicitly reads

$$p_\mu \to U(p_\mu) = \frac{p_\mu}{1 - \lambda p_0}\,. \tag{4.5}$$

The Lorentz generators $M_\mu{}^\nu = p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu}$ are correspondingly transformed to $M_\mu{}^\nu \to U M_\mu{}^\nu U^{-1}$ (cf. the adjoint action $ad_U(M_\mu{}^\nu)$ in (2.2)), explicitly $M_i{}^j \to M_i{}^j$ and $M_0{}^i \to M_0{}^i + \lambda p_i p_\mu \frac{\partial}{\partial p_\mu}$ [hint: use $U\frac{\partial}{\partial p_\mu}U^{-1} = (1-\lambda p_0)(\frac{\partial}{\partial p_\mu} - \delta_{\mu 0}\lambda p_\nu \frac{\partial}{\partial p_\nu})$]. Under the modified action of the Lorentz generators the usual quadratic expression $\eta^{\mu\nu}p_\mu p_\nu$ is no more invariant, the new invariant is

$$U(\eta^{\mu\nu}p_\mu p_\nu) = \frac{\eta^{\mu\nu}p_\mu p_\nu}{(1-\lambda p_0)^2}\,. \tag{4.6}$$

Upon the identification $\frac{1}{\kappa} = -\lambda$, and observing that the translation generators P_μ act in momentum space via multiplication by p_μ, we see from equation (3.9) that the twist quantization map on momenta

$$P_\mu \to P_\mu^{\mathcal{F}} = \mathcal{D}(P_\mu) = \frac{P_\mu}{1 + \frac{1}{\kappa}P_0} \tag{4.7}$$

equals the U transformation map. This holds more in general for functions of momenta, indeed $(P_{\mu_1}P_{\mu_2}\ldots P_{\mu_n})^{\mathcal{F}} := \mathcal{D}(P_{\mu_1}P_{\mu_2}\ldots P_{\mu_n}) = P^{\mathcal{F}}_{\mu_1}P^{\mathcal{F}}_{\mu_2}\ldots P^{\mathcal{F}}_{\mu_n}$ (for a proof just recall the derivation of (3.36)), so that $(fh)^{\mathcal{F}} = f^{\mathcal{F}} h^{\mathcal{F}}$ for f and h functions of P_μ, i.e. we recover the algebra property of U: $U(fh) = U(f)U(h)$.

We have shown that the quantization map $\,^{\mathcal{F}} = \mathcal{D}$ corresponding to the Jordanian twist (3.6) equals, on momentum space, the U transformation introduced in [7]. However the action of $\mathcal{D}$ is independent from the representation used ($\mathcal{D}$ acts on the Poincaré-Weyl generators irrespectively of their representation in momentum or position space), therefore the present result allows to extend the construction in DSR2 from operators in momentum space to operators in position space.

In DSR2 the change to the new invariant $\eta^{\mu\nu} p_\mu p_\nu \to \frac{\eta^{\mu\nu} p_\mu p_\nu}{(1-\lambda p_0)^2}$ is interpreted as giving rise to modified dispersion relations because the physical momenta are considered to be the usual undeformed ones, see also [37, 41]. Otherwise stated, the transformation $Q \to UQU^{-1}$ applies only to the Lorentz generators, it does not apply to the whole Poincaré generators.[5] In the present paper we similarly consider the change to the new invariant $\eta^{\mu\nu} p_\mu p_\nu \to \frac{\eta^{\mu\nu} p_\mu p_\nu}{(1-\lambda p_0)^2}$, it is written as $\Box \to \Box^{\mathcal{F}}$ in (3.32), (3.36); and $\Box^{\mathcal{F}}$ is proven to be an invariant under quantum Poincaré transformations in (3.33), (3.34). However we follow a different route from that of DSR theories. We have first singled out the quantum Lie algebra (3.13)–(3.27) of the quantum group of Poincaré-Weyl (i.e., of the twist deformed universal enveloping algebra of Poincaré-Weyl). This is generated by $P^{\mathcal{F}}, M^{\mathcal{F}}_{\mu\nu}, D^{\mathcal{F}}$. Then we have shown in (3.45) that the differential calculus on κ-Minkowski space is defined in terms of the generators $P^{\mathcal{F}}$, that hence have not only the Lie-algebraic but also the differential geometry meaning of generators of translations. We therefore conclude that these are the generators of the Poincaré-Weyl group that encode the physics of energy and momentum. As shown at the beginning of this section, in terms of these momenta the dispersion relations for massless and massive particles are undeformed. We hence arrive at different conclusions with respect to [7], where energy-momentum dispersion relations for massless particles are undeformed, but for massive particles are deformed.

5 Conclusions

Deformed Special Relativity theories have been considered as phenomenological models providing an effective description of quantum spacetime. The associated wave equations can be obtained as wave equations arising in κ-Minkowski spacetime. The relation between noncommutative spacetimes, dispersion relations and DSR theories depends however on the different choices and realization of the algebra of momenta and coordinates in noncommutative space. This issue has emerged in particular considering κ-Minkowski as homogeneous space under the κ-Poincaré quantum symmetry group. There different nonlinearly related sets of generators of the κ-Poincaré algebra lead to different dispersion relations [42] as well as different realizations of the κ-Minkowski coordinates [17], see also [43, 44] and [40]. We resolved the ambiguities associated with the choice of the basis of momenta and coordinates

[5]Since P_μ on momentum space acts simply as multiplication by p_μ, the transformation $Q \to UQU^{-1}$ on momenta reads $P_\mu \to UP_\mu U^{-1} = \frac{P_\mu}{1-\lambda P_0}$, indeed $UP_\mu U^{-1}(f) = U(p_\mu U^{-1}(f)) = U(p_\mu)f = \frac{p_\mu}{1-\lambda p_0} f = \frac{P_\mu}{1-\lambda P_0}(f)$.

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and their realization by singling out the quantum Lie algebra, eq. (2.8), of the given quantum universal enveloping algebra (quantum symmetry group), and by constructing a coordinate independent noncommutative differential geometry (build on the exterior derivative, the $*^{\mathcal{F}}$ Hodge star operator, the twisted d'Alembertian $\Box^{\mathcal{F}} = *^{\mathcal{F}} \mathrm{d} *^{\mathcal{F}} \mathrm{d}$). We exemplified the construction in the case of the Jordanian twist deformed Poincaré-Weyl group with κ-Minkowski spacetime as its homogeneous space. While the twist depends on the dilatation generator, all relevant formulae for the differential geometry on κ-Minkowski spacetime and the dispersion relations depend only the quantum Lie algebra of translations $P^{\mathcal{F}}_\mu$, summarized in (3.30). The quantum Lie algebra construction can in principle be carried out also for the κ-Poincaré group, however it is not canonically determined as in the case of quantum groups obtained via twists. Strikingly, the example considered leads to the wave equation studied in DSR2, hence extending it from momentum space to position space. It would be interesting to further investigate DSR theories and the associated relative locality principle [45], including the corresponding interpretation of the dispersion relations (see [16] for a critical discussion), with the perspectives and techniques provided in this paper.

Acknowledgments

The authors would like to thank L. Castellani and J. Lukierski for useful discussions. This work begun during a three month visit of A. B. at Università del Piemonte Orientale supported by the university funds for internationalization. P. A. is partially supported by INFN, CSN4, iniziativa specifica GSS and is affiliated to INdAM, GNFM (Istituto Nazionale di Alta Matematica, Gruppo Nazionale di Fisica Matematica). A. B is supported by NCN (Polish National Science Center) project 2014/13/B/ST2/04043. P. A. and A. B. acknowledge also the support from COST (European Cooperation in Science and Technology) Action MP1405 QSPACE. A. P. acknowledges the funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 660061.

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