General Optimized Lower and Upper Bounds for Discrete and Continuous Arithmetic Asian Options

Gianluca Fusai
Dipartimento SEI, Università del Piemonte Orientale & Cass Business School, City University London, fusai@eco.unipmn.it & Gianluca.Fusai.1@city.ac.uk

Ioannis Kyriakou
Cass Business School, City University London, Ioannis.Kyriakou@city.ac.uk

We propose an accurate method for pricing arithmetic Asian options on the discrete or continuous average in a general model setting by means of a lower bound approximation. In particular, we derive analytical expressions for the lower bound in the Fourier domain. This is then recovered by a single univariate inversion and sharpened using an optimization technique. In addition, we derive an upper bound to the error from the lower bound price approximation. Our proposed method can be applied to computing the prices and price sensitivities of Asian options with fixed or floating strike price, discrete or continuous averaging, under a wide range of stochastic dynamic models, including exponential Lévy models, stochastic volatility models, and the constant elasticity of variance diffusion. Our extensive numerical experiments highlight the notable performance and robustness of our optimized lower bound for different test cases.

Key words: arithmetic Asian options; CEV diffusion; stochastic volatility models; Lévy processes; discrete average; continuous average
MSC2000 subject classification: Primary: 91B70, 91B25, 60J25; Secondary: 60H30, 91B24
OR/MS subject classification: Primary: asset pricing, diffusion, Markov processes, stochastic model applications

1. Introduction. We develop accurate analytical pricing formulae for discretely and continuously monitored arithmetic Asian options under general stochastic asset models, including exponential Lévy models, stochastic volatility models, and the constant elasticity of variance diffusion. The payoff of the arithmetic Asian option depends on the arithmetic average price of the underlying asset monitored over a pre-specified period. For more than two decades, much effort has been put into the research on efficient methodologies for computing the price of this option or, in general, expected values of functionals of the average value, under different model assumptions for the underlying. Developing such methods is of considerable practical importance as arithmetic averages see wide application in many fields of finance. Amongst others, we mention uses in computing net present value in project valuation (see [72]), optimal capacity planning under average demand uncertainty for a single firm (see [32]) and stock-swap merger proposals (see [60]). Weighted arithmetic averages also appear in technical analysis and in algorithmic trading; for example, we recall the moving average trading rule and its use from an asset allocation perspective (see [75]). Moving average automatic trading strategies set buying and selling orders depending on the position of the average price for a given period with respect to the current market price (see [50]). Finally, weighted arithmetic average indexes are used as trading benchmarks in pension plans (see [11]).

Arithmetic Asian options are very popular among derivatives traders and risk managers. Their appeal stems mainly from the fact that averaging smooths possible market manipulations occurring near the expiry date. Moreover, averaging provides suitable volatility reduction and better cash-flow matching to firms facing streams of cash flows. Due to these nice features, Asian options are particularly appropriate for currency, energy, metal, agricultural and freight markets and, unsurprisingly, represent a large fraction of the options traded in these markets.

To be able to reproduce stylized properties of the asset prices in the various markets, it is necessary in many cases to depart from the basic lognormal model by incorporating, for example, random jumps, stochastic volatility and mean-reversion in the price dynamics. Unluckily, the pricing of arithmetic Asian options does not admit true analytical solutions, even under the lognormal model, as the distribution law of the arithmetic average is not known analytically. In fact, the advantage of a more realistic model specification is often offset by the need to implement computationally expensive numerical procedures, where available. It is, thus, the main objective of this paper to present a simple, accurate and fast pricing formula, albeit approximate, for arithmetic Asian options allowing flexible modelling of the underlying asset price dynamics, filling this way an important long-standing gap in the literature.

A large volume of publications is devoted to the pricing of Asian options, mainly, with continuous monitoring in the Black–Scholes economy. Without being exhaustive, we mention Rogers and Shi [61], who are the first to provide the Asian option price by solving a partial differential equation (PDE) in one space dimension, but also Večeř [69] and Zhang [74] with improved PDE-based solutions in low-volatility settings; Linetsky [53] derives an elegant spectral expansion for the option price; Geman and Yor [42] are the first to give a solution in terms of a single-Laplace transform, which is re-derived in Dewynne and Shaw [31] where it is shown how to treat the low-volatility case more effectively. An also accurate double-transform at low volatility levels is suggested in Fusai [38], whereas Cai and Kou [14] generalize to a double-Laplace transform under the hyperexponential jump diffusion model, encompassing the Gaussian model and Kou's double exponential jump diffusion as special cases. Večeř and Xu [70] show that the option price satisfies a partial integro-differential equation (PIDE) in the case of exponential Lévy price dynamics, which is solved numerically in Bayraktar and Xing [9] for the special case of jump diffusion models. Finally, Ewald et al. [36] propose a solution under the Heston model by means of a PDE and a Monte Carlo simulation method, whereas Yamazaki [71] a pricing formula based on the Gram-Charlier expansion. Another stream in the literature is concerned with pricing discretely monitored Asian options. We mention, amongst others, contributed works by Andreasen [3] and Večeř [69] based on PDE approaches under lognormal asset price dynamics and the Fourier transform-based recursive convolution of Carverhill and Clewlow [20] on a reduced state space, whereas enhanced variations of the latter under general exponential Lévy dynamics appear, for example, in Benhamou [10], Fusai and Meucci [40], Černý and Kyriakou [21] and Zhang and Oosterlee [73]. Beyond the Lévy framework, Dassios and Nagaradjasarma [29] and Fusai et al. [39] obtain explicit prices for Asian options under the square root asset price dynamics ([55] consider a modified version with independent jumps added), whereas Cai et al. [15] and Sesana et al. [64] develop, respectively, an efficient asymptotic expansion and a quadrature method applicable to the generalized constant elasticity of variance (CEV) diffusion.

In this paper, we present an analytical approximation to the price of the Asian option in the form of a sharp lower bound under discrete or continuous monitoring and general model assumptions. The idea of such an approximation dates back to the celebrated works of Curran [27, 28] and Rogers and Shi [61] who derive a lower bound to the option price in the lognormal model using properties of conditional expectations, where the conditioning variable is represented by the geometric average with an analytically tractable law. In addition, Rogers and Shi [61] prove an upper bound to the option price which is considerably strengthened later by Nielsen and Sandmann [58] and Thompson [67]. Lord [54] revisits these earlier contributions and provides important enhancements, including identifying and fixing convergence issues in Curran [28] when the strike price of the option tends to zero or infinity. Novikov and Kordzakhia [59] extend to the case of volume-weighted average options, where the volume process is independent of the underlying asset price process, and also to the cases of non-random deterministic or independent stochastic interest rates. Finally, Lemmens et al. [52] apply the conditional expectation technique to obtain lower bounds for the prices of discrete arithmetic Asian options in the general exponential Lévy asset price model.

Although earlier methods relying on Lévy log-increments of the underlying are found to be impressively fast and accurate (e.g., see [40], [21] and [73]), lack of such an assumption, as, for example, in the case of stochastic volatility models or the CEV diffusion, poses nontrivial mathematical and computational challenges. Our lower bound price approximation aims to tackle these difficulties efficiently. Following Curran [28], Rogers and Shi [61], Thompson [67] and Lemmens et al. [52], we devise suitable conditioning variables under different stochastic dynamic models and monitoring frequencies (discrete or continuous). In general, our method relies on identifying suitable averages of the underlying asset prices, for use as conditioning variables, which have analytically tractable laws and are close proxies to the original arithmetic average so that a tight lower bound is eventually obtained. More precisely, for exponential Lévy and stochastic volatility models we use the (log) geometric mean, whereas for the CEV diffusion we resort to the generalized (power) mean. Using only knowledge of the underlying asset price law (jointly with the stochastic volatility where assumed) via the associated characteristic function, we provide a general explicit recursive algorithm which gives us access to the bivariate characteristic function of the asset price and the new average. Given this, and beyond the original contributions of Curran [28] and Rogers and Shi [61], we derive an analytical solution for the lower bound in the Fourier domain, similarly to Lemmens et al. [52] under exponential Lévy models. This is then recovered by a single univariate inversion and sharpened using simple optimization.

Our proposed method is distinguished from other pricing methodologies for Asian options due to a number of appealing features. First, it can be applied flexibly to a wide range of non-Gaussian models, such as pure jump Lévy models, Merton's normal and Cai and Kou's generalized hyperexponential jump diffusions, models with/out jumps in the asset price/volatility dynamics, and the CEV diffusion, without restricting to models admitting time changed Brownian (Lévy) representations which may not be always common or straightforward to use. Second, we provide interesting theoretical findings related to the pricing of Asian options in the CEV diffusion model, which requires special treatment due to its distinct distributional properties. Third, in the absence of symmetry relations between fixed and floating strike Asian options beyond the exponential Lévy asset price model (see [34]), by a slight modification of the conditioning variables and a change of numéraire, we are able to switch from fixed to floating strike option price results. Fourth, with slight modification of the pricing formulae, we can obtain the option price sensitivities with respect to parameters of interest. Moreover, for first time in the literature, we provide a formulation which applies also to continuous Asian options under general model assumptions. The final line of research that we contribute to in this paper is concerned with deriving a theoretical upper bound to the error made in our optimized lower bound price approximation that can be calculated numerically.

To verify the efficiency of the proposed methodology, extensive numerical experiments are conducted to compare the accuracy of our optimized lower bound with existing methods in the literature against benchmarks generated by a very accurate control variate Monte Carlo simulation strategy which uses as control variate the lower bound itself. To make concrete our analysis, we investigate the numerical performance of our lower bound in a wide range of Lévy models, volatility models and the CEV diffusion, for options with varying moneyness and monitoring frequency (monthly, weekly, daily, continuously). In summary, our numerical experiments demonstrate the high accuracy of our optimized lower bound, with notable performance even for extremely low volatilities, and its robustness in all the aforementioned test cases. The method is also fast and simple to implement requiring a single univariate transform inversion, while the problem dimension remains unaffected by the additional random volatility factor.

The structure of this paper is as follows. Section 2 summarizes the various market models considered in this study and provides preliminary transform results for discrete and continuous averages.

Section 3 presents our transform representations of the lower bounds to the Asian option prices under different contract specifications (fixed strike or floating strike price), monitoring frequencies and model specifications. For the same cases, we present in Section 4 expressions for the error in our lower bound price approximation. Section 5 is devoted to our numerical study. Section 6 concludes.

2. Market models and preliminary results. Assume that the price of the underlying asset S is observed at the equally spaced discrete times $t_0 \equiv 0, t_1 \equiv \Delta, \dots, t_j \equiv \Delta j, \dots, t_N \equiv \Delta N = T$, where T is a fixed time horizon. We assume a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} \equiv (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ where $\mathbb{P}$ is the risk neutral probability measure.

2.1. Lévy models. Assume that $X \equiv \ln S$ is represented by a Lévy process, i.e., satisfies the general form

$$dX_t = \varepsilon dt + \sigma dW_t + dL_t^x,$$

where $\varepsilon \in \mathbb{R}$ is deterministic, $\sigma \geq 0$ constant, W a standard Brownian motion and L^x a purely discontinuous random process. Consider the log-increments of the underlying

$$\ln S_{\Delta j} - \ln S_{\Delta(j-1)} = X_{\Delta j} - X_{\Delta(j-1)} \equiv Z_j^{\Delta}, \tag{1}$$

so that the price of the underlying asset at t_j is

$$S_{\Delta j} = S_0 e^{Z_1^{\Delta} + \cdots + Z_j^{\Delta}}$$

for $j = 1, \dots, N$. At the level of risk neutral modelling, exponential Lévy asset price models allow to generate implied volatility smiles and skews similar to the ones observed in market prices. Under such model assumptions, the increments (1) are independent and identically distributed. By the celebrated Lévy–Khintchine formula, the characteristic function of Z_j^{Δ} has the form

$$\mathbb{E}[\exp(iuZ_j^{\Delta})] \equiv \exp(\psi_{\Delta}(u)) \equiv \exp(iu\varepsilon\Delta + \varphi_{\Delta}(u)) \tag{2}$$

for all j, where $\varphi_{\Delta}(u) = \psi_{\Delta}(u) - iu\varepsilon\Delta$. Further, we choose

$$\varepsilon = r - q - \frac{1}{\Delta}\varphi_{\Delta}(-i) = r - q - \varphi_1(-i), \tag{3}$$

where $r \geq 0$ and $q \geq 0$ denote respectively the constant instantaneous risk-free interest rate and dividend yield, to ensure that the discounted asset price is a martingale under the probability measure $\mathbb{P}$, e.g., see Schoutens [62], which is necessary for the risk neutral pricing of derivatives.

For our purposes, it is also necessary to specify the asset price dynamics under the measure $\overline{\mathbb{P}}$ where the underlying itself represents the numéraire, see Geman et al. [41]. From the numéraire change formula, the characteristic function of Z_j^{Δ} under the measure $\overline{\mathbb{P}}$ has the form

$$\overline{\mathbb{E}}[\exp(iuZ_j^{\Delta})] = \mathbb{E}[\exp(-r\Delta + i(u-i)Z_j^{\Delta})] = \exp(-r\Delta + \psi_{\Delta}(u-i)). \tag{4}$$

In Table 1, we list several Lévy processes with the associated characteristic exponents φ, including the variance gamma (VG), normal inverse Gaussian (NIG), Carr–Geman–Madan–Yor (CGMY), Merton jump diffusion (MJD), Kou double exponential jump diffusion (DEJD) and Meixner models. For extensive descriptions of the various models, we refer to Schoutens [62] and Cont and Tankov [23].

2.2. Affine stochastic volatility (ASV) models. When the risk neutral dynamics of the log-price X is given by a Lévy process, the implied volatility surface follows a deterministic evolution (see [23]). Stochastic volatility models can tackle this difficulty. More specifically, diffusion-based volatility models account for dependence in increments and long-term smiles and skews, but cannot give rise to realistic short-term implied volatility patterns. This shortcoming can be overcome by introducing jumps in the returns and in the evolution of the volatility. To this end, a number of ASV models have been introduced in the literature. Popular examples include the time changed Lévy processes proposed by [18] and [19], where the time change is given by integrated Ornstein–Uhlenbeck (OU) or square root variance processes, with special cases being these of the Heston and Barndorff–Nielsen–Shephard (BNS) models based on time changed arithmetic Brownian motions (e.g., see [49]), the Bates and Duffie–Pan–Singleton (DPS) models, the Stein–Stein and Schöbel–Zhu model with mean-reverting Gaussian volatility dynamics, which is affine with the state vector augmented by the squared volatility (e.g., see [48]), but also members from the affine GARCH class (see [56]). While our proposed method (see Section 3) can be readily applied to pricing Asian options under the aforementioned model assumptions requiring only knowledge of the characteristic function of the driving affine process, for ease of exposition we focus here attention on the Heston, Bates, DPS and BNS cases.

Under the risk neutral measure, the Heston model [46] is described by the following stochastic differential equations

$$\begin{aligned} dX_t &= (r-q-V_t/2)dt + \sqrt{V_t}(\rho dW_t + \sqrt{1-\rho^2}dB_t), \\ dV_t &= \alpha(\beta - V_t)dt + \gamma\sqrt{V_t}dW_t, \end{aligned}$$

where B, W are independent standard Brownian motions, α, β, γ are positive constants and $\rho \in [-1,1]$ is the instantaneous correlation coefficient between the log-asset price process X and the variance process V. The Bates model [8] is an extension of the Heston model to include jumps in the (log) asset price dynamics

$$dX_t = (r-q-lk(1)-V_t/2)dt + \sqrt{V_t}(\rho dW_t + \sqrt{1-\rho^2}dB_t) + dL_t^x,$$

where L^x is a time-homogeneous compound Poisson process with intensity $l>0$ and normal distribution of jump sizes ξ_x with mean $\mu_x \in \mathbb{R}$ and standard deviation $\sigma_x \geq 0$ and $k(u) \equiv \exp(\mu_x u + \sigma_x^2 u^2/2) - 1$. The DPS model introduced in Duffie et al. [33], in addition to the jumps in the (log) asset price process, includes contemporaneous jumps in the variance process. The governing equations are

$$\begin{aligned} dX_t &= (r-q-lk(1,0)-V_t/2)dt + \sqrt{V_t}(\rho dW_t + \sqrt{1-\rho^2}dB_t) + dL_t^x, \\ dV_t &= \alpha(\beta - V_t)dt + \gamma\sqrt{V_t}dW_t + dL_t^v, \end{aligned}$$

where L^v and L^x are driven by a common Poisson process, hence jumps occur concurrently in both processes, however the jump sizes ξ_v have exponential distribution with mean $\mu_v > 0$. Also, the magnitudes of the jumps ξ_x and ξ_v have a correlation determined by the parameter $\rho_{x,v}$; given ξ_v, the jump sizes ξ_x are normally distributed with mean $(\mu_x + \rho_{x,v}\xi_v)$ and variance σ_x^2. Under this model specification, $k(u_1,u_2) \equiv \exp(\mu_x u_1 + \sigma_x^2 u_1^2/2)/(1-\rho_{x,v}\mu_v u_1 - \mu_v u_2) - 1$.

Lévy-driven positive OU processes are also of particular interest in the context of stochastic volatility modelling. Barndorff-Nielsen and Shephard [6] and Barndorff-Nielsen et al. [5] suggest the so-called BNS model of the form

$$\begin{aligned} dX_t &= (r-q-lk(\rho)-V_t/2)dt + \sqrt{V_t}dW_t + \rho dL_t, \\ dV_t &= -lV_t dt + dL_t, \end{aligned} \tag{5}$$

where parameters $\rho \le 0$, $l > 0$, W is a standard Brownian motion and L is the background driving Lévy process, which is a subordinator without drift and is independent of W. We consider two popular parametric specifications of the BNS model, namely the BNS-Γ and BNS-IG model, where the OU process (5) respectively has a gamma (Γ) stationary distribution with $k(u) \equiv \nu u/(\alpha - u)$ for some $\nu, \alpha > 0$, and an inverse Gaussian (IG) stationary distribution with $k(u) \equiv \nu u/\sqrt{\alpha^2 - 2u}$ for some $\nu, \alpha > 0$.

Affinity of the volatility models described above implies that the characteristic function of the pair (V, X) has exponentially affine dependence on V and X, i.e., there exist $\varphi_\Delta, \psi_\Delta : i\mathbb{R}^2 \to \mathbb{C}$ such that

$$\mathbb{E}[\exp\{iuV_{\Delta(j+1)} + ivX_{\Delta(j+1)}\}|\mathcal{F}_{\Delta j}] = \exp\{\varphi_\Delta(iu, iv) + \psi_\Delta(iu, iv)V_{\Delta j} + ivX_{\Delta j}\},$$

or, equivalently, from (1)

$$\mathbb{E}[\exp\{iuV_{\Delta(j+1)} + ivZ^\Delta_{j+1}\}|\mathcal{F}_{\Delta j}] = \exp\{\varphi_\Delta(iu, iv) + \psi_\Delta(iu, iv)V_{\Delta j}\}. \tag{6}$$

Using the numéraire change formula, we also obtain under the measure $\overline{\mathbb{P}}$

$$\overline{\mathbb{E}}[\exp\{iuV_{\Delta(j+1)} + ivZ^\Delta_{j+1}\}|\mathcal{F}_{\Delta j}] = \exp\{-r\Delta + \varphi_\Delta(iu, i(v-i)) + \psi_\Delta(iu, i(v-i))V_{\Delta j}\}. \tag{7}$$

In Table 1, we provide the functions φ, ψ for the Heston, Bates, DPS and BNS models.

2.3. CEV diffusion model. Among the one-dimensional Markov processes, the CEV diffusion of Cox [24] is an important asset price model which has interesting analytical properties and can flexibly provide good fits to various shapes of implied volatility curves observed in the marketplace by varying the elasticity parameter γ. Despite its economic importance, the CEV diffusion has been studied less in the literature of Asian options pricing. In this model, the underlying asset price dynamics under the risk neutral measure is given by

$$dS_t = (r - q)S_t dt + \sigma S_t^{\gamma/2} dW_t, \quad \sigma \ge 0, \gamma \in \mathbb{R}.$$

Cox [24] originally studied the case $\gamma < 2$, whereas Emanuel and MacBeth [35] extended his analysis to the case $\gamma > 2$ (see also [63]). The special case of $\gamma = 1$ corresponds to the well known square root process of Cox and Ross [25], whereas when $\gamma = 2$ we obtain the lognormal model. The CEV diffusion with $\gamma = 1$ has been applied to the pricing of arithmetic Asian options in Dassios and Nagaradjasarma [29], Fusai et al. [39] and, more recently, in its general form in Cai et al. [15], Sesana et al. [64] and Cai et al. [16].

A useful property of this model for the purposes of our application in Section 3 is the following: for $X \equiv S^{2-\gamma}$, we get from Itô's lemma

$$dX_t = (r-q)(\gamma-2)\left(\frac{\sigma^2(\gamma-1)}{2(r-q)} - X_t\right)dt + \sigma(2-\gamma)\sqrt{X_t}dW_t. \tag{8}$$

Model (8) is affine in the state variable and can be characterized by its moment generating function (see [51, Proposition 6.2.4]),

$$\mathbb{E}[e^{-\mu X_{\Delta(j+1)}}|\mathcal{F}_{\Delta j}] = \exp\{\varphi_\Delta(0, \mu) - \psi_\Delta(0, \mu)X_{\Delta j}\}, \tag{9}$$

where

$$\varphi_\Delta(\nu, \mu) \equiv \frac{\gamma-1}{\gamma-2}\ln\left(\frac{2\theta e^{((r-q)(\gamma-2)-\theta)\Delta/2}}{(\sigma^2(2-\gamma)^2\mu + (r-q)(\gamma-2))(1-e^{-\theta\Delta}) + \theta(1+e^{-\theta\Delta})}\right), \tag{10}$$

$$\psi_\Delta(\nu, \mu) \equiv \frac{(\theta(1+e^{-\theta\Delta}) - (r-q)(\gamma-2)(1-e^{-\theta\Delta}))\mu + 2(1-e^{-\theta\Delta})\nu}{(\sigma^2(2-\gamma)^2\mu + (r-q)(\gamma-2))(1-e^{-\theta\Delta}) + \theta(1+e^{-\theta\Delta})} \tag{11}$$

and $\theta \equiv \theta(\nu) \equiv |\gamma - 2|\sqrt{(r-q)^2 + 2\nu\sigma^2}$.

2.4. Discrete average. In light of the lack of analytical tractability of the law of the discrete arithmetic average of the asset prices $\frac{1}{N+1}\sum_{j=0}^{N} S_{\Delta j}$, we look for close proxies with known distributional properties. More specifically, such a proxy is given by

$$Y_{\Delta N} \equiv \frac{1}{N+1}\sum_{j=0}^{N} X_{\Delta j}, \tag{12}$$

where $X = \ln S$ in the case of the Lévy and ASV models of Sections 2.1 and 2.2, whereas $X = S^{2-\gamma}$ in the case of the CEV diffusion of Section 2.3. Important to our arithmetic Asian option pricing framework of Section 3 is knowledge of the distribution law of the new average (12). In Propositions 1, 2 and 3, we derive key results under the Lévy, ASV and CEV diffusion models for use in Section 3.

PROPOSITION 1. *Define*

$$\eta_j(u,\upsilon) = \begin{cases} \upsilon\left(1-\frac{j}{N+1}\right), & k\vee n < j \le N \\ u+\upsilon\left(1-\frac{j}{N+1}\right), & k\wedge n < j \le k\vee n \\ 2u+\upsilon\left(1-\frac{j}{N+1}\right), & 0 < j \le k\wedge n \end{cases} \tag{13}$$

and

$$\phi_{k,n,N}(u,\upsilon) = \mathbb{E}[\exp\{iu(X_{\Delta k}+X_{\Delta n}) + i\upsilon Y_{\Delta N}\}]$$

under the risk neutral measure.

(i) (LÉVY MODELS). *Under the assumption of increments Z_j^{Δ} satisfying (2), define*

$$\Psi_{h,\Delta}(u,\upsilon) = \sum_{j=h+1}^{N} \psi_{\Delta}(\eta_j(u,\upsilon)),$$

where $h = 0, k\wedge n, k\vee n$. Then,

$$\phi_{k,n,N}(u,\upsilon) = \exp\{i(2u+\upsilon)X_0 + \Psi_{0,\Delta}(u,\upsilon)\}. \tag{14}$$

(ii) (ASV MODELS). *Under the assumption of increments Z_j^{Δ} satisfying (6), define*

$$\Psi_{h,\Delta}(u,\upsilon;V_{\Delta h}) = \sum_{j=h+1}^{N} \varphi_{\Delta}(\vartheta_j(u,\upsilon), i\eta_j(u,\upsilon)) + \psi_{\Delta}(\vartheta_{h+1}(u,\upsilon), i\eta_{h+1}(u,\upsilon))V_{\Delta h},$$

where $h = 0, k\wedge n, k\vee n$ and ϑ_j satisfies the recursive equation

$$\vartheta_j \equiv \psi_{\Delta}(\vartheta_{j+1}, i\eta_{j+1}) \tag{15}$$

for $j = N-1,\ldots,1$ with $\vartheta_N \equiv 0$. Then,

$$\phi_{k,n,N}(u,\upsilon) = \exp\{i(2u+\upsilon)X_0 + \Psi_{0,\Delta}(u,\upsilon;V_0)\}. \tag{16}$$

Proof. See Appendix A. □

PROPOSITION 2. *Define*

$$\bar{\eta}_j(u,\upsilon) = \begin{cases} -2u-\upsilon\frac{j}{N+1}, & k\vee n < j \le N \\ -u-\upsilon\frac{j}{N+1}, & k\wedge n < j \le k\vee n \\ -\upsilon\frac{j}{N+1}, & 0 < j \le k\wedge n \end{cases} \tag{17}$$

and

$$\bar{\phi}_{k,n,N}(u,\upsilon) = \overline{\mathbb{E}}[\exp\{iu(X_{\Delta k}+X_{\Delta n}-2X_{\Delta N}) + i\upsilon(Y_{\Delta N}-X_{\Delta N})\}]$$

under the measure $\overline{\mathbb{P}}$.

(i) (Lévy models). *Under the assumption of increments Z_j^Δ satisfying (4), define*

$$\bar{\Psi}_{h,\Delta}(u,\upsilon)=\exp\left\{-r\Delta(N-h)+\sum\nolimits_{j=h+1}^{N}\psi_\Delta(\bar{\eta}_j(u,\upsilon)-i)\right\},$$

where $h=0,k\wedge n,k\vee n$. Then,

$$\bar{\phi}_{k,n,N}(u,\upsilon)=\bar{\Psi}_{0,\Delta}(u,\upsilon). \tag{18}$$

(ii) (ASV models). *Under the assumption of increments Z_j^Δ satisfying (7), define*

$$\begin{aligned}\bar{\Psi}_{h,\Delta}(u,\upsilon;V_{\Delta h})=\exp\Big\{&-r\Delta(N-h)+\sum\nolimits_{j=h+1}^{N}\varphi_\Delta(\bar{\vartheta}_j(u,\upsilon),i(\bar{\eta}_j(u,\upsilon)-i))\\&+\psi_\Delta(\bar{\vartheta}_{h+1}(u,\upsilon),i(\bar{\eta}_{h+1}(u,\upsilon)-i))V_{\Delta h}\Big\},\end{aligned}$$

where $h=0,k\wedge n,k\vee n$ and $\bar{\vartheta}_j$ satisfies the recursive equation

$$\bar{\vartheta}_j\equiv\psi_\Delta(\bar{\vartheta}_{j+1},i\bar{\eta}_{j+1})$$

for $j=N-1,\ldots,1$ with $\bar{\vartheta}_N\equiv 0$. Then,

$$\bar{\phi}_{k,n,N}(u,\upsilon)=\bar{\Psi}_{0,\Delta}(u,\upsilon;V_0). \tag{19}$$

Proof. See Appendix A. □

Proposition 3. (CEV model). *Define recursive equations*

$$\vartheta_j(\mu)=\psi_\Delta(0,\vartheta_{j+1}(\mu))+\frac{\mu}{N+1}$$

for $j=N-1,\ldots,0$ with $\vartheta_N(\mu)\equiv\mu/(N+1)$ and ψ given by (11).

(i) The moment generating function of $Y_{\Delta(N-k-1)}\equiv\frac{1}{N+1}\sum_{j=k+1}^{N}X_{\Delta j}$ under the risk neutral measure is given by

$$\mathbb{E}[e^{-\mu Y_{\Delta(N-k-1)}}|\mathcal{F}_{\Delta k}]=\exp\left\{\sum\nolimits_{j=k+1}^{N}\varphi_\Delta(0,\vartheta_j(\mu))-\psi_\Delta(0,\vartheta_{k+1}(\mu))X_{\Delta k}\right\}, \tag{20}$$

where φ is given by (10).

(ii) In addition,

$$\mathbb{E}\left[X_{\Delta k}^{\frac{1}{2-\gamma}}e^{-\mu Y_{\Delta N}}\right]=X_0^{\frac{1}{2-\gamma}}\exp\left\{r\Delta k+\sum\nolimits_{j=k+1}^{N}\varphi_\Delta(0,\vartheta_j(\mu))+\sum\nolimits_{j=1}^{k}\bar{\varphi}_\Delta(0,\vartheta_j(\mu))-\vartheta_0(\mu)X_0\right\}, \tag{21}$$

where

$$\bar{\varphi}_\Delta(\nu,\mu)\equiv\frac{\gamma-3}{\gamma-1}\varphi_\Delta(\nu,\mu) \tag{22}$$

and φ is given by (10).

Proof. See Appendix A. □

2.5. Continuous average. In the case of the continuous average, the quantity of interest is

$$Y_t \equiv \frac{1}{T}\int_0^t X_s ds,$$

where $X = \ln S$ in the case of Lévy and ASV models, whereas $X = S^{2-\gamma}$ in the case of the CEV diffusion.

In the general Lévy model case, it is possible to obtain characteristic functions for the pairs $(X_t + X_z, Y_T)$ and $(X_t + X_z - 2X_T, Y_T - X_T)$, with $t, z \in [0,T]$, similarly to (14) and (18) in Propositions 1 and 2 based, instead, on the discrete average for N monitoring dates. This is possible if we let N approach infinity while the time spacing Δ approaches zero, so that $T = N\Delta$ remains constant. This way we get under the risk neutral measure

$$\begin{aligned}\phi_{t,z,T}(u,v) &= \mathbb{E}[\exp\{iu(X_t + X_z) + ivY_T\}]\\ &= \exp\left\{i(2u+v)X_0 + \int_0^T \psi\left(u(\mathbf{1}_{[0,t\wedge z]}(s) + \mathbf{1}_{[0,t\vee z]}(s)) + v\left(1 - \frac{s}{T}\right)\right) ds\right\}\end{aligned} \tag{23}$$

where $\psi \equiv \psi_1$. The integrals on the exponent can be computed analytically for several Lévy models, see Table 2, using any symbolic computation package such as Mathematica. Under the measure $\overline{\mathbb{P}}$,

$$\begin{aligned}\bar{\phi}_{t,z,T}(u,v) &= \overline{\mathbb{E}}[\exp\{iu(X_t + X_z - 2X_T) + iv(Y_T - X_T)\}]\\ &= \exp\left\{-rT + \int_0^T \psi\left(-u(\mathbf{1}_{[t\wedge z,T]}(s) + \mathbf{1}_{[t\vee z,T]}(s)) - v\frac{s}{T} - i\right) ds\right\}\end{aligned} \tag{24}$$

holds.

In the case of the ASV models considered in this study, obtaining the characteristic functions based on the continuous average by applying the same limiting argument as in the Lévy model case on the discrete average-based characteristic functions (16) and (19) is not trivial. Alternatively, it is necessary to derive first the characteristic function of the triple (V, X, Y) (see [47] for the case of ASV models). Given this and by iterated expectations, we can then obtain expressions for the characteristic functions of $(X_t + X_z, Y_T)$ under the risk neutral measure and $(X_t + X_z - 2X_T, Y_T - X_T)$ under the measure $\overline{\mathbb{P}}$.

Finally, in the case of the CEV diffusion, the continuous-time analogues of (20) and (21) are given by

$$\mathbb{E}\left[e^{-\mu(Y_T - Y_t)}\middle|\mathcal{F}_t\right] = \exp\left\{\varphi_{T-t}\left(\frac{\mu}{T}, 0\right) - \psi_{T-t}\left(\frac{\mu}{T}, 0\right) X_t\right\} \tag{25}$$

(see [51, Proposition 6.2.4]) and

$$\mathbb{E}\left[X_t^{\frac{1}{2-\gamma}} e^{-\mu Y_T}\right] = X_0^{\frac{1}{2-\gamma}} \exp\left\{rt + \varphi_{T-t}\left(\frac{\mu}{T}, 0\right) + \bar{\varphi}_t\left(\frac{\mu}{T}, \psi_{T-t}\left(\frac{\mu}{T}, 0\right)\right) - \psi_t\left(\frac{\mu}{T}, \psi_{T-t}\left(\frac{\mu}{T}, 0\right)\right) X_0\right\} \tag{26}$$

which follows from (25) by iterated expectations and a change to the measure $\overline{\mathbb{P}}$ with $\varphi, \psi, \bar{\varphi}$ given by (10), (11), (22).

3. General lower bounds. In this section we derive an optimal lower bound formula for the price of the arithmetic Asian call option with fixed or floating strike price. Price results for put-type options can then be obtained via standard put-call parity. The idea for the derivation of the bound is given in Curran [28] and Rogers and Shi [61] under elementary Black–Scholes market assumptions, and it is generalized here to Lévy and ASV models and the CEV diffusion. We note that our results are consistent with those of Lemmens et al. [52] in the case of fixed strike discrete Asian options under Lévy models. We present first our framework for the discrete average, while results for the continuous average follow based on the same principles.

3.1. Lower bounds for discrete Asian options. In the case of the discrete average, the payoff of the arithmetic Asian call option with time to maturity T has form

$$\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - \bar{K} S_{\Delta N} - K\right)^{+} \equiv \left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - \bar{K} S_{\Delta N} - K\right) \mathbf{1}_{A} \tag{27}$$

consisting of the fixed strike price $K \geq 0$ and coefficient $\bar{K} \geq 0$ for floating strike options, with

$$A \equiv \left\{ \frac{1}{N+1} \sum_{k=0}^{N} S_{\Delta k} > \bar{K} S_{\Delta N} + K \right\}. \tag{28}$$

The time-0 value of this option, P_0, satisfies

$$P_0 = e^{-rT} \mathbb{E}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - \bar{K} S_{\Delta N} - K\right) \mathbf{1}_{A}\right] \geq \mathrm{LB}_0 \equiv e^{-rT} \mathbb{E}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - \bar{K} S_{\Delta N} - K\right) \mathbf{1}_{A'}\right]$$

for any $A' \subset \Omega$ as $\frac{1}{N+1} \sum_{k=0}^{N} S_{\Delta k} \leq \bar{K} S_{\Delta N} + K$ in $A' \setminus A$. Therefore, the value of the option with fixed strike (i.e., $K > 0, \bar{K} = 0$) or floating strike (i.e., $K = 0, \bar{K} > 0$) satisfies respectively

$$P_{\mathrm{fix},0} \geq \mathrm{LB}_{\mathrm{fix},0} = e^{-rT} \mathbb{E}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - K\right) \mathbf{1}_{A'}\right], \tag{29}$$

$$P_{\mathrm{fl},0} \geq \mathrm{LB}_{\mathrm{fl},0} = e^{-rT} \mathbb{E}\left[S_{\Delta N}\left(\frac{\sum_{k=0}^{N} S_{\Delta k} S_{\Delta N}^{-1}}{N+1} - \bar{K}\right) \mathbf{1}_{A'}\right] = S_0 \overline{\mathbb{E}}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k} S_{\Delta N}^{-1}}{N+1} - \bar{K}\right) \mathbf{1}_{A'}\right], \tag{30}$$

where the last equality in (30) follows by a change to the $\overline{\mathbb{P}}$ measure.

Thus, the choice of a A' gives us a lower bound for the option price. The idea is that the chosen A' relates as closely as possible to the true A, so that the distance between the lower bound and the true option price is minimized, while at the same time makes the problem more analytically tractable compared to the original A. In what follows, we explain how A' is determined depending on the model choice for the underlying asset price dynamics. For consistency with (29) and (30), we define a parameter m taking value 0 (1) in the case of the fixed (floating) strike option.

3.1.1. The case of Lévy and ASV models. Given that

$$\frac{1}{N+1} \sum_{k=0}^{N} S_{\Delta k} S_{\Delta N}^{-m} \geq \left(\prod_{k=0}^{N} S_{\Delta k} S_{\Delta N}^{-m}\right)^{1/(N+1)} \tag{31}$$

(e.g., see [1, §3.2.1]), we choose

$$A' \equiv \left\{ \left(\prod_{k=0}^{N} S_{\Delta k} S_{\Delta N}^{-m}\right)^{1/(N+1)} > \exp(\lambda) \right\} \equiv \left\{ \frac{1}{N+1} \sum_{k=0}^{N} \ln S_{\Delta k} - m \ln S_{\Delta N} > \lambda \right\} \tag{32}$$

in the lower bounds (29) and (30) where λ is a real parameter whose value will be determined in Theorem 1. The choice of A' based on the (log) geometric average was originally applied in Curran [28] and Rogers and Shi [61] in the lognormal model and turned out to be a very accurate choice due to the high correlation between the arithmetic and (log) geometric averages. The fact that the log-geometric average also has favourable distributional properties under Lévy and ASV models (see Propositions 1 and 2), as opposed to the arithmetic average, further motivates its choice.

3.1.2. The case of the CEV diffusion model. CEV diffusion is treated separately from Lévy and ASV models as of interest in this case is the quantity $\frac{1}{N+1}\sum_{k=0}^{N}(S_{\Delta k}S_{\Delta N}^{-m})^{2-\gamma}$ with useful distributional properties, as opposed to $\frac{1}{N+1}\sum_{k=0}^{N}S_{\Delta k}S_{\Delta N}^{-m}$. Inequalities

$$\frac{1}{N+1}\sum_{k=0}^{N}S_{\Delta k}S_{\Delta N}^{-m} \gtrless \left(\frac{1}{N+1}\sum_{k=0}^{N}(S_{\Delta k}S_{\Delta N}^{-m})^{2-\gamma}\right)^{1/(2-\gamma)} \tag{33}$$

which hold for $\gamma \gtrless 1$ (e.g., see [1, §3.2.4]) motivate the following choice of A' for $\lambda > 0$

$$A' \equiv \left\{\left(\frac{1}{N+1}\sum_{k=0}^{N}\left(\frac{S_{\Delta k}}{S_{\Delta N}^{m}}\right)^{2-\gamma}\right)^{\frac{1}{2-\gamma}} > \lambda^{\frac{1}{2-\gamma}}\right\} \equiv \begin{matrix}\left\{\frac{1}{N+1}\sum_{k=0}^{N}\left(\frac{S_{\Delta k}}{S_{\Delta N}^{m}}\right)^{2-\gamma} > \lambda\right\}\mathbf{1}_{\{\gamma<1\}} \\ \cup\left\{\frac{1}{N+1}\sum_{k=0}^{N}\left(\frac{S_{\Delta k}}{S_{\Delta N}^{m}}\right)^{2-\gamma} > \lambda\right\}\mathbf{1}_{\{1<\gamma<2\}} \\ \cup\left\{\frac{1}{N+1}\sum_{k=0}^{N}\left(\frac{S_{\Delta k}}{S_{\Delta N}^{m}}\right)^{2-\gamma} < \lambda\right\}\mathbf{1}_{\{\gamma>2\}}\end{matrix}. \tag{34}$$

The averages $\frac{1}{N+1}\sum_{k=0}^{N}S_{\Delta k}S_{\Delta N}^{-m}$ and $\frac{1}{N+1}\sum_{k=0}^{N}(S_{\Delta k}S_{\Delta N}^{-m})^{2-\gamma}$ are positively correlated for $\gamma < 2$, whereas the correlation becomes negative for $\gamma > 2$.

3.2. Lower bound optimization and transform representations for discrete Asian options. Due to the correlation between the two types of average, i.e., $\frac{1}{N+1}\sum_{k=0}^{N}S_{\Delta k}S_{\Delta N}^{-m}$ and $\frac{1}{N+1}\sum_{k=0}^{N}\ln(S_{\Delta k}S_{\Delta N}^{-m})$ for Lévy and ASV models or $\frac{1}{N+1}\sum_{k=0}^{N}S_{\Delta k}S_{\Delta N}^{-m}$ and $\frac{1}{N+1}\sum_{k=0}^{N}(S_{\Delta k}\, S_{\Delta N}^{-m})^{2-\gamma}$ for the CEV model, by replacing A by A' and additionally optimizing the parameter λ we minimize the error in the lower bound. In Section 4 we prove an estimate for the error, whereas in Section 5 we demonstrate the effect of the optimal parameter λ in various numerical examples. Next, we determine the value of parameter λ which maximizes the lower bounds (29) and (30).

THEOREM 1. (OPTIMALITY CONDITIONS). *Consider the random variables* $Y_{\Delta N} = \frac{1}{N+1}\sum_{k=0}^{N}X_{\Delta k}$ *and* $\bar{Y}_{\Delta N}$, *where* $X = \ln S$, $\bar{Y}_{\Delta N} \equiv Y_{\Delta N} - X_{\Delta N}$ *under Lévy and ASV models, and* $X = S^{2-\gamma}$, $\bar{Y}_{\Delta N} \equiv Y_{\Delta N}X_{\Delta N}^{-1}$ *under the CEV diffusion. Then, the optimal lower bound is given for*

$$\lambda^* \equiv \arg\max_{\lambda} \mathrm{LB}_0(\lambda)$$

which satisfies the optimality conditions

$$\mathbb{E}\left(\frac{\sum_{k=0}^{N}S_{\Delta k}}{N+1}\middle| Y_{\Delta N} = \lambda^*\right) = K \text{ and } \bar{\mathbb{E}}\left(\frac{\sum_{k=0}^{N}S_{\Delta k}S_{\Delta N}^{-1}}{N+1}\middle| \bar{Y}_{\Delta N} = \lambda^*\right) = \bar{K}, \tag{35}$$

respectively, for a fixed and a floating strike option under Lévy, ASV and the CEV models.

Proof. We consider the case of the fixed strike option (the floating strike case is proved similarly). From (29) and the definitions of A' given in (32) and (34)

$$\mathbb{E}\left[\left(\frac{\sum_{k=0}^{N}S_{\Delta k}}{N+1} - K\right)\mathbf{1}_{\{Y_{\Delta N}>\lambda\}}\right] = \frac{1}{N+1}\sum_{k=0}^{N}\mathbb{E}[\mathbb{E}[S_{\Delta k} - K|Y_{\Delta N}]\mathbf{1}_{\{Y_{\Delta N}>\lambda\}}]$$

(note that opposite inequality sign applies for CEV elasticity $\gamma > 2$). Differentiating w.r.t. λ and interchanging with the expectation yields

$$\frac{1}{N+1}\sum_{k=0}^{N}\mathbb{E}\left[\mathbb{E}\left[S_{\Delta k} - K|Y_{\Delta N}\right]\frac{d}{d\lambda}\mathbf{1}_{\{Y_{\Delta N}>\lambda\}}\right] = \frac{-1}{N+1}\sum_{k=0}^{N}\mathbb{E}(S_{\Delta k} - K|Y_{\Delta N} = \lambda)f_N(\lambda), \tag{36}$$

where the last equality follows from $d\mathbf{1}_{\{Y_{\Delta N}>\lambda\}}/d\lambda = -\delta(Y_{\Delta N} - \lambda)$ with δ representing the Dirac delta function and f_N the density function of $Y_{\Delta N}$ under the risk neutral measure. Then (35) follows from (36) by setting this equal to zero. □

We now proceed to derive the transform representations of the lower bounds (29) and (30) with respect to λ, which can then be inverted to retrieve the lower bounds.

THEOREM 2. (FIXED AND FLOATING STRIKE DISCRETE ASIAN OPTIONS).
(i) (LÉVY AND ASV MODELS). *Suppose $X=\ln S$. The Fourier transform of the lower bound (29) w.r.t. λ is*

$$\begin{aligned}\Phi(u;\delta) &\equiv \int_{\mathbb{R}} e^{iu\lambda+\delta\lambda}\left\{\frac{e^{-rT}}{N+1}\sum_{k=0}^{N}\mathbb{E}[(e^{X_{\Delta k}}-K)\mathbf{1}_{\{Y_{\Delta N}>\lambda\}}]\right\}d\lambda \\ &= \frac{e^{-rT}}{iu+\delta}\left\{\frac{1}{N+1}\sum_{k=0}^{N}\phi_{k,k,N}(-i/2,u-i\delta)-K\phi_{N,N,N}(0,u-i\delta)\right\}, \end{aligned} \tag{37}$$

where constant $\delta>0$ ensures integrability and ϕ is given in (14) and (16), respectively, for Lévy and ASV models.

The Fourier transform of the lower bound (30) w.r.t. λ is

$$\begin{aligned}\bar{\Phi}(u;\delta) &\equiv \int_{\mathbb{R}} e^{iu\lambda+\delta\lambda}\left\{\frac{e^{X_0}}{N+1}\sum_{k=0}^{N}\overline{\mathbb{E}}[(e^{X_{\Delta k}-X_{\Delta N}}-\bar{K})\mathbf{1}_{\{Y_{\Delta N}-X_{\Delta N}>\lambda\}}]\right\}d\lambda \\ &= \frac{e^{X_0}}{iu+\delta}\left\{\frac{1}{N+1}\sum_{k=0}^{N}\bar{\phi}_{k,k,N}(-i/2,u-i\delta)-\bar{K}\bar{\phi}_{N,N,N}(0,u-i\delta)\right\}, \end{aligned} \tag{38}$$

where $\bar{\phi}$ is given in (18) and (19) for the relevant model cases.

(ii) (CEV MODEL). *Suppose $X=S^{2-\gamma}$. The (bilateral) Laplace transform of the lower bound (29) w.r.t. λ is*

$$\begin{aligned}\Phi(i\mu;\delta) &\equiv \int_{\mathbb{R}} e^{-\mu\lambda+\delta\lambda}\left\{\frac{e^{-rT}}{N+1}\sum_{k=0}^{N}\mathbb{E}\left[(X_{\Delta k}^{1/(2-\gamma)}-K)\mathbf{1}_{\{Y_{\Delta N}\gtrless\lambda\}}\right]\right\}d\lambda \\ &= \frac{\operatorname{sgn}(\delta)e^{-rT}}{-\mu+\delta}\left\{\frac{1}{N+1}\sum_{k=0}^{N}\mathbb{E}\left[X_{\Delta k}^{1/(2-\gamma)}e^{-(\mu-\delta)Y_{\Delta N}}\right]-K\mathbb{E}[e^{-(\mu-\delta)Y_{\Delta N}}]\right\}, \end{aligned} \tag{39}$$

where $\mu\in i\mathbb{R}$, constant $\delta\gtrless 0$ for $\gamma\lessgtr 2$, sgn denotes the signum function and $\mathbb{E}\left[X_{\Delta k}^{1/(2-\gamma)}e^{-(\mu-\delta)Y_{\Delta N}}\right]$ and $\mathbb{E}[e^{-(\mu-\delta)Y_{\Delta N}}]$ are given in (21) and (20), respectively.

The (bilateral) Laplace transform of the lower bound (30) w.r.t. λ is

$$\begin{aligned}\bar{\Phi}(i\mu;\delta) &\equiv \int_{\mathbb{R}} e^{-\mu\lambda+\delta\lambda}\left\{\frac{X_0^{1/(2-\gamma)}}{N+1}\sum_{k=0}^{N}\overline{\mathbb{E}}\left[((X_{\Delta k}X_{\Delta N}^{-1})^{1/(2-\gamma)}-\bar{K})\mathbf{1}_{\{Y_{\Delta N}X_{\Delta N}^{-1}\gtrless\lambda\}}\right]\right\}d\lambda \\ &= \frac{\operatorname{sgn}(\delta)X_0^{1/(2-\gamma)}}{-\mu+\delta}\left\{\frac{1}{N+1}\sum_{k=0}^{N}\overline{\mathbb{E}}\left[(X_{\Delta k}X_{\Delta N}^{-1})^{1/(2-\gamma)}e^{-(\mu-\delta)Y_{\Delta N}X_{\Delta N}^{-1}}\right]-\bar{K}\overline{\mathbb{E}}[e^{-(\mu-\delta)Y_{\Delta N}X_{\Delta N}^{-1}}]\right\}, \end{aligned} \tag{40}$$

where the expected values in (40) are computed numerically[1]*.*

(iii) The lower bounds (29) and (30) are given in terms of the inversion formulae

$$\mathrm{LB}_{\mathrm{fix},0}(\lambda)=\frac{e^{-\delta\lambda}}{2\pi}\int_{\mathbb{R}}e^{-iu\lambda}\Phi(u;\delta)du \text{ and } \mathrm{LB}_{\mathrm{fl},0}(\lambda)=\frac{e^{-\delta\lambda}}{2\pi}\int_{\mathbb{R}}e^{-iu\lambda}\bar{\Phi}(u;\delta)du, \tag{41}$$

where Φ (resp. $\bar{\Phi}$) is given in (37) (resp. 38) for Lévy and ASV models and (39) (resp. 40) for the CEV model.

[1] Derivation of explicit expressions for the expected values in (40) is not trivial. Instead, in principle, these can be computed numerically; such a computation requires derivation of the moment generating function of the triple $(X_{\Delta k}, X_{\Delta N}, Y_{\Delta N})$ under the measure $\overline{\mathbb{P}}$ using iterated expectations (details are omitted here for brevity) and transform inversion using the multivariate version of the algorithm proposed in Choudhury et al. [22].

Proof. See Appendix A. □

REMARK 1 (PRICE SENSITIVITIES). One of the advantages of our method is that, at essentially no additional computational cost, it lends itself to computing the option price sensitivity with respect to some parameter κ of interest, e.g., the initial value or volatility of the underlying asset, the risk-free rate, etc. This is possible: (i) assuming that the interchange of differentiation and integration in (41) is allowed, a usual assumption in option pricing via Fourier transform; and (ii) resorting to the envelope theorem[2] (see [66, p. 160]). For example, in the case of the fixed strike Asian option we compute the price sensitivities by

$$\frac{\partial \mathrm{LB}_{\mathrm{fix},0}(\lambda^*;\kappa)}{\partial \kappa} = \frac{e^{-\delta\lambda^*}}{2\pi}\int_{\mathbb{R}} e^{-iu\lambda^*}\frac{\partial \Phi(u;\delta,\kappa)}{\partial \kappa}du, \tag{42}$$

where λ^* satisfies (35) (we change slightly our notation here to make explicit the dependence of Φ and, hence, $\mathrm{LB}_{\mathrm{fix},0}$ on κ). We highlight that (42) is the derivative of the lower bound for the option price in (41) w.r.t. κ and does not imply a bound for the corresponding sensitivity.

By sake of exemplification, to compute the delta we require

$$\frac{\partial \Phi\left(u;\delta,X_0\right)}{\partial S_0} = \frac{\partial \Phi\left(u;\delta,X_0\right)}{\partial X_0}\frac{\partial X_0}{\partial S_0},$$

where $\partial\Phi\left(u;\delta,X_0\right)/\partial X_0$ is computed using (37) and (14) for Lévy models; (37) and (16) for ASV models; (39) and (20)–(21) for the CEV model. In addition, $\partial X_0/\partial S_0 = \exp(-X_0)$ for Lévy and ASV models; $\partial X_0/\partial S_0 = (2-\gamma)X_0^{-1}$ for the CEV model.

Based on the same principles, gamma can be computed by the second derivative of (41) w.r.t. S_0.

REMARK 2 (AUSTRALIAN OPTIONS). It is worth noting that our construction can be adapted easily to the case of Australian options, i.e., options whose payoff depends on $\frac{1}{N+1}\sum_{k=0}^{N} S_{\Delta k}S_{\Delta N}^{-1}$ (e.g., see [36]). Pricing in this case is similar to that of floating strike options (see expression 30, however with the expected values taken under the risk neutral measure and S_0 replaced by e^{-rT}).

3.3. Lower bounds for continuous Asian options. The results for the continuous average case are derived with a straightforward application of the same passages as in Section 3.1. In more details,

$$e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] \geq e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)\mathbf{1}_{\{Y_T>\lambda\}}\right], \tag{43}$$

$$e^{-rT}\mathbb{E}\left[S_T\left(\frac{1}{T}\int_0^T S_t S_T^{-1} dt - \bar{K}\right)^+\right] \geq S_0\overline{\mathbb{E}}\left[\left(\frac{1}{T}\int_0^T S_t S_T^{-1} dt - \bar{K}\right)\mathbf{1}_{\{\bar{Y}_T>\lambda\}}\right], \tag{44}$$

where $Y_T = \frac{1}{T}\int_0^T X_t dt$, $\bar{Y}_T \equiv Y_T - X_T$ under Lévy and ASV models, $\bar{Y}_T \equiv Y_T X_T^{-1}$ under the CEV diffusion. Note that opposite inequality sign applies in the indicator functions in (43)–(44) for CEV elasticity $\gamma > 2$. The maximum lower bounds are given for $\lambda = \lambda^*$ satisfying

$$\mathbb{E}\left(\frac{1}{T}\int_0^T S_t dt\middle| Y_T = \lambda^*\right) = K \text{ and } \overline{\mathbb{E}}\left(\frac{1}{T}\int_0^T S_t S_T^{-1} dt\middle| \bar{Y}_T = \lambda^*\right) = \bar{K}. \tag{45}$$

[2] Changes in the parameters may cause changes in the optimal value λ^* in Theorem 1 and the maximum lower bound $\mathrm{LB}_0(\lambda^*)$. The envelope theorem guarantees that changes in λ^* due to changes in the parameters do not contribute to changes in $\mathrm{LB}_0(\lambda^*)$.

Theorem 3. (Fixed and floating strike continuous Asian options).

(i) (Lévy and ASV models). *Suppose $X = \ln S$. The Fourier transform of the lower bound (43) w.r.t. λ is*

$$\begin{aligned}\Phi(u;\delta) &\equiv \int_{\mathbb{R}} e^{iu\lambda+\delta\lambda}\left\{\frac{e^{-rT}}{T}\int_0^T \mathbb{E}[(e^{X_t}-K)\mathbf{1}_{\{Y_T>\lambda\}}]dt\right\}d\lambda \\ &= \frac{e^{-rT}}{iu+\delta}\left\{\frac{1}{T}\int_0^T \phi_{t,t,T}(-i/2,u-i\delta)dt - K\phi_{T,T,T}(0,u-i\delta)\right\}, \end{aligned} \tag{46}$$

where constant $\delta>0$ ensures integrability and ϕ is given in (23) for Lévy models.

The Fourier transform of the lower bound (44) w.r.t. λ is

$$\begin{aligned}\bar{\Phi}(u;\delta) &\equiv \int_{\mathbb{R}} e^{i(u-i\delta)\lambda}\left\{\frac{e^{X_0}}{T}\int_0^T \bar{\mathbb{E}}[(e^{X_t-X_T}-K)\mathbf{1}_{\{Y_T-X_T>\lambda\}}]dt\right\}d\lambda \\ &= \frac{e^{X_0}}{iu+\delta}\left\{\frac{1}{T}\int_0^T \bar{\phi}_{t,t,T}(-i/2,u-i\delta)dt - K\bar{\phi}_{T,T,T}(0,u-i\delta)\right\}, \end{aligned} \tag{47}$$

where $\bar{\phi}$ is given in (24) for Lévy models[3].

(ii) (CEV model). *Suppose $X = S^{2-\gamma}$. The (bilateral) Laplace transform of the lower bound (43) w.r.t. λ is*

$$\begin{aligned}\Phi(i\mu;\delta) &\equiv \int_{\mathbb{R}} e^{-\mu\lambda+\delta\lambda}\left\{\frac{e^{-rT}}{T}\int_0^T \mathbb{E}\left[(X_t^{1/(2-\gamma)}-K)\mathbf{1}_{\{Y_T\gtrless\lambda\}}\right]dt\right\}d\lambda \\ &= \frac{\operatorname{sgn}(\delta)e^{-rT}}{-\mu+\delta}\left\{\frac{1}{T}\int_0^T \mathbb{E}\left[X_t^{1/(2-\gamma)}e^{-(\mu-\delta)Y_T}\right]dt - K\mathbb{E}[e^{-(\mu-\delta)Y_T}]\right\}, \end{aligned} \tag{48}$$

where $\mu\in i\mathbb{R}$, constant $\delta\gtrless 0$ for $\gamma\lessgtr 2$, sgn denotes the signum function and $\mathbb{E}\left[X_t^{1/(2-\gamma)}e^{-(\mu-\delta)Y_T}\right]$ and $\mathbb{E}[e^{-(\mu-\delta)Y_T}]$ are given in (26) and (25), respectively.

The (bilateral) Laplace transform of the lower bound (44) w.r.t. λ is

$$\begin{aligned}\bar{\Phi}(i\mu;\delta) &\equiv \int_{\mathbb{R}} e^{-\mu\lambda+\delta\lambda}\left\{\frac{X_0^{1/(2-\gamma)}}{T}\int_0^T \bar{\mathbb{E}}\left[((X_tX_T^{-1})^{1/(2-\gamma)}-\bar{K})\mathbf{1}_{\{Y_TX_T^{-1}\gtrless\lambda\}}\right]dt\right\}d\lambda \\ &= \frac{\operatorname{sgn}(\delta)X_0^{1/(2-\gamma)}}{-\mu+\delta}\left\{\frac{1}{T}\int_0^T \bar{\mathbb{E}}\left[(X_tX_T^{-1})^{1/(2-\gamma)}e^{-(\mu-\delta)Y_TX_T^{-1}}\right]dt - \bar{K}\bar{\mathbb{E}}[e^{-(\mu-\delta)Y_TX_T^{-1}}]\right\}. \end{aligned} \tag{49}$$

(iii) The lower bounds (43) and (44) are given in terms of the inversion formulae

$$\mathrm{LB}_{\mathrm{fix},0}(\lambda) = \frac{e^{-\delta\lambda}}{2\pi}\int_{\mathbb{R}} e^{-iu\lambda}\Phi(u;\delta)du \text{ and } \mathrm{LB}_{\mathrm{fl},0}(\lambda) = \frac{e^{-\delta\lambda}}{2\pi}\int_{\mathbb{R}} e^{-iu\lambda}\bar{\Phi}(u;\delta)du, \tag{50}$$

where Φ and $\bar{\Phi}$ are given in (46)–(49) for the relevant models and types of options.

Proof. The proof proceeds along the same lines as that of Theorem 2. □

The option price sensitivities can be obtained as explained in Remark 1.

Remark 3 (Generalized weighted sums). It is worth noting that our lower bound results can be extended to payoffs which depend on some generalized weighted sum of asset prices $\int_0^T S_t\mu(t)dt$, where $\mu(t)$ represents a measure on the time interval $[0,T]$ on which we monitor the underlying asset price process; this encompasses the cases of the continuous Asian option with $\mu(t)\equiv\frac{1}{T}$ and discrete Asian option with $\mu(t)\equiv\frac{1}{N+1}\sum_{k=0}^N \delta_{(\Delta k)}(t)$ where δ denotes the Dirac delta. Our lower bound results are readily extendible to other candidates for μ of interest.

[3] The derivation of expressions for ϕ and $\bar{\phi}$ under ASV models is briefly discussed in Section 2.5.

4. An estimate for the error of the lower bound. In what follows, we derive an estimate, in the form of an upper bound, for the error when approximating the true Asian option price using the general lower bound of Section 3 based on the principles set out in Rogers and Shi [61] and Nielsen and Sandmann [58] in the Gaussian model setting and Lemmens et al. [52] under Lévy models.

THEOREM 4. (ERROR BOUNDS). *Consider the random variables* $Y_{\Delta N}=\frac{1}{N+1}\sum_{k=0}^{N}X_{\Delta k}$ *and* $\bar{Y}_{\Delta N}$, *where* $X=\ln S$, $\bar{Y}_{\Delta N}=Y_{\Delta N}-X_{\Delta N}$ *under Lévy and ASV models, and* $\bar{Y}_{\Delta N}=Y_{\Delta N}X_{\Delta N}^{-1}$ *with* $X=S^{2-\gamma}$ *under the CEV diffusion. In addition,* $Y_T=\frac{1}{T}\int_0^T X_t dt$ *and* $\bar{Y}_T=Y_T-X_T$ *or* $\bar{Y}_T=Y_TX_T^{-1}$ *for the relevant model case.*

(i) (DISCRETE AVERAGE, FIXED AND FLOATING STRIKE OPTIONS). *The error* ϵ *from approximating the price of the Asian call option on the discrete average with payoff (27) for a fixed strike (i.e.,* $K>0,\bar{K}=0$*) or a floating strike (i.e.,* $K=0,\bar{K}>0$*) by the corresponding lower bound (29) or (30) is bounded, respectively, by*

$$0\le\epsilon_{\text{fix}}\le\frac{e^{-rT}}{2(N+1)}\mathbb{E}\left\{\left[\sum_{k,n=1}^{N}\mathbb{E}(S_{\Delta k}S_{\Delta n}|Y_{\Delta N})-\left(\sum_{k=1}^{N}\mathbb{E}(S_{\Delta k}|Y_{\Delta N})\right)^2\right]^{\frac{1}{2}}\mathbf{1}_{\{Y_{\Delta N}\le\lambda^*\}}\right\},\tag{51}$$

$$0\le\epsilon_{\text{fl}}\le\frac{S_0}{2(N+1)}\overline{\mathbb{E}}\left\{\left[\sum_{k,n=1}^{N}\overline{\mathbb{E}}(S_{\Delta k}S_{\Delta n}S_{\Delta N}^{-2}|\bar{Y}_{\Delta N})-\left(\sum_{k=1}^{N}\overline{\mathbb{E}}(S_{\Delta k}S_{\Delta N}^{-1}|\bar{Y}_{\Delta N})\right)^2\right]^{\frac{1}{2}}\mathbf{1}_{\{\bar{Y}_{\Delta N}\le\lambda^*\}}\right\},\tag{52}$$

where the expectations in (51) and (52) are taken under the $\mathbb{P}$ *and* $\overline{\mathbb{P}}$ *measures, respectively.*

(ii) (CONTINUOUS AVERAGE, FIXED AND FLOATING STRIKE OPTIONS). *The error from approximating the price of the fixed or floating strike Asian call option on the continuous average by the corresponding lower bound (43) or (44) is bounded, respectively, by*

$$0\le\epsilon_{\text{fix}}\le\frac{e^{-rT}}{2T}\mathbb{E}\left\{\left[\int_{[0,T]^2}\mathbb{E}(S_tS_z|Y_T)d(t,z)-\left(\int_0^T\mathbb{E}\left(S_t|Y_T\right)dt\right)^2\right]^{\frac{1}{2}}\mathbf{1}_{\{Y_T\le\lambda^*\}}\right\},\tag{53}$$

$$0\le\epsilon_{\text{fl}}\le\frac{S_0}{2T}\overline{\mathbb{E}}\left\{\left[\int_{[0,T]^2}\overline{\mathbb{E}}(S_tS_zS_T^{-2}|\bar{Y}_T)d(t,z)-\left(\int_0^T\overline{\mathbb{E}}(S_tS_T^{-1}|\bar{Y}_T)dt\right)^2\right]^{\frac{1}{2}}\mathbf{1}_{\{\bar{Y}_T\le\lambda^*\}}\right\},\tag{54}$$

where the expectations in (53) and (54) are taken under the $\mathbb{P}$ *and* $\overline{\mathbb{P}}$ *measures, respectively. (Note that for CEV elasticity* $\gamma>2$, *the inequality sign in the indicator functions in (51)–(54) is reversed.)*

Proof. See Appendix A. □

From (51)–(54) (see also 67) it is obvious that the more information the conditioning average contains about the original arithmetic average, the smaller the conditional variance of the arithmetic average, hence the smaller the error in the lower bound price approximation, becomes. This confirms our choice of the conditioning average and of A' to substitute for A in Section 3.

It is possible to obtain analytical expressions for the error bounds (51) and (52). More specifically, we get

$$\mathbb{E}(S_{\Delta k}S_{\Delta n}^{m}|Y_{\Delta N}=y)=\frac{\int_{\mathbb{R}}e^{-ivy}\mathbb{E}(S_{\Delta k}S_{\Delta n}^{m}e^{ivY_{\Delta N}})dv}{\int_{\mathbb{R}}e^{-ivy}\mathbb{E}(e^{ivY_{\Delta N}})dv},\tag{55}$$

$$\overline{\mathbb{E}}(S_{\Delta k}S_{\Delta n}^{m}S_{\Delta N}^{-1-m}|\bar{Y}_{\Delta N}=y)=\frac{\int_{\mathbb{R}}e^{-ivy}\overline{\mathbb{E}}(S_{\Delta k}S_{\Delta n}^{m}S_{\Delta N}^{-1-m}e^{iv\bar{Y}_{\Delta N}})dv}{\int_{\mathbb{R}}e^{-ivy}\overline{\mathbb{E}}(e^{iv\bar{Y}_{\Delta N}})dv}\tag{56}$$

for $m=0,1$ (see [7, p. 62–63]). Consider the functions ϕ and $\bar{\phi}$ given respectively by the expressions (14) and (18) for X driven by a Lévy model; (16) and (19) for X driven by an ASV model. Then, we have that

$$\mathbb{E}(S_{\Delta k}S_{\Delta n}^{m}e^{ivY_{\Delta N}})\quad=\quad\phi_{k,m(n-k)+k,N}(-i/(2-m),v),$$

$$\overline{\mathbb{E}}(S_{\Delta k} S_{\Delta n}^{m} S_{\Delta N}^{-1-m} e^{iv\bar{Y}_{\Delta N}}) \;=\; \bar{\phi}_{k,m(n-k)+k,N}(-i/(2-m), v),$$
$$\mathbb{E}(e^{ivY_{\Delta N}}) = \phi_{N,N,N}(0,v) \;\text{ and }\; \overline{\mathbb{E}}(e^{iv\bar{Y}_{\Delta N}}) = \bar{\phi}_{N,N,N}(0,v).$$

In the case of the CEV model, for $m=0$

$$\mathbb{E}(S_{\Delta k} e^{ivY_{\Delta N}}) = \mathbb{E}({X_{\Delta k}}^{1/(2-\gamma)} e^{ivY_{\Delta N}}) \text{ and } \mathbb{E}(e^{ivY_{\Delta N}})$$

are given explicitly in (21) and (20), whereas the remaining expected values in (55)–(56) can be computed numerically or using fractional calculus techniques as $1/(2-\gamma) \in \mathbb{R}$, see Cressie and Borkent [26], subject to certain regularity conditions.

In the case of the continuous average, the error bounds (53) and (54) can also be computed by means of explicit conditional expectation representations, similar to (55)–(56) for the discrete average, to the extent these are available for different model specifications under continuous averaging: for example, see (23)–(24) for Lévy models; (25)–(26) for the CEV model; discussion in Section 2.5 for ASV models.

5. Numerical study. In order to illustrate the performance of our maximum lower bound (MLB) with optimal parameter $\lambda = \lambda^*$ satisfying (35) or (45), respectively for a discrete or a continuous average, we perform an extensive pricing exercise across a wide range of stochastic dynamic models (see Table 3) for varying strike price K and monitoring frequency. In addition, we consider a suboptimal lower bound (SLB) in which we fix the parameter $\lambda = \ln K$, for Lévy and ASV models, and $\lambda = K^{2-\gamma}$, for the CEV diffusion; these choices follow from a comparison of A given in (28), based on the original average, and A' in (32) and (34), based on the correlated average. (In the case of floating strike options, K should be replaced by $\bar{K}$.) Formulae (41) and (50), respectively for a discrete and a continuous average, are computed using the (fractional) fast Fourier transform algorithm (e.g., `fft` or `czt` in Matlab, see [21] for more details) which outputs: (i) lower bound values on a fine, equally spaced grid of parameter λ values, including the MLB for optimal $\lambda = \lambda^*$ as well as the SLB for $\lambda = \ln K$ or $\lambda = K^{2-\gamma}$; and (ii) the optimal value λ^*. Computed lower bounds are then compared with benchmark prices generated by a very accurate control variate Monte Carlo (CVMC) simulation method using as control variates the lower bounds themselves[4]; we call these optimized CVMC (with the MLB as CV) and suboptimal CVMC (with the SLB as CV). We employ CVMC setup with the CV coefficient estimated in a pilot run, e.g., see Glasserman [43] and Cont and Tankov [23]. The choice of the CVMC method is justified by its high accuracy and applicability under various model assumptions for the underlying asset price process. In Appendix B, we summarize the simulation methodologies we have used in our numerical study. Note that when estimating the price of the option on the continuous average, we correct the discretization bias inherent in the simulation using our maximum lower bound based on the continuous average as the control variate[5]. The sets of parameter values used for the Lévy models are from the calibrations of Schoutens [62] and Fusai and Meucci [40]; the ASV models' parameter values are from the calibrations of Duffie et al. [33] (see also [13], [2] and [68]) and Nicolato and Venardos [57]; the CEV parameter sets are taken from Cai et al. [15].

In addition, where applicable, we compare with numerical results from other important methods in the literature, including, for example, Geman and Yor [42], Zhang [74], Dewynne and Shaw [31], Cai and Kou [14], Bayraktar and Xing [9], Ewald et al. [36], Cai et al. [15] and Sesana et al.

[4] Lower bound price approximations have also been used as control variates in Caldana and Fusai [17] in simulating spread option prices with substantial variance reduction effect.

[5] Fu et al. [37] have implemented previously a similar efficient simulation approach for pricing continuous arithmetic Asian options in the basic Gaussian model using, instead, the analytically tractable continuous geometric Asian option as the control variate.

[64]. We point out that the relevant results consist of various model settings for the underlying, parameter values for the volatility, interest rate and dividend yield, contract parameters such as the strike price, and monitoring frequencies (discrete and continuous).

Our computations are conducted on a desktop PC with an Intel Core 2 Duo 2.93 GHz processor and 2.00 GB of RAM. As our computing platforms, we have chosen Matlab R2010a and, for the method of Bayraktar and Xing [9], Visual C++ 2010.

5.1. Performance comparisons against Monte Carlo simulation benchmarks. First, we consider the case of the discrete average. Tables 4–5 give numerical results of prices of Asian options under various Lévy models (Gaussian, VG, NIG, CGMY, MJD, DEJD, Meixner) and ASV models (Heston, Bates, DPS, BNS-Γ, BNS-IG). We let the strike vary from 90 to 110 with an increment of 10 and consider different monitoring frequencies: monthly ($N = 12$), weekly ($N = 50$) and daily ($N = 250$). It can be seen that the optimized CVMC systematically produces estimates with lower standard errors than the suboptimal CVMC. For this, in what follows we consider only the optimized CVMC estimates. In fact, our reported optimized CVMC price estimates are accurate to $4-5$ decimal places (at the 95% confidence level), hence can serve as benchmarks to the numerical outcomes from alternative pricing methods. However, despite its high accuracy, Monte Carlo simulation can be computationally intensive, in general, for high monitoring frequencies, but also in model-specific cases, such as the Meixner, Bates, BNS-Γ and BNS-IG models, by construction of their simulation procedures with CPU times in excess of 1000 seconds per price estimate for $N = 12$. A comparison in terms of the average absolute % relative error (AAPRE) of the MLB and SLB against the optimized CVMC estimates across strikes and number of monitoring dates indicates that the MLB generates AAPREs in the range 0.01% − 0.05% under the Lévy models and 0.02% − 0.03% under stochastic volatility, whereas the SLB produces AAPREs of 0.04% − 0.09% and 0.1% − 0.16% respectively. In summary, Tables 4–5 suggest that the MLB is consistently more accurate and robust than the SLB across different models, contract specifications and monitoring frequencies. More importantly, given its high accuracy level and ease of use, the MLB can be used itself as an efficient and power-saving substitute to the CVMC price.

In Table 6, we extend our analysis to the CEV asset price model. More specifically, we compare our MLB and SLB results for different elasticities γ with those obtained through the quadrature method of Sesana et al. [64] and the asymptotic expansion approach of Cai et al. [15]. Based on the AAPREs computed against the optimized CVMC estimates, the SLB is shown to perform consistently worse than the MLB. It can be seen that for $\gamma = 1.5$ the MLB produces the lowest AAPRE and also the lowest maximum absolute % relative error (MAPRE), i.e., 0.004% (0.009%), followed by the quadrature method with 0.005% (0.035%), the SLB with 0.013% (0.028%) and the expansion formula with 0.051% (0.098%); for $\gamma = 2.5$, the AAPREs (MAPREs) are found to be 0.009% (0.037%), 0.040% (0.087%), 0.045% (0.078%), 0.118% (0.265%), respectively, for the quadrature method, the MLB, the expansion formula and the SLB. Although very accurate, the quadrature method by Sesana et al. [64] turns out be computationally expensive requiring approximately 90 seconds, as opposed to 0.15 seconds required by our MLB, for $N = 12$, with the CPU times further increasing with N. On the other hand, the expansion formula of Cai et al. [15] is very fast, generating one result in less than 0.5 seconds almost independently of N, however its applicability is limited to one-dimensional diffusion models.

In Table 7, we present numerical results of price sensitivities of Asian options computed as explained in Remark 1. More specifically, we compare our deltas and gammas under the CEV model with accurate results from the method of Sesana et al. [64], whereas in the cases of Heston and CGMY models (for brevity) we compare our deltas with unbiased likelihood-ratio optimized CVMC estimates (see [12] for the Heston model using conditional Monte Carlo; [44] for the CGMY model using combined Monte Carlo with transform inversion). Our reports confirm the high precision of our method also in this computation: for example, this generates AAPRE of 0.005% under CEV with $\gamma = 1.5$, which is an improvement to the asymptotic expansion approach of Cai et al. [15] with AAPRE of 0.028% in this case, whereas both methods perform the same under CEV with $\gamma = 2.5$ with AAPRE of 0.013%. Our approximation seems less accurate in the case of the gamma sensitivity (still acceptable for practical applications) with AAPRE of 0.5% (approx.) for both $\gamma = 1.5, 2.5$.

Next, we assess the performance of our MLB in the case of the continuous average. In light of recent advances on the pricing of continuous Asian options, we compare with the numerical prices from the inversion of the double-Laplace transform of Cai and Kou [14] under the DEJD model; the PIDE implementation of Bayraktar and Xing [9] under the MJD and DEJD models; and the implementation of the PDE developed in Ewald et al. [36], as well as the second-order Taylor 2.0 Monte Carlo price estimates of Ewald et al. [36] under the Heston model. A more detailed analysis under the Gaussian model dynamics is deferred to Section 5.2. Using the same sets of parameters as in the aforementioned works, it is shown in Table 8 that the MLB and the double-Laplace transform algorithm generate nearby AAPREs under the DEJD model with jump arrival rate $l = 3$ ($l = 5$), i.e., 0.02% and 0.01% (0.03% and 0.06%), as opposed to the PIDE method of Bayraktar and Xing [9] with a persistently higher AAPRE of 0.07% (0.15%). Also, the reported MAPREs follow the same pattern across the different methods. The CPU times per result are approximately 4, 6 and 6 seconds for the MLB, the double-Laplace transform algorithm and the PIDE method. Under the MJD model with $l = 1$, the PIDE method's AAPRE improves to 0.03%, whereas the AAPRE of our MLB is only marginally affected and is no greater than 0.04%. In Table 9, we present numerical results for continuous Asian options under the Heston model. Given the simulation error (RMSE) reports for the second-order Taylor scheme implemented in Ewald et al. [36], we reach that the relevant price estimates are accurate to $1-2$ decimal places (at the 90% confidence level) with each estimate taking 310 seconds to compute. In addition, implementing the PDE of Ewald et al. [36] takes approximately 1000 seconds to generate a price result accurate to 3 decimal places, whereas our MLB requires approximately 10 seconds for the same level of precision.

5.2. Pricing under the Gaussian model: comparisons for varying volatility. In this section, we focus attention on the special case of the continuous Asian option under the basic Gaussian model. To check the accuracy of our MLB, we adhere to the test cases considered in Cai and Kou [14] and compare against existing methods devoted to the Gaussian dynamics and the continuous average case.

Table 10 shows that in most of the cases of moderate to high volatility, $0.1 \leq \sigma \leq 0.5$, our MLB results agree to 4 decimal places with the prices obtained from the methods of Geman and Yor [42], Linetsky [53], Cai and Kou [14] (accurate to 10 decimal places), and Večeř [69], Zhang [74] (accurate to 6 decimal places). Although the MLB seems less accurate than the other approaches (still sufficiently accurate for practical applications), its performance improves substantially with decreasing volatility to extremely low levels in which case many numerical methods for Asian options perform poorly. Here, we extend the previous analysis of Cai and Kou [14] on cross-comparisons of behaviours for reasonably low volatilities, e.g., $\sigma = 0.05$, and extremely low volatilities, $\sigma \leq 0.01$, to include also our MLB method. Further, we study different cases $q < r$, $q = r$, $q > r$. Table 11 shows that, for $\sigma = 0.05$, our MLB, the double-Laplace transform algorithm of Cai and Kou [14], the Geman–Yor Laplace transform formula implemented as in Shaw [65], the matched asymptotic expansion of Dewynne and Shaw [31] and the PDE of Zhang [74] coincide with one another to 6 decimal places. When $\sigma \leq 0.01$, Cai and Kou's algorithm and Shaw's GY implementations fail, whereas Dewynne and Shaw's expansion, Zhang's PDE and our MLB work notably well agreeing with each other to 7, 8 and 9 decimal places for $q < r$, $q > r$ and $q = r$, respectively. Besides, to produce one MLB requires about 0.03 seconds, which is no more than what it takes with the alternative methods.

In summary, our results confirm the efficiency of our MLB which becomes remarkably sharp at low volatility levels.

5.3. Upper bounds for Asian option prices. In Section 4, we have derived theoretical upper bounds to the error ϵ from approximating the true option prices by the lower bounds (see Theorem 4). For the model cases of Tables 4–6, we report in Table 12 our upper error bounds computed numerically using the fast Fourier transform algorithm for (i) our maximum lower bound and (ii) the suboptimal lower bound; we call these optimized and suboptimal error bounds, respectively. Note that, when added to the lower bounds, these error estimates give us also access to upper bounds to the true option prices.

We observe that in all cases the optimized upper bounds are tighter than the suboptimal ones. It also appears that the error is an increasing function of the strike price, whereas the impact of increasing number of monitoring dates to the error is trivial; this is consistent with the results of Nielsen and Sandmann [58] based on a lognormal underlying asset price and an improvement over the more conservative upper bound of Rogers and Shi [61] which is independent of the strike price level. It is worth noting that the same behaviour across strikes and number of monitoring dates is observed also in the experimental absolute % relative errors reported in Tables 4–6. In addition, the experimental relative differences between the reported MLBs and the benchmark CVMC estimates in Tables 4–6 across all strikes and monitoring dates are no greater than 0.0067 for the Lévy models, 0.0004 for the ASV models and 0.0026 for the CEV diffusion, whereas the corresponding computed theoretical error estimates presented in Table 12 appear higher (in particular, these are smaller than 0.2390, 0.054 and 0.148) in consistency with the theoretical analysis as the latter represent upper bounds to the true error.

6. Conclusions. In this paper, an approximation in the form of an acute lower bound is proposed to price discretely or continuously monitored Asian options, with fixed or floating strike price, in a general model setting with jumps allowed or not in the underlying asset returns and in the evolution of the volatility process. Special attention is given to the CEV diffusion model with distinct distributional properties. An estimate to the error from the lower bound approximation is also obtained. Extensive numerical experiments across a wide range of stochastic dynamic models, monitoring frequencies and option moneyness indicate that our proposed method is easy to implement, fast and accurate compared to other important methods in the literature; and it performs remarkably well even in the case of extremely low volatilities.

We note that our approach can be applied to exponential Lévy-driven mean-reverting models, which, although we have only briefly mentioned in the paper, are nevertheless empirically accepted by several studies as important models for commodity price dynamics. Following a change to the forward measure, it can be also extended to the case of affine models with stochastic interest rates, which become more essential in the case of long-dated contracts. Finally, our approximation can be applied to in-progress options and functions of generalized weighted (discrete or continuous) sums of asset prices.

Appendix A: Proofs.

Proof of Proposition 1. From (1) we have that for any $0 \le a < b$,

$$X_{\Delta b} = X_{\Delta a} + \sum_{j=a+1}^{b} Z_j^{\Delta}, \tag{57}$$

$$\sum_{j=a+1}^{b} X_{\Delta j} = (b-a)X_{\Delta a} + \sum_{j=a+1}^{b} (b+1-j)Z_j^{\Delta} \tag{58}$$

hold. Based on (57)–(58) we are able to write

$$Y_{\Delta N} = \frac{1}{N+1}\sum_{j=0}^{N} X_{\Delta j} = X_0 + \sum_{j=1}^{N}\left(1-\frac{j}{N+1}\right) Z_j^{\Delta}, \tag{59}$$

$$X_{\Delta(k\wedge n)} + X_{\Delta(k\vee n)} = 2X_0 + 2\sum_{j=1}^{k\wedge n} Z_j^{\Delta} + \sum_{j=k\wedge n+1}^{k\vee n} Z_j^{\Delta}. \tag{60}$$

Given that $X_{\Delta k} + X_{\Delta n} \equiv X_{\Delta(k\wedge n)} + X_{\Delta(k\vee n)}$, we get from (59)–(60)

$$\begin{aligned}
&\mathbb{E}[\exp\{iu(X_{\Delta k}+X_{\Delta n})+ivY_{\Delta N}\}] = \mathbb{E}[\exp\{iu(X_{\Delta(k\wedge n)}+X_{\Delta(k\vee n)})+ivY_{\Delta N}\}]\\
&= \exp\{i(2u+v)X_0\}\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{k\wedge n}\left(2u+v\left(1-\frac{j}{N+1}\right)\right)Z_j^{\Delta}\right.\right.\\
&\left.\left.+i\sum_{j=k\wedge n+1}^{k\vee n}\left(u+v\left(1-\frac{j}{N+1}\right)\right)Z_j^{\Delta}+i\sum_{j=k\vee n+1}^{N} v\left(1-\frac{j}{N+1}\right)Z_j^{\Delta}\right\}\right].
\end{aligned}$$

(i) Given (2) and (13), (14) follows by stochastic independence of Z_j^{Δ}.
(ii) From (6) we have

$$\mathbb{E}[\exp\{\vartheta_N V_{\Delta N} + i\eta_N Z_N^{\Delta}\}|\mathcal{F}_{\Delta(N-1)}] = \exp\{\varphi_{\Delta}(\vartheta_N, i\eta_N) + \psi_{\Delta}(\vartheta_N, i\eta_N)V_{\Delta(N-1)}\} \tag{61}$$

and for $j = N-1, \ldots, k\vee n, \ldots, k\wedge n, \ldots, 1$

$$\begin{aligned}
&\mathbb{E}[\exp\{\psi_{\Delta}(\vartheta_{j+1}, i\eta_{j+1})V_{\Delta j} + i\eta_j Z_j^{\Delta}\}|\mathcal{F}_{\Delta(j-1)}]\\
&= \exp\{\varphi_{\Delta}(\psi_{\Delta}(\vartheta_{j+1}, i\eta_{j+1}), i\eta_j) + \psi_{\Delta}(\psi_{\Delta}(\vartheta_{j+1}, i\eta_{j+1}), i\eta_j)V_{\Delta(j-1)}\}\\
&= \exp\{\varphi_{\Delta}(\vartheta_j, i\eta_j) + \psi_{\Delta}(\vartheta_j, i\eta_j)V_{\Delta(j-1)}\},
\end{aligned} \tag{62}$$

where the last equality follows from (15). Then, given (59)–(60) and using iterated expectations applying (61)–(62), we obtain (16).
□

Proof of Proposition 2. Based on (57)–(58) we are able to write

$$Y_{\Delta N} - X_{\Delta N} = \frac{1}{N+1}\sum_{j=0}^{N} X_{\Delta j} - X_{\Delta N} = \frac{-1}{N+1}\sum_{j=1}^{N} jZ_j^{\Delta}, \tag{63}$$

$$X_{\Delta(k\wedge n)} + X_{\Delta(k\vee n)} - 2X_{\Delta N} = -\sum_{j=k\wedge n+1}^{k\vee n} Z_j^{\Delta} - 2\sum_{j=k\vee n+1}^{N} Z_j^{\Delta}. \tag{64}$$

From (63)–(64) we get

$$\begin{aligned}
&\overline{\mathbb{E}}[\exp\{iu(X_{\Delta k}+X_{\Delta n}-2X_{\Delta N})+iv(Y_{\Delta N}-X_{\Delta N})\}]\\
&= \overline{\mathbb{E}}[\exp\{iu(X_{\Delta(k\wedge n)}+X_{\Delta(k\vee n)}-2X_{\Delta N})+iv(Y_{\Delta N}-X_{\Delta N})\}]\\
&= \overline{\mathbb{E}}\left[\exp\left\{-i\sum_{j=1}^{k\wedge n} v\frac{j}{N+1}Z_j^{\Delta}\right.\right.\\
&\left.\left.-i\sum_{j=k\wedge n+1}^{k\vee n}\left(u+v\frac{j}{N+1}\right)Z_j^{\Delta} - i\sum_{j=k\vee n+1}^{N}\left(2u+v\frac{j}{N+1}\right)Z_j^{\Delta}\right\}\right].
\end{aligned}$$

(i)–(ii) Given the arguments above, the proof follows that of parts (i)–(ii) of Proposition 1 using (4) and (7) instead of (2) and (6).
□

Proof of Proposition 3. (i) The proof of (20) follows that of Proposition 2 in Fusai et al. [39] given the expression (9) and using iterated expectations.

(ii) We have

$$\mathbb{E}\left[X_{\Delta k}^{1/(2-\gamma)} e^{-\mu Y_{\Delta N}}\right] = \mathbb{E}\left[X_{\Delta k}^{1/(2-\gamma)} e^{-\mu Y_{\Delta k}} \mathbb{E}\left[e^{-\mu Y_{\Delta(N-k-1)}} \middle| \mathcal{F}_{\Delta k}\right]\right]$$
$$= X_0^{1/(2-\gamma)} \exp\left\{r\Delta k + \sum_{j=k+1}^{N} \varphi_\Delta(0, \vartheta_j(\mu))\right\} \overline{\mathbb{E}}\left[e^{-\mu Y_{\Delta k} - \psi_\Delta(0,\vartheta_{k+1}(\mu)) X_{\Delta k}}\right], \tag{65}$$

where the second equality follows from a change to the measure $\overline{\mathbb{P}}$ and expression (20). Evaluating the expected value (65) iteratively under $\overline{\mathbb{P}}$ yields the expression (21).

□

Proof of Theorem 2. (i)–(ii) Applying Fubini's theorem yields

$$\frac{e^{-rT}}{N+1} \sum_{k=0}^{N} \mathbb{E}\left[(e^{X_{\Delta k}} - K) \int_{-\infty}^{Y_{\Delta N}} e^{i(u-i\delta)\lambda} d\lambda\right]$$
$$= \frac{e^{-rT}}{iu+\delta}\left\{\frac{1}{N+1}\sum_{k=0}^{N} \mathbb{E}[e^{X_{\Delta k} + i(u-i\delta)Y_{\Delta N}}] - K\mathbb{E}[e^{i(u-i\delta)Y_{\Delta N}}]\right\},$$

from which (37) follows;

$$\frac{e^{X_0}}{N+1} \sum_{k=0}^{N} \overline{\mathbb{E}}\left[(e^{X_{\Delta k} - X_{\Delta N}} - \bar{K}) \int_{-\infty}^{Y_{\Delta N} - X_{\Delta N}} e^{i(u-i\delta)\lambda} d\lambda\right]$$
$$= \frac{e^{X_0}}{iu+\delta}\left\{\frac{1}{N+1}\sum_{k=0}^{N} \overline{\mathbb{E}}[e^{X_{\Delta k} - X_{\Delta N} + i(u-i\delta)(Y_{\Delta N} - X_{\Delta N})}] - \bar{K}\overline{\mathbb{E}}[e^{i(u-i\delta)(Y_{\Delta N} - X_{\Delta N})}]\right\},$$

from which (38) follows; for $\gamma \lessgtr 2$

$$\frac{e^{-rT}}{N+1} \sum_{k=0}^{N} \mathbb{E}\left[(X_{\Delta k}^{1/(2-\gamma)} - K) \int_{-\infty}^{\infty} e^{-\mu\lambda + \delta\lambda} \mathbf{1}_{\{Y_{\Delta N} \gtrless \lambda\}} d\lambda\right]$$
$$= \frac{\mathrm{sgn}(\delta) e^{-rT}}{-\mu+\delta}\left\{\frac{1}{N+1}\sum_{k=0}^{N} \mathbb{E}\left[X_{\Delta k}^{1/(2-\gamma)} e^{-(\mu-\delta)Y_{\Delta N}}\right] - K\mathbb{E}[e^{-(\mu-\delta)Y_{\Delta N}}]\right\}$$

and

$$\frac{X_0^{1/(2-\gamma)}}{N+1} \sum_{k=0}^{N} \overline{\mathbb{E}}\left[((X_{\Delta k} X_{\Delta N}^{-1})^{1/(2-\gamma)} - \bar{K}) \int_{-\infty}^{\infty} e^{-\mu\lambda + \delta\lambda} \mathbf{1}_{\{Y_{\Delta N} X_{\Delta N}^{-1} \gtrless \lambda\}} d\lambda\right]$$
$$= \frac{\mathrm{sgn}(\delta) X_0^{1/(2-\gamma)}}{-\mu+\delta}\left\{\frac{1}{N+1}\sum_{k=0}^{N} \overline{\mathbb{E}}\left[(X_{\Delta k} X_{\Delta N}^{-1})^{1/(2-\gamma)} e^{-(\mu-\delta)Y_{\Delta N} X_{\Delta N}^{-1}}\right] - \bar{K}\overline{\mathbb{E}}[e^{-(\mu-\delta)Y_{\Delta N} X_{\Delta N}^{-1}}]\right\},$$

from which the proof of parts (i)–(ii) is completed.

(iii) Application of standard inversion formula, see [45, Theorem 5C].

□

Proof of Theorem 4. For brevity we restrict our attention to the case of the fixed strike Asian call option on the discrete average, as the extensions to the continuous average and the floating strike option are straightforward. From Jensen's inequality

$$0 \le \epsilon_{\text{fix}} \equiv e^{-rT}\mathbb{E}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - K\right)^+\right] - e^{-rT}\mathbb{E}\left\{\left[\mathbb{E}\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} \middle| Y_{\Delta N}\right) - K\right]^+\right\}.$$

From (35)

$$\mathbb{E}\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} \middle| Y_{\Delta N} = y\right) - K \tag{66}$$

is equal to zero when $y = \lambda^*$. In addition, assuming that the function (66) is strictly increasing in y (this assumption is numerically verified in all our experiments; more results can be made available upon request) implies that it is positive for $y > \lambda^*$. Hence,

$$\epsilon_{\text{fix}} = e^{-rT}\mathbb{E}\left[\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - K\right)^+\right] - e^{-rT}\mathbb{E}\left\{\left[\mathbb{E}\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1}\middle| Y_{\Delta N}\right) - K\right]\mathbf{1}_{\{Y_{\Delta N}>\lambda^*\}}\right\}$$

coincides with the error from approximating the true option price by the lower bound. As noted in Rogers and Shi [61] and Nielsen and Sandmann [58],

$$\epsilon_{\text{fix}} \leq \frac{e^{-rT}}{2}\mathbb{E}\left[\text{Var}\left(\frac{\sum_{k=0}^{N} S_{\Delta k}}{N+1} - K\middle| Y_{\Delta N}\right)^{\frac{1}{2}}\mathbf{1}_{\{Y_{\Delta N}\leq\lambda^*\}}\right], \tag{67}$$

from which inequality (51) follows by definition of the variance. □

Appendix B: Monte Carlo simulation methods. In this section, we provide more details about the simulation methodologies we consider for the purposes of our numerical study. In particular, the VG and NIG model trajectories are simulated exactly by exploiting their representations as subordinated arithmetic Brownian motions (see [43], [23]); for the CGMY model we use the joint Monte Carlo-Fourier transform sampling scheme of Ballotta and Kyriakou [4]; for the MJD and DEJD models the improved algorithm in Cont and Tankov [23, Section 6.1]; for the Meixner model the exact acceptance-rejection sampling method of Devroye [30].

Stochastic volatility models are proved harder to simulate accurately and various approaches have been proposed to this end (e.g., see discussion in [68]). In particular, for the Heston model we implement the quadratic-exponential method of Andersen [2] to simulate the square root variance diffusion process with central discretization of the integrated variance, which is the most suitable for a large number of averaging points (e.g., $N = 12, 50, 250$) and also easily extensible to the models of Bates [8] and Duffie et al. [33] (see [13, Section 6]). In addition, the variance process in the BNS-Γ model is given by an OU process driven by a compound Poisson subordinator with exponential jump size distribution which can be simulated exactly; to simulate the variance process in the BNS-IG model we use an approximate series representation (see [62]); we approximate the integrated variance in either BNS model using central discretization.

Finally, we refer to Glasserman [43, Section 3.4.4] for the simulation of the CEV process.

Acknowledgments. We would like to thank Ning Cai, Daniele Marazzina, Olaf Menkens, Debora Sesana, Sai Hung Marten Ting and Hao Xing for kindly sharing their code material with us for the purposes of our numerical experiments. We also thank Marco Santoli for his contribution to an earlier version of the paper. Finally, we thank Steven Kou, Dilip Madan, Pietro Millossovich and other participants of the 8th Conference in Actuarial Science & Finance on Samos for useful feedback. Usual caveat applies.

References

[1] Abramowitz, M., I. A. Stegun. 1968. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover Books on Mathematics, Dover, New York.

[2] Andersen, L. 2008. Simple and efficient simulation of the Heston stochastic volatility model. *Journal of Computational Finance* **11**(3) 1–42.

[3] Andreasen, J. 1998. The pricing of discretely sampled Asian and lookback options: a change of numéraire approach. *Journal of Computational Finance* **2**(1) 5–30.

[4] Ballotta, L., I. Kyriakou. 2014. Monte Carlo simulation of the CGMY process and option pricing. *Journal of Futures Markets* **34**(12) 1095–1121.

[5] Barndorff-Nielsen, O. E., E. Nicolato, N. Shephard. 2002. Some recent developments in stochastic volatility modelling. *Quantitative Finance* **2**(1) 11–23.

[6] Barndorff-Nielsen, O. E., N. Shephard. 2001. Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **63**(2) 167–241.

[7] Bartlett, M. S. 1938. The characteristic function of a conditional statistic. *Journal of the London Mathematical Society* **13**(1) 62–67.

[8] Bates, D. S. 1996. Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options. *Review of Financial Studies* **9**(1) 69–107.

[9] Bayraktar, E., H. Xing. 2011. Pricing Asian options for jump diffusion. *Mathematical Finance* **21**(1) 117–143.

[10] Benhamou, E. 2002. Fast Fourier transform for discrete Asian options. *Journal of Computational Finance* **6**(1) 49–68.

[11] Białkowski, J., S. Darolles, G. Le Fol. 2008. Improving VWAP strategies: A dynamic volume approach. *Journal of Banking and Finance* **32**(9) 1709–1722.

[12] Broadie, M., Ö. Kaya. 2004. Exact simulation of option Greeks under stochastic volatility and jump diffusion models. R. G. Ingalls, M. D. Rossetti, J. S. Smith, B. A. Peters, eds., *Proceedings of the 2004 Winter Simulation Conference*. 1607–1615.

[13] Broadie, M., Ö. Kaya. 2006. Exact simulation of stochastic volatility and other affine jump diffusion processes. *Operations Research* **54**(2) 217–231.

[14] Cai, N., S. Kou. 2012. Pricing Asian options under a hyper-exponential jump diffusion model. *Operations Research* **60**(1) 64–77.

[15] Cai, N., C. Li, C. Shi. 2014. Closed-form expansions of discretely monitored Asian options in diffusion models. *Mathematics of Operations Research* **39**(3) 789–822.

[16] Cai, N., Y. Song, S. Kou. forthcoming. A general framework for pricing Asian options under Markov processes. *Operations Research* **0**(0) null. doi:10.1287/opre.2015.1385. URL http://dx.doi.org/10.1287/opre.2015.1385.

[17] Caldana, R., G. Fusai. 2013. A general closed-form spread option pricing formula. *Journal of Banking and Finance* **37**(12) 4893–4906.

[18] Carr, P., H. Geman, D. B. Madan, M. Yor. 2003. Stochastic volatility for Lévy processes. *Mathematical Finance* **13**(3) 345–382.

[19] Carr, P., L. Wu. 2004. Time-changed Lévy processes and option pricing. *Journal of Financial Economics* **71**(1) 113–141.

[20] Carverhill, A., L. Clewlow. 1990. Flexible convolution. *Risk* **3**(4) 25–29.

[21] Černý, A., I. Kyriakou. 2011. An improved convolution algorithm for discretely sampled Asian options. *Quantitative Finance* **11**(3) 381–389.

[22] Choudhury, G. L., D. M. Lucantoni, W. Whitt. 1994. Multidimensional transform inversion with applications to the transient M/G/1 queue. *The Annals of Applied Probability* **4**(3) 719–740.

[23] Cont, R., P. Tankov. 2004. *Financial Modelling With Jump Processes*. Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton.

[24] Cox, J. C. 1975. Notes on option pricing I: Constant elasticity of variance diffusions. Working paper, Stanford University (reprinted in Journal of Portfolio Management, 1996, 23(1), 15–17).

[25] Cox, J. C., S. A. Ross. 1976. The valuation of options for alternative stochastic processes. *Journal of Financial Economics* **3**(1–2) 145–166.

[26] Cressie, N., M. Borkent. 1986. The moment generating function has its moments. *Journal of Statistical Planning and Inference* **13** 337–344.

[27] Curran, M. 1992. Beyond average intelligence. *Risk* **5**(10) 60.

[28] Curran, M. 1994. Valuing Asian and portfolio options by conditioning on the geometric mean price. *Management Science* **40**(12) 1705–1711.

[29] Dassios, A., J. Nagaradjasarma. 2006. The square-root process and Asian options. *Quantitative Finance* **6**(4) 337–347.

[30] Devroye, L. 1981. On the computer generation of random variables with a given characteristic function. *Computers and Mathematics with Applications* **7**(6) 547–552.

[31] Dewynne, J. N., W. T. Shaw. 2008. Differential equations and asymptotic solutions for arithmetic Asian options: "Black–Scholes formulae" for Asian rate calls. *European Journal of Applied Mathematics* **19**(4) 353–391.

[32] Driouchi, T., D. Bennett, G. Battisti. 2006. Capacity planning under uncertainty: an Asian option approach. *10th Annual Internat. Conf. Real Options: Theory Meets Practice*.

[33] Duffie, D., J. Pan, K. Singleton. 2000. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica* **68**(6) 1343–1376.

[34] Eberlein, E., A. Papapantoleon. 2005. Equivalence of floating and fixed strike Asian and lookback options. *Stochastic Processes and their Applications* **115**(1) 31–40.

[35] Emanuel, D. C., J. D. MacBeth. 1982. Further results on the constant elasticity of variance call option pricing model. *Journal of Financial and Quantitative Analysis* **17**(4) 533–554.

[36] Ewald, C.-O., O. Menkens, S. H. M. Ting. 2013. Asian and Australian options: A common perspective. *Journal of Economic Dynamics and Control* **37**(5) 1001–1018.

[37] Fu, M. C., D. B. Madan, T. Wang. 1999. Pricing continuous Asian options: a comparison of Monte Carlo and Laplace transform inversion methods. *Journal of Computational Finance* **2**(2) 49–74.

[38] Fusai, G. 2004. Pricing Asian options via Fourier and Laplace transforms. *Journal of Computational Finance* **7**(3) 87–106.

[39] Fusai, G., M. Marena, A. Roncoroni. 2008. Analytical pricing of discretely monitored Asian-style options: Theory and application to commodity markets. *Journal of Banking and Finance* **32**(10) 2033–2045.

[40] Fusai, G., A. Meucci. 2008. Pricing discretely monitored Asian options under Lévy processes. *Journal of Banking and Finance* **32**(10) 2076–2088.

[41] Geman, H., N. El Karoui, J.-C. Rochet. 1995. Changes of numéraire, changes of probability measure and option pricing. *Journal of Applied Probability* **32**(2) 443–458.

[42] Geman, H., M. Yor. 1993. Bessel processes, Asian options, and perpetuities. *Mathematical Finance* **3**(4) 349–375.

[43] Glasserman, P. 2004. *Monte Carlo Methods in Financial Engineering*. Stochastic Modelling and Applied Probability, Springer–Verlag, New York.

[44] Glasserman, P., Z. Liu. 2010. Sensitivity estimates from characteristic functions. *Operations Research* **58**(6) 1611–1623.

[45] Goldberg, R. R. 1961. *Fourier Transforms*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 52, Cambridge University Press, Cambridge.

[46] Heston, S. L. 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* **6**(2) 327–343.

[47] Hubalek, F., M. Keller-Ressel, C. Sgarra. 2014. Geometric Asian option pricing in general affine stochastic volatility models with jumps. Working paper.

[48] Kallsen, J. 2006. A didactic note on affine stochastic volatility models. Y. Kabanov, R. Liptser, J. Stoyanov, eds., *From Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift*. Springer Berlin Heidelberg, 343–368.

[49] Kallsen, J., A. Pauwels. 2010. Variance-optimal hedging in general affine stochastic volatility models. *Advances in Applied Probability* **42**(1) 83–105.

[50] Kim, K. 2007. *Electronic and Algorithmic Trading Technology: The Complete Guide*. Complete Technology Guides for Financial Services Series, Elsevier.

[51] Lamberton, D., B. Lapeyre. 2008. *Introduction to Stochastic Calculus Applied to Finance*. Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton.

[52] Lemmens, D., L.Z.J. Liang, J. Tempere, A. De Schepper. 2010. Pricing bounds for discrete arithmetic Asian options under Lévy models. *Physica A: Statistical Mechanics and its Applications* **389**(22) 5193–5207.

[53] Linetsky, V. 2004. Spectral expansions for Asian (average price) options. *Operations Research* **52**(6) 856–867.

[54] Lord, R. 2006. Partially exact and bounded approximations for arithmetic Asian options. *Journal of Computational Finance* **10**(2) 1–52.

[55] Marena, M., A. Roncoroni, G. Fusai. 2013. Asian options with jumps. *Argo – New Frontiers in Practical Risk Management* **1** 48–55.

[56] Mercuri, L. 2011. Pricing Asian options in affine Garch models. *International Journal of Theoretical and Applied Finance* **14**(2) 313–333.

[57] Nicolato, E., E. Venardos. 2003. Option pricing in stochastic volatility models of the Ornstein–Uhlenbeck type. *Mathematical Finance* **13**(4) 445–466.

[58] Nielsen, J. Aase, K. Sandmann. 2003. Pricing bounds on Asian options. *Journal of Financial and Quantitative Analysis* **38**(2) 449–473.

[59] Novikov, A.A., N.E. Kordzakhia. 2014. Lower and upper bounds for prices of Asian-type options. *Proceedings of the Steklov Institute of Mathematics* **287**(1) 225–231.

[60] Officer, M. S. 2006. The market pricing of implicit options in merger collars. *The Journal of Business* **79**(1) 115–136.

[61] Rogers, L. C. G., Z. Shi. 1995. The value of an Asian option. *Journal of Applied Probability* **32**(4) 1077–1088.

[62] Schoutens, W. 2003. *Lévy Processes in Finance: Pricing Financial Derivatives*. Wiley Series in Probability and Statistics, Wiley, Chichester, West Sussex.

[63] Schroder, M. 1989. Computing the constant elasticity of variance option pricing formula. *Journal of Finance* **44**(1) 211–219.

[64] Sesana, D., D. Marazzina, G. Fusai. 2014. Pricing exotic derivatives exploiting structure. *European Journal of Operational Research* **236**(1) 369–381.

[65] Shaw, W. T. 1998. *Modelling Financial Derivatives with Mathematica*. Cambridge University Press, Cambridge.

[66] Takayama, A. 1974. *Mathematical Economics*. Cambridge University Press, Cambridge.

[67] Thompson, G. W. P. 1999. Fast narrow bounds on the value of Asian options. Working paper, Cambridge University.

[68] Tse, S. T., J. W. L. Wan. 2013. Low-bias simulation scheme for the Heston model by Inverse Gaussian approximation. *Quantitative Finance* **13**(6) 919–937.

[69] Večeř, J. 2002. Unified Asian pricing. *Risk* **15**(6) 113–116.

[70] Večeř, J., M. Xu. 2004. Pricing Asian options in a semimartingale model. *Quantitative Finance* **4**(2) 170–175.

[71] Yamazaki, A. 2014. Pricing average options under time-changed Lévy processes. *Review of Derivatives Research* **17**(1) 79–111.

[72] Zahra, S., R. Reza. 2012. Asian real option: New approach to project economic valuation. *2012 Internat. Conf. on Information Management, Innovation Management and Industrial Engineering (ICIII)*, vol. 2. 475–479.

[73] Zhang, B., C. W. Oosterlee. 2013. Efficient pricing of European-style Asian options under exponential Lévy processes based on Fourier cosine expansions. *SIAM Journal on Financial Mathematics* **4**(1) 399–426.

[74] Zhang, J. E. 2001. A semi-analytical method for pricing and hedging continuously sampled arithmetic average rate options. *Journal of Computational Finance* **5**(1) 1–20.

[75] Zhu, Y., G. Zhou. 2009. Technical analysis: An asset allocation perspective on the use of moving averages. *Journal of Financial Economics* **92**(3) 519–544.

Table 1. Characteristic exponents.

Model	$\varphi_\Delta(u,\upsilon)$	$\psi_\Delta(u,\upsilon)$
Gaussian	$-\frac{1}{2}\sigma^2u^2\Delta$	-
VG	$-\frac{\Delta}{\nu}\ln(1-\theta\nu iu+\nu\sigma^2u^2/2)$	-
NIG	$-\delta\Delta(\sqrt{a^2-(b+iu)^2}-\sqrt{a^2-b^2})$	-
CGMY	$C\Delta\Gamma(-Y)((M-iu)^Y-M^Y+(G+iu)^Y-G^Y)$	-
MJD	$l\Delta(\exp(iu\mu_x-\frac{1}{2}u^2\sigma_x^2)-1)-\frac{1}{2}\sigma^2u^2\Delta$	-
DEJD (Kou)	$l\Delta\left(\frac{p\eta_1}{\eta_1-iu}+\frac{(1-p)\eta_2}{\eta_2+iu}-1\right)-\frac{1}{2}\sigma^2u^2\Delta$	-
Meixner	$2\delta\Delta\ln\left(\frac{\cos(b/2)}{\cosh((au-ib)/2)}\right)$	-
Heston	$\upsilon\left(r-q-\frac{\rho\alpha\beta}{\gamma}\right)\Delta$ $+\frac{\alpha\beta}{\gamma^2}\left[(\alpha-\omega_2(\upsilon))\Delta-2\ln\left(\frac{\omega_1(u,\upsilon)e^{-\omega_2(\upsilon)\Delta}-1}{\omega_1(u,\upsilon)-1}\right)\right]$	$\frac{\alpha-\upsilon\rho\gamma-\omega_2(\upsilon)-\omega_1(u,\upsilon)e^{-\omega_2(\upsilon)\Delta}(\alpha-\upsilon\rho\gamma+\omega_2(\upsilon))}{(1-\omega_1(u,\upsilon)e^{-\omega_2(\upsilon)\Delta})\gamma^2}$
Bates	$\varphi_\Delta^{\text{Hes}}(u,\upsilon)+l(k(\upsilon)-k(1)\upsilon)\Delta$	$\psi_\Delta^{\text{Hes}}(u,\upsilon)$
DPS	$\upsilon\left(r-q-\frac{\rho\alpha\beta}{\gamma}-lk(1,0)\right)\Delta$ $+\frac{\alpha\beta}{\gamma^2}\left[\alpha\Delta+\ln\left(\frac{\omega_2^2(\upsilon)(1+\omega_{1,\Delta}^2(u,\upsilon))}{\gamma^2(2(\upsilon\rho\gamma-\alpha)u+\gamma^2u^2+\upsilon(\upsilon-1))}\right)\right]$ $+l\int_0^\Delta k(\upsilon,\psi_s(u,\upsilon))ds$	$\frac{1}{\gamma^2}(\omega_{1,\Delta}(u,\upsilon)\omega_2(\upsilon)+\alpha-\upsilon\rho\gamma)$
BNS	$(r-q-lk(\rho))\Delta\upsilon+l\int_0^\Delta k(\psi_s(u,\upsilon)+\rho\upsilon)ds$	$e^{-l\Delta}u+\frac{1-e^{-l\Delta}}{2l}(\upsilon-1)\upsilon$

Notes. Heston/Bates: $\omega_1(u,\upsilon)\equiv\frac{\alpha-\upsilon\rho\gamma-\omega_2(\upsilon)-u\gamma^2}{\alpha-\upsilon\rho\gamma+\omega_2(\upsilon)-u\gamma^2}$, $\omega_2(\upsilon)\equiv\sqrt{(\alpha-\upsilon\rho\gamma)^2+(1-\upsilon)\upsilon\gamma^2}$; DPS: $\omega_{1,\Delta}(u,\upsilon)\equiv\tan\left(\frac{\omega_2(\upsilon)\Delta}{2}+\arctan\left(\frac{\upsilon\rho\gamma-\alpha+u\gamma^2}{\omega_2(\upsilon)}\right)\right)$, $\omega_2(\upsilon)\equiv\sqrt{(\upsilon-1)\upsilon\gamma^2-(\alpha-\upsilon\rho\gamma)^2}$.

Table 2. Integrated characteristic exponents of some Lévy models for use in expression (23).

Model	$I(s,c,u,\upsilon)$
Gaussian	$i\varepsilon(cu+\upsilon-\frac{s\upsilon}{2T})s-\frac{\sigma^2}{2}(cu+\upsilon)(cu+\upsilon-\frac{s\upsilon}{T})s-\frac{s^3\sigma^2\upsilon^2}{6T^2}$
VG	$i\varepsilon(cu+\upsilon-\frac{s\upsilon}{2T})s+\frac{T}{\nu\upsilon}(cu+\upsilon-\frac{s\upsilon}{T})\left[\ln\left(\frac{\nu\sigma^2}{2}(cu+\upsilon-\frac{s\upsilon}{T})^2-i\nu\theta(cu+\upsilon-\frac{s\upsilon}{T})+1\right)-2\right]$ $-\frac{iT\theta}{2\nu\sigma^2\upsilon}\ln\left(\nu^2\sigma^4(cu+\upsilon-\frac{s\upsilon}{T})^4+4\nu(\nu\theta^2+\sigma^2)(cu+\upsilon-\frac{s\upsilon}{T})^2+4\right)$ $+\frac{T}{\nu\sigma^2\upsilon}\left[2\sqrt{\theta^2+\frac{2\sigma^2}{\nu}}\tan^{-1}\left(\frac{\sigma^2(cu+\upsilon-\frac{s\upsilon}{T})-i\theta}{\sqrt{\theta^2+\frac{2\sigma^2}{\nu}}}\right)-\theta\tan^{-1}\left(\frac{2\nu\theta(cu+\upsilon-\frac{s\upsilon}{T})}{\nu\sigma^2(cu+\upsilon-\frac{s\upsilon}{T})^2+2}\right)\right]$
NIG	$i\varepsilon(cu+\upsilon-\frac{s\upsilon}{2T})s+\frac{i\delta T}{\upsilon}\left[\sqrt{a^2-b^2}-\frac{1}{2}\sqrt{a^2-(b+i(cu+\upsilon-\frac{s\upsilon}{T}))^2}\right](b+i(cu+\upsilon-\frac{s\upsilon}{T}))$ $-\frac{ia^2\delta T}{2\upsilon}\tan^{-1}\left[\frac{b+i(cu+\upsilon-\frac{s\upsilon}{T})}{\sqrt{a^2-(b+i(cu+\upsilon-\frac{s\upsilon}{T}))^2}}\right]$
CGMY	$i\varepsilon(cu+\upsilon-\frac{s\upsilon}{2T})s-C\Gamma(-Y)(G^Y+M^Y)s$ $+\frac{iT}{\upsilon(Y+1)}C\Gamma(-Y)\left[(G+i(cu+\upsilon-\frac{s\upsilon}{T}))^{Y+1}-(M-i(cu+\upsilon-\frac{s\upsilon}{T}))^{Y+1}\right]$
MJD	$i\varepsilon(cu+\upsilon-\frac{s\upsilon}{2T})s+\frac{\sigma^2T}{6\upsilon}(cu+\upsilon-\frac{s\upsilon}{T})^3-ls+\frac{i\sqrt{2\pi}lT}{2\sigma_x\upsilon}\exp\left(-\frac{\mu_x^2}{2\sigma_x^2}\right)\text{erfi}\left[\frac{\sqrt{2}}{2}\left(\frac{\mu_x}{\sigma_x}+i\sigma_x(cu+\upsilon-\frac{s\upsilon}{T})\right)\right]$
DEJD (Kou)	$i\varepsilon(cu+\upsilon-\frac{s\upsilon}{2T})s+\frac{\sigma^2T}{6\upsilon}(cu+\upsilon-\frac{s\upsilon}{T})^3-ls+\frac{lTp\eta_1}{\upsilon}\left[\tan^{-1}\left(\frac{\eta_1}{cu+\upsilon-\frac{s\upsilon}{T}}\right)-\frac{i}{2}\ln((cu+\upsilon-\frac{s\upsilon}{T})^2T^2+\eta_1^2T^2)\right]$ $+\frac{lT(1-p)\eta_2}{\upsilon}\left[\tan^{-1}\left(\frac{\eta_2}{cu+\upsilon-\frac{s\upsilon}{T}}\right)+\frac{i}{2}\ln((cu+\upsilon-\frac{s\upsilon}{T})^2T^2+\eta_2^2T^2)\right]$
Meixner	$i\varepsilon(cu+\upsilon-\frac{s\upsilon}{2T})s+\frac{\delta T}{2a\upsilon}(b+ia(cu+\upsilon-\frac{s\upsilon}{T}))^2$ $+\frac{2\delta T}{a\upsilon}\text{Li}_2\left(-e^{-a(cu+\upsilon-\frac{s\upsilon}{T})+ib}\right)-\frac{2\delta T}{a\upsilon}(a(cu+\upsilon-\frac{s\upsilon}{T})-ib)\ln\left[\frac{\cos\left(\frac{b}{2}\right)\left(1+e^{-a(cu+\upsilon-\frac{s\upsilon}{T})+ib}\right)}{\cos\left(\frac{1}{2}(b+ia(cu+\upsilon-\frac{s\upsilon}{T}))\right)}\right]$

Notes. $\psi(u)=iu\varepsilon+\varphi(u)$, where φ is the characteristic exponent of the demeaned Lévy process (see Table 1) for $\Delta=1$ and ε is the drift (see Eq. 3). Then, from (23), $\int_0^T\psi(u(\mathbf{1}_{[0,t\wedge z]}(s)+\mathbf{1}_{[0,t\vee z]}(s))+\upsilon(1-\frac{s}{T}))ds=\int_0^{t\wedge z}\psi(2u+\upsilon(1-\frac{s}{T}))ds+\int_{t\wedge z}^{t\vee z}\psi(u+\upsilon(1-\frac{s}{T}))ds+\int_{t\vee z}^T\psi(\upsilon(1-\frac{s}{T}))ds=I(t\wedge z,2,u,\upsilon)-I(0,2,u,\upsilon)+I(t\vee z,1,u,\upsilon)-I(t\wedge z,1,u,\upsilon)+I(T,0,u,\upsilon)-I(t\vee z,0,u,\upsilon)$. MJD: $\text{erfi}(z)\equiv-i\text{erf}(iz)$ is the imaginary error function; Meixner: $\text{Li}_2(z)$ is the polylogarithm of order 2 and argument z.

Table 3. Model parameter sets.

Model	Parameters				
Gaussian	σ				
	0.17801				
VG	ν	θ	σ		
	0.736703	−0.136105	0.180022		
NIG	a	b	δ		
	6.1882	−3.8941	0.1622		
CGMY	C	G	M	Y	
	0.0244	0.0765	7.5515	1.2945	
MJD	σ	l	μ_x	σ_x	
	0.126349	0.174814	−0.390078	0.338796	
DEJD (Kou)	σ	l	p	η_1	η_2
	0.120381	0.330966	0.20761	9.65997	3.13868
Meixner	a	b	δ		
	0.3977	−1.494	0.3462		
Heston	α	β	γ	ρ	$\sqrt{V_0}$
	6.21	0.019	0.61	−0.7	0.101
Bates	α	β	γ	ρ	$\sqrt{V_0}$
	3.99	0.014	0.27	−0.79	0.094
	l	μ_x	σ_x		
	0.11	−0.1391	0.15		
DPS	α	β	γ	ρ	$\sqrt{V_0}$
	3.46	0.008	0.14	−0.82	0.087
	l	μ_x	σ_x	μ_v	$\rho_{x,v}$
	0.47	−0.0865	0.0001	0.05	−0.38
BNS-Γ	l	ν	α	ρ	$\sqrt{V_0}$
	1.6787	1.0071	116.01	−4.4617	0.065883
BNS-IG	l	ν	α	ρ	$\sqrt{V_0}$
	2.4958	0.0872	11.98	−4.7039	0.064262
CEV	γ	σ			
	1.5, 2.5	$0.25S_0^{1-\gamma/2}$			

Table 4. Prices of arithmetic Asian options with discrete monitoring under Lévy models.

		Lower bound					CVMC prices			
N	K	MLB	$\exp(\lambda^*)$	Abs. rel err. (%)	SLB	Abs. rel err. (%)	CV MLB	Std. err. $\times 10^{-5}$	CV SLB	Std. err. $\times 10^{-5}$
Gaussian model										
12	90	11.90462	89.74	0.0025	11.90386	0.0088	11.90491	0.848	11.90493	1.861
12	100	4.88168	99.84	0.0059	4.88121	0.0155	4.88197	0.790	4.88197	1.372
12	110	1.36255	109.70	0.0346	1.36143	0.1165	1.36302	1.217	1.36299	2.479
50	90	11.93265	89.75	0.0025	11.93199	0.0080	11.93294	0.853	11.93294	1.726
50	100	4.93693	99.84	0.0054	4.93646	0.0149	4.93720	0.734	4.93722	1.359
50	110	1.40204	109.72	0.0358	1.40105	0.1061	1.40254	1.282	1.40248	2.339
250	90	11.94027	89.76	0.0023	11.93964	0.0076	11.94054	0.809	11.94056	1.704
250	100	4.95189	99.84	0.0053	4.95142	0.0148	4.95215	0.735	4.95215	1.336
250	110	1.41289	109.72	0.0336	1.41194	0.1009	1.41337	1.222	1.41337	2.356
AAPRE				0.014		0.044				
VG model										
12	90	12.52729	89.42	0.0162	12.52572	0.0287	12.52932	4.806	12.52959	6.916
12	100	5.09210	99.87	0.0197	5.09180	0.0255	5.09310	2.898	5.09321	3.300
12	110	1.00625	109.83	0.0598	1.00569	0.1151	1.00685	1.986	1.00729	2.679
50	90	12.56931	89.44	0.0162	12.56791	0.0274	12.57135	4.783	12.57159	6.770
50	100	5.13879	99.88	0.0183	5.13850	0.0239	5.13973	2.749	5.13983	3.120
50	110	1.02897	109.85	0.0612	1.02855	0.1020	1.02960	1.958	1.02999	2.553
250	90	12.58064	89.45	0.0162	12.57929	0.0270	12.58269	4.879	12.58293	6.750
250	100	5.15158	99.88	0.0183	5.15130	0.0237	5.15252	2.826	5.15260	3.168
250	110	1.03535	109.87	0.0598	1.03497	0.0971	1.03597	1.966	1.03633	2.507
AAPRE				0.032		0.052				
NIG model										
12	90	12.61912	89.42	0.0112	12.61729	0.0257	12.62053	6.593	12.61932	8.607
12	100	5.05926	99.85	0.0136	5.05889	0.0208	5.05994	3.225	5.05964	3.566
12	110	1.01328	109.77	0.0296	1.01231	0.1248	1.01358	1.668	1.01298	2.174
50	90	12.65800	89.44	0.0075	12.65633	0.0208	12.65896	6.590	12.65771	8.284
50	100	5.10239	99.85	0.0107	5.10203	0.0178	5.10294	2.815	5.10261	3.086
50	110	1.03743	109.79	0.0291	1.03661	0.1079	1.03773	1.730	1.03720	2.093
250	90	12.66852	89.45	0.0081	12.66688	0.0210	12.66954	6.815	12.66873	8.333
250	100	5.11425	99.85	0.0107	5.11389	0.0176	5.11480	3.219	5.11446	3.468
250	110	1.04421	109.79	0.0271	1.04343	0.1017	1.04449	1.592	1.04405	2.123
AAPRE				0.016		0.051				
CGMY model										
12	90	12.70022	89.45	0.0516	12.69871	0.0635	12.70678	62.492	12.70675	65.195
12	100	5.03301	99.88	0.0344	5.03273	0.0402	5.03475	20.262	5.03470	20.589
12	110	1.02053	109.76	0.0617	1.01954	0.1587	1.02116	7.138	1.02116	9.467
50	90	12.73374	89.47	0.0527	12.73237	0.0635	12.74046	63.739	12.74031	65.902
50	100	5.07405	99.88	0.0480	5.07376	0.0537	5.07649	33.347	5.07644	33.527
50	110	1.04613	109.77	0.0754	1.04529	0.1561	1.04692	12.203	1.04693	13.531
250	90	12.74288	89.49	0.0518	12.74155	0.0623	12.74949	56.644	12.74995	61.557
250	100	5.08542	99.88	0.0377	5.08512	0.0435	5.08734	28.555	5.08727	28.968
250	110	1.05332	109.79	0.0664	1.05251	0.1431	1.05402	11.692	1.05400	12.927
AAPRE				0.053		0.087				
MJD model										
12	90	12.70606	89.37	0.0363	12.70440	0.0493	12.71067	9.542	12.71070	11.930
12	100	5.00959	99.88	0.0345	5.00929	0.0405	5.01132	5.364	5.01133	5.709
12	110	1.05101	109.76	0.0589	1.05003	0.1518	1.05163	2.366	1.05160	3.110
50	90	12.73639	89.42	0.0343	12.73493	0.0457	12.74076	9.370	12.74081	11.562
50	100	5.05080	99.88	0.0324	5.05050	0.0385	5.05244	5.302	5.05242	5.575
50	110	1.07898	109.77	0.0557	1.07814	0.1331	1.07958	2.201	1.07960	2.978
250	90	12.74465	89.43	0.0360	12.74324	0.0471	12.74924	9.719	12.74916	11.606
250	100	5.06218	99.88	0.0329	5.06187	0.0389	5.06384	5.219	5.06385	5.519
250	110	1.08679	109.77	0.0553	1.08599	0.1288	1.08739	2.406	1.08739	3.099
AAPRE				0.042		0.075				

Table 4. Continued.

N	K	Lower bound					CVMC prices			
		MLB	$\exp(\lambda^*)$	Abs. rel err. (%)	SLB	Abs. rel err. (%)	CV MLB	Std. err. $\times 10^{-5}$	CV SLB	Std. err. $\times 10^{-5}$
DEJD model										
12	90	12.70750	89.38	0.0387	12.70583	0.0519	12.71242	11.664	12.71244	13.708
12	100	5.01540	99.88	0.0369	5.01510	0.0429	5.01725	6.436	5.01726	6.689
12	110	1.04083	109.76	0.0558	1.03986	0.1495	1.04141	2.281	1.04144	3.109
50	90	12.73911	89.41	0.0403	12.73762	0.0520	12.74424	11.979	12.74421	13.667
50	100	5.05648	99.88	0.0335	5.05618	0.0395	5.05818	5.891	5.05820	6.208
50	110	1.06821	109.77	0.0572	1.06738	0.1352	1.06883	2.615	1.06883	3.201
250	90	12.74770	89.42	0.0389	12.74626	0.0503	12.75267	11.472	12.75256	12.971
250	100	5.06782	99.88	0.0353	5.06752	0.0412	5.06961	6.524	5.06962	6.856
250	110	1.07587	109.79	0.0554	1.07507	0.1295	1.07647	2.542	1.07649	3.167
AAPRE				0.044		0.077				
Meixner model										
12	90	12.59519	89.41	0.0090	12.59325	0.0243	12.59632	5.688	12.59484	7.743
12	100	5.06225	99.85	0.0124	5.06186	0.0202	5.06288	2.554	5.06253	2.805
12	110	1.01599	109.76	0.0338	1.01497	0.1347	1.01634	1.586	1.01569	2.153
50	90	12.63551	89.43	0.0023	12.63373	0.0164	12.63580	5.491	12.63376	7.411
50	100	5.10569	99.85	0.0128	5.10530	0.0205	5.10635	2.962	5.10600	3.272
50	110	1.03957	109.79	0.0301	1.03870	0.1138	1.03988	1.513	1.03934	2.040
250	90	12.64640	89.43	0.0070	12.64466	0.0208	12.64728	5.943	12.64648	7.659
250	100	5.11763	99.85	0.0127	5.11724	0.0203	5.11828	2.811	5.11791	3.046
250	110	1.04619	109.79	0.0290	1.04536	0.1085	1.04650	1.697	1.04598	2.139
AAPRE				0.017		0.053				

Notes. Maximum LB (MLB) are optimized lower bounds to the option prices, λ^* are the corresponding optimal bound parameters; suboptimal LB (SLB) are lower bounds to the option prices obtained for $\lambda = \ln K$; CVMC prices are Monte Carlo simulation estimates based on 1,000,000 paths simulations with (i) MLB or (ii) SLB used as control variate, std. err. are the standard errors of the CVMC price estimates. Abs. rel. err. (%) are absolute percentage relative errors computed against the CVMC price estimates with MLB used as control variate; AAPRE are average errors. Model parameters used: see Table 3, other parameters: $S_0 = 100$, $r = 0.0367$, $T = 1$, $q = 0$.

Table 5. Prices of arithmetic Asian options with discrete monitoring under ASV models.

N	K	Lower bound MLB	$\exp(\lambda^*)$	Abs. rel err. (%)	SLB	Abs. rel err. (%)	CVMC CV MLB	Std. err. $\times 10^{-5}$	CV SLB	Std. err. $\times 10^{-5}$
					Heston model					
12	90	11.74399	89.67	0.0035	11.74342	0.0083	11.74439	1.402	11.74444	2.357
12	100	3.71351	99.90	0.0052	3.71330	0.0107	3.71370	0.648	3.71367	0.918
12	110	0.19750	109.83	0.0613	0.19697	0.3322	0.19763	0.429	0.19768	1.127
50	90	11.75795	89.69	0.0036	11.75745	0.0079	11.75838	1.464	11.75840	2.293
50	100	3.74216	99.90	0.0050	3.74196	0.0104	3.74235	0.615	3.74231	0.842
50	110	0.20565	109.84	0.0503	0.20521	0.2625	0.20575	0.366	0.20578	0.987
250	90	11.76189	89.69	0.0037	11.76141	0.0078	11.76232	1.454	11.76231	2.260
250	100	3.75021	99.90	0.0046	3.75001	0.0100	3.75038	0.567	3.75034	0.787
250	110	0.20805	109.84	0.0530	0.20764	0.2511	0.20816	0.393	0.20821	0.988
AAPRE				0.021		0.100				
					Bates model					
12	90	11.74225	89.65	0.0040	11.74164	0.0092	11.74271	1.611	11.74268	2.484
12	100	3.69266	99.91	0.0054	3.69250	0.0097	3.69286	0.756	3.69287	0.969
12	110	0.17619	109.83	0.0554	0.17570	0.3350	0.17629	0.449	0.17636	1.113
50	90	11.75670	89.67	0.0040	11.75616	0.0086	11.75717	1.653	11.75717	2.542
50	100	3.72152	99.91	0.0055	3.72137	0.0097	3.72173	0.815	3.72173	1.010
50	110	0.18445	109.85	0.0544	0.18405	0.2722	0.18455	0.407	0.18460	0.947
250	90	11.76070	89.68	0.0038	11.76018	0.0083	11.76115	1.541	11.76111	2.387
250	100	3.72956	99.91	0.0055	3.72940	0.0096	3.72976	0.814	3.72977	0.993
250	110	0.18683	109.85	0.0542	0.18645	0.2579	0.18693	0.404	0.18697	0.910
AAPRE				0.021		0.102				
					DPS model					
12	90	11.76552	89.64	0.0038	11.76485	0.0095	11.76596	1.491	11.76593	2.513
12	100	3.73504	99.91	0.0057	3.73488	0.0098	3.73525	0.707	3.73527	0.948
12	110	0.17587	109.83	0.0749	0.17531	0.3919	0.17600	0.591	0.17602	1.205
50	90	11.78004	89.66	0.0037	11.77945	0.0087	11.78047	1.513	11.78042	2.359
50	100	3.76404	99.91	0.0055	3.76389	0.0096	3.76425	0.710	3.76426	0.938
50	110	0.18462	109.84	0.0691	0.18416	0.3170	0.18474	0.510	0.18475	1.085
250	90	11.78408	89.67	0.0039	11.78350	0.0087	11.78453	1.560	11.78452	2.410
250	100	3.77214	99.91	0.0056	3.77199	0.0097	3.77235	0.737	3.77238	0.984
250	110	0.18713	109.84	0.0677	0.18670	0.2990	0.18726	0.549	0.18726	1.076
AAPRE				0.027		0.118				
					BNS-Γ model					
12	90	11.60442	89.69	0.0018	11.60404	0.0051	11.60463	0.922	11.60464	1.657
12	100	3.22232	99.95	0.0041	3.22225	0.0064	3.22246	0.503	3.22248	0.644
12	110	0.06973	109.83	0.0933	0.06940	0.5557	0.06979	0.329	0.06975	0.796
50	90	11.61364	89.71	0.0016	11.61331	0.0044	11.61383	0.789	11.61383	1.494
50	100	3.24434	99.95	0.0043	3.24426	0.0066	3.24448	0.511	3.24449	0.635
50	110	0.07431	109.85	0.0837	0.07406	0.4256	0.07437	0.321	0.07434	0.685
250	90	11.61619	89.71	0.0018	11.61587	0.0045	11.61640	0.875	11.61639	1.504
250	100	3.25048	99.95	0.0041	3.25040	0.0065	3.25061	0.509	3.25062	0.639
250	110	0.07562	109.85	0.0917	0.07538	0.4074	0.07569	0.357	0.07566	0.703
AAPRE				0.032		0.158				
					BNS-IG model					
12	90	11.61748	89.68	0.0020	11.61710	0.0043	11.61771	1.414	11.61735	1.533
12	100	3.22174	99.95	0.0021	3.22167	0.0053	3.22180	0.405	3.22173	0.418
12	110	0.07043	109.83	0.0496	0.07010	0.5219	0.07046	0.213	0.07019	0.537
50	90	11.62717	89.70	0.0019	11.62683	0.0043	11.62739	1.338	11.62707	1.441
50	100	3.24406	99.95	0.0020	3.24398	0.0048	3.24412	0.402	3.24405	0.417
50	110	0.07495	109.84	0.0607	0.07469	0.4126	0.07500	0.277	0.07477	0.507
250	90	11.62983	89.70	0.0019	11.62951	0.0043	11.63005	1.313	11.62974	1.405
250	100	3.25026	99.95	0.0020	3.25019	0.0046	3.25033	0.398	3.25025	0.409
250	110	0.07624	109.85	0.0683	0.07600	0.3939	0.07630	0.317	0.07608	0.498
AAPRE				0.021		0.151				

Notes. See Table 4.

Table 6. Comparison of accuracy for discrete monitoring under the CEV model.

N	K	MLB	$\lambda^{*\frac{1}{2-\gamma}}$	Abs. rel. err. (%)	SLB	Abs. rel. err. (%)	AE-CLS	Abs. rel. err. (%)	QUAD-SMF	Abs. rel. err. (%)	$\frac{\text{CVMC}}{\text{CV MLB}}$	Std. err. $\times 10^{-5}$
CEV model: case of $\gamma = 1.5$												
12	90	13.20307	89.79	0.0016	13.20262	0.0050	13.20050	0.0210	13.20327	0.0000	13.20327	2.309
12	100	6.75420	99.85	0.0031	6.75389	0.0077	6.75168	0.0404	6.75439	0.0003	6.75441	2.114
12	110	2.84932	109.77	0.0096	2.84879	0.0283	2.84678	0.0989	2.84959	0.0003	2.84960	2.538
50	90	13.25690	89.79	0.0013	13.25648	0.0044	13.25443	0.0199	13.25708	0.0001	13.25707	1.840
50	100	6.83031	99.85	0.0028	6.82999	0.0075	6.82790	0.0381	6.83049	0.0001	6.83050	2.072
50	110	2.91426	109.77	0.0087	2.91377	0.0255	2.91184	0.0918	2.91453	0.0006	2.91452	2.666
250	90	13.27147	89.79	0.0019	13.27107	0.0049	13.26903	0.0202	13.27227	0.0042	13.27171	2.544
250	100	6.85091	99.85	0.0024	6.85059	0.0071	6.84853	0.0372	6.85064	0.0064	6.85108	1.651
250	110	2.93202	109.77	0.0076	2.93154	0.0239	2.92962	0.0893	2.93121	0.0351	2.93224	2.178
AAPRE				0.004		0.013		0.051		0.005		
CEV model: case of $\gamma = 2.5$												
12	90	13.08727	89.41	0.0137	13.08360	0.0417	13.08660	0.0188	13.08919	0.0010	13.08906	10.326
12	100	6.74471	99.53	0.0247	6.74179	0.0680	6.74351	0.0425	6.74621	0.0024	6.74637	9.700
12	110	2.99163	109.29	0.0873	2.98630	0.2653	2.99188	0.0789	2.99454	0.0098	2.99424	14.809
50	90	13.14150	89.44	0.0123	13.13814	0.0378	13.14086	0.0171	13.14330	0.0015	13.14311	9.547
50	100	6.82080	99.53	0.0266	6.81786	0.0697	6.81969	0.0428	6.82224	0.0055	6.82261	10.591
50	110	3.05435	109.33	0.0797	3.04942	0.2409	3.05453	0.0739	3.05709	0.0097	3.05679	13.639
250	90	13.15613	89.44	0.0133	13.15286	0.0381	13.15550	0.0181	13.15892	0.0079	13.15788	10.652
250	100	6.84142	99.53	0.0243	6.83849	0.0672	6.84034	0.0401	6.84262	0.0068	6.84308	10.477
250	110	3.07163	109.33	0.0774	3.06681	0.2342	3.07180	0.0720	3.07286	0.0373	3.07401	13.656
AAPRE				0.040		0.118		0.045		0.009		

Notes. See Table 4. In addition, results of the asymptotic expansion (AE) approach are from Cai et al. [15]; numerical quadrature (QUAD) results are from the approach of Sesana et al. [64]. Model parameters used: see Table 3, other parameters: $S_0 = 100$, $r = 0.05$, $T = 1$, $q = 0$.

Table 7. Deltas and gammas of arithmetic Asian options with discrete monitoring.

N	K	MLB-Delta	Abs. rel. err. (%)	AE-CLS Delta	Abs. rel. err. (%)	QUAD-SMF Delta	MLB-Gamma	Abs. rel. err. (%)	QUAD-SMF Gamma
				CEV model: case of $\gamma = 1.5$					
12	90	0.80975	0.0000	0.80930	0.0551	0.80975	0.01769	0.4660	0.01777
12	100	0.57142	0.0008	0.57138	0.0054	0.57141	0.02669	0.5048	0.02683
12	110	0.31663	0.0033	0.31669	0.0208	0.31662	0.02421	0.5476	0.02435
50	90	0.80687	0.0000	0.80644	0.0528	0.80687	0.01756	0.4815	0.01765
50	100	0.57145	0.0009	0.57141	0.0058	0.57144	0.02631	0.5133	0.02644
50	110	0.31970	0.0029	0.31976	0.0210	0.31969	0.02402	0.5602	0.02416
250	90	0.80610	0.0069	0.80568	0.0588	0.80615	0.01753	0.4016	0.01760
250	100	0.57145	0.0202	0.57141	0.0264	0.57156	0.02621	0.5170	0.02634
250	110	0.32052	0.0082	0.32057	0.0087	0.32054	0.02397	0.6112	0.02411
AAPRE			0.005		0.028			0.511	
				CEV model: case of $\gamma = 2.5$					
12	90	0.82316	0.0012	0.82328	0.0136	0.82317	0.01706	0.3406	0.01701
12	100	0.58150	0.0040	0.58164	0.0279	0.58148	0.02671	0.4604	0.02659
12	110	0.33176	0.0316	0.33170	0.0140	0.33165	0.02454	0.4302	0.02443
50	90	0.82053	0.0019	0.82065	0.0130	0.82054	0.01693	0.4221	0.01686
50	100	0.58222	0.0042	0.58235	0.0267	0.58219	0.02633	0.4824	0.02620
50	110	0.33525	0.0276	0.33521	0.0151	0.33516	0.02440	0.4561	0.02429
250	90	0.81981	0.0120	0.81995	0.0046	0.81991	0.01690	0.5762	0.01680
250	100	0.58239	0.0246	0.58252	0.0018	0.58253	0.02623	0.4886	0.02610
250	110	0.33618	0.0100	0.33614	0.0000	0.33614	0.02436	0.3934	0.02427
AAPRE			0.013		0.013			0.450	
		MLB-Delta	Abs. rel. err. (%)	CVMC / CV MLB-Delta		Std. err. $\times 10^{-5}$		MLB-Gamma	
				Heston model					
12	90	0.93056	0.0000	0.93056		4.010		0.00984	
12	100	0.67760	0.0779	0.67707		5.236		0.04811	
12	110	0.09691	0.5497	0.09744		4.639		0.03572	
50	90	0.92924	0.0027	0.92927		4.062		0.00993	
50	100	0.67756	0.0727	0.67707		5.183		0.04762	
50	110	0.09972	0.2729	0.09999		4.652		0.03622	
AAPRE			0.163						
		MLB-Delta	Abs. rel. err. (%)	CVMC / CV MLB-Delta		Std. err. $\times 10^{-4}$		MLB-Gamma	
				CGMY model					
12	90	0.90073	0.1302	0.89956		0.048		0.00893	
12	100	0.67506	0.0171	0.67495		0.032		0.03971	
12	110	0.21942	0.0000	0.21942		0.014		0.03980	
50	90	0.90018	0.0563	0.89967		6.966		0.00895	
50	100	0.67549	0.0328	0.67571		6.540		0.03941	
50	110	0.22236	0.2188	0.22285		6.084		0.03984	
AAPRE			0.076						

Notes. MLB-Deltas and MLB-Gammas are computed first and second derivatives w.r.t. S_0 of the optimized lower bound for the option price (see Remark 1); asymptotic expansion (AE) deltas are from Cai et al. [15] and numerical quadrature (QUAD) deltas and gammas from Sesana et al. [64]; CVMC deltas are likelihood-ratio Monte Carlo simulation estimates with MLB-Deltas used as control variates, std. err. are the standard errors. Abs. rel. err. (%) are absolute percentage relative errors computed against QUAD (case of CEV model) or CVMC (cases of Heston and CGMY models); AAPRE are average errors. Model parameters used: see Tables 3–6.

TABLE 8. Comparison of accuracy for continuous monitoring under jump diffusion models.

σ	K	MLB	$\exp(\lambda^*)$	Abs. rel. err. (%)	DL-CK	Abs. rel. err. (%)	PIDE-BX	Abs. rel. err. (%)	CVMC CV cts. MLB	Std. err. $\times 10^{-5}$
DEJD model: case of $l=3$, $p=0.6$, $\eta_1=\eta_2=25$, $S_0=100$, $r=0.09$, $T=1$, $q=0$										
0.05	90	13.42081	89.80	0.0003	13.41924	0.0120	13.43610	0.1137	13.42085	1.453
0.05	95	8.98763	94.90	0.0034	8.98812	0.0020	8.99711	0.1021	8.98794	1.259
0.05	100	4.95758	99.95	0.0058	4.95673	0.0229	4.96267	0.0969	4.95786	1.895
0.05	105	2.13522	104.92	0.0095	2.13611	0.0322	2.13661	0.0556	2.13542	2.074
0.05	110	0.83077	109.79	0.0294	0.83091	0.0122	0.83077	0.0291	0.83101	3.900
0.1	90	13.47563	89.80	0.0007	13.48451	0.0652	13.49220	0.1223	13.47572	1.635
0.1	95	9.20611	94.89	0.0013	9.20478	0.0157	9.21706	0.1177	9.20623	1.626
0.1	100	5.53646	99.91	0.0025	5.53662	0.0004	5.54257	0.1079	5.53660	1.573
0.1	105	2.88851	104.87	0.0103	2.88896	0.0053	2.89095	0.0742	2.88881	2.467
0.1	110	1.33764	109.77	0.0388	1.33809	0.0053	1.33810	0.0046	1.33816	3.918
0.2	90	14.03477	89.69	0.0035	14.03280	0.0175	14.05240	0.1221	14.03526	4.795
0.2	95	10.32434	94.75	0.0045	10.32293	0.0182	10.33600	0.1084	10.32481	3.895
0.2	100	7.21337	99.76	0.0069	7.21244	0.0197	7.21976	0.0817	7.21386	3.985
0.2	105	4.78558	104.71	0.0160	4.78516	0.0248	4.78788	0.0321	4.78634	5.676
0.2	110	3.02260	109.62	0.0358	3.02270	0.0325	3.02208	0.0530	3.02368	7.865
0.3	90	15.19499	89.46	0.0098	15.19639	0.0006	15.21340	0.1113	15.19648	9.825
0.3	95	11.92790	94.51	0.0110	11.92926	0.0004	11.94010	0.0913	11.92921	9.212
0.3	100	9.14625	99.51	0.0157	9.14769	0.0000	9.15304	0.0585	9.14769	10.252
0.3	105	6.85880	104.45	0.0226	6.86049	0.0021	6.86072	0.0055	6.86034	10.386
0.3	110	5.03821	109.35	0.0394	5.04029	0.0019	5.03663	0.0708	5.04020	11.867
0.4	90	16.68659	89.13	0.0199	16.68984	0.0004	16.70640	0.0988	16.68990	18.760
0.4	95	13.73072	94.17	0.0222	13.73384	0.0006	13.74490	0.0811	13.73376	17.742
0.4	100	11.16798	99.16	0.0319	11.17115	0.0035	11.17800	0.0578	11.17154	20.618
0.4	105	8.98770	104.09	0.0409	8.99114	0.0026	8.99450	0.0347	8.99138	21.617
0.4	110	7.16426	108.98	0.0577	7.16816	0.0033	7.16714	0.0176	7.16840	23.322
0.5	90	18.34752	88.72	0.0339	18.35379	0.0002	18.36960	0.0863	18.35375	31.256
0.5	95	15.62209	93.74	0.0389	15.62810	0.0005	15.63680	0.0552	15.62817	30.663
0.5	100	13.22262	98.71	0.0456	13.22860	0.0004	13.23150	0.0215	13.22865	29.777
0.5	105	11.13321	103.63	0.0518	11.13944	0.0042	11.13740	0.0142	11.13898	29.670
0.5	110	9.33127	108.51	0.0675	9.33799	0.0044	9.33092	0.0713	9.33758	32.709
AAPRE				0.023		0.010		0.070		
DEJD model: case of $l=5$, $p=0.6$, $\eta_1=\eta_2=25$, $S_0=100$, $r=0.09$, $T=1$, $q=0$										
0.05	90	13.47297	89.79	0.0618	13.47952	0.1104	13.50430	0.2944	13.46466	1.891
0.05	95	9.16287	94.88	0.0561	9.16588	0.0890	9.18250	0.2705	9.15773	1.702
0.05	100	5.38649	99.94	0.0256	5.38761	0.0463	5.39632	0.2081	5.38512	1.868
0.05	105	2.72611	104.89	0.0093	2.72681	0.0166	2.72878	0.0888	2.72636	2.982
0.05	110	1.28307	109.76	0.1083	1.28264	0.1421	1.28220	0.1764	1.28447	4.536
0.1	90	13.56081	89.78	0.0406	13.55964	0.0320	13.59170	0.2685	13.55530	2.213
0.1	95	9.42292	94.87	0.0361	9.41962	0.0011	9.44366	0.2563	9.41952	2.372
0.1	100	5.91773	99.89	0.0266	5.91537	0.0132	5.92970	0.2290	5.91615	2.288
0.1	105	3.35275	104.85	0.0029	3.35071	0.0636	3.35803	0.1547	3.35284	3.526
0.1	110	1.75079	109.74	0.0943	1.74896	0.1989	1.75186	0.0334	1.75245	4.726
0.2	90	14.17290	89.66	0.0444	14.17380	0.0507	14.19070	0.1700	14.16661	5.469
0.2	95	10.53738	94.72	0.0380	10.53795	0.0435	10.54910	0.1493	10.53337	5.061
0.2	100	7.48774	99.73	0.0254	7.48805	0.0295	7.49282	0.0933	7.48584	5.312
0.2	105	5.08980	104.68	0.0003	5.09001	0.0043	5.08871	0.0212	5.08979	6.454
0.2	110	3.32040	109.58	0.0606	3.32061	0.0544	3.31423	0.2465	3.32242	8.924
0.3	90	15.33263	89.43	0.0421	15.33688	0.0699	15.36350	0.2436	15.32617	9.800
0.3	95	12.10384	94.48	0.0349	12.10723	0.0629	12.12330	0.1957	12.09962	8.758
0.3	100	9.35060	99.48	0.0193	9.35336	0.0488	9.35985	0.1183	9.34879	10.578
0.3	105	7.07816	104.42	0.0008	7.08059	0.0335	7.07893	0.0100	7.07822	10.749
0.3	110	5.25878	109.32	0.0418	5.26109	0.0021	5.25264	0.1585	5.26098	14.554
0.4	90	16.80822	89.11	0.0371	16.81490	0.0768	16.84130	0.2340	16.80199	18.688
0.4	95	13.87427	94.15	0.0226	13.87995	0.0635	13.89550	0.1756	13.87114	20.746
0.4	100	11.32758	99.13	0.0085	11.33257	0.0525	11.33860	0.1058	11.32662	24.205
0.4	105	9.15669	104.06	0.0096	9.16131	0.0409	9.15873	0.0127	9.15757	23.436
0.4	110	7.33609	108.95	0.0365	7.34063	0.0254	7.33031	0.1152	7.33877	24.117
0.5	90	18.45211	88.69	0.0262	18.46259	0.0831	18.45030	0.0165	18.44726	35.097
0.5	95	15.74077	93.71	0.0203	15.75006	0.0794	15.73200	0.0354	15.73757	30.114
0.5	100	13.35180	98.68	0.0035	13.36027	0.0669	13.33820	0.0984	13.35133	35.205
0.5	105	11.26912	103.60	0.0190	11.27716	0.0523	11.25170	0.1736	11.27127	36.524
0.5	110	9.47033	108.47	0.0429	9.47826	0.0408	9.44911	0.2669	9.47439	35.904
AAPRE				0.033		0.056		0.154		

Table 8. Continued.

σ	K	MLB	$\exp(\lambda^*)$	Abs. rel. err. (%)	PIDE-BX	Abs. rel. err. (%)	CVMC CV cts. MLB	Std. err. $\times 10^{-5}$
MJD model: case of $l = 1$, $\mu_x = -0.1$, $\sigma_x = 0.3$, $S_0 = 100$, $r = 0.15$, $T = 1$, $q = 0$								
0.1	90	16.99518	88.92	0.0216	16.99660	0.0132	16.99884	22.998
0.1	100	10.06073	99.60	0.0299	10.06160	0.0212	10.06374	20.211
0.1	110	4.83616	109.68	0.0596	4.83594	0.0642	4.83904	26.621
0.2	90	17.34403	89.16	0.0285	17.34590	0.0178	17.34898	31.231
0.2	100	10.95847	99.51	0.0375	10.95900	0.0327	10.96258	27.557
0.2	110	6.30210	109.44	0.0655	6.30264	0.0569	6.30623	28.661
AAPRE				0.040		0.034		

Notes. MLB are optimized lower bounds to the option prices, λ^* are the corresponding optimal bound parameters; "DL-CK" numbers are from Cai and Kou [14, Tables 5–6]; "PIDE-BX" numbers correspond to the PIDE implementation of Bayraktar and Xing [9]; CVMC prices are Monte Carlo simulation estimates based on 1,000,000 paths simulations and daily monitoring with the continuous-time (cts.) MLB used as control variate, std. err. are the standard errors of the CVMC price estimates. Abs. rel. err. (%) are absolute percentage relative errors computed against the CVMC price estimates; AAPRE are average errors.

Table 9. Comparison of accuracy for continuous monitoring under the Heston model.

S_0	K	r	q	T	V_0	α	β	γ	ρ	MLB	PDE-EMT	Aus Taylor 2.0-EMT	RMSE
100	100	0.04	0	1	0.0483	4.75	0.0483	0.55	−0.569	5.854	5.853	5.851	0.0059
100	100	0.1	0	1	0.1	3	0.1	0.2	0.7	9.323	9.323	9.321	0.0108
100	100	0.07	0	1	0.075	7	0.05	0.3	−0.4	7.123	7.123	7.123	0.0078

Notes. MLB are optimized lower bounds to the option prices; "PDE-EMT" and "Aus Taylor 2.0-EMT" numbers correspond to the PDE implementation of Ewald et al. [36] and the Monte Carlo simulation estimates (with RMSE reported) of Ewald et al. [36] using a second-order Taylor 2.0 scheme for 1,000,000 paths simulations with 1,000 time steps.

Table 10. Comparison of accuracy with existing methods for continuous monitoring under the Gaussian model.

S_0	K	r	q	σ	T	MLB	$\exp(\lambda^*)$	DL-CK, GY-S, EE-Linetsky	PDE-Večeř, PDE-Zhang
2	2	0.02	0	0.1	1	0.055985	2.00	0.0559860415	0.055986
2	2	0.18	0	0.3	1	0.218366	1.99	0.2183875466	0.218388
2	2	0.0125	0	0.25	2	0.172226	1.99	0.1722687410	0.172269
1.9	2	0.05	0	0.5	1	0.193060	1.97	0.1931737903	0.193174
2	2	0.05	0	0.5	1	0.246298	1.98	0.2464156905	0.246416
2.1	2	0.05	0	0.5	1	0.306094	1.97	0.3062203648	0.306220
2	2	0.05	0	0.5	2	0.349779	1.95	0.3500952190	0.350095

Notes. MLB are optimized lower bounds to the option prices, λ^* are the corresponding optimal bound parameters; results of Cai and Kou's inversion of the double-Laplace (DL) transform are from Cai and Kou [14, Table 2]; results of Linetsky's eigenfunction expansion (EE) method are from Linetsky [53, Table 3]; "GY-S", "PDE-Večeř" and "PDE-Zhang" numbers are from the table in Dewynne and Shaw [31, p. 383] and correspond to the Geman–Yor (GY) Laplace transform formula run with the code of Shaw [65], the PDEs of Večeř [69] and Zhang [74], respectively.

Table 11. Comparison of accuracy for continuous monitoring under the Gaussian model for different test cases & volatility levels when $q<r$, $q>r$ or $q=r$.

Case of $q<r$ & extremely small σ

S_0	K	r	q	σ	T	MLB	$\exp(\lambda^*)$	DL-CK	GY-S	MAE3-DS	PDE-Zhang
2	2	0.02	0	0.1	1	0.0559851	1.999	0.0559860	0.0559860	0.0559860	0.0559860
2	2	0.02	0	0.05	1	0.0339411	2.000	0.0339412	0.0339412	0.0339412	0.0339412
2	2	0.02	0	0.01	1	0.0199278	2.000	n.a.	n.a.	0.0199278	0.0199278
2	2	0.02	0	0.005	1	0.0197357	1.992	n.a.	n.a.	0.0197357	0.0197357
2	2	0.02	0	0.001	1	0.0197353	1.992	n.a.	n.a.	0.0197353	0.0197353

Case of $q>r$ & extremely small σ

S_0	K	r	q	σ	T	MLB	$\exp(\lambda^*)$	DL-CK	GY-S-full	MAE3-DS	PDE-Zhang
2	2	0.02	0.04	0.1	1	0.0357844	1.999	0.0357853	0.0357853	0.0357854	0.0357853
2	2	0.02	0.04	0.05	1	0.0140246	2.000	0.0140247	0.0140247	0.0140248	0.0140247
2	2	0.02	0.04	0.01	1	0.000190248	2.000	n.a.	n.a.	0.000190254	0.000190254
2	2	0.02	0.04	0.005	1	3.7991E-07	1.998	n.a.	n.a.	3.7991E-07	3.7993E-07
2	2	0.02	0.04	0.001	1	o(1E-67)	1.998	n.a.	n.a.	o(1E-70)	o(1E-72)

Case of $q=r$ & extremely small σ

S_0	K	r	q	σ	T	MLB	$\exp(\lambda^*)$	DL-CK	CIBess	MAE3-DS	PDE-Zhang
2	2	0.02	0.02	0.1	1	0.0451421	1.9990	0.0451431	0.0451431	0.0451431	0.0451431
2	2	0.02	0.02	0.05	1	0.0225754	1.9998	0.0225755	0.0225755	0.0225755	0.0225755
2	2	0.02	0.02	0.01	1	0.00451533	2.0000	n.a.	n.a.	0.00451536	0.00451536
2	2	0.02	0.02	0.005	1	0.00225768	2.0000	n.a.	n.a.	0.00225768	0.00225768
2	2	0.02	0.02	0.001	1	0.000451537	2.0000	n.a.	n.a.	0.000451537	0.000451537

Notes. MLB are optimized lower bounds to the option prices, λ^* are the corresponding optimal bound parameters; "DL-CK" numbers are from Cai and Kou [14, Table 4]; "GY-S", "GY-S-full", "CIBess", "MAE3-DS" and "PDE-Zhang" numbers are from the tables in Dewynne and Shaw [31, §6.1–6.3]. Note that "GY-S-full" and "CIBess" correspond to variants of the "GY-S" method, and "MAE3-DS" to the matched asymptotic expansion method of Dewynne and Shaw [31].

TABLE 12. Upper error bounds.

N	K	Gaussian model		VG model		NIG model		CGMY model		MJD model	
		Opt. Error UB	Subopt. Error UB	Opt. Error UB	Subopt. Error UB	Opt. Error UB	Subopt. Error UB	Opt. Error UB	Subopt. Error UB	Opt. Error UB	Subopt. Error UB
12	90	0.01166	0.01214	0.05888	0.05981	0.08728	0.08960	0.14385	0.14631	0.09034	0.09302
12	100	0.03297	0.03333	0.09689	0.09915	0.13250	0.13344	0.19559	0.19620	0.14660	0.14727
12	110	0.05772	0.05847	0.15517	0.16041	0.17260	0.17329	0.23630	0.23705	0.18907	0.18983
50	90	0.01205	0.01249	0.06017	0.06079	0.08885	0.09104	0.14480	0.14715	0.09190	0.09439
50	100	0.03288	0.03323	0.09732	0.09962	0.13331	0.13428	0.19714	0.19777	0.14752	0.14817
50	110	0.05699	0.05768	0.15634	0.16038	0.17313	0.17376	0.23899	0.23968	0.18938	0.19009
		DEJD model		Meixner model		Heston model		Bates model		DPS model	
		Opt. Error UB	Subopt. Error UB	Opt. Error UB	Subopt. Error UB	Opt. Error UB	Subopt. Error UB	Opt. Error UB	Subopt. Error UB	Opt. Error UB	Subopt. Error UB
12	90	0.10985	0.11253	0.07346	0.07575	0.01316	0.01364	0.01314	0.01364	0.0130	0.0140
12	100	0.16539	0.16604	0.11667	0.11714	0.03033	0.03073	0.03224	0.03262	0.0330	0.0340
12	110	0.20774	0.20850	0.15266	0.15281	0.04852	0.04909	0.05307	0.05358	0.0540	0.0550
50	90	0.11137	0.11387	0.07496	0.07716	0.01334	0.01372	0.01340	0.01387	0.0140	0.0150
50	100	0.16613	0.16678	0.11762	0.11805	0.03069	0.03095	0.03310	0.03330	0.0330	0.0340
50	110	0.20806	0.20876	0.15265	0.15315	0.04876	0.04917	0.05403	0.05430	0.0530	0.0540
		BNS-Γ model		BNS-IG model		CEV model: $\gamma = 1.5$			CEV model: $\gamma = 2.5$		
		Opt. Error UB	Subopt. Error UB	Opt. Error UB	Subopt. Error UB	Opt. Error UB		Subopt. Error UB	Opt. Error UB		Subopt. Error UB
12	90	0.00646	0.00672	0.00758	0.00789	0.0190		0.0210	0.0490		0.0520
12	100	0.02609	0.02631	0.02758	0.02777	0.0350		0.0370	0.0970		0.1000
12	110	0.04012	0.04040	0.04045	0.04072	0.0520		0.0530	0.1480		0.1520
50	90	0.00654	0.00678	0.00772	0.00799	0.0200		0.0210	0.0500		0.0530
50	100	0.02069	0.02080	0.02281	0.02292	0.0350		0.0370	0.0970		0.1000
50	110	0.03992	0.04006	0.04110	0.04119	0.0510		0.0530	0.1460		0.1500

Notes. Opt. (Subopt.) upper error bounds are upper bounds to the differences between the true option prices and the corresponding MLBs (SLBs, i.e., for $\lambda = \ln K$ for Lévy and ASV models; $\lambda = K^{2-\gamma}$ for the CEV model) computed using result (51). Model parameters used: see Table 3, other parameters: $S_0 = 100$, $r = 0.0367$, $T = 1$, $q = 0$.