

Deformation quantization of principal bundles*

Paolo Aschieri

Dipartimento di Scienze e Innovazione Tecnologica
and INFN Torino, Gruppo collegato di Alessandria,
Università del Piemonte Orientale, Viale T. Michel 11, 15121 Alessandria, Italy.
(e-mail: aschieri@to.infn.it)

November 4, 2016

Abstract

We outline how Drinfeld twist deformation techniques can be applied to the deformation quantization of principal bundles into noncommutative principal bundles, and more in general to the deformation of Hopf-Galois extensions. First we twist deform the structure group in a quantum group, and this leads to a deformation of the fibers of the principal bundle. Next we twist deform a subgroup of the group of automorphisms of the principal bundle, and this leads to a noncommutative base space. Considering both deformations we obtain noncommutative principal bundles with noncommutative fiber and base space as well.

Keywords: noncommutative geometry, noncommutative principal bundles, Hopf-Galois extensions, cocycle twisting

MSC 2010: 16T05, 16T15, 53D55, 81R50, 81R60

1 Introduction

There are many approaches to noncommutative geometry, one of this is based on deformation quantization of the algebra of smooth functions on commutative manifolds: the usual pointwise product is there deformed into a \star -product, and the corresponding noncommutative algebra is then thought as the algebra of functions on a quantum (or noncommutative) manifold. If we consider the algebra of function on a Lie group L it is natural to deform the product to a \star -product that is obtained via the action of left and right invariant vector fields, hence the \star -product is defined by elements of the Lie algebra \mathfrak{l} of the Lie Group L ; more precisely, following Drinfeld [13], by a twist (or twisting element) \mathcal{F} that is a formal power series in a deformation parameter \hbar of elements in $U(\mathfrak{l}) \otimes U(\mathfrak{l})$, where $U(\mathfrak{l})$ is the universal enveloping algebra of the Lie algebra \mathfrak{l} .

*Based on joint work with Pierre Bieliavsky, Chiara Pagani and Alexander Schenkel

Furthermore, if the group L acts on a manifold M we have an action of the Lie algebra \mathfrak{l} on the algebra B of smooth function on M , then the action of the twist \mathcal{F} defines a \star -product deformation of B . Thus Drinfeld twist deformation is a powerful method, based on first deforming a Lie group and then its representations. This method has been extended to deform vector bundles over M that carry an action of a Lie group L (i.e., to L -equivariant vector bundles), and in [5] to their differential geometry, leading in particular to a theory of arbitrary (i.e., not necessarily equivariant) connections on B -bimodules and on their tensor products that generalizes the notion of bimodule connection introduced in [21, 14]. (Vector bundles are here described by their sections that form a B -bimodule, B being the algebra of functions on M). The construction is categorical, and in particular commutative connections can be canonically quantized to noncommutative connections.

Here we further extend these techniques and provide a general deformation theory of principal bundles; we refer to [1] for an exhaustive presentation that complements the present one: here we first present a pedagogical and shorter route to the notion of Hopf-Galois extension (that captures the algebraic aspects of principality of a bundle) and then lead the reader through the basic key points and proofs of the general deformation theory. As we explain, G -principal bundles are described in terms of G -equivariant maps between A -bimodules where now A is the algebra of functions on the total space of the G -principal bundle. When the Lie group L used for the Drinfeld twist deformation is G itself, then a corresponding Drinfeld twist \mathcal{F} deforms the fibers of the principal bundle, when the Lie group L is not G but a subgroup of the group of automorphisms of the G -principal bundle then we obtain twist deformations of the base space. In general we have A -bimodules that carry both an action of the structure group G as well as of a subgroup L of the group of automorphisms. We can therefore consider Drinfeld twists associated with G as well as with L and thus obtain noncommutative bundles with both noncommutative fibers and base space.

The categorical setting we develop is very promising in order to study the notions of gauge group in noncommutative geometry and that of connection on noncommutative principal bundle. Indeed these forthcoming projects are main motivations for the present study. In particular, gauge transformations in noncommutative geometry are typically $GL(n)$ or $U(N)$ valued, while we foresee the gauge group of a twist deformed G -principal bundle to give twist deformed G -valued gauge transformations (in the spirit of [3]). This would allow to consider gauge theories with arbitrary twist deformed gauge groups, not just $GL(n)$ or $U(N)$ ones.

We further explain the content of the paper by outlining each chapter: In §2 we show how the algebras A and B of functions on the total space and on the base space of a principal bundle define a Hopf-Galois extension $B \subset A$ that captures the algebraic aspects of the principal bundle. Thus noncommutative principal bundles are described by noncommutative Hopf-Galois extensions in the same way that noncommutative manifolds are described by noncommutative algebras of smooth functions. In §3 we recall the theory of Drinfeld twist deformation in the dual language, used throughout the paper, of twist deformation by 2-cocycles. In §4 we consider twist deformations of Hopf-Galois extensions: in §4.1 Hopf-Galois extensions with Hopf algebra H (for example corresponding to principal bundles with structure group G) are twist deformed in new Hopf-Galois extensions with twisted Hopf algebra H_γ (corresponding to noncommutative principal bundles with a quantum group G_γ and noncommutative fibers). In §4.2 we consider twist deformations of the base space.

We also recover, as a relevant example of the general theory, the instanton bundle on the noncommutative 4-sphere S_θ^4 of Connes-Landi. In this case the total space, base space and structure group are affine algebraic varieties, so that the \star -products obtained by Drinfeld twist deformation are well defined (on the algebras of coordinate functions on these varieties) also when the formal deformation parameter \hbar (called θ) becomes nonformal and is valued in \mathbb{R} . In §4.3 we consider both base and fiber twist deformations and present the example of formal deformations of G -principal bundles. We conclude outlining the noncommutative deformation of the frame bundle of a Lorentzian manifold that is the first step to a global geometric study of a noncommutative theory of gravity in the vierbein formulation.

Acknowledgments

The present contribution to the volume in memory of Professor Mauro Francaviglia would not have been possible without his teaching and enthusiasm that determined my scientific background and interests as a student during his Lezioni di Meccanica Razionale. Indeed it was in his course, and in his book [15], that I was introduced to the theory of fiber bundles and to geometric methods in theoretical physics. More recently I profited from the many discussions on gravity and on noncommutative geometry, and from his support on these studies because of their potential in providing a more general (noncommutative spacetime) setting for the formulation of gauge and gravity theories.

The author is member of the COST Action MP1405 QSPACE, supported by COST (European Cooperation in Science and Technology) and is affiliated to INdAM, GNFM (Istituto Nazionale di Alta Matematica, Gruppo Nazionale di Fisica Matematica).

2 From principal bundles to Hopf-Galois extensions

We briefly recall the definition of principal bundle (cf. [18], and also [9]) presenting it in a form readily generalizable to the noncommutative case. Replacing manifolds (algebraic varieties) with their algebras of (coordinate) functions we arrive at the definition of Hopf-Galois extension. Then it is shown how Hopf-Galois extensions are understood in the category of A -bimodules that are also H -comodules (A -bimodules that are G -equivariant).

We recall that given a topological group G , a topological space E is a G -space if there is a continuous map $E \times G \rightarrow G, (e, g) \mapsto eg$ that is a right action of the group G on E , i.e. $e(gg') = (eg)g', e1_G = e$ for all $e \in E$ and $g, g' \in G$, where 1_G is the identity in G .

A G -bundle $E \rightarrow M$ is then a bundle $\pi : E \rightarrow M$ as well as a G -space E , these two structures being compatible, i.e., the G -action being fiber preserving: $\pi(eg) = \pi(e)$. In this case the projection $\pi : E \rightarrow M$ is canonically induced on the quotient $E/G \rightarrow M$. It is then natural to further ask $E/G \rightarrow M$ to be a homeomorphism. Let's now consider the case where the G -action $E \times G \rightarrow E$ is free (i.e., if $eg = e$ then $g = 1_G$) and the induced map $E/G \rightarrow M$ is indeed a homeomorphism. Freeness of the action then implies injectivity of the map

$$\begin{aligned} F : E \times G &\longrightarrow E \times_M E \\ (e, g) &\longmapsto (e, eg) \end{aligned} \tag{2.1}$$

where $E \times_M E = \{(e, e') \in E \times E; \pi(e) = \pi(e')\}$, (the map F is well defined since the G -action is fiber preserving). The map is furthermore surjective because $M \simeq E/G$ implies that if $\pi(e) = \pi(e')$ then there exists an element $g \in G$ such that $e = e'g$. Continuity of F follows from

that of the G -action (we assume $E \times_M E$ closed in $E \times E$), requiring the continuous bijection F to be a homeomorphism we hence arrive at

Definition 2.1. A principal G -bundle (E, M, π, G) is a G -bundle $\pi : E \rightarrow M$ where the induced map $E/G \rightarrow M$ as well as the map F in (2.1) are homeomorphisms.

Consider now the principal bundle (E, M, π, G) where E and M are affine algebraic varieties, G is an affine algebraic group (e.g. $GL(n), SL(n), O(n), SO(n), \dots$) and $M = E/G$. Denote by $H = \mathcal{O}(G)$, $A = \mathcal{O}(E)$, $B = \mathcal{O}(M)$ the coordinate rings of the corresponding complex valued algebraic functions. Then $\mathcal{O}(G \times G) \simeq \mathcal{O}(G) \otimes \mathcal{O}(G)$ so that $H = \mathcal{O}(G)$ is a Hopf algebra with coproduct, counit and antipode respectively defined by, for $g, g' \in G$,

$$\begin{aligned} \Delta : H &\rightarrow H \otimes H & , & \quad \Delta(h)(g, g') = h(gg') , \\ \epsilon : H &\rightarrow \mathbb{C} & , & \quad \epsilon(h)(g) = h(1_G) , \\ S : H &\rightarrow H & , & \quad S(h)(g) = h(g^{-1}) . \end{aligned}$$

Similarly, since $\mathcal{O}(E \times G) \simeq \mathcal{O}(E) \otimes \mathcal{O}(G)$, we have that the right G -action on E pulls-back to a right H -coaction $\delta^A : A \rightarrow A \otimes H$ that is also an algebra map: $\delta^A(aa') = \delta^A(a)\delta^A(a')$ (with $(a \otimes h)(a' \otimes h') = aa' \otimes hh'$ for all $a \otimes h, a' \otimes h' \in A \otimes H$). Furthermore $B = \mathcal{O}(M) = \mathcal{O}(E/G)$ is the subalgebra of functions on E that are constants on the fibers, i.e. $B = \{a \in A; a(eg) = a(e), \text{ for all } e \in E, g \in G\}$, or equivalently, it is the subalgebra of coinvariant elements under the coaction $\delta^A : A \rightarrow A \otimes H$, i.e.,

$$B = A^{coH} = \{b \in A \mid \delta^A(b) = b \otimes 1\} . \quad (2.2)$$

Finally we also have $A \otimes_B A \simeq \mathcal{O}(E \times_{E/G} E)$ where \otimes_B is the tensor product over the algebra B , and that the algebraic structure of the principal G -bundle $E \rightarrow E/G$, i.e., bijectivity of the map F , is equivalently captured by the bijectivity of the pull back of F .

The above construction is formalized and generalized to the noncommutative case in the definitions that follows. Let \mathbb{K} denote the field of complex numbers \mathbb{C} , or the the ring of formal power series $\mathbb{C}[[\hbar]]$; with slight abuse of notation a \mathbb{K} -module will be simply called a vector space or linear space.

Definition 2.2. Let H be a Hopf-algebra. A right H -comodule is a vector space V with a linear map $\delta^V : V \rightarrow V \otimes H$ (called a right H -coaction) such that

$$(\text{id} \otimes \Delta) \circ \delta^V = (\delta^V \otimes \text{id}) \circ \delta^V , \quad (\text{id} \otimes \epsilon) \circ \delta^V = \text{id} . \quad (2.3)$$

The coaction on an element $v \in V$ is written in Sweedler notation as $\delta^V(v) = v_{(0)} \otimes v_{(1)}$ (sum understood), so that, for all $v \in V$, $(\text{id} \otimes \Delta) \circ \delta^V(v) = (\delta^V \otimes \text{id}) \circ \delta^V(v) = v_{(0)} \otimes v_{(1)} \otimes v_{(2)}$ and $v_{(0)} \epsilon(v_{(1)}) = v$. A morphism $\psi : V \rightarrow W$ of H -comodules is a linear map compatible with the H -coactions:

$$\delta^W(\psi(v)) = (\psi \otimes \text{id}) \delta^V(v) , \quad (2.4)$$

for all $v \in V$. We denote by \mathcal{M}^H the category of H -comodules.

Definition 2.3. A (right) H -comodule algebra A is a right H -comodule A that is also an algebra (unital and associative, possibly noncommutative), with the two structures that are compatible, i.e., for all $a, a' \in A$,

$$\delta^A(aa') = \delta^A(a) \delta^A(a') \quad , \quad \delta^A(1_A) = 1_A \otimes 1_H . \quad (2.5)$$

(where $A \otimes H$ has the tensor product algebra structure).

Definition 2.4. Let H be a Hopf algebra with invertible antipode, and A an H -comodule algebra. Let $B \subset A$ be the subalgebra of coinvariants, i.e., $B := A^{coH} = \{b \in A \mid \delta^A(b) = b \otimes 1_H\}$. The map

$$\begin{aligned} \chi : A \otimes_B A &\longrightarrow A \otimes H, \\ a \otimes_B a' &\longmapsto aa'_{(0)} \otimes a'_{(1)} \end{aligned} \quad (2.6)$$

is called the *canonical map*. The extension $B \subset A$ is an H -**Hopf-Galois extension** if the canonical map is bijective.

In order to study the properties of the canonical map we have to study tensor products of H -comodules. Given right H -comodules V and W , the tensor product $V \otimes W$ is an H -comodule with the right H -coaction

$$\begin{aligned} \delta^{V \otimes W} : V \otimes W &\longrightarrow V \otimes W \otimes H, \\ v \otimes w &\longmapsto v_{(0)} \otimes w_{(0)} \otimes v_{(1)} w_{(1)}. \end{aligned} \quad (2.7)$$

With this tensor product, H -comodules form a monoidal category (the unit object being \mathbb{K}). In particular $A \otimes A$ is a right H -comodule, and this structure is induced to the quotient $A \otimes_B A$. The relevant H -comodule structure on $A \otimes H$ is obtained by considering the H -adjoint coaction on H itself: we denote by \underline{H} the H -comodule that equals H as vector space and that has right H -adjoint coaction

$$\delta^{\underline{H}} = \text{Ad} : \underline{H} \longrightarrow \underline{H} \otimes H, \quad h \longmapsto h_{(2)} \otimes S(h_{(1)}) h_{(3)}, \quad (2.8)$$

(the notation \underline{H} is in order to distinguish this structure from the Hopf algebra structure). The tensor product of H -comodules $A \otimes \underline{H}$ is an H -comodule with right H -coaction $\delta^{A \otimes \underline{H}} : A \otimes \underline{H} \rightarrow A \otimes \underline{H} \otimes H$ given by (cf. (2.7)), for all $a \in A$, $h \in \underline{H}$,

$$\delta^{A \otimes \underline{H}}(a \otimes h) = a_{(0)} \otimes h_{(2)} \otimes a_{(1)} S(h_{(1)}) h_{(3)} \in A \otimes \underline{H} \otimes H. \quad (2.9)$$

The H -comodules $A \otimes_B A$ and $A \otimes \underline{H}$ are furthermore trivially left A -modules, where the left A -action is just multiplication from the left on the first component of the tensor product; they are also right A -modules: the right A -action on $A \otimes_B A$ is just multiplication from the right on the second component, while on $A \otimes \underline{H}$ the right A -action is given by

$$\begin{aligned} \triangleleft_{A \otimes \underline{H}} : A \otimes \underline{H} \otimes A &\longrightarrow A \otimes \underline{H}, \\ a \otimes h \otimes c &\longmapsto ac_{(0)} \otimes hc_{(1)}. \end{aligned} \quad (2.10)$$

The left and right A -actions are compatible (commute) so that $A \otimes_B A$ and $A \otimes \underline{H}$ are A -bimodules. These A -actions are also compatible with the H -coaction, explicitly, an H -comodule V has a compatible A -bimodule structure (where A is an H -module algebra) if, for all $a \in A$ and $v \in V$,

$$(a \triangleright_V v)_{(0)} \otimes (a \triangleright_V v)_{(1)} = a_{(0)} \triangleright_V v_{(0)} \otimes a_{(1)} v_{(1)}, \quad (2.11)$$

$$(v \triangleleft_V a)_{(0)} \otimes (v \triangleleft_V a)_{(1)} = v_{(0)} \triangleleft_V a_{(0)} \otimes v_{(1)} a_{(1)} \quad (2.12)$$

where \triangleright_V and \triangleleft_V denote the left and right A -actions on V . By definition an (H, A) -**relative Hopf module** is an H -comodule that has a compatible A -bimodule structure.

We denote the category of (H, A) -relative Hopf modules by ${}_A \mathcal{M}_A^H$; morphisms in this category are morphisms of right H -comodules which are also morphisms of A -bimodules. We have just seen that $A \otimes_B A$ and $A \otimes \underline{H}$ are (H, A) -relative Hopf modules,

Proposition 2.5. *The canonical map $\chi : A \otimes_B A \rightarrow A \otimes \underline{H}$ is a morphism of (H, A) -relative Hopf modules.*

Proof. We show that the canonical map is a morphism of right H -comodules, for all $a, a' \in A$,

$$\begin{aligned} \delta^{A \otimes \underline{H}}(\chi(a \otimes_B a')) &= \delta^{A \otimes \underline{H}}(a a'_{(0)} \otimes a'_{(1)}) = a_{(0)} a'_{(0)} \otimes a'_{(3)} \otimes a_{(1)} a'_{(1)} S(a'_{(2)}) a'_{(4)} \\ &= a_{(0)} a'_{(0)} \otimes a'_{(1)} \otimes a_{(1)} a'_{(2)} = (\chi \otimes \text{id})((a_{(0)} \otimes_B a'_{(0)}) \otimes a_{(1)} a'_{(1)}) \\ &= (\chi \otimes \text{id})(\delta^{A \otimes_B A}(a \otimes_B a')). \end{aligned}$$

It is immediate to see that χ is a morphism of left and right A -modules. \square

Example 2.6. Let as before (E, M, π, G) be a principal bundle where E and M are affine algebraic varieties, G is an affine algebraic group and $M = E/G$. Let furthermore E' be an affine algebraic variety and a G -space. The tensor product of H -comodules (2.7) corresponds to the cartesian product $E \times E'$ that is a G -space with the diagonal G -action $(e, e')g = (eg, e'g)$. The right G -adjoint action on G pulls back to the right adjoint H -coaction $\delta^{\underline{H}} = \text{Ad}$ on \underline{H} , see (2.8). *Proof:* $(h_{(2)} \otimes S(h_{(1)}) h_{(3)})(g, g') = h_{(2)}(g) S(h_{(1)})(g') h_{(3)}(g') = h_{(2)}(g) h_{(1)}(g'^{-1}) h_{(3)}(g') = h(g'^{-1} g g')$, for all $g, g' \in G, h \in H$. The map $F : E \times G \rightarrow E \times_M E$ in (2.1) is compatible with the diagonal right G -actions $(E \times G) \times G \rightarrow (E \times G)$, $(e, g)g' = (eg', g'^{-1} g g')$ and $(E \times_{E/G} E) \times G \rightarrow (E \times_{E/G} E)$, $(e, e')g' = (eg', e'g')$, i.e., it is G -equivariant¹: $(F(e, g))g' = F((e, g)g')$, so that its pull back $\chi : A \otimes_B A \rightarrow A \otimes \underline{H}$ is an H -comodule map. Furthermore the A -bimodule structure on $A \otimes H$ corresponds to the maps $E \times G \rightarrow E \times E \times G$, $(e, g) \mapsto (e, e, g)$, and $E \times G \rightarrow E \times G \times E$, $(e, g) \mapsto (e, g, eg)$; similarly, the A -bimodule structure on $A \otimes_B A$ corresponds to the diagonal maps $E \times_{E/G} E \rightarrow E \times_{E/G} E \times_{E/G} E$, $(e, e') \mapsto (e, e, e')$ and $E \times_{E/G} E \rightarrow E \times_{E/G} E \times_{E/G} E$, $(e, e') \mapsto (e, e', e')$ for all $e, e' \in E$ with $\pi(e) = \pi(e')$. Compatibility of the map F with these maps implies that $\chi : A \otimes_B A \rightarrow A \otimes \underline{H}$ is an (H, A) -relative Hopf module map. Of course this result follows from Proposition 2.5, however we have here derived it from the geometric properties of the map $F : E \times G \rightarrow E \times_M E$, thus providing geometric intuition for its pull back $\chi : A \otimes_B A \rightarrow A \otimes \underline{H}$.

Example 2.7. (Fréchet Hopf-Galois extension). Let (E, M, π, G) be a principal bundle in the smooth category (E and M are smooth manifolds, G is a Lie group) and $M = E/G$. The space of smooth functions $C^\infty(E)$ is a (nuclear) Fréchet space with respect to the usual smooth topology. It is furthermore a unital Fréchet algebra with (continuous) product $m := \text{diag}_E^* : A \widehat{\otimes} A \rightarrow A$, where $A \widehat{\otimes} A \simeq C^\infty(P \times P)$ denotes the completed tensor product. Similarly $H = C^\infty(G)$ is a Fréchet Hopf algebra, i.e., a Hopf algebra were product, antipode, counit and coproduct $\Delta : H \rightarrow H \widehat{\otimes} H$ are continuous maps. The right G -action $E \times G \rightarrow E$ pulls back to a continuous right H -coaction $\delta^A : A \rightarrow A \widehat{\otimes} H$, so that A becomes a Fréchet H -comodule algebra. The H -coinvariant subalgebra is $B = A^{coH} = C^\infty(E/G) = C^\infty(M)$, and the canonical map is the pull back of the isomorphism $F : E \times G \rightarrow E \times_M E$ in (2.1),² $\chi : A \widehat{\otimes}_B A \rightarrow A \widehat{\otimes} H$, hence $B \subset A$ is a Fréchet H -Hopf-Galois extension. As in the previous example the right G -adjoint action on G pulls back to a right H -coaction on \underline{H} , so that \underline{H} is a

¹The map F is also compatible with the G -actions $(E \times G) \times G \rightarrow (E \times G)$, $(e, g) \cdot g' = (e, g g')$ and $(E \times_{E/G} E) \times G \rightarrow (E \times_{E/G} E)$, $(e, e') \cdot g' = (e, e' g')$. For a trivial bundle $E \simeq M \times G$ it is equivalent to state compatibility of F with respect to these actions or to the actions defined in the main text. Indeed the isomorphism $(M \times G) \times G \rightarrow (M \times G) \times G$, $(m, \tilde{g}, g) \mapsto (m, \tilde{g}, \tilde{g}g)$ intertwines these two actions (i.e., it is G -equivariant).

²The topological tensor product over B is the quotient $A \widehat{\otimes}_B A := A \widehat{\otimes} A / \text{Im}(m \widehat{\otimes} \text{id} - \text{id} \widehat{\otimes} m)$, where $m \widehat{\otimes} \text{id}$ and $\text{id} \widehat{\otimes} m$ are maps from $A \widehat{\otimes} B \widehat{\otimes} A$ to $A \widehat{\otimes} A$ that describe the right and respectively left action of B on A , and where the overline $\overline{\quad}$ denotes the closure in the Fréchet space $A \widehat{\otimes} A$. It can be shown that $A \widehat{\otimes}_B A \simeq C^\infty(P \times_M P)$.

Fréchet H -comodule; furthermore $A \widehat{\otimes}_B A$ and $A \widehat{\otimes} \underline{H}$ are Fréchet (H, A) -relative Hopf modules and the canonical map $\chi : A \widehat{\otimes}_B A \rightarrow A \widehat{\otimes} \underline{H}$ is a homeomorphism of Fréchet (H, A) -relative Hopf modules.

We conclude this section recalling that an H -Hopf-Galois extension $B := A^{coH} \subset A$ is said to be trivial (or to have the normal basis property or to be cleft) if there exists an isomorphism $A \simeq B \otimes H$ of left B -modules and right H -comodules (where $B \otimes H$ is a left B -module via $m_B \otimes \text{id}$ and a right H -comodule via $\text{id} \otimes \Delta$). This condition captures the algebraic aspect of triviality of a principal bundle.

We have recalled that both topological and algebraic structures combine in the definition of principal G -bundle. In the definition of Hopf-Galois extension we have implemented the algebraic properties of a principal bundle, considering their richer structure of topological spaces leads to a refinement of the notion of Hopf-Galois extension,

Definition 2.8. Let H be a Hopf algebra with invertible antipode over a field \mathbb{K} . A **principal H -comodule algebra** A is an H -comodule algebra A such that $B := A^{coH} \subset A$ is an H -Hopf-Galois extension and A is equivariantly projective as a left B -module, i.e. there exists a left B -module and right H -comodule morphism $s : A \rightarrow B \otimes A$ that is a section of the (restricted) product $m : B \otimes A \rightarrow A$, i.e. such that $m \circ s = \text{id}_A$.

The condition of equivariant projectivity of A is equivalent to that of faithful flatness of A [22] (we assume the antipode of H is invertible). From the characterization of faithfully flat extensions [23] it follows that if H is cosemisimple then surjectivity of the canonical map is sufficient to prove its bijectivity and principality of A .

3 Drinfeld twists and 2-cocycles deformations

We first recall the notion of 2-cocycle [12] and the dual notion of Drinfeld twist [13]. We then review Hopf algebra deformations via 2-cocycles and present the corresponding deformations of H -comodules, H -comodule algebras A , and (H, A) -relative Hopf-modules.

3.1 2-Cocycles, twists and Hopf algebra deformations

Let H be a Hopf algebra and recall that $H \otimes H$ is canonically a coalgebra with coproduct $\Delta_{H \otimes H}(h \otimes k) = h_{(1)} \otimes k_{(1)} \otimes h_{(2)} \otimes k_{(2)}$ and counit $\varepsilon_{H \otimes H}(h \otimes k) = \varepsilon(h)\varepsilon(k)$, for all $h, k \in H$. In particular, we can consider the convolution product of \mathbb{K} -linear maps $H \otimes H \rightarrow \mathbb{K}$.

Definition 3.1. A linear map $\gamma : H \otimes H \rightarrow \mathbb{K}$ is called a **2-cocycle**, provided that:

i) it satisfies, for all $g, h, k \in H$,

$$\gamma(g_{(1)} \otimes h_{(1)})\gamma(g_{(2)}h_{(2)} \otimes k) = \gamma(h_{(1)} \otimes k_{(1)})\gamma(g \otimes h_{(2)}k_{(2)}) , \quad (3.1)$$

ii) it is convolution invertible, i.e., there exists $\bar{\gamma} : H \otimes H \rightarrow \mathbb{K}$ such that $\bar{\gamma} * \gamma = \gamma * \bar{\gamma} = \varepsilon_{H \otimes H}$ (where the convolution product explicitly reads $\bar{\gamma} * \gamma(h \otimes k) = \bar{\gamma}(h_{(1)} \otimes k_{(1)})\gamma(h_{(2)} \otimes k_{(2)})$),

iii) it is unital, i.e. $\gamma(h \otimes 1) = \varepsilon(h) = \gamma(1 \otimes h)$, for all $h \in H$.

Remark 3.2 (Twists and 2-cocycles). Let H' be another Hopf algebra, a twist on H' is an invertible element $\mathcal{F} \in H' \otimes H'$ such that $(\varepsilon_{H'} \otimes \text{id})(\mathcal{F}) = 1 = (\text{id} \otimes \varepsilon_{H'})(\mathcal{F})$ and

$$(\mathcal{F} \otimes 1)[(\Delta_{H'} \otimes \text{id})(\mathcal{F})] = (1 \otimes \mathcal{F})[(\text{id} \otimes \Delta_{H'})(\mathcal{F})] . \quad (3.2)$$

Let further H' and H be dually paired Hopf algebras, with pairing $\langle \cdot, \cdot \rangle : H' \times H \rightarrow \mathbb{K}$, i.e., for all $\xi, \zeta \in H'$ and $h, k \in H$ we have $\langle \xi \zeta, h \rangle = \langle \xi, h_{(1)} \rangle \langle \zeta, h_{(2)} \rangle$, $\langle \xi, hk \rangle = \langle \xi_{(1)}, h \rangle \langle \xi_{(2)}, k \rangle$, $\langle \xi, 1_H \rangle = \varepsilon_{H'}(\xi)$, $\langle 1_{H'}, h \rangle = \varepsilon_H(h)$. Then to each twist $\mathcal{F} = \mathbf{f}^\alpha \otimes \mathbf{f}_\alpha \in H' \otimes H'$ (sum over α understood) there corresponds a 2-cocycle $\gamma_{\mathcal{F}} : H \otimes H \rightarrow \mathbb{K}$ on H defined by

$$\gamma_{\mathcal{F}}(h \otimes k) := \langle \mathcal{F}, h \otimes k \rangle = \langle \mathbf{f}^\alpha, h \rangle \langle \mathbf{f}_\alpha, k \rangle, \quad (3.3)$$

for all $h, k \in H$. The 2-cocycle condition for $\gamma_{\mathcal{F}}$ follows from the twist condition for \mathcal{F} and similarly the remaining properties *ii*) and *iii*) of γ follow from invertibility of \mathcal{F} and its normalization $(\varepsilon_{H'} \otimes \text{id})(\mathcal{F}) = 1 = (\text{id} \otimes \varepsilon_{H'})(\mathcal{F})$.

Examples of dually paired Hopf algebras are the Hopf algebra $H = \mathcal{O}(G)$ of an affine algebraic group G and the universal enveloping algebra $U(\mathfrak{g})$ of its Lie algebra \mathfrak{g} . A Lie algebra element $v \in \mathfrak{g}$ is equivalently a left invariant vector field v on G and the pairing with a function $f \in \mathcal{O}(G)$ is given by applying the vector field to the function and then evaluating at the unit element 1_G of the group: $\langle v, f \rangle = v(f)|_{1_G}$. The pairing is then extended to all $U(\mathfrak{g})$ using the coproduct of $\mathcal{O}(G)$ and by linearity. Twists associated with $U(\mathfrak{g})$ were studied by Drinfeld (in the form of formal power series, cf. also Example 4.16) and as outlined in this remark lead to 2-cocycles on $H = \mathcal{O}(G)$.

Proposition 3.3. *Let $\gamma : H \otimes H \rightarrow \mathbb{K}$ be a 2-cocycle. Then*

$$m_\gamma(h \otimes k) := h \cdot_\gamma k := \gamma(h_{(1)} \otimes k_{(1)}) h_{(2)} k_{(2)} \bar{\gamma}(h_{(3)} \otimes k_{(3)}), \quad (3.4)$$

for all $h, k \in H$, defines a new associative product on H . The resulting algebra $H_\gamma := (H, m_\gamma, 1_H)$ is a Hopf algebra when endowed with the unchanged coproduct Δ and counit ε and with the new antipode $S_\gamma := u_\gamma * S * \bar{u}_\gamma$, where $u_\gamma : H \rightarrow \mathbb{K}$, $h \mapsto \gamma(h_{(1)} \otimes S(h_{(2)}))$, with inverse $\bar{u}_\gamma : H \rightarrow \mathbb{K}$, $h \mapsto \bar{\gamma}(S(h_{(1)}) \otimes h_{(2)})$. We call H_γ the twisted Hopf algebra of H by γ .

We refer for example to [12] for a proof of this standard result. Notice that the twisted Hopf algebra H_γ can be ‘untwisted’ by using the convolution inverse $\bar{\gamma} : H \otimes H \rightarrow \mathbb{K}$; indeed, $\bar{\gamma}$ is a 2-cocycle for H_γ and the twisted Hopf algebra of H_γ by $\bar{\gamma}$ is isomorphic to H via the identity map. Finally, among the identities satisfied by 2-cocycles we will later use

$$\gamma(gh_{(1)} \otimes S(h_{(2)})k) = \bar{\gamma}(g_{(1)} \otimes h_{(1)}) u_\gamma(h_{(2)}) \bar{\gamma}(S(h_{(3)}) \otimes k_{(1)}) \gamma(g_{(2)} \otimes k_{(2)}), \quad (3.5)$$

for all $g, h, k \in H$, that is for example proven in [1].

3.2 Twist deformation of right H -comodules

Given a 2-cocycle $\gamma : H \otimes H \rightarrow \mathbb{K}$ not only we have a new Hopf algebra H_γ but also corresponding comodules. Indeed if $V \in \mathcal{M}^H$ is a right H -comodule with coaction $\delta^V : V \rightarrow V \otimes H$, then V with the same coaction, but now thought of as a map with values in $V \otimes H_\gamma$, is a right H_γ -comodule. This is the case simply because the Definition 2.2 of H -comodule only involves the coalgebra structure of H , and H_γ coincides with H as a coalgebra. When considering V as an object in \mathcal{M}^{H_γ} we will denote it by V_γ and the coaction by $\delta^{V_\gamma} : V_\gamma \rightarrow V_\gamma \otimes H_\gamma$. Moreover, any morphism $\psi : V \rightarrow W$ in \mathcal{M}^H can be thought as a morphism $\psi : V_\gamma \rightarrow W_\gamma$ in \mathcal{M}^{H_γ} ; indeed, H -equivariance of $\psi : V \rightarrow W$ implies H_γ -equivariance of $\psi : V_\gamma \rightarrow W_\gamma$ since by construction the right H -coaction in V agrees

with the right H_γ -coaction in V_γ . Hence we have a functor between the categories of right H -comodules and of right H_γ -comodules,

$$\Gamma : \mathcal{M}^H \rightarrow \mathcal{M}^{H_\gamma}, \quad (3.6)$$

defined by $\Gamma(V) := V_\gamma$ and $\Gamma(\psi) := \psi : V_\gamma \rightarrow W_\gamma$. Furthermore this functor Γ induces an equivalence of categories because we can use the convolution inverse $\bar{\gamma}$ in order to twist back H_γ to $(H_\gamma)_{\bar{\gamma}} = H$ and V_γ to $(V_\gamma)_{\bar{\gamma}} = V$.

We denote by $(\mathcal{M}^{H_\gamma}, \otimes^\gamma)$ the monoidal category corresponding to the Hopf algebra H_γ . Explicitly, for all objects $V_\gamma, W_\gamma \in \mathcal{M}^{H_\gamma}$ (with coactions $\delta^{V_\gamma} : V_\gamma \rightarrow V_\gamma \otimes H_\gamma$ and $\delta^{W_\gamma} : W_\gamma \rightarrow W_\gamma \otimes H_\gamma$), the right H_γ -coaction on $V_\gamma \otimes^\gamma W_\gamma$, according to (2.7), is given by

$$\begin{aligned} \delta^{V_\gamma \otimes^\gamma W_\gamma} : V_\gamma \otimes^\gamma W_\gamma &\longrightarrow V_\gamma \otimes^\gamma W_\gamma \otimes H_\gamma, \\ v \otimes^\gamma w &\longmapsto v_{(0)} \otimes^\gamma w_{(0)} \otimes v_{(1)} \cdot_\gamma w_{(1)}. \end{aligned} \quad (3.7)$$

The equivalence between the categories \mathcal{M}^H and \mathcal{M}^{H_γ} extends to their monoidal structure:

Theorem 3.4. *The functor $\Gamma : \mathcal{M}^H \rightarrow \mathcal{M}^{H_\gamma}$ induces an equivalence between the monoidal categories (\mathcal{M}^H, \otimes) and $(\mathcal{M}^{H_\gamma}, \otimes^\gamma)$ that is given by the isomorphisms*

$$\begin{aligned} \varphi_{V,W} : V_\gamma \otimes^\gamma W_\gamma &\longrightarrow (V \otimes W)_\gamma, \\ v \otimes^\gamma w &\longmapsto v_{(0)} \otimes w_{(0)} \bar{\gamma}(v_{(1)} \otimes w_{(1)}), \end{aligned} \quad (3.8)$$

in the category \mathcal{M}^{H_γ} of right H_γ -comodules for all objects $V, W \in \mathcal{M}^H$.

Proof. The invertibility of $\varphi_{V,W}$ follows immediately from the invertibility of the cocycle γ . The fact that it is a morphism in the category \mathcal{M}^{H_γ} is easily shown as follows:

$$\begin{aligned} (\varphi_{V,W} \otimes \text{id}) \left(\delta^{V_\gamma \otimes^\gamma W_\gamma}(v \otimes^\gamma w) \right) &= v_{(0)} \otimes w_{(0)} \bar{\gamma}(v_{(1)} \otimes w_{(1)}) \otimes \gamma(v_{(2)} \otimes w_{(2)}) v_{(3)} w_{(3)} \bar{\gamma}(v_{(4)} \otimes w_{(4)}) \\ &= v_{(0)} \otimes w_{(0)} \otimes v_{(1)} w_{(1)} \bar{\gamma}(v_{(2)} \otimes w_{(2)}) \\ &= \delta^{(V \otimes W)_\gamma}(v_{(0)} \otimes w_{(0)}) \bar{\gamma}(v_{(1)} \otimes w_{(1)}) \\ &= \delta^{(V \otimes W)_\gamma}(\varphi_{V,W}(v \otimes^\gamma w)), \end{aligned}$$

where the coaction $\delta^{(V \otimes W)_\gamma}$ is given by $\delta^{(V \otimes W)_\gamma} : v \otimes w \mapsto v_{(0)} \otimes w_{(0)} \otimes v_{(1)} w_{(1)}$ (cf. (2.7)). Hence $(\Gamma, \varphi) : (\mathcal{M}^H, \otimes) \rightarrow (\mathcal{M}^{H_\gamma}, \otimes^\gamma)$ is a monoidal functor.

The monoidal categories are equivalent (actually they are isomorphic) because $\bar{\gamma}$ twists back H_γ to H and V_γ to V so that the monoidal functor (Γ, φ) has an inverse $(\bar{\Gamma}, \bar{\varphi})$, where $\bar{\Gamma} : \mathcal{M}^{H_\gamma} \rightarrow \mathcal{M}^H$ is the inverse of the functor Γ and $\bar{\varphi}_{V_\gamma, W_\gamma} : (V_\gamma)_{\bar{\gamma}} \otimes (W_\gamma)_{\bar{\gamma}} \rightarrow (V_\gamma \otimes^\gamma W_\gamma)_{\bar{\gamma}}$, $v \otimes w \mapsto v_{(0)} \otimes^\gamma w_{(0)} \gamma(v_{(1)} \otimes w_{(1)})$. \square

Given a 2-cocycle γ on H , the H -comodule algebra A is also deformed in an H_γ -comodule algebra A_γ . The H_γ -comodule structure is just the H -comodule structure now thought as an H_γ -structure, the product in A_γ is given by

$$m_\gamma : A_\gamma \otimes^\gamma A_\gamma \longrightarrow A_\gamma, \quad a \otimes^\gamma a' \longmapsto a_{(0)} a'_{(0)} \bar{\gamma}(a_{(1)} \otimes a'_{(1)}) =: a \bullet_\gamma a' \quad (3.9)$$

(and the unit is undeformed). Associativity of this product follows from the cocycle condition (3.1). Using the convolution inverse $\bar{\gamma}$ of γ we can twist back A_γ to A . This implies

that the functor that deforms H -comodule algebras into H_γ -comodule algebras induces an equivalence between H and H_γ -comodule algebras.

By a similar construction one obtains the functor $\Gamma : {}_A\mathcal{M}_A^H \rightarrow {}_{A_\gamma}\mathcal{M}_{A_\gamma}^{H_\gamma}$ between relative (H, A) and (H_γ, A_γ) -Hopf modules. If V is a relative (H, A) -Hopf module, then it is an H_γ -comodule V_γ that becomes a relative (H_γ, A_γ) -Hopf module with the deformed left and right A_γ -actions:

$$\begin{aligned} \triangleright_{V_\gamma} : A_\gamma \otimes^\gamma V_\gamma &\longrightarrow V_\gamma, \\ a \otimes^\gamma v &\longmapsto (a_{(0)} \triangleright_V v_{(0)}) \bar{\gamma}(a_{(1)} \otimes v_{(1)}), \\ \triangleleft_{V_\gamma} : V_\gamma \otimes^\gamma A_\gamma &\longrightarrow V_\gamma, \\ v \otimes^\gamma a &\longmapsto (v_{(0)} \triangleleft_V a_{(0)}) \bar{\gamma}(v_{(1)} \otimes a_{(1)}). \end{aligned} \tag{3.10}$$

Moreover the maps $\varphi_{V,W}$ in (3.8) are isomorphisms in the category ${}_{A_\gamma}\mathcal{M}_{A_\gamma}^{H_\gamma}$ of (H_γ, A_γ) -relative Hopf modules.

4 Twist deformations of Hopf-Galois extensions

We first deform H -Hopf-Galois extensions via a 2-cocycle on H , then via a 2-cocycle on a Hopf algebra K associated with an external symmetry of the Hopf-Galois extension and finally combine both deformations. If the initial Hopf-Galois extension is given by a G -principal bundle the first twist deformation is a deformation of the structure group and of the fiber of the principal bundle, while the second is a deformation of the base space. With abuse of language, also for arbitrary H -Hopf-Galois extensions $B = A^{coH} \subset A$ we speak of deformations of the “structure group” H and of the “base space” B .

4.1 Deformation of the “structure group” H via a 2-cocycle on H

Given an H -comodule algebra A and a twist γ on H we can consider the canonical map $\chi : A \otimes_B A \rightarrow A \otimes \underline{H}$ as well as the canonical map on the twist deformed structures $\chi_\gamma : A_\gamma \otimes_B^\gamma A_\gamma \rightarrow A_\gamma \otimes^\gamma \underline{H}_\gamma$. We show that χ is invertible iff χ_γ is invertible, i.e., that Hopf-Galois extensions are deformed into Hopf-Galois extensions. In particular if $\chi : A \otimes_B A \rightarrow A \otimes \underline{H}$ is associated to a commutative principal bundle as in Example 2.6 and 2.7, we obtain noncommutative (or quantum) principal bundles described by the Hopf Galois extension $\chi_\gamma : A_\gamma \otimes_B^\gamma A_\gamma \rightarrow A_\gamma \otimes^\gamma \underline{H}_\gamma$. In order to relate χ to χ_γ we first observe that bijectivity of the (H, A) -relative Hopf module map $\chi : A \otimes_B A \rightarrow A \otimes \underline{H}$ is equivalent to bijectivity of the (H_γ, A_γ) -relative Hopf module map $\Gamma(\chi) : (A \otimes_B A)_\gamma \rightarrow (A \otimes \underline{H})_\gamma$ (recall that as a linear map $\Gamma(\chi) = \chi$). Next we relate $\Gamma(\chi) : (A \otimes_B A)_\gamma \rightarrow (A \otimes \underline{H})_\gamma$ to $\chi_\gamma : A_\gamma \otimes_B^\gamma A_\gamma \rightarrow A_\gamma \otimes^\gamma \underline{H}_\gamma$ via the

diagram,

$$\begin{array}{ccc}
A_\gamma \otimes_B^\gamma A_\gamma & \xrightarrow{\chi_\gamma} & A_\gamma \otimes^\gamma \underline{H}_\gamma \\
\downarrow \varphi_{A,A} & & \downarrow \text{id} \otimes^\gamma \mathfrak{G} \\
(A \otimes_B A)_\gamma & \xrightarrow{\Gamma(\chi)} & A_\gamma \otimes^\gamma \underline{H}_\gamma \\
& & \downarrow \varphi_{A,\underline{H}} \\
& & (A \otimes \underline{H})_\gamma
\end{array} \tag{4.1}$$

where, as we now explain, the vertical arrows are H_γ -comodule isomorphisms. We will show that this is a commutative diagram.

From Theorem 3.4 the maps $\varphi_{A,A} : A_\gamma \otimes^\gamma \underline{H}_\gamma \rightarrow (A \otimes \underline{H})_\gamma$ and $\varphi_{A,A} : A_\gamma \otimes^\gamma A_\gamma \rightarrow (A \otimes A)_\gamma$ and hence the induced map on the quotients $\varphi_{A,A} : A_\gamma \otimes_B^\gamma A_\gamma \rightarrow (A \otimes_B A)_\gamma$ are all H -comodule isomorphisms. We are left with the description of the map $\mathfrak{G} : \underline{H}_\gamma \rightarrow \underline{H}_\gamma$, between the H_γ -comodule \underline{H}_γ with H_γ -adjoint coaction

$$\delta^{\underline{H}_\gamma} = \text{Ad}_\gamma : \underline{H}_\gamma \longrightarrow \underline{H}_\gamma \otimes H_\gamma, \quad h \longmapsto h_{(2)} \otimes S_\gamma(h_{(1)}) \cdot_\gamma h_{(3)} \tag{4.2}$$

and the H_γ -comodule \underline{H}_γ that has H_γ -coaction (cf. (2.8))

$$\delta^{\underline{H}_\gamma} = \text{Ad} : \underline{H}_\gamma \longrightarrow \underline{H}_\gamma \otimes H_\gamma, \quad h \longmapsto h_{(2)} \otimes S(h_{(1)})h_{(3)}; \tag{4.3}$$

while \underline{H}_γ is the deformation of the H -comodule \underline{H} , in \underline{H}_γ we first deform the Hopf algebra H to H_γ and then regard it as an H_γ -comodule.

Theorem 4.1. *The \mathbb{K} -linear map*

$$\mathfrak{G} : \underline{H}_\gamma \longrightarrow \underline{H}_\gamma, \quad h \longmapsto h_{(3)} u_\gamma(h_{(1)}) \bar{\gamma}(S(h_{(2)}) \otimes h_{(4)}) \tag{4.4}$$

is an isomorphism of right H_γ -comodules, with inverse

$$\mathfrak{G}^{-1} : \underline{H}_\gamma \longrightarrow \underline{H}_\gamma, \quad h \longmapsto h_{(3)} \bar{u}_\gamma(h_{(2)}) \gamma(S(h_{(1)}) \otimes h_{(4)}) . \tag{4.5}$$

Proof. It is easy to prove by a direct calculation that \mathfrak{G}^{-1} is the inverse of \mathfrak{G} . We now show that \mathfrak{G} is a right H_γ -comodule morphism, for all $h \in \underline{H}_\gamma$,

$$\begin{aligned}
(\mathfrak{G} \otimes \text{id})(\text{Ad}_\gamma(h)) &= \\
&= \mathfrak{G}(h_{(2)}) \otimes S_\gamma(h_{(1)}) \cdot_\gamma h_{(3)} \\
&= \mathfrak{G}(h_{(4)}) \otimes u_\gamma(h_{(1)}) S(h_{(2)}) \bar{u}_\gamma(h_{(3)}) \cdot_\gamma h_{(5)} \\
&= u_\gamma(h_{(6)}) h_{(8)} \bar{\gamma}(S(h_{(7)}) \otimes h_{(9)}) \otimes u_\gamma(h_{(1)}) \bar{u}_\gamma(h_{(5)}) \gamma(S(h_{(4)}) \otimes h_{(10)}) S(h_{(3)}) h_{(11)} \bar{\gamma}(S(h_{(2)}) \otimes h_{(12)}) \\
&= h_{(6)} \bar{\gamma}(S(h_{(5)}) \otimes h_{(7)}) \otimes u_\gamma(h_{(1)}) \gamma(S(h_{(4)}) \otimes h_{(8)}) S(h_{(3)}) h_{(9)} \bar{\gamma}(S(h_{(2)}) \otimes h_{(10)}) \\
&= u_\gamma(h_{(1)}) h_{(4)} \otimes S(h_{(3)}) h_{(5)} \bar{\gamma}(S(h_{(2)}) \otimes h_{(6)}) = \text{Ad}(\mathfrak{G}(h)),
\end{aligned}$$

were in the fourth passage we used $u_\gamma(h_{(6)}) \bar{u}_\gamma(h_{(5)}) = \varepsilon(h_{(5)})$, and in the fifth $h_{(6)} \bar{\gamma}(S(h_{(5)}) \otimes h_{(7)}) \gamma(S(h_{(4)}) \otimes h_{(8)}) = h_{(5)} \varepsilon(h_{(4)}) \varepsilon(h_{(6)})$. \square

Remark 4.2. If we dualize this picture by considering a dually paired Hopf algebra H' (and dual modules on dual vector spaces), then the right H -adjoint coaction dualizes into the right H' -adjoint action, $\zeta \blacktriangleleft \xi = S(\xi_{(1)})\zeta\xi_{(2)}$ for all $\zeta, \xi \in H'$. If we further consider a mirror construction by using left adjoint actions rather than right ones, then the analogue of the isomorphism \mathfrak{G} is the isomorphism D studied in [4] and more in general in [5]. Explicitly the isomorphism \mathfrak{G} is dual to the isomorphism D relative to the Hopf algebra $H'^{op\ cop}$ with opposite product and coproduct; it follows from [6] that this latter is a component of a natural transformation determining the equivalence of the closed monoidal categories of left $H'^{op\ cop}$ -modules and left $(H'_\gamma)^{op\ cop}$ -modules.

Theorem 4.3. *Let H be a Hopf algebra and A an H -comodule algebra. Consider the algebra extension $B = A^{coH} \subset A$ and the associated canonical map $\chi : A \otimes_B A \rightarrow A \otimes H$. Given a 2-cocycle $\gamma : H \otimes H \rightarrow \mathbb{K}$ the diagram (4.1) is a commutative diagram of H -comodules.*

Proof. We prove that the diagram (4.1) commutes. We obtain for the composition $(\text{id} \otimes^\gamma \mathfrak{G}) \circ \chi_\gamma$ the following expression

$$\begin{aligned} (\text{id} \otimes^\gamma \mathfrak{G})(\chi_\gamma(a \otimes_B^\gamma a')) &= a_{(0)}a'_{(0)} \otimes^\gamma \mathfrak{G}(a'_{(2)})\bar{\gamma}(a_{(1)} \otimes a'_{(1)}) \\ &= a_{(0)}a'_{(0)} \otimes^\gamma a'_{(4)}u_\gamma(a'_{(2)})\bar{\gamma}(S(a'_{(3)}) \otimes a'_{(5)})\bar{\gamma}(a_{(1)} \otimes a'_{(1)}) . \end{aligned}$$

On the other hand, from (3.8) and (2.8) we have

$$\varphi_{A,\underline{H}}^{-1}(a \otimes h) = a_{(0)} \otimes^\gamma h_{(2)}\gamma(a_{(1)} \otimes S(h_{(1)})h_{(3)}) ,$$

so that for the composition $\varphi_{A,\underline{H}}^{-1} \circ \Gamma(\chi) \circ \varphi_{A,A}$ we obtain (recalling that $\Gamma(\chi) = \chi$)

$$\begin{aligned} &\varphi_{A,\underline{H}}^{-1}(\Gamma(\chi)(\varphi_{A,A}(a \otimes_B^\gamma a'))) \\ &= \varphi_{A,\underline{H}}^{-1}(a_{(0)}a'_{(0)} \otimes a'_{(1)})\bar{\gamma}(a_{(1)} \otimes a'_{(2)}) \\ &= a_{(0)}a'_{(0)} \otimes^\gamma a'_{(3)}\gamma(a_{(1)}a'_{(1)} \otimes S(a'_{(2)})a'_{(4)})\bar{\gamma}(a_{(2)} \otimes a'_{(5)}) \\ &= a_{(0)}a'_{(0)} \otimes^\gamma a'_{(4)}\bar{\gamma}(a_{(1)} \otimes a'_{(1)})u_\gamma(a'_{(2)})\bar{\gamma}(S(a'_{(3)}) \otimes a'_{(5)})\gamma(a_{(2)} \otimes a'_{(6)})\bar{\gamma}(a_{(3)} \otimes a'_{(7)}) , \end{aligned}$$

where we have used (3.5). Since $\bar{\gamma}$ is the convolution inverse of γ , the last two terms simplify, giving the desired identity. From the properties of the canonical map (Proposition 2.5) and of the natural isomorphisms φ all arrows in the diagram are H_γ -comodule maps. \square

Since all vertical arrows in diagram (4.1) are isomorphisms, as immediate corollary of this theorem (and recalling that $\Gamma(\chi) = \chi$ as linear map) we have that χ is bijective iff χ_γ is bijective. Hence we conclude that

Corollary 4.4. *Hopf-Galois extensions are twist deformed in Hopf-Galois extensions.*

Moreover if the Hopf-Galois extension $\chi : A \otimes_B A \rightarrow A \otimes \underline{H}$ is trivial (i.e. has the normal basis property or equivalently is cleft) we have a left B -module and right H -comodule isomorphism $\theta : A \rightarrow B \otimes H$. This same linear map, now seen as a map $A_\gamma \rightarrow B \otimes_\gamma H_\gamma$ is a left B -module and right H_γ -comodule isomorphism that determines triviality of the Hopf-Galois extension $\chi_\gamma : A_\gamma \otimes_B^\gamma A_\gamma \rightarrow A_\gamma \otimes^\gamma \underline{H}_\gamma$.

Remark 4.5. As shown in [1], there is a canonical relative (H_γ, A_γ) -Hopf module structure on $A_\gamma \otimes' \underline{H}_\gamma$, so that diagram (4.1) becomes a commutative diagram of relative (H_γ, A_γ) -Hopf modules, i.e., a diagram in the category ${}_{A_\gamma}\mathcal{M}_{A_\gamma}^{H_\gamma}$.

Remark 4.6. Montgomery and Schneider in [20, Th. 5.3] prove the above corollary by using that as vector spaces $A \otimes_B A = A_\gamma \otimes_B A_\gamma$ and $A \otimes H = A_\gamma \otimes H_\gamma$, and showing that the canonical map χ is the composition of χ_γ with an invertible map. The proof is not within the natural categorical setting of twists of Hopf-Galois extensions that we consider, and that we have motivated in the introduction to be relevant for the study of the differential geometry of noncommutative principal bundles.

Finally, recalling from Definition 2.8 the notion of principal H -comodule algebra it is easy to show that deformations by 2-cocycles $\gamma : H \otimes H \rightarrow \mathbb{K}$ preserve this structure, the key point being that given a section $s : A \rightarrow B \otimes A$ we have the section $s_\gamma := \varphi_{B,A}^{-1} \circ \Gamma(s) : A_\gamma \rightarrow B_\gamma \otimes' A_\gamma$ hence (cf. [1]),

Corollary 4.7. *A is a principal H -comodule algebra if and only if A_γ is a principal H_γ -comodule algebra.*

4.2 Deformation of the “base space” B via a 2-cocycle on an external symmetry K

Let L be Lie group acting via diffeomorphisms on both the total manifold and the base manifold of a bundle $E \rightarrow M$, these actions being compatible with the bundle projection (hence L acts via automorphisms of $E \rightarrow M$). We say that L is an external symmetry of $E \rightarrow M$. If $E \rightarrow M$ is a G -bundle then we also require G -equivariance of the L -action on the total manifold, i.e., we require the L -action to commute with the G -action.

Considering algebras rather than manifolds (cf. Example 2.7, or Example 2.6 if L is an algebraic group and its action is via morphisms of affine algebraic varieties), we say that a Hopf algebra K is an external symmetry of the extension $B = A^{coH} \subset A$ if A is a (K, H) -bicomodule algebra, i.e., if A is a left K -comodule algebra and the K -coaction on A , $\rho^A : A \rightarrow K \otimes A$, commutes with the right H -coaction $\delta^A : A \rightarrow A \otimes H$ on A

$$(\rho^A \otimes \text{id}) \circ \delta^A = (\text{id} \otimes \delta^A) \circ \rho^A. \quad (4.6)$$

Due to this compatibility the vector subspace $B = A^{coH} \subset A$ of H -coinvariant elements of A is also a K -comodule, the K -coaction on B is just the restriction of that on A and we assume it to be nontrivial (this corresponds to a nontrivial action of L on M).

We have seen that the tensor product of H -comodules is again an H -comodule, similarly the tensor product of K -comodules is again a K -comodule, in particular $A \otimes A$ is a K -comodule with K -coaction

$$\begin{aligned} \rho^{A \otimes A} : A \otimes A &\longrightarrow K \otimes A \otimes A \\ a \otimes a' &\longmapsto a_{(-1)} a'_{(-1)} \otimes a_{(0)} \otimes a'_{(0)}, \end{aligned} \quad (4.7)$$

where we used the notation $\rho^A(a) = a_{(-1)} \otimes a_{(0)}$. Recalling that $A \otimes A$ is also an H -comodule (cf. (2.7)), it is not difficult to show that $A \otimes A$ is a (K, H) -bicomodule. Moreover this (K, H) -bicomodule structure is induced on the quotient $A \otimes_B A$. Similarly H is trivially a K -comodule (with coaction $\rho^H : H \rightarrow K \otimes H$, $\rho^H(h) = 1_K \otimes h$) so that $A \otimes H$ is a (K, H) -bicomodule with K -coaction

$$\rho^{A \otimes H} : A \otimes \underline{H} \longrightarrow K \otimes A \otimes \underline{H}, \quad a \otimes h \longmapsto a_{(-1)} \otimes a_{(0)} \otimes h. \quad (4.8)$$

Furthermore $A \otimes_B A$ and $A \otimes H$ are A -bimodules and this structure, that is compatible with the H -comodule structure, is also compatible with the K -comodule structure, hence $A \otimes_B A$ and $A \otimes H$ are relative Hopf modules in the category ${}^K_A \mathcal{M}_A^H$ of (K, H) -bicomodules with compatible A -bimodule structure (where A is a (K, H) -bicomodule algebra). The canonical map preserves this additional structure:

Proposition 4.8. *Let A be a (K, H) -bicomodule algebra, then the canonical map $\chi : A \otimes_B A \rightarrow A \otimes H$, where $B = A^{coH}$, is a morphism in ${}^K_A \mathcal{M}_A^H$.*

Proof. Since from Proposition 2.5 we know that the canonical map χ is a morphism in ${}^K_A \mathcal{M}_A^H$, we just have to show that it preserves the left K -coactions, i.e. $\rho^{A \otimes H} \circ \chi = (\text{id} \otimes \chi) \circ \rho^{A \otimes_B A}$. This is indeed the case:

$$\begin{aligned} \rho^{A \otimes H}(\chi(a \otimes_B c)) &= (ac_{(0)})_{(-1)} \otimes (ac_{(0)})_{(0)} \otimes c_{(1)} \\ &= a_{(-1)}(c_{(0)})_{(-1)} \otimes a_{(0)}(c_{(0)})_{(0)} \otimes c_{(1)} \\ &= a_{(-1)}c_{(-1)} \otimes a_{(0)}c_{(0)} \otimes c_{(1)} \\ &= a_{(-1)}c_{(-1)} \otimes \chi(a_{(0)} \otimes_B c_{(0)}) \\ &= (\text{id} \otimes \chi)(\rho^{A \otimes_B A}(a \otimes_B c)), \end{aligned}$$

where we have used the compatibility condition (4.6). \square

Let us now briefly present the twist deformation theory of left K -modules that parallels that of right H -modules studied in §3.2. Given a 2-cocycle $\sigma : K \otimes K \rightarrow \mathbb{K}$ on K we deform according to Proposition 3.3 the Hopf algebra K into the Hopf algebra K_σ . Every left K -comodule V is also a left ${}_\sigma K$ -comodule that we denote by ${}_\sigma V$ (with coaction $\rho^{\sigma V} : {}_\sigma V \rightarrow K_\sigma \otimes {}_\sigma V$, that as a linear map is the same as the coaction $\rho^V : V \rightarrow K \otimes V$). As in (3.6) we have a functor $\Sigma : {}^K \mathcal{M} \rightarrow {}^{K_\sigma} \mathcal{M}$ between the categories of left K -comodules and left ${}_\sigma K$ -comodules. It is defined on objects by $\Sigma(V) = {}_\sigma V$ and on morphisms $\psi : V \rightarrow W$ by $\Sigma(\psi) := \psi : {}_\sigma V \rightarrow {}_\sigma W$. Similarly to Theorem 3.4 we have

Theorem 4.9. *The functor $\Sigma : {}^K \mathcal{M} \rightarrow {}^{K_\sigma} \mathcal{M}$ induces an equivalence between the monoidal categories $({}^K \mathcal{M}, \otimes)$ and $({}^{K_\sigma} \mathcal{M}, {}^\sigma \otimes)$ that is given by the isomorphisms*

$$\begin{aligned} \varphi_{V,W}^\ell : {}_\sigma V {}^\sigma \otimes {}_\sigma W &\longrightarrow {}_\sigma(V \otimes W), \\ v {}^\sigma \otimes w &\longmapsto \sigma(v_{(-1)} \otimes w_{(-1)}) v_{(0)} \otimes w_{(0)}, \end{aligned} \quad (4.9)$$

for all objects $V, W \in {}^K \mathcal{M}$.

Similarly (K, H) -bicomodules are deformed in $({}_\sigma K, H)$ -bicomodules so that the corresponding functor $\Sigma : {}^K \mathcal{M}^H \rightarrow {}^{K_\sigma} \mathcal{M}^H$ induces as well an equivalence between the monoidal categories $({}^K \mathcal{M}^H, \otimes)$ and $({}^{K_\sigma} \mathcal{M}^H, {}^\sigma \otimes)$. This equivalence is given by the isomorphisms (4.9) that now are isomorphisms in ${}^{K_\sigma} \mathcal{M}^H$, i.e., (K_σ, H) -bicomodule isomorphisms.

The left (K, H) -comodule algebra A is also deformed into a left (K_σ, H) -comodule algebra ${}_\sigma A$, with product

$$\sigma m : {}_\sigma A {}^\sigma \otimes {}_\sigma A \longrightarrow {}_\sigma A, \quad a {}^\sigma \otimes a' \longmapsto \sigma(a_{(-1)} \otimes a'_{(-1)}) a_{(0)} a'_{(0)} =: a {}_\sigma \bullet a'. \quad (4.10)$$

Consequently relative Hopf modules $V \in {}^K_A \mathcal{M}_A^H$ are deformed in relative Hopf modules ${}_\sigma V \in {}^{K_\sigma} {}_{\sigma A} \mathcal{M}_{\sigma A}^H$ so that the corresponding functor $\Sigma : {}^K_A \mathcal{M}_A^H \rightarrow {}^{K_\sigma} {}_{\sigma A} \mathcal{M}_{\sigma A}^H$ induces and

equivalence of the categories ${}^K_A\mathcal{M}_A^H$ and ${}^{K_\sigma}{}_{\sigma A}\mathcal{M}_{\sigma A}^H$. The left and right σA -actions explicitly read (cf. (3.10))

$$\begin{aligned} \triangleright_{\sigma V} : \sigma A \overset{\sigma}{\otimes} \sigma V &\longrightarrow \sigma V, \\ a \overset{\sigma}{\otimes} v &\longmapsto \sigma(a_{(-1)} \otimes v_{(-1)}) a_{(0)} \triangleright_V v_{(0)}, \\ \triangleleft_{\sigma V} : \sigma V \overset{\sigma}{\otimes} \sigma A &\longrightarrow \sigma V, \\ v \overset{\sigma}{\otimes} a &\longmapsto \sigma(v_{(-1)} \otimes a_{(-1)}) v_{(0)} \triangleleft_V a_{(0)}. \end{aligned} \quad (4.11)$$

Given the (K_σ, H) -bicomodule algebra σA we consider the subalgebra of H -coinvariant elements $(\sigma A)^{coH}$ that is easily seen to equal $\sigma B := \sigma(A^{coH})$, the twist deformation of the K -subcomodule algebra $B \subset A$ of H -coinvariant elements, i.e., the deformed “base space”. As a consequence we have the twisted canonical map $\sigma\chi : \sigma A \overset{\sigma}{\otimes} \sigma B \sigma A \rightarrow \sigma A \overset{\sigma}{\otimes} \underline{H}$, which by Proposition 4.8 is a morphism in ${}^{K_\sigma}{}_{\sigma A}\mathcal{M}_{\sigma A}^H$. We now relate the twisted canonical map $\sigma\chi$ with the original canonical map χ .

Theorem 4.10. *Let A be a (K, H) -bicomodule algebra, and $B = A^{coH}$. Given a 2-cocycle $\sigma : K \otimes K \rightarrow \mathbb{K}$ the diagram*

$$\begin{array}{ccc} \sigma A \overset{\sigma}{\otimes} \sigma B \sigma A & \xrightarrow{\sigma\chi} & \sigma A \overset{\sigma}{\otimes} \underline{H} \\ \varphi_{A,A}^\ell \downarrow & & \downarrow \varphi_{A,H}^\ell \\ \sigma(A \otimes_B A) & \xrightarrow{\Sigma(\chi)} & \sigma(A \otimes \underline{H}) \end{array} \quad (4.12)$$

in ${}^{K_\sigma}{}_{\sigma A}\mathcal{M}_{\sigma A}^H$ commutes.

Proof. First we notice that the left vertical arrow is the induction to the quotient of the isomorphism $\varphi_{A,A}^\ell : \sigma A \overset{\sigma}{\otimes} \sigma A \rightarrow \sigma(A \otimes A)$ defined in (4.9); it is well defined thanks to the cocycle condition (3.1) for σ . Next let us observe that $\varphi_{A,H}^\ell$ is the identity; indeed, since \underline{H} is equipped with the trivial left K -coaction $h \mapsto 1_K \otimes h$ and σ is unital, we have

$$\varphi_{A,H}^\ell(a \overset{\sigma}{\otimes} h) = \sigma(a_{(-1)} \otimes h_{(-1)}) a_{(0)} \otimes h_{(0)} = \sigma(a_{(-1)} \otimes 1_K) a_{(0)} \otimes h = a \otimes h,$$

for all $a \in \sigma A$ and $h \in \underline{H}$. These vertical arrows are easily seen to be morphisms in ${}^{K_\sigma}{}_{\sigma A}\mathcal{M}_{\sigma A}^H$. Furthermore the horizontal arrows in the diagram are also morphism in ${}^{K_\sigma}{}_{\sigma A}\mathcal{M}_{\sigma A}^H$ (cf. Proposition 4.8) so that all arrows are morphisms in ${}^{K_\sigma}{}_{\sigma A}\mathcal{M}_{\sigma A}^H$. It remains to prove the commutativity of the diagram:

$$\begin{aligned} \chi(\varphi_{A,A}^\ell(a \overset{\sigma}{\otimes} \sigma B a')) &= \sigma(a_{(-1)} \otimes a'_{(-1)}) \chi(a_{(0)} \otimes_B a'_{(0)}) \\ &= \sigma(a_{(-1)} \otimes a'_{(-1)}) a_{(0)}(a'_{(0)})_{(0)} \otimes (a'_{(0)})_{(1)} \\ &= \sigma(a_{(-1)} \otimes (a'_{(0)})_{(-1)}) a_{(0)}(a'_{(0)})_{(0)} \otimes a'_{(1)} \\ &= a \bullet_{\sigma} a'_{(0)} \otimes a'_{(1)} \\ &= \sigma\chi(a \overset{\sigma}{\otimes} \sigma B a'), \end{aligned}$$

for all $a, a' \in \sigma A$. □

Since the vertical arrows $\varphi_{A,H}$ and $\varphi_{A,A}$ in diagram (4.12) are isomorphisms then we immediately have that an horizontal arrow in (4.12) is an isomorphism if and only if the other horizontal arrow is, i.e.,

Corollary 4.11. $B \subset A$ is an H -Hopf-Galois extension if and only if ${}_{\sigma}B \subset {}_{\sigma}A$ is an H -Hopf-Galois extension.

Finally it is also possible to prove that

Corollary 4.12. A is a principal H -comodule algebra if and only if ${}_{\sigma}A$ is a principal H -comodule algebra.

The key part of the proof is to show that given a section $s : A \rightarrow B \otimes A$ of the multiplication map $m : B \otimes A \rightarrow A$, as in Definition 2.8, then ${}_{\sigma}s := \mathfrak{S}(s) \circ (\varphi_{B,A}^{\ell})^{-1} : {}_{\sigma}A \rightarrow {}_{\sigma}B^{\sigma} \otimes {}_{\sigma}A$ is a section of the deformed multiplication map ${}_{\sigma}B^{\sigma} \otimes {}_{\sigma}A \rightarrow {}_{\sigma}A$. Here $\mathfrak{S}(s) : {}_{\sigma}A \rightarrow {}_{\sigma}(B \otimes A)$ is defined by, for all $a \in {}_{\sigma}A$,

$$\mathfrak{S}(s)(a) = \sigma\left(a_{(-2)} \otimes S(a_{(-1)}) m(s(a_{(0)}))_{(-1)}\right) s(a_{(0)}) \quad (4.13)$$

and, similarly to Remark 4.2, is related to the natural isomorphism proving that the categories of Hopf modules and of twisted Hopf modules are equivalent as closed monoidal categories.

Example 4.13 (The instanton bundle on the noncommutative sphere S_{θ}^4). In this example we describe the $SU(2)$ -principal bundle $S^7 \rightarrow S^4$ as an Hopf-Galois extension and then twist deform it to the Hopf-Galois extension describing the instanton bundle on the noncommutative sphere S_{θ}^4 [17, 8].

Let $A := \mathcal{O}(S^7)$ be the algebra over \mathbb{C} of coordinate functions on the 7-sphere S^7 , it is generated by the elements $\{z_i, z_i^*, i = 1, \dots, 4\}$ modulo the relation $\sum z_i^* z_i = 1$. It is a $*$ -algebra with involution $*$: $z \mapsto z^*$ extended as an antilinear and antimultiplicative map to all of $\mathcal{O}(S^7)$. Let $H := \mathcal{O}(SU(2))$ be the Hopf algebra of coordinate functions on $SU(2)$ realized as the $*$ -algebra generated by commuting elements $\{w_i, w_i^*, i = 1, 2\}$ with $\sum w_i^* w_i = 1$ and standard Hopf algebra structure induced from the group structure of $SU(2)$, i.e., setting

$$T = (T^i_j) = \begin{pmatrix} w_1 & -w_2^* \\ w_2 & w_1^* \end{pmatrix}, \quad (4.14)$$

$\Delta(T^i_j) = T^i_k \otimes T^k_j$ (sum over k understood), that we rewrite in matrix notation as $\Delta(T) = T \dot{\otimes} T$ (where $\dot{\otimes}$ denotes tensor product and matrix multiplication), $\varepsilon(T^i_j) = \delta_j^i$ and $S(T^i_j) = (T^{-1})^i_j$, i.e., in matrix notation $S(T) = T^{-1}$. The action of $SU(2)$ on S^7 pulls back to the right coaction of $\mathcal{O}(SU(2))$ on $\mathcal{O}(S^7)$:

$$\delta^{\mathcal{O}(S^7)} : \mathcal{O}(S^7) \longrightarrow \mathcal{O}(S^7) \otimes \mathcal{O}(SU(2)); \quad (4.15)$$

on the matrix of generators of $\mathcal{O}(S^7)$

$$u := \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ -z_2^* & z_1^* & -z_4^* & z_3^* \end{pmatrix}^t$$

it simply reads $\delta^{\mathcal{O}(S^7)}(u) = u \dot{\otimes} T$ and is extended to the whole $\mathcal{O}(S^7)$ as a $*$ -algebra morphism. The subalgebra $B := \mathcal{O}(S^7)^{\text{co}(\mathcal{O}(SU(2)))} \subset \mathcal{O}(S^7)$ of coinvariants under the coaction $\delta^{\mathcal{O}(S^7)}$ is generated by the elements

$$\alpha := 2(z_1 z_3^* + z_2^* z_4), \quad \beta := 2(z_2 z_3^* - z_1^* z_4), \quad x := z_1 z_1^* + z_2 z_2^* - z_3 z_3^* - z_4 z_4^*, \quad (4.16)$$

and their $*$ -conjugated α^*, β^* . Form the 7-sphere relation $\sum z_i^* z_i = 1$ it follows that they satisfy

$$\alpha^* \alpha + \beta^* \beta + x^2 = 1$$

and therefore, as expected, these elements generate the algebra of coordinate functions on the 4-sphere S^4 ; thus the subalgebra B of coinvariants is isomorphic to the algebra $\mathcal{O}(S^4)$ of coordinate functions on S^4 . Since $S^7 \rightarrow S^4$ is a principal $SU(2)$ -bundle then $\mathcal{O}(S^4) \subset \mathcal{O}(S^7)$ is a Hopf-Galois extension. Moreover, since $\mathcal{O}(SU(2))$ is cosemisimple and has a bijective antipode, then $\mathcal{O}(S^7)$ is a principal comodule algebra (recall the paragraph after Definition 2.8).

We twist deform this Hopf-Galois extension by using as external symmetry of the instanton bundle the (abelian) Lie group \mathbb{T}^2 . Let $K := \mathcal{O}(\mathbb{T}^2)$ be the corresponding commutative Hopf algebra of functions with generators t_j , $t_j^* = t_j^{-1}$, $j = 1, 2$ and co-structures $\Delta(t_i) = t_i \otimes t_i$, $\varepsilon(t_i) = 1$, $S(t_i) = t_i^{-1} = t_i^*$. The action of \mathbb{T}^2 on S^7 pulls back to a left coaction of $\mathcal{O}(\mathbb{T}^2)$ on the algebra $\mathcal{O}(S^7)$: it is given on the generators as

$$\rho^{\mathcal{O}(S^7)} : \mathcal{O}(S^7) \longrightarrow \mathcal{O}(\mathbb{T}^2) \otimes \mathcal{O}(S^7), \quad z_i \longmapsto \tau_i \otimes z_i, \quad (4.17)$$

where $(\tau_i) := (t_1, t_1^*, t_2, t_2^*)$, and it is extended to the whole of $\mathcal{O}(S^7)$ as a $*$ -algebra homomorphism. It is easy to prove that the $SU(2)$ and the \mathbb{T}^2 coactions $\delta^{\mathcal{O}(S^7)}$ and $\rho^{\mathcal{O}(S^7)}$ satisfy the compatibility condition (4.6), hence they structure $\mathcal{O}(S^7)$ as a $(\mathcal{O}(\mathbb{T}^2), \mathcal{O}(SU(2)))$ -bicomodule algebra. The subalgebra $\mathcal{O}(S^4)$ of $\mathcal{O}(SU(2))$ -coinvariants is a $\mathcal{O}(\mathbb{T}^2)$ -subcomodule algebra with $\mathcal{O}(\mathbb{T}^2)$ -coaction

$$\alpha \longmapsto t_1 t_2^* \otimes \alpha, \quad \beta \longmapsto t_1^* t_2 \otimes \beta, \quad x \longmapsto 1 \otimes x. \quad (4.18)$$

Let σ be the 2-cocycle on K defined on the generators by:

$$\sigma(t_j \otimes t_k) = \exp(i\pi \Theta_{jk}) , \quad \Theta = \frac{1}{2} \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} , \quad \theta \in \mathbb{R} \quad (4.19)$$

and extended to the whole algebra by requiring $\sigma(ab \otimes c) = \sigma(a \otimes c_{(1)}) \sigma(b \otimes c_{(2)})$ and $\sigma(a \otimes bc) = \sigma(a_{(1)} \otimes c) \sigma(a_{(2)} \otimes b)$, for all $a, b, c \in \mathcal{O}(\mathbb{T}^n)$.

We can now apply the theory of deformation by 2-cocycles to both the comodule algebras $\mathcal{O}(S^7)$ and $\mathcal{O}(S^4)$. The resulting noncommutative algebras, denoted respectively by $\mathcal{O}(S_\theta^7)$ and $\mathcal{O}(S_\theta^4)$, are two representatives of the class of θ -spheres in [11]. In particular, the Hopf-Galois extension $\mathcal{O}(S^4) \simeq \mathcal{O}(S^7)^{coH} \subset \mathcal{O}(S^7)$ deforms to the Hopf-Galois extension $\mathcal{O}(S_\theta^4) \simeq \mathcal{O}(S_\theta^7)^{coH} \subset \mathcal{O}(S_\theta^7)$ with undeformed structure Hopf algebra $H = \mathcal{O}(SU(2))$. Actually, from Corollary 4.12, we further obtain

Proposition 4.14. *The algebra $\mathcal{O}(S_\theta^7)$ is a principal $\mathcal{O}(SU(2))$ -comodule algebra.*

The noncommutative bundle so obtained is the quantum Hopf bundle on the Connes-Landi sphere $\mathcal{O}(S_\theta^4)$ that was originally constructed in [17], and further studied in the context of 2-cocycles deformation in [8]. The principality of the algebra inclusion $\mathcal{O}(S_\theta^4) \subset \mathcal{O}(S_\theta^7)$ was first proven in [17, §5] by explicit construction of the inverse of the canonical map. Proposition 4.14 follows instead as a straightforward result of the general theory developed in the present section (out of the principality of the underlying classical bundle).

4.3 Deformations of both the “structure group” H and the “base space” B

We now consider the combination of the previous two deformations. This leads to Hopf-Galois extensions in which the structure Hopf algebra, total space and base space are all deformed.

As before, we let H and K be Hopf algebras and A be a (K, H) -bicomodule algebra, with $B = A^{coH}$. Let $\sigma : K \otimes K \rightarrow \mathbb{K}$ and $\gamma : H \otimes H \rightarrow \mathbb{K}$ be 2-cocycles and denote by K_σ and H_γ the twisted Hopf algebras and by ${}_\sigma A_\gamma := {}_\sigma(A_\gamma) = ({}_\sigma A)_\gamma$ the deformed (K_σ, H_γ) -bicomodule algebra. We also have the deformed (K_σ, H_γ) -bicomodule algebra ${}_\sigma B_\gamma = ({}_\sigma A_\gamma)^{coH_\gamma} \subset {}_\sigma A_\gamma$ of H_γ -coinvariants in ${}_\sigma A_\gamma$. Notice that ${}_\sigma B_\gamma = ({}_\sigma A_\gamma)^{coH_\gamma} = {}_\sigma(A_\gamma^{coH_\gamma}) = {}_\sigma(A^{coH}) = {}_\sigma B$. Hence we can consider the canonical map ${}_\sigma \chi_\gamma : {}_\sigma A_\gamma \overset{\sigma}{\otimes} {}_\sigma B_\gamma \rightarrow {}_\sigma A_\gamma \overset{\sigma}{\otimes} H_\gamma$ that is a ${}^{K_\sigma}{}_{\sigma A_\gamma} \mathcal{M}_{\sigma A_\gamma}^{H_\gamma}$ -morphism because of Proposition 4.8. There are two ways to relate ${}_\sigma \chi_\gamma$ to the canonical map $\chi : A \otimes_B A \rightarrow A \otimes \underline{H}$. We can first relate χ to χ_γ and then χ_γ to ${}_\sigma \chi_\gamma$, or first χ to ${}_\sigma \chi$ and then ${}_\sigma \chi$ to ${}_\sigma \chi_\gamma$. It can be proven that these two constructions are equivalent because the left K -coaction commutes with the right H -coaction. More precisely we can apply the functor Σ to the commutative diagram (4.1) of Theorem 4.3 and then top the resulting diagram with the analogue of the commutative diagram (4.12) of Theorem 4.10, or we can first apply the functor Γ to (4.12) and then top it with the analogue of (4.1). Either of these equivalent procedures leads to commutative diagrams in ${}^{K_\sigma}{}_{\sigma A_\gamma} \mathcal{M}_{\sigma A_\gamma}^{H_\gamma}$ and to the following result

Theorem 4.15. *Given two Hopf algebras K and H , a (K, H) -bicomodule algebra A and two 2-cocycles $\sigma : K \otimes K \rightarrow \mathbb{K}$ and $\gamma : H \otimes H \rightarrow \mathbb{K}$, we have*

- (i) $B = A^{coH} \subset A$ is an H -Hopf-Galois extension if and only if ${}_\sigma B \subset {}_\sigma A_\gamma$ is an H_γ -Hopf-Galois extension.
- (ii) A is a principal H -comodule algebra if and only if ${}_\sigma A_\gamma$ is a principal H_γ -comodule algebra.

Example 4.16 (Formal deformation quantization). Recall from Example 2.7, that if $(E, M = E/G, \pi, G)$ is a principal bundle in the smooth category (i.e., if E and M are smooth manifolds, G is a Lie group) we have a Fréchet H -Hopf-Galois extension $B = C^\infty(M) = A^{coH} \subset A = C^\infty(E)$ with $H = C^\infty(G)$ and $\mathbb{K} = \mathbb{C}$. Let us further consider a finite dimensional Lie group L that is a Lie subgroup of the automorphism group of (E, M, π, G) , so that together with L we have a canonical smooth left action of L on E and M that commutes with the right G -action. The left L -actions on E and M pull-back to a Fréchet left $K = C^\infty(L)$ -comodule structure on A and B , which is compatible with the right H -coaction on A and the canonical map, i.e. $A = C^\infty(E)$ is a Fréchet $(K = C^\infty(L), H = C^\infty(G))$ -bicomodule algebra.

We consider formal deformations of H , K and A because in this context 2-cocycles are easily obtained, cf. Remark 3.2, from formal Drinfeld twists on the universal enveloping algebras $U(\mathfrak{g})$ and $U(\mathfrak{l})$, where \mathfrak{g} and \mathfrak{l} are the Lie algebras of G and L respectively. Therefore we consider the formal power series extension in a deformation parameter \hbar of the \mathbb{C} -modules H , A , B and K , that we denote as usual $H[[\hbar]]$, $A[[\hbar]]$, $B[[\hbar]]$ and $K[[\hbar]]$. The natural topology on these $\mathbb{C}[[\hbar]]$ -modules is a combination of the original Fréchet topology in each order of \hbar together with the \hbar -adic topology, see e.g. [16, Chapter XVI]. The canonical map induces a continuous $\mathbb{C}[[\hbar]]$ -linear isomorphism (denoted with abuse of notation by the same symbol)

$$\chi : A[[\hbar]] \widehat{\otimes}_{B[[\hbar]]} A[[\hbar]] \simeq C^\infty(E \times_M E)[[\hbar]] \longrightarrow A[[\hbar]] \widehat{\otimes}_{\underline{H}[[\hbar]]} A[[\hbar]] \simeq C^\infty(E \times G)[[\hbar]], \quad (4.20)$$

where now $\widehat{\otimes}$ denotes the completion of the algebraic tensor product with respect to the natural topologies described above. Hence we have obtained a topological $H[[\hbar]]$ -Hopf-Galois extension $B[[\hbar]] = A[[\hbar]]^{coH[[\hbar]]} \subset A[[\hbar]]$. The existence of continuous 2-cocycles $\gamma : H[[\hbar]] \widehat{\otimes} H[[\hbar]] \rightarrow \mathbb{K}[[\hbar]]$ and $\sigma : K[[\hbar]] \widehat{\otimes} K[[\hbar]] \rightarrow \mathbb{K}[[\hbar]]$ follows from the existence of Drinfeld twist deformations of the universal enveloping algebras $U(\mathfrak{g})[[\hbar]]$ and $U(\mathfrak{l})[[\hbar]]$. We now twist the $\mathbb{C}[[\hbar]]$ -modules $H[[\hbar]]$, $A[[\hbar]]$, $B[[\hbar]]$ and $K[[\hbar]]$ as described in general in

Section 3, and obtain a noncommutative topological $H[[\hbar]]_\gamma$ -Hopf-Galois extension ${}_\sigma B[[\hbar]] = {}_\sigma A[[\hbar]]_\gamma^{coH[[\hbar]]_\gamma} \subset {}_\sigma A[[\hbar]]_\gamma$, in particular the structure group G has been deformed in a quantum group G_γ that is described by the Hopf algebra $H[[\hbar]]_\gamma$, and similarly the base space M is deformed in a noncommutative base space ${}_\sigma M$ decribed by the algebra ${}_\sigma B[[\hbar]]$.

Example 4.16 is very general and it is interesting to specialize it to specific cases. For example deformations of homogenous spaces into quantum homogeneous spaces are obtained via this combined twist deformation of the structure group and of the base space [1]. Another application is in the formulation of gravity on noncommutative spacetime. We consider a 4-dimensional manifold which admits a Lorentzian metric. We correspondingly have the principal $SO(3,1)$ -bundle of orthonormal frames and also the principal $ISO(3,1)$ bundle of orthonormal affine frames. Hence we can consider Drinfeld twists of the universal enveloping algebras $U(so(3,1))$ and $U(iso(3,1))$ of the Lorentz and Poincaré groups, for example the abelian twists discussed in [2], [19] or even the nonabelian one (of extended Jordanian type) studied in [7, §V]. These twists give deformations of the structure groups of the principal bundles relevant in gravity. Gravity theories on commutative spacetime in the vierbein formalism obtained by gauging a quantum Poincaré group have been studied in [10]. The present construction would allow to consider also non local (globally nontrivial) aspects of these gravity theories. It is interesting to further twist deform the base space of these principal bundles, this is a first step in order to obtain a vierbein gravity theory on noncommutative spacetime with quantum Lorentz group invariance.

References

- [1] P. Aschieri, P. Bieliavsky, C. Pagani and A. Schenkel, *Noncommutative principal bundles through twist deformation*, arXiv:1604.03542 [math.QA]. To appear in Commun. Math. Phys.
- [2] P. Aschieri and L. Castellani, *R-matrix formulation of the quantum inhomogeneous groups $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$* , Lett. Math. Phys. **36** (1996) 197
- [3] P. Aschieri, M. Dimitrijevic, F. Meyer, S. Schraml and J. Wess, *Twisted gauge theories*, Lett. Math. Phys. **78** (2006) 61
- [4] P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, *Noncommutative geometry and gravity*, Class. Quant. Grav. **23** (2006) 1883
- [5] P. Aschieri and A. Schenkel, *Noncommutative connections on bimodules and Drinfeld twist deformation*, Adv. Theor. Math. Phys. **18** (2014) 513
- [6] G. E. Barnes, A. Schenkel and R. J. Szabo, *Nonassociative geometry in quasi-Hopf representation categories I: Bimodules and their internal homomorphisms*, J. Geom. Phys. **89** (2014) 111
- [7] A. Borowiec and A. Pachol, *Unified description for κ -deformations of orthogonal groups*, Eur. Phys. J. C **74** (2014) no.3, 2812
- [8] S. Brain and G. Landi, *Moduli spaces of non-commutative instantons: gauging away non-commutative parameters*, Q. J. Math. **63** (2012) 41

- [9] T. Brzeziński, G. Janelidze and T. Maszczyk, *Galois structures*, in P.M. Hajac (Ed.) Lecture Notes on Noncommutative Geometry and Quantum Groups. Available at <http://www.mimuw.edu.pl/~pwit/toknotes/toknotes.pdf>, T. Brzeziński and S. A. Fairfax *Bundles over Quantum Real Weighted Projective Spaces*, *Axioms* 2012, 1, 201
- [10] L. Castellani, *Differential calculus on ISO- $q(N)$, quantum Poincare algebra and q gravity*, *Commun. Math. Phys.* **171** (1995) 383 and *The Lagrangian of q Poincare gravity*, *Phys. Lett. B* **327** (1994) 22
- [11] A. Connes and G. Landi, *Noncommutative manifolds: the instanton algebra and isospectral deformations*, *Commun. Math. Phys.* **221** (2001) 141–159.
- [12] Y. Doi, *Braided bialgebras and quadratic bialgebras*, *Comm. Algebra* **21** (1993) 1731
- [13] V. G. Drinfeld, *On constant quasiclassical solutions of the Yang-Baxter quantum equation*, *Soviet Math. Dokl.* **28** (1983) 667 and *Hopf algebras and the quantum Yang-Baxter equation*, *Soviet Math. Dokl.* **32** (1985) 254
- [14] M. Dubois-Violette and T. Masson, *On the first order operators in bimodules*, *Lett. Math. Phys.* **37** (1996) 467
- [15] M. Francaviglia, *Element of Differential and Riemannian Geometry*, Bibliopolis (1988)
- [16] C. Kassel, *Quantum Groups*, Springer-Verlag, New York (1995)
- [17] G. Landi and W. van Suijlekom, *Principal fibrations from noncommutative spheres*, *Commun. Math. Phys.* **260** (2005) 203
- [18] D. Husemoller, *Fibre Bundles*, GTM, Springer (1993)
- [19] J. Lukierski and M. Woronowicz, *New Lie-algebraic and quadratic deformations of Minkowski space from twisted Poincare symmetries*, *Phys. Lett. B* **633** (2006) 116
- [20] S. Montgomery and H.J. Schneider, *Krull relations in Hopf Galois extensions: lifting and twisting*, *J. Algebra* **288** (2005) 364
- [21] J. Mourad, *Linear connections in noncommutative geometry*, *Class. Quant. Grav.* **12** (1995) 96
- [22] P. Schauenburg and H.J. Schneider, *On generalized Hopf Galois extensions*, *J. Pure App. Algebra* **202** (2005) 168
- [23] H.J. Schneider, *Principal homogeneous spaces for arbitrary Hopf algebras*, *Israel J. Math.* **72** (1990) 167