Modular anomaly equation, heat kernel and S-duality in $\mathcal{N} = 2$ theories

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ABSTRACT: We investigate ϵ-deformed $\mathcal{N} = 2$ superconformal gauge theories in four dimensions, focusing on the $\mathcal{N} = 2^*$ and $N_f = 4$ SU(2) cases. We show how the modular anomaly equation obeyed by the deformed prepotential can be efficiently used to derive its non-perturbative expression starting from the perturbative one. We also show that the modular anomaly equation implies that S-duality is implemented by means of an exact Fourier transform even for arbitrary values of the deformation parameters, and then we argue that it is possible, perturbatively in the deformation, to choose appropriate variables such that it reduces to a Legendre transform.

Contents

1 Introduction

Gauge theories with rigid $\mathcal{N} = 2$ supersymmetry in four dimensions represent one of the main areas where substantial progress toward a non-perturbative description has been made, starting with the seminal papers of Seiberg and Witten (SW) [1, 2]. In recent years much attention has been devoted to the deformation of these theories by means of the so-called ϵ (or Ω) background. This anti-symmetric tensor background breaks Lorentz invariance and can be generically characterized by two parameters ϵ_1 and ϵ_2. Originally, it was used by Nekrasov [3–6] as a regulator in the explicit computations, by means of localization techniques, of the multi-instanton contributions to the partition function[1] $Z(a; \epsilon)$ and to the prepotential of the low-energy effective theory

$$F(a; \epsilon) = F_{\mathrm{cl}}(a) - \epsilon_1 \epsilon_2 \log Z(a; \epsilon) \tag{1.1}$$

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[1]Here a stands or the vacuum expectation values of the adjoint complex scalar in the $\mathcal{N} = 2$ gauge multiplet along the Cartan directions, which parametrize the moduli space in the Coulomb branch; in this paper we will be concerned with rank one cases, so that we will have a single a. In presence of matter multiplets, the expressions depend on the masses of the latter as well. They will also depend on the dynamically generated scale Λ or (in the conformal cases) on the bare coupling τ_0.

where F_{cl} is the classical term. The prepotential can be expanded in powers of ϵ, and it is convenient to organize such an expansion as follows:

$$F(a;\epsilon) = F_{\text{cl}}(a) + \sum_{n,g=0}^{\infty} (\epsilon_1+\epsilon_2)^{2n}\,(\epsilon_1\epsilon_2)^g\,F^{(n,g)}(a)\,, \qquad (1.2)$$

where the coefficients $F^{(n,g)}$ account for the perturbative and non-perturbative contributions. The usual prepotential $\mathcal{F}$ of the SW theory is obtained by setting the ϵ regulators to zero, that is $\mathcal{F}(a) \equiv F_{\text{cl}}(a) + F^{(0,0)}(a)$ in the above notation. In this way one can derive the explicit instanton expansion and successfully compare it with the expression obtained by geometrizing the monodromy and the duality features of the low-energy theory in terms of a SW curve. In the latter treatment, the vacuum expectation values a and their S-duals[2] $\tilde{a}$ arise as periods of a suitable meromorphic differential λ along a symplectic basis (A, B) of one-cycles of the SW curve, and the prepotential $\mathcal{F}$ is such that

$$2\pi\mathrm{i}\,\tilde{a} = \frac{\partial\mathcal{F}}{\partial a}\,. \qquad (1.3)$$

The duality group of the effective theory is contained in the $\text{Sp}(2r,\mathbb{Z})$ redefinitions of the symplectic basis, that reduce to $\text{Sl}(2,\mathbb{Z})$ in the rank $r=1$ case. The $\mathcal{S}$ generator of this group exchanges a and $\tilde{a}$, and the S-dual description of the theory is given through the Legendre transform of the prepotential:

$$\widetilde{\mathcal{F}}(\tilde{a}) = \mathcal{F}(a) - 2\pi\mathrm{i}\tilde{a}\,a\,, \qquad (1.4)$$

where in the right hand side a has to be expressed in terms of $\tilde{a}$ by inverting (1.3).

It is natural to wonder about the physical meaning of the prepotential $F(a;\epsilon)$ at finite values of the ϵ-deformation. This question was first addressed in the particular case $\epsilon_2 = -\epsilon_1$, when only the $n=0$ terms in (1.2) contribute. In [3–6] it was proposed that they correspond to gravitational F-terms in the effective action of the form $F^{(0,g)}\,\mathcal{W}^{2g}$, where $\mathcal{W}$ is the chiral Weyl superfield containing the graviphoton field strength as its lowest component [7]. This statement can be understood by realizing the $\mathcal{N}=2$ gauge theories on the world-volume of stacks of D-branes in backgrounds with four orbifolded internal directions. In such a microscopic string set-up it is possible to compute non-perturbative corrections provided by D-instantons [8] and show that the ϵ-background corresponds to a Ramond-Ramond three-form which appears in the effective theory as the graviphoton [9]; the above gravitational F-terms are thus directly accounted for.[3]

On the other hand, the low-energy $\mathcal{N}=2$ theories can be "geometrically engineered" in Type II string theory [12–14] in terms of closed strings on suitable non-compact "local" Calabi-Yau (CY) manifolds whose complex structure moduli t encode the gauge-invariant moduli of the SW theory (and are thus non-trivially related to the Coulomb branch parameters a). In such CY compactifications, the graviphoton F-terms of the four-dimensional

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[2]Throughout this paper, we will denote the S-dual of a quantity X as $\mathcal{S}[X]$ or, whenever typographically clear, as $\widetilde{X}$.

[3]For another string theory interpretation of the Ω background, see [10, 11].

effective action are captured by topological string amplitudes as $F_{\text{top}}^{(g)}(t)\,\mathcal{W}^{2g}$ [7, 15]. For local CY realizations of $\mathcal{N}=2$ gauge theories, the coefficients $F^{(0,g)}(a)$ are therefore identified with the topological amplitudes $F_{\text{top}}^{(g)}(t)$ and the expansion of the deformed prepotential $F(a;\epsilon)$ in powers of $\epsilon_1\epsilon_2$ corresponds to the genus expansion of the topological string vacuum amplitude with a string coupling constant

$$g_s^2 = \epsilon_1\epsilon_2\,. \tag{1.5}$$

The link to the topological string brings an important bonus: the amplitudes $F_{\text{top}}^{(g)}$ satisfy a "holomorphic anomaly equation" [15, 16]. In fact, despite their apparent holomorphicity in t, they also develop a dependence on $\bar{t}$ due to contributions from the boundary of the genus g moduli space, and thus should be more properly denoted as $F_{\text{top}}^{(g)}(t,\bar{t})$. The holomorphic anomaly equation, which relates amplitudes at different genera, can be written [17] as a linear equation, similar in structure to the heat equation, for the topological partition function

$$Z_{\text{top}}(t,\bar{t};g_s) = \exp\left(-\sum_{g=0}^{\infty} g_s^{2g-2}\, F_{\text{top}}^{(g)}(t,\bar{t})\right)\,, \tag{1.6}$$

which in the local CY case can be connected to the deformed partition function $Z(a;\epsilon)$ of the corresponding $\mathcal{N}=2$ theory.

The holomorphic anomaly can be understood by assuming that the moduli space of complex structures has to be quantized and that the partition function behaves as a wave function, with g_s^2 playing the rôle of $\hbar$ [17–19]. This assumption requires that the partition function transforms consistently under canonical transformations acting on the moduli phase space. Parametrizing the phase space by the periods a and $\tilde{a}$, on which the duality symmetries act as symplectic transformations, one can thus determine the effect of the latter on $Z(a;\epsilon)$. In particular, the S-duality exchanging a and $\tilde{a}$ should be represented on $Z(a;\epsilon)$ as a Fourier transform:

$$\widetilde{Z}(\tilde{a};\epsilon) \simeq \int dx\; \mathrm{e}^{\frac{2\pi\mathrm{i}\tilde{a}\,x}{g_s^2}}\, Z(x;\epsilon)\,. \tag{1.7}$$

Using (1.1) and (1.2), one can evaluate the integral in the saddle-point approximation for small values of g_s and show that only at the leading order the S-duality reduces to a Legendre transform as in (1.4). By comparing the description based on $(a,\tilde{a})$ with the one employing the global cooordinates $(t,\bar{t})$, it is possible to relate the non-holomorphicity of the amplitudes $F_{\text{top}}^{(g)}(t,\bar{t})$ to the failure of modularity for the $F^{(0,g)}(a)$ and to argue that the amplitudes should be given either in terms of modular but almost holomorphic expressions (in the global coordinates description) or in terms of holomorphic but quasi-modular forms (in the period basis) [18, 19].

The general picture we have outlined above has been pursued and applied to many $\mathcal{N}=2$ models, ranging from SU(2) to higher rank pure gauge theories, as well as to theories with matter in various representations of the gauge group. It is of course a highly non-trivial task to *explicitly* compute the deformed prepotential in a given model, and the various approaches can cooperate to this aim. On the microscopic side, efficient tools have

been devised to implement the localization techniques and compute the non-perturbative corrections [20–28]. On the topological side, results can be obtained from a direct integration of the holomorphic anomaly equation, supplemented with appropriate boundary conditions [29–33].

In the case of $\mathcal{N} = 2$ *superconformal* gauge theories, an extremely interesting interpretation of the ϵ-deformation, going under the name of AGT relation, has been uncovered [34–38]: the generalized prepotential $F(a; \epsilon)$ equals the logarithm of conformal blocks in a two-dimensional Liouville theory. The central charge of the Liouville theory and the dimensions of the operators are ϵ-dependent and the number of inserted operators, as well as the genus of the two-dimensional surface on which the conformal blocks are defined, depend on the gauge group and matter content. In order to realize a generic central charge for the Liouville theory, it is necessary to consider the case with non-vanishing

$$s = \epsilon_1 + \epsilon_2 \,, \tag{1.8}$$

which means that the full expansion (1.2) and not just its $n = 0$ terms must be taken into account. The $F^{(n,g)}(a)$ coefficients with $n \neq 0$ correspond to so-called "refined" topological amplitudes $F_{\mathrm{top}}^{(n,g)}(t, \bar{t})$. They appear in F-terms of the form $F_{\mathrm{top}}^{(n,g)}(t, \bar{t}) \, \mathcal{Y}^{2n} \, \mathcal{W}^{2g}$, involving a chiral superfield $\mathcal{Y}$ composed out of extra vector multiplets [39, 40], and obey a generalized holomorphic anomaly equation [30–33]. The AGT relation maps the duality properties, and in particular the strong/weak-coupling S-duality, of the $\mathcal{N} = 2$ superconformal theories into the modular transformation properties of the Liouville conformal blocks; this makes even more interesting to study the duality properties of these superconformal theories at generic values of the ϵ-parameters.

The simplest canonical examples of four-dimensional superconformal gauge theories are given by SU(2) SYM theory with either $N_f = 4$ adjoint matter hypermultiplets or with one adjoint hypermultiplet, the latter case being also known as the $\mathcal{N} = 2^*$ theory.[4] These theories have vanishing β-function, but, when the hypermultiplets are massive, they receive both perturbative and non-perturbative corrections. For vanishing ϵ-deformations, via the SW description, it is possible to obtain the prepotential $\mathcal{F}$ as an exact function of the bare coupling τ_0, and show that under S-duality, which on the bare coupling acts by $\tau_0 \to -1/\tau_0$, the prepotential and its S-dual are related precisely by a Legendre transform, as in (1.4) so that also the effective coupling τ is mapped by $\mathcal{S}$ into $-1/\tau$ [41, 42].

Using the various approaches described above, much progress has been made in obtaining exact expressions for the generalized prepotential terms $F^{(n,g)}(a, \epsilon)$ and in deriving their modular properties. In particular, using the topological string point of view, the generalized holomorphic anomaly equation that applies to these superconformal cases has been analyzed in [43–45]. This holomorphic anomaly translates into an anomalous modular behavior of the $F^{(n,g)}(a, \epsilon)$'s with respect to the bare coupling τ_0 which can only occur through the second Eisenstein series $E_2(\tau_0)$. The dependence on E_2 is described by a

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[4]We believe, however, that the methods we describe in this paper should easily extend to other theories as well.

modular anomaly equation of the form [43]–[46]

$$\partial_{E_2} F^{(n,g)} = -\frac{1}{24k} \sum_{n_1=0}^{n} \sum_{g_1=0}^{g} \partial_a F^{(n_1,g_1)} \, \partial_a F^{(n-n_1,g-g_1)} + \frac{1}{24k} \partial_a^2 F^{(n,g-1)} \,, \qquad (1.9)$$

with $k = 2$ for the $\mathcal{N} = 2^*$ theory, and $k = 1$ for the $N_f = 4$ one. The last term in the right hand side of (1.9) is absent when $g = 0$. This case happens in the so-called Nekrasov-Shatashvili (NS) limit $\epsilon_2 \to 0$ with ϵ_1 finite [47], which selects precisely the coefficients $F^{(n,0)}$. Via the modular anomaly equation, exact expressions in terms of modular forms have been obtained for the first few $F^{(n,g)}$ coefficients at generic values of the ϵ parameters in the massless cases [43, 44], and for the massive $\mathcal{N} = 2^*$ theory in the NS limit [45]. Explicit results have been obtained also form the microscopic point of view, computing the non-perturbative instanton corrections by means of localization techniques. In particular, in [46] the expression in terms of modular forms of the first few $F^{(n,g)}$'s, for generic deformation and masses, was inferred from their instanton expansion; these findings suggest a recursion relation among the coefficient of the expansion of the prepotential for large a which is equivalent to the modular anomaly equation (1.9). Finally, to obtain explicit expressions and uncover the modular properties of the deformed theories, one can exploit the AGT relation, as recently done in [48, 49], or the deformed matrix models [50].

In this paper we address various issues. In section 2 we present an efficient method to obtain the exact expressions of the $F^{(n,g)}$'s based on the modular anomaly equation. We show that this equation is equivalent to the heat equation on the non-classical part of the partition function, and use the heat kernel to express the prepotential in terms of its "boundary" value obtained by disregarding all terms that involve the Eisenstein series E_2. With the knowledge of the boundary value obtained from the perturbative 1-loop result (and possibly the very first few instanton terms), one can derive the exact expression in terms of modular forms of the prepotential up to a very high order in the expansion for large a.

The knowledge of the explicit form of the prepotential is crucial in analyzing its modularity properties, and in particular the way S-duality is implemented in the full deformed quantum theory. It is readily established that the generalized prepotential $F(a; \epsilon)$ and its S-dual $\widetilde{F}(\tilde{a}; \epsilon)$ are no longer related to each other by a Legendre transform when $g_s \neq 0$. Recently in [51] it has been proposed that for generic values of the ϵ-parameters the S-duality acts as a modified Fourier transform on the deformed partition function; however, more recently, in [52] it has been conjectured, also on the basis of the explicit results of [46], that it is in fact exactly given by the exact Fourier transform (1.7), which would thus be valid also for $\epsilon_1 + \epsilon_2 \neq 0$ and for massive superconformal $\mathcal{N} = 2$ theories. In section 3 we show that the modular anomaly equation (1.9) implies that this conjecture is indeed correct.

In section 4 we take a further step and show that it is possible to introduce a modified prepotential $\widehat{F}(a; \epsilon)$, determined order by order in g_s^2, in such a way that the S-duality gets reformulated as a Legendre transform of this new quantity. This possibility had already been pointed out in [46], where $\widehat{F}$ was determined by a rather involved procedure imposing order by order the consistency condition that $\mathcal{S}^2[a] = -a$. Here this result follows much

more naturally from the Fourier transform property of $Z(a; \epsilon)$. We find this observation quite intriguing, even if we do not yet have a clear physical interpretation of $\widehat{F}$, and in section 4 and in the Conclusions we further comment on it. Finally, we collect in the appendices several technical details that are useful for the explicit calculations.

2 Solution of the modular anomaly equation and heat kernel

In this section we discuss how to solve the modular anomaly equation (1.9) in two significant models: the mass-deformed $\mathcal{N} = 4$ SU(2) SYM theory, also known as $\mathcal{N} = 2^*$ theory, and the $\mathcal{N} = 2$ SU(2) SYM theory with $N_f = 4$ fundamental flavors. In both cases we show how to reconstruct the generalized effective prepotential, including its non-perturbative terms, from the perturbative ones.[5]

2.1 The $\mathcal{N} = 2^*$ SU(2) theory

The $\mathcal{N} = 2^*$ SYM theory with gauge group SU(2) describes the interactions of an $\mathcal{N} = 2$ gauge vector multiplet with a massive $\mathcal{N} = 2$ hypermultiplet in the adjoint representation of SU(2). After giving a vacuum expectation value to the scalar field ϕ of the vector multiplet:

$$\langle \phi \rangle = \mathrm{diag}\, (a, -a) \,, \tag{2.1}$$

and introducing the parameters ϵ_1 and ϵ_2 of the Nekrasov background [3–5], the deformed prepotential F takes the form (1.2) where

$$F_{\mathrm{cl}} = 2\pi \mathrm{i} \tau_0 \, a^2 \,, \tag{2.2}$$

τ_0 being the bare gauge coupling constant. The coefficients $F^{(n,g)}$ of the ϵ-expansion account for the perturbative and non-perturbative contributions, and are functions of a, of the hypermultiplet mass m and of τ_0 through the Eisenstein series E_2, E_4 and E_6.[6] As shown in [45, 46], they satisfy the modular anomaly equation (1.9) with $k = 2$. Defining

$$\varphi_0 \equiv F_{\mathrm{cl}} - F = - \sum_{n,g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} \, (\epsilon_1 \epsilon_2)^g \, F^{(n,g)} \,, \tag{2.3}$$

it is immediate to show that (1.9) becomes

$$\partial_{E_2} \varphi_0 = \frac{1}{48} (\partial_a \varphi_0)^2 + \frac{\epsilon_1 \epsilon_2}{48} \, \partial_a^2 \varphi_0 \tag{2.4}$$

which is the homogeneous Kardar-Parisi-Zhang (KPZ) equation in one space dimension [55]. To write it in the standard form, namely

$$\partial_t \varphi_0 = \frac{1}{2} \left(\partial_x \varphi_0 \right)^2 + \nu \, \partial_x^2 \varphi_0 \,, \tag{2.5}$$

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[5]A related technique was illustrated in [41] for the $\mathcal{N} = 2^*$ theory at vanishing ϵ; a similar question can be addressed also in the matrix model approach [53, 54].

[6]For these modular functions and for the Jacobi θ-functions appearing in the next subsection, we use the conventions given in appendix A.

it is enough to set

$$t = \frac{E_2}{24}, \quad x = a \quad \text{and} \quad \nu = \frac{\epsilon_1 \epsilon_2}{2}. \tag{2.6}$$

By taking a further derivative of (2.5) with respect to x, one finds the viscous Burgers equation [56]

$$\partial_t u + u\, \partial_x u = \nu\, \partial_x^2 u \tag{2.7}$$

where $u \equiv -\partial_x \varphi_0$. It is well-known that the non-linear Burgers equation (2.7) can be mapped into a linear parabolic equation by means of the Hopf-Cole transformation [57, 58]. The same is true also for the non-linear KPZ equation (2.5). Indeed, writing

$$\varphi_0 = \epsilon_1 \epsilon_2 \log \Psi, \tag{2.8}$$

one easily obtains

$$\partial_t \Psi - \frac{\epsilon_1 \epsilon_2}{2}\, \partial_a^2 \Psi = 0 \tag{2.9}$$

which is the linear heat conduction equation. Note that, using (2.3), we can rewrite (2.8) as

$$\Psi = \exp\left(\frac{\varphi_0}{\epsilon_1 \epsilon_2}\right) = \exp\left(-\frac{F - F_{\text{cl}}}{\epsilon_1 \epsilon_2}\right), \tag{2.10}$$

from which we read that Ψ is the non-classical part of the generalized partition function of the theory. The fact that the partition function is related to a solution of the heat equation was already noticed in [17], even if in a different context, and more recently also in [43].

The general solution of the heat equation (2.9) can be written as the convolution of the heat kernel

$$G(x; t) = \frac{1}{\sqrt{2\pi\epsilon_1 \epsilon_2 t}}\, \exp\left(-\frac{x^2}{2\epsilon_1 \epsilon_2 t}\right), \tag{2.11}$$

with an "initial" condition $\Psi_0 \equiv \Psi|_{t=0}$, namely as

$$\Psi(a; t) = \big(G * \Psi_0\big)(a; t) \tag{2.12}$$

or, more explicitly, as

$$\exp\left(\frac{\varphi_0(a; t)}{\epsilon_1 \epsilon_2}\right) = \frac{1}{\sqrt{2\pi\epsilon_1 \epsilon_2 t}} \int_{-\infty}^{+\infty} dy\, \exp\left(-\frac{(a - y)^2}{2\epsilon_1 \epsilon_2 t} + \frac{\varphi_0(y; 0)}{\epsilon_1 \epsilon_2}\right). \tag{2.13}$$

This formula allows us to reconstruct the dependence on E_2 of the non-classical part of the generalized prepotential starting from φ_0 evaluated at $E_2 = 0$. Let us now give some details.

As discussed in [46] (see also [43]) the perturbative part of the generalized prepotential is

$$F_{\text{pert}} \equiv -\varphi_0\big|_{\text{pert}} = \epsilon_1 \epsilon_2 \Big[\gamma_{\epsilon_1,\epsilon_2}(2a) + \gamma_{\epsilon_1,\epsilon_2}(-2a) - \gamma_{\epsilon_1,\epsilon_2}(2a + \widetilde{m}) - \gamma_{\epsilon_1,\epsilon_2}(-2a + \widetilde{m})\Big] \tag{2.14}$$

where

$$\widetilde{m} = m + \frac{\epsilon_1 + \epsilon_2}{2} \tag{2.15}$$

is the equivariant mass parameter [59], and

$$\gamma_{\epsilon_1,\epsilon_2}(x) = \frac{d}{ds}\left(\frac{\Lambda^s}{\Gamma(s)}\int_0^\infty \frac{dy}{y}\frac{y^s\,\mathrm{e}^{-yx}}{(\mathrm{e}^{-\epsilon_1 y}-1)(\mathrm{e}^{-\epsilon_2 y}-1)}\right)\Bigg|_{s=0} \tag{2.16}$$

is related to the logarithm of the Barnes double Γ-function [5, 20, 43]. Choosing a branch for this logarithm and expanding for small values of ϵ_1 and ϵ_2, one finds [46]

$$\varphi_0\big|_{\mathrm{pert}} = -\frac{1}{2}\,h_0\log\frac{4a^2}{\Lambda^2} + \sum_{\ell=1}^\infty \frac{h_\ell^{(0)}}{2^{\ell+1}\,\ell}\frac{1}{a^{2\ell}} \tag{2.17}$$

where the first few coefficients are

$$h_0 = \frac{1}{4}\big(4m^2-s^2\big)\,, \tag{2.18}$$

$$h_1^{(0)} = \frac{1}{12}h_0\big(h_0+\epsilon_1\epsilon_2\big)\,, \tag{2.19}$$

$$h_2^{(0)} = \frac{1}{120}h_0\big(h_0+\epsilon_1\epsilon_2\big)\big(2h_0-s^2+3\epsilon_1\epsilon_2\big)\,, \tag{2.20}$$

$$h_3^{(0)} = \frac{1}{672}h_0\big(h_0+\epsilon_1\epsilon_2\big)\big(3h_0^2-4h_0s^2+11h_0\epsilon_1\epsilon_2+10(\epsilon_1\epsilon_2)^2-10s^2\epsilon_1\epsilon_2+2s^4\big)\,, \tag{2.21}$$

with $s = \epsilon_1 + \epsilon_2$. In appendix B.1 we also give the expression for $h_4^{(0)}$.

Among the various terms in (2.17), the logarithmic one plays a distinguished rôle because it is exact at 1-loop and does not receive any non-perturbative correction.[7] Therefore, it is natural to expect that if we take it as the "initial condition" at $t = 0$, we can generate all structures of the prepotential that only depend on E_2. Indeed, if in (2.13) we take

$$\varphi_0(y;0) \simeq -\frac{1}{2}\,h_0\log\frac{4y^2}{\Lambda^2}\,, \tag{2.22}$$

then

$$\exp\left(\frac{\varphi_0(a;t)}{\epsilon_1\epsilon_2}\right) \simeq \frac{1}{\sqrt{2\pi\epsilon_1\epsilon_2 t}}\int_{-\infty}^{+\infty}dy\,\exp\left(-\frac{(a-y)^2}{2\epsilon_1\epsilon_2 t}\right)\left(\frac{2y}{\Lambda}\right)^{-\frac{h_0}{\epsilon_1\epsilon_2}}\,. \tag{2.23}$$

With a suitable change of variables, we can recognize in (2.23) the integral representation of the parabolic cylinder functions (see appendix C). Carrying out the integration over y, we can organize the result as an expansion in inverse powers of a as follows

$$\exp\left(\frac{\varphi_0(a;t)}{\epsilon_1\epsilon_2}\right) \simeq \left(\frac{2a}{\Lambda}\right)^q \sum_{\ell=0}^\infty \frac{q(q-1)\cdots(q-2\ell+1)}{2^\ell\,\ell!}\frac{(\epsilon_1\epsilon_2 t)^\ell}{a^{2\ell}} \tag{2.24}$$

where for convenience we have defined

$$q = -\frac{h_0}{\epsilon_1\epsilon_2}\,. \tag{2.25}$$

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[7]This is the reason why the coefficient h_0 does not carry the superscript $^{(0)}$. On the contrary the terms proportional to $a^{-2\ell}$ get corrected by instantons and their exact coefficients h_ℓ's will be the non-perturbative completion of the $h_\ell^{(0)}$'s given in (2.19)–(2.21).

Taking the logarithm of (2.24) and reinstating E_2 according to (2.6), after simple algebra we have

$$\varphi_0 \simeq -\frac{1}{2}\, h_0 \log \frac{4a^2}{\Lambda^2} + \sum_{\ell=1}^{\infty} \frac{h_\ell}{2^{\ell+1}\,\ell}\, \frac{1}{a^{2\ell}} \tag{2.26}$$

where the first few coefficients are[8]

$$h_1 \simeq \frac{1}{12} h_0 \big(h_0 + \epsilon_1\epsilon_2\big)\, E_2\,, \tag{2.27}$$

$$h_2 \simeq \frac{1}{144} h_0 \big(h_0 + \epsilon_1\epsilon_2\big)\big(2h_0 + 3\epsilon_1\epsilon_2\big)\, E_2^2\,, \tag{2.28}$$

$$h_3 \simeq \frac{1}{1728} h_0 \big(h_0 + \epsilon_1\epsilon_2\big)\big(5h_0^2 + 17 h_0\, \epsilon_1\epsilon_2 + 15(\epsilon_1\epsilon_2)^2\big)\, E_2^3\,, \tag{2.29}$$

and so on and so forth. As expected, the h_ℓ's turn out to be quasi-modular forms of weight 2ℓ made only of powers of E_2. Since in this model there is only one form of weight 2, namely E_2 itself, the above expression for h_1 must be exact. This observation is further confirmed by the fact that in the perturbative limit $E_2 \to 1$ (2.27) correctly reduces to $h_1^{(0)}$ given in (2.19). Indeed

$$\Theta_1 \equiv h_1^{(0)} - h_1\big|_{E_2\to1} = 0 \tag{2.30}$$

and in (2.27) we can replace $\simeq$ with $=$.

On the contrary, the h_ℓ's with $\ell \geq 2$ given above are not exact, since there are other modular forms of weight 2ℓ beside E_2^ℓ that could or should be present. For instance, for $\ell = 2$ we have also the Eisenstein series E_4. Moreover, the perturbative limit of (2.28) does not coincide with $h_2^{(0)}$ given in (2.20):

$$\Theta_2 \equiv h_2^{(0)} - h_2\big|_{E_2\to1} = \frac{1}{720} h_0 \big(h_0 + \epsilon_1\epsilon_2\big)\big(2h_0 + 3\epsilon_1\epsilon_2 - 6s^2\big)\,. \tag{2.31}$$

To take this fact into account, we have to change the initial condition in our heat-kernel formula (2.13) and use

$$\varphi_0(y;0) \simeq -\frac{1}{2}\, h_0 \log \frac{4y^2}{\Lambda^2} + \frac{\Theta_2\, E_4}{16\, y^4} \tag{2.32}$$

instead of (2.22). If we do this, after simple algebra we obtain again (2.26) with h_1 as in (2.27) but with h_2 and h_3 replaced by

$$h_2 \simeq \frac{1}{144} h_0 \big(h_0 + \epsilon_1\epsilon_2\big)\left[\big(2h_0 + 3\epsilon_1\epsilon_2\big)\, E_2^2 + \frac{1}{5}\big(2h_0 + 3\epsilon_1\epsilon_2 - 6s^2\big)\, E_4\right]\,, \tag{2.33}$$

$$h_3 \simeq \frac{1}{1728} h_0 \big(h_0 + \epsilon_1\epsilon_2\big)\bigg[\big(5h_0^2 + 17 h_0\, \epsilon_1\epsilon_2 + 15(\epsilon_1\epsilon_2)^2\big)\, E_2^3$$

$$+ \frac{3}{5}\big(2h_0 + 5\epsilon_1\epsilon_2\big)\big(2h_0 + 3\epsilon_1\epsilon_2 - 6s^2\big)\, E_2 E_4\bigg]\,. \tag{2.34}$$

---

[8]It is worth noting that, while the coefficients of $a^{-2\ell}$ in the expansion of $\exp\left(\frac{\varphi_0(a;t)}{\epsilon_1\epsilon_2}\right)$ given in (2.24) are polynomials of degree 2ℓ in h_0, the coefficients h_ℓ in the expansion of φ_0 are polynomials of degree $\ell+1$ in h_0. This is simply due to dimensional reasons. The cancellations that occur in passing from $\exp\left(\frac{\varphi_0(a;t)}{\epsilon_1\epsilon_2}\right)$ to φ_0 are a consequence of the properties of the parabolic cylinder functions as explained in appendix C.

In this way we generate all terms depending on E_2 and those that are linear in E_4. Actually, since there are no other possible forms of weight 4 other than E_2^2 and E_4, the above expression for h_2 is exact and its perturbative limit indeed coincides with $h_2^{(0)}$ given in (2.20). Thus, in (2.33) we can substitute $\simeq$ with $=$.

Instead, the expression (2.34) for h_3 is still incomplete since there is a further modular structure of weight 6 that has not yet appeared, namely E_6. Moreover, the perturbative limit of (2.34) does not reproduce the 1-loop result:

$$\Theta_3 \equiv h_3^{(0)} - h_3\big|_{E_2,E_4\to 1} \tag{2.35}$$
$$= \frac{1}{60480}h_0\big(h_0 + \epsilon_1\epsilon_2\big)\big(11h_0^2 + 59h_0\epsilon_1\epsilon_2 + 60(\epsilon_1\epsilon_2)^2 - 108h_0 s^2 - 270 s^2\epsilon_1\epsilon_2 + 180 s^4\big)\,.$$

This fact forces us to change once again the initial condition for the heat kernel and use in (2.13)

$$\varphi_0(y;0) \simeq -\frac{1}{2}\,h_0 \log\frac{4y^2}{\Lambda^2} + \frac{\Theta_2\,E_4}{16\,y^4} + \frac{\Theta_3\,E_6}{48\,y^6} \tag{2.36}$$

instead of (2.32). By doing this, the resulting expression for φ_0 has the form (2.26) with h_1 and h_2 as in (2.27) and (2.33) respectively, and with h_3 given by

$$h_3 = \frac{1}{1728}\,h_0\big(h_0 + \epsilon_1\epsilon_2\big)\Big[\big(5h_0^2 + 17h_0\,\epsilon_1\epsilon_2 + 15(\epsilon_1\epsilon_2)^2\big)\,E_2^3$$
$$+ \frac{3}{5}\big(2h_0 + 5\epsilon_1\epsilon_2\big)\big(2h_0 + 3\epsilon_1\epsilon_2 - 6s^2\big)\,E_2 E_4 \tag{2.37}$$
$$+ \frac{1}{35}\big(11h_0^2 + 59h_0\epsilon_1\epsilon_2 + 60(\epsilon_1\epsilon_2)^2 - 108h_0 s^2 - 270 s^2\epsilon_1\epsilon_2 + 180 s^4\big)E_6\Big]\,.$$

Using (2.18) to express h_0 in terms of the hypermultiplet mass, one can check that our results perfectly match those we obtained in [46] from explicit multi-instanton calculations combined with the requirement of quasi-modularity.

This procedure can be further iterated with no conceptual difficulties to obtain the exact expressions of the higher coefficients h_ℓ. In appendix B.1 we provide some details for the calculation of h_4. Of course, the algebraic complexity increases with ℓ but still this method remains computationally very efficient. In particular we would like to remark that the knowledge of the 1-loop prepotential (2.14) together with the heat-kernel formula (2.13) allows one to obtain the full non-perturbative expressions of h_ℓ up to $\ell = 5$. In h_6, which is a quasi-modular form of weight 12, there are two independent structures of weight 12 not involving E_2, namely E_6^2 and E_4^3, and thus the perturbative information is not sufficient to fix the relative coefficients. However, combining this with the 1-instanton corrections [46], one is able to resolve the ambiguity and find h_6. In the same way one obtains all coefficients up to h_{11}; in order to find h_{12}, which contains three different structures of weight 24 independent of E_2, namely E_6^4, $E_6^2E_4^3$ and E_4^6, the 2-instanton results become necessary. This structure keeps repeating itself.

Thus, we may conclude that the 1-loop and the first instanton corrections combined with the heat-kernel equation permit to reconstruct the exact generalized prepotential of the theory to a very high degree of accuracy in a systematic and algebraic fashion, generalizing the method and the results of [41] to the case of arbitrary values of ϵ_1 and ϵ_2.

2.2 The SU(2) theory with $N_f = 4$

We now repeat this analysis in the $\mathcal{N} = 2$ SU(2) SYM theory with four fundamental flavors. As is well-known, this model has a vanishing 1-loop β-function and its conformal invariance is broken only by the flavor masses m_f ($f = 1, \ldots, 4$). After giving a vacuum expectation value to the adjoint scalar field as in (2.1), the deformed prepotential takes the form (1.2) with

$$F_{\text{cl}} = \pi i \tau_0 \, a^2 \,. \tag{2.38}$$

Here we have used the standard normalization [2] for the classical term (which differs by a factor of 2 from the $\mathcal{N} = 2^*$ one). The coefficients $F^{(n,g)}$ of the ϵ-expansion depend on a, on the bare coupling τ_0 through the Eisenstein series E_2, E_4, E_6 and the Jacobi θ-functions, and also on the hypermultiplet masses through the SO(8) flavor invariants

$$
\begin{aligned}
R &= \frac{1}{2} \sum_f m_f^2 \,, \\
T_1 &= \frac{1}{12} \sum_{f<f'} m_f^2 m_{f'}^2 - \frac{1}{24} \sum_f m_f^4 \,, \\
T_2 &= -\frac{1}{24} \sum_{f<f'} m_f^2 m_{f'}^2 + \frac{1}{48} \sum_f m_f^4 - \frac{1}{2} \prod_f m_f \,, \\
N &= \frac{3}{16} \sum_{f<f'<f''} m_f^2 m_{f'}^2 m_{f''}^2 - \frac{1}{96} \sum_{f \neq f'} m_f^2 m_{f'}^4 + \frac{1}{96} \sum_f m_f^6 \,.
\end{aligned}
\tag{2.39}
$$

As shown in [46] (see also [43]), the coefficients $F^{(n,g)}$ satisfy the modular anomaly equation (1.9) with $k = 1$. We can therefore follow the same steps described in section 2.1 and prove that

$$\exp\left(\frac{\varphi_0(a;t)}{\epsilon_1 \epsilon_2}\right) = \frac{1}{\sqrt{2\pi\epsilon_1\epsilon_2 t}} \int_{-\infty}^{+\infty} dy \, \exp\left(-\frac{(a-y)^2}{2\epsilon_1\epsilon_2 t} + \frac{\varphi_0(y;0)}{\epsilon_1 \epsilon_2}\right). \tag{2.40}$$

This is the same as the $\mathcal{N} = 2^*$ equation (2.13), but with

$$t = \frac{E_2}{12} \tag{2.41}$$

instead of $t = E_2/24$.

As before, we use this equation to reconstruct $\varphi_0(a;t)$ starting from an "initial" condition at $t = 0$. The latter can be taken from the $N_f = 4$ perturbative prepotential

$$\varphi_0\big|_{\text{pert}} = -\epsilon_1\epsilon_2 \left[\gamma_{\epsilon_1\epsilon_2}(2a) + \gamma_{\epsilon_1\epsilon_2}(-2a) - \sum_{f=1}^{4}\left(\gamma_{\epsilon_1\epsilon_2}(a + \widetilde{m}_f) + \gamma_{\epsilon_1\epsilon_2}(-a + \widetilde{m}_f)\right)\right] \tag{2.42}$$

where $\gamma_{\epsilon_1\epsilon_2}$ is given in (2.16) and the equivariant masses $\tilde{m}_f$ are defined as in (2.15). Expanding for large values of a, we get

$$\varphi_0\big|_{\text{pert}} = -\frac{1}{2} h_0 \log \frac{a^2}{\Lambda^2} + \sum_{\ell=1}^{\infty} \frac{h_\ell^{(0)}}{2^{\ell+1} \ell} \frac{1}{a^{2\ell}} \tag{2.43}$$

where the first few coefficients are [46]

$$h_0 = \frac{1}{2}\big(4R - s^2 + \epsilon_1\epsilon_2\big),\tag{2.44}$$

$$h_1^{(0)} = \frac{1}{6}h_0(h_0 + \epsilon_1\epsilon_2) - 4T_1,\tag{2.45}$$

$$h_2^{(0)} = \frac{1}{240}\big(16h_0^3 + 56h_0^2\epsilon_1\epsilon_2 - 16h_0^2 s^2 - 960h_0 T_1 + 28h_0(\epsilon_1\epsilon_2)^2$$
$$- 16h_0 s^2\epsilon_1\epsilon_2 - 1440T_1\epsilon_1\epsilon_2 + 480T_1 s^2 - 3s^2(\epsilon_1\epsilon_2)^2 + 768N\big).\tag{2.46}$$

In appendix B.2 we also give the expression for $h_3^{(0)}$. All coefficients, except h_0, receive non-perturbative corrections due to instantons which can be explicitly computed using localization methods. Using as initial condition

$$\varphi_0(y; 0) \simeq -\frac{1}{2}\, h_0 \log \frac{y^2}{\Lambda^2}\,,\tag{2.47}$$

and following the same steps described in the $\mathcal{N} = 2^*$ theory, from (2.40) we generate all terms in the h_ℓ's which only depend on E_2. For example we find

$$h_1 \simeq \frac{1}{6}h_0\big(h_0 + \epsilon_1\epsilon_2\big) E_2\,,\tag{2.48}$$

$$h_2 \simeq \frac{1}{36}h_0\big(h_0 + \epsilon_1\epsilon_2\big)\big(2h_0 + 3\epsilon_1\epsilon_2\big) E_2^2\,.\tag{2.49}$$

In the perturbative limit, when $E_2 \to 1$, these expressions do not reduce to (2.45) and (2.46), signaling the fact that other structures have to be considered in the initial condition. Indeed, in the $N_f = 4$ theory combining the usual modular properties of the Jacobi θ-functions with the triality transformations of the mass invariants (2.39), one can construct several modular forms. For example, taking into account that the generators $\mathcal{T}$ and $\mathcal{S}$ of the modular group act on T_1 and T_2 as [2]

$$\begin{aligned} \mathcal{T} &: & T_1 &\to T_1\,, & T_2 &\to -T_1 - T_2\,, \\ \mathcal{S} &: & T_1 &\to T_2\,, & T_2 &\to T_1\,, \end{aligned}\tag{2.50}$$

and on the θ-functions as

$$\begin{aligned} \mathcal{T} &: & \theta_2^4 &\to -\theta_2^4\,, & \theta_3^4 &\to \theta_4^4\,, & \theta_4^4 &\to \theta_3^4\,, \\ \mathcal{S} &: & \theta_2^4 &\to -\tau_0^2\theta_2^4\,, & \theta_3^4 &\to -\tau_0^2\theta_3^4\,, & \theta_4^4 &\to -\tau_0^2\theta_4^4\,, \end{aligned}\tag{2.51}$$

one can easily check that the combination

$$T_1\,\theta_4^4 - T_2\,\theta_2^4\tag{2.52}$$

is a modular form of weight 2. As such it could/should be present in the exact expression of h_1. In the perturbative limit, $\theta_4 \to 1$ and $\theta_2 \to 0$, it simply reduces to T_1, and thus by comparing with $h_1^{(0)}$ in (2.45) we are led to change the initial condition (2.47) into

$$\varphi_0(y; 0) \simeq -\frac{1}{2}\, h_0 \log \frac{y^2}{\Lambda^2} - \frac{T_1\,\theta_4^4 - T_2\,\theta_2^4}{y^2}\,.\tag{2.53}$$

Inserting this into the heat-kernel formula (2.40) we obtain the final expression for h_1:

$$h_1 = \frac{1}{6}h_0(h_0 + \epsilon_1\epsilon_2) E_2 - 4(T_1\,\theta_4^4 - T_2\,\theta_2^4)\,, \tag{2.54}$$

and the following expression for h_2:

$$h_2 \simeq \frac{1}{36}h_0(h_0 + \epsilon_1\epsilon_2)(2h_0 + 3\epsilon_1\epsilon_2) E_2^2 - \frac{4}{3}(2h_0 + 3\epsilon_1\epsilon_2)(T_1\,\theta_4^4 - T_2\,\theta_2^4) E_2\,. \tag{2.55}$$

This fails to reproduce the perturbative limit (2.46) since

$$h_2^{(0)} - h_2\big|_{E_2\to 1, \theta_4\to 1, \theta_2\to 0} = \Theta_2 + \Theta_2' T_1 \tag{2.56}$$

with

$$\Theta_2 = \frac{1}{720}\big(8h_0^3 + 68h_0^2\epsilon_1\epsilon_2 - 48h_0^2 s^2 + 24h_0(\epsilon_1\epsilon_2)^2 - 48h_0 s^2\epsilon_1\epsilon_2 - 9s^2(\epsilon_1\epsilon_2)^2 + 2304N\big)\,,$$
$$\Theta_2' = -\frac{2}{3}\big(2h_0 - 3s^2 + 3\epsilon_1\epsilon_2\big)\,. \tag{2.57}$$

Note, however, that in the $N_f = 4$ theory there are two modular forms of weight 4 that should be considered: the Eisenstein series E_4 (that can be multiplied by any modular invariant term made up with R, N, s^2 and $\epsilon_1\epsilon_2$) and the following combination

$$T_1\theta_4^8 + 2(T_1 + T_2)\theta_2^4\theta_4^4 + T_2\theta_2^8\,, \tag{2.58}$$

whose perturbative limits are, respectively, 1 and T_1 Thus, we can easily fix the problem by changing once more the initial condition in the heat-kernel formula and use

$$\varphi_0(y;0) \simeq -\frac{1}{2}\,h_0\log\frac{y^2}{\Lambda^2} - \frac{T_1\,\theta_4^4 - T_2\,\theta_2^4}{y^2} - \frac{\Theta_2 E_4 + \Theta_2'\big(T_1\theta_4^8 + 2(T_1 + T_2)\theta_2^4\theta_4^4 + T_2\theta_2^8\big)}{16y^4}\,. \tag{2.59}$$

In this way we obtain the exact expression for h_2:

$$h_2 = \frac{1}{36}h_0(h_0 + \epsilon_1\epsilon_2)(2h_0 + 3\epsilon_1\epsilon_2) E_2^2 - \frac{4}{3}(2h_0 + 3\epsilon_1\epsilon_2)(T_1\,\theta_4^4 - T_2\,\theta_2^4) E_2$$
$$+ \frac{1}{720}\big(8h_0^3 + 68h_0^2\epsilon_1\epsilon_2 - 48h_0^2 s^2 + 24h_0(\epsilon_1\epsilon_2)^2 - 48h_0 s^2\epsilon_1\epsilon_2 - 9s^2(\epsilon_1\epsilon_2)^2 + 2304N\big) E_4$$
$$- \frac{2}{3}\big(2h_0 - 3s^2 + 3\epsilon_1\epsilon_2\big)\big(T_1\theta_4^8 + 2(T_1 + T_2)\theta_2^4\theta_4^4 + T_2\theta_2^8\big)\,. \tag{2.60}$$

One can check that (2.54) and (2.60) precisely match eq.s (3.20) and (3.21) of [46]. Of course we can continue iteratively this way to determine the exact expressions of the higher coefficients h_ℓ's and hence reconstruct term by term the exact generalized prepotential. In appendix B.2 we provide some details for the calculation of h_3, which shows the consistency and the efficiency of the entire procedure.

3 S-duality as a Fourier transform

The modular anomaly equation is a very powerful tool for several different purposes: in fact it can be used not only to determine the generalized prepotential F, as we have seen in the previous section, but also to investigate its behavior under the S-duality transformation and thus its properties at strong coupling, as we are going to see in this section. In particular, exploiting the modular anomaly equation or its equivalent heat-kernel version, we will prove the following general result: for arbitrary values of ϵ_1 and ϵ_2, the generalized prepotential F and its S-dual $\widetilde{F} = \mathcal{S}[F]$ are related through an *exact* Fourier transform, up to an important normalization factor, namely

$$\exp\Big(-\frac{\widetilde{F}(\tilde{a})}{\epsilon_1\epsilon_2}\Big) = \sqrt{\frac{\mathrm{i}\tau_0}{\epsilon_1\epsilon_2}} \int_{-\infty}^{+\infty} dx \, \exp\Big(\frac{2\pi\mathrm{i}\,\tilde{a}\,x - F(x)}{\epsilon_1\epsilon_2}\Big) \tag{3.1}$$

where $\tilde{a} = \mathcal{S}[a]$. For simplicity, we will discuss only the $N_f = 4$ case, but it is obvious that our derivation works in the $\mathcal{N} = 2^*$ theory as well.

Our starting point is the heat-kernel formula (2.40), which for convenience we rewrite here as

$$\Psi(a;t) = \big(G * \Psi_0\big)(a;t) \tag{3.2}$$

using the same convolution notation introduced in (2.10)–(2.12). We then apply to it an S-duality transformation with the minimal assumption that[9]

$$\mathcal{S}[\epsilon_1\epsilon_2] = \epsilon_1\epsilon_2\,, \tag{3.3}$$

and get

$$\exp\Big(\frac{\widetilde{\varphi}_0(\tilde{a};\tilde{t})}{\epsilon_1\epsilon_2}\Big) = \frac{1}{\sqrt{2\pi\epsilon_1\epsilon_2\tilde{t}}} \int_{-\infty}^{+\infty} d\tilde{y} \, \exp\Big(-\frac{(\tilde{a}-\tilde{y})^2}{2\epsilon_1\epsilon_2\tilde{t}} + \frac{\widetilde{\varphi}_0(\tilde{y};0)}{\epsilon_1\epsilon_2}\Big)\,. \tag{3.4}$$

Here we have set $\widetilde{\varphi}_0 = \mathcal{S}[\varphi_0]$, $\tilde{y} = \mathcal{S}[y]$ and

$$\tilde{t} = \mathcal{S}[t] = \tau_0^2\Big(t + \frac{1}{2\pi\mathrm{i}\tau_0}\Big)\,. \tag{3.5}$$

Note that this last equation is simply a consequence of the anomalous transformation properties of the second Eisenstein series E_2 under $\tau_0 \to \mathcal{S}[\tau_0] = -1/\tau_0$. Also the expression of $\widetilde{\varphi}_0(\tilde{y};0)$, appearing in the right hand side of (3.4), can be easily computed. In fact, $\varphi_0(y;0)$ is the part of the prepotential with E_2 set to zero and, up to the logarithmic term, is a sum of terms depending on powers of E_4, E_6 and the Jacobi functions θ_j^4. As one can see from the examples worked out in the previous section (see, *e.g.* (2.53) and (2.59)), such terms are typically of the form

$$\varphi_0(y;0) \propto \frac{E_4^\alpha\,E_6^\beta\,(\theta_j^4)^\gamma\,M^{4\alpha+6\beta+2\gamma+2}}{y^{4\alpha+6\beta+2\gamma}} \tag{3.6}$$

---

[9]Actually, following the considerations made in [60] for the ϵ-deformed conformal Chern-Simons theory in three dimensions, one can show that $\mathcal{S}$ acts as an exchange of ϵ_1 and ϵ_2.

where α, β and γ are non-negative integers and M stands for a generic mass structure which is needed for dimensional reasons. Taking into account possible exchanges of the Jacobi θ-functions among themselves and of the mass invariants T_1 and T_2 as described in (2.51) and (2.52), the numerator of (3.6) transforms as a modular form of weight $4\alpha + 6\beta + 2\gamma$, and thus

$$\widetilde{\varphi}_0(\tilde{y};0) \ \propto \ \tau_0^{4\alpha+6\beta+2\gamma} \, \frac{E_4^\alpha \, E_6^\beta \, (\theta_j^4)^\gamma \, M^{4\alpha+6\beta+2\gamma+2}}{\tilde{y}^{\,4\alpha+6\beta+2\gamma}} \, . \tag{3.7}$$

Hence we simply have

$$\widetilde{\varphi}_0(\tilde{y};0) = \varphi_0\Big(\frac{\tilde{y}}{\tau_0};0\Big) \, . \tag{3.8}$$

Plugging this into (3.4), using (3.5) and changing the integration variable $\tilde{y} \to \tau_0 \, y$, we end up with

$$\exp\Big(\frac{\widetilde{\varphi}_0(\,\tilde{a};\tilde{t}\,)}{\epsilon_1\epsilon_2}\Big) = \frac{1}{\sqrt{2\pi\epsilon_1\epsilon_2 \, \tilde{t}/\tau_0^2}} \ \int_{-\infty}^{+\infty} dy \ \exp\Big(-\frac{\big(\frac{\tilde{a}}{\tau_0}-y\big)^2}{2\epsilon_1\epsilon_2 \, \tilde{t}/\tau_0^2} + \frac{\varphi_0(y;0))}{\epsilon_1\epsilon_2}\Big) \, . \tag{3.9}$$

The right hand side has the same structure as the original equation (2.40); in fact it is the convolution of the "initial" condition $\Psi_0 = \exp\big(\frac{\varphi_0(y;0)}{\epsilon_1\epsilon_2}\big)$ with a gaussian heat kernel $\widetilde{G}$ with a rescaled parameter:

$$\widetilde{G}(x;\tilde{t}) = \frac{1}{\sqrt{2\pi\epsilon_1\epsilon_2 \, \tilde{t}/\tau_0^2}} \, \exp\Big(-\frac{x^2}{2\epsilon_1\epsilon_2 \, \tilde{t}/\tau_0^2}\Big) \, . \tag{3.10}$$

Defining

$$\Psi_{\mathcal{S}}\Big(\frac{\tilde{a}}{\tau_0};\tilde{t}\Big) \equiv \exp\Big(\frac{\widetilde{\varphi}_0(\tilde{a};\tilde{t})}{\epsilon_1\epsilon_2}\Big) \, , \tag{3.11}$$

we can rewrite (3.9) more compactly as

$$\Psi_{\mathcal{S}}\Big(\frac{\tilde{a}}{\tau_0};\tilde{t}\Big) = \big(\widetilde{G} * \Psi_0\big)\Big(\frac{\tilde{a}}{\tau_0};\tilde{t}\Big) \, . \tag{3.12}$$

The similarity between (3.2) and (3.12) suggests to take the Fourier transform $\mathfrak{F}$,[10] of both equations. Recalling that the Fourier transform of a convolution of two functions f and g is

$$\mathfrak{F}[f * g] = \mathfrak{F}[f] \, \mathfrak{F}[g] \, , \tag{3.13}$$

and that the Fourier transform of a gaussian is

$$\mathfrak{F}\left[\exp(-\alpha x^2)\right](k) = \sqrt{\frac{\pi}{\alpha}} \, \exp\Big(-\frac{\pi^2}{\alpha}k^2\Big) \, , \tag{3.14}$$

---

[10]Our conventions for the Fourier transform $\mathfrak{F}$ and its inverse $\mathfrak{F}^{-1}$ are

$$\mathfrak{F}[f](k) = \int_{-\infty}^{+\infty} dx \ \mathrm{e}^{-2\pi \mathrm{i}\, x\, k} \, f(x) \, , \quad \mathfrak{F}^{-1}[g](x) = \int_{-\infty}^{+\infty} dk \ \mathrm{e}^{+2\pi \mathrm{i}\, x\, k} \, g(k) \, .$$

from (3.2) and (3.12) we find

$$\mathfrak{F}[\Psi](k) = \exp\left(-2\pi^2\epsilon_1\epsilon_2 t\, k^2\right)\, \mathfrak{F}[\Psi_0](k)\,,$$

$$\mathfrak{F}\left[\Psi_{\mathcal{S}}\right](k) = \exp\left(-\frac{2\pi^2\epsilon_1\epsilon_2\,\tilde{t}}{\tau_0^2}\, k^2\right)\mathfrak{F}[\Psi_0](k) \tag{3.15}$$

$$= \exp\left(-2\pi^2\epsilon_1\epsilon_2\big(t + (2\pi\mathrm{i}\,\tau_0)^{-1}\big)\, k^2\right)\mathfrak{F}[\Psi_0](k)$$

where in the last step we have used (3.5). By taking the ratio of these two equations, we can eliminate the boundary factor $\mathfrak{F}[\Psi_0]$ and also all explicit dependence on t, obtaining

$$\mathfrak{F}\left[\Psi_{\mathcal{S}}\right](k) = \exp\left(\frac{\mathrm{i}\pi\epsilon_1\epsilon_2}{\tau_0}\, k^2\right)\, \mathfrak{F}[\Psi](k)\,. \tag{3.16}$$

Finally, to get $\Psi_{\mathcal{S}}$ we apply to (3.16) the inverse Fourier transform which yields

$$\begin{aligned}
\Psi_{\mathcal{S}}(\tilde{x}, \tilde{t}) &= \int_{-\infty}^{+\infty} dk\; \mathrm{e}^{+2\mathrm{i}\pi k\,\tilde{x}}\; \mathfrak{F}\left[\Psi_{\mathcal{S}}\right](k) \\
&= \int_{-\infty}^{+\infty} dk\; \mathrm{e}^{+2\mathrm{i}\pi k\,\tilde{x}}\; \mathrm{e}^{\frac{\mathrm{i}\pi\epsilon_1\epsilon_2}{\tau_0} k^2} \int_{-\infty}^{+\infty} dx\; \mathrm{e}^{-2\mathrm{i}\pi k\,x}\; \mathrm{e}^{\frac{\varphi_0(x;t)}{\epsilon_1\epsilon_2}} \\
&= \sqrt{\frac{\mathrm{i}\tau_0}{\epsilon_1\epsilon_2}}\; \mathrm{e}^{-\frac{\mathrm{i}\pi\tau_0}{\epsilon_1\epsilon_2}\tilde{x}^2} \int_{-\infty}^{+\infty} dx\; \mathrm{e}^{\frac{2\pi\mathrm{i}\tau_0\,\tilde{x}\,x - \pi\mathrm{i}\tau_0 x^2 + \varphi_0(x,t)}{\epsilon_1\epsilon_2}}
\end{aligned} \tag{3.17}$$

where the square-root normalization factor in the last line originates from the integral over k.

According to (3.11) we must evaluate $\Psi_{\mathcal{S}}$ at $\tilde{x} = \tilde{a}/\tau_0$. If we do so, we get

$$\exp\left(\frac{\widetilde{\varphi}_0(\tilde{a};\tilde{t})}{\epsilon_1\epsilon_2}\right) = \sqrt{\frac{\mathrm{i}\tau_0}{\epsilon_1\epsilon_2}}\; \mathrm{e}^{-\frac{\mathrm{i}\pi}{\tau_0\,\epsilon_1\epsilon_2}\tilde{a}^2} \int_{-\infty}^{+\infty} dx\; \mathrm{e}^{\frac{2\pi\mathrm{i}\,\tilde{a}\,x - \pi\mathrm{i}\tau_0 x^2 + \varphi_0(x,t)}{\epsilon_1\epsilon_2}}\,, \tag{3.18}$$

which can be put in a more transparent form if we observe that the two terms quadratic in x and $\tilde{a}$ appearing in the right hand side are, respectively, the classical parts of the prepotential F and of its S-dual $\widetilde{F}$; indeed

$$\begin{aligned}
F(x) &= \pi\mathrm{i}\tau_0\, x^2 - \varphi_0(x, t)\,, \\
\widetilde{F}(\tilde{a}) &= \frac{\pi\mathrm{i}}{\tau_0}\, \tilde{a}^2 - \widetilde{\varphi}_0(\tilde{a};\tilde{t})\,.
\end{aligned} \tag{3.19}$$

Taking this into account, we then obtain the announced result (3.1), which, for future convenience, we rewrite here after setting $\epsilon_1\epsilon_2 = g_s^2$:

$$\exp\left(-\frac{\widetilde{F}(\tilde{a})}{g_s^2}\right) = \sqrt{\frac{\mathrm{i}\tau_0}{g_s^2}} \int_{-\infty}^{+\infty} dx\; \exp\left(\frac{2\pi\mathrm{i}\,\tilde{a}\,x - F(x)}{g_s^2}\right)\,. \tag{3.20}$$

A few comments are in order. Recently, in [51] it has been argued that for generic values of the deformation parameters ϵ_1 and ϵ_2, the S-duality acts as a modified Fourier transform on the prepotential of the $N_f = 4$ SYM theory for some particular values of the flavor masses, while more recently in [52] it has been conjectured, building also on the

explicit results of [46], that the S-duality must act precisely as a Fourier transformation. Our present analysis provides a general derivation and a proof of this result based on the modular anomaly equation and the minimal (and natural) assumption (3.3). On the other hand, the fact that the generalized prepotential and its S-dual are related by an exact Fourier transform is perfectly consistent with the interpretation of $\exp\left(-\frac{F(a)}{\epsilon_1 \epsilon_2}\right)$ as a wave function, of a and $\tilde{a} = \mathcal{S}[a]$ as a pair of canonically conjugate variables and, correspondingly, of the S-duality as a canonical transformation, in complete analogy with what has been observed in topological string models and local CY compactifications of Type II string theories (see for example [18] and references therein).

Finally, we observe that by inserting in (3.20) the expansion of F for large values of x

$$F(x) = \pi \mathrm{i} \tau_0 \, x^2 + \frac{1}{2} \, h_0 \log \frac{x^2}{\Lambda^2} - \sum_{\ell=1}^{\infty} \frac{h_\ell}{2^{\ell+1} \, \ell} \frac{1}{x^{2\ell}} \tag{3.21}$$

and by carrying out the integration over x using the parabolic cylinder functions as described in section 2, we can derive the S-duality transformation properties of the coefficients h_ℓ and check that they indeed are modular forms of weight 2ℓ with with anomalous terms due to the presence of the second Eisenstein series E_2, in perfect agreement with the explicit expressions derived from multi-instanton calculations.

4   S-duality in the saddle-point approximation

In this section we consider the Fourier transform relation (3.20) between the generalized prepotential F and its S-dual $\widetilde{F}$ and evaluate it in the saddle-point approximation for small values of g_s^2. To do so, it is convenient to reorganize the generalized prepotential F in powers of g_s^2 and write

$$F = \sum_{g=0}^{\infty} g_s^{2g} \, F_g \quad \text{with} \quad F_g = F_{\mathrm{cl}} \, \delta_{g,0} + \sum_{s=0}^{\infty} s^{2n} \, F^{(n,g)} \,. \tag{4.1}$$

As is well-known, the coefficients F_g are related to the refined topological string amplitudes at genus g. Note that the genus-zero term F_0 corresponds to the NS prepotential and that, in the limit $s \to 0$, it reduces to $\mathcal{F} = F_{\mathrm{cl}} + F^{(0,0)}$, which is the prepotential of the SW theory.

For small g_s^2, the integral in (3.20) is dominated by the "classical" value $x = a_0$ that extremizes the exponent $\left(2\pi \mathrm{i} \tilde{a} x - F_0(x)\right)/g_s^2$, that is

$$2\pi \mathrm{i} \tilde{a} = \partial F_0(a_0) \,. \tag{4.2}$$

Computing the fluctuations up to second order in g_s^2 and using the standard saddle-point method, we easily obtain

$$\widetilde{F}(\tilde{a}) = F(a_0) + g_s^2 \, W_1(a_0) + g_s^4 \, W_2(a_0) - \partial F_0(a_0) \, a_0 \tag{4.3}$$

where

$$\begin{aligned}
W_1 &= \frac{1}{2} \log \frac{\partial^2 F_0}{2\pi \mathrm{i} \tau_0} \,, \\
W_2 &= \frac{1}{2} \frac{\partial^2 F_1}{\partial^2 F_0} + \frac{1}{8} \frac{\partial^4 F_0}{\left(\partial^2 F_0\right)^2} - \frac{1}{2} \frac{\left(\partial F_1\right)^2}{\partial^2 F_0} - \frac{1}{2} \frac{\partial F_1 \partial^3 F_0}{\left(\partial^2 F_0\right)^2} - \frac{5}{24} \frac{\left(\partial^3 F_0\right)^2}{\left(\partial^2 F_0\right)^3} \,,
\end{aligned} \tag{4.4}$$

and where a_0 has to be thought as a function of $\tilde{a}$ through the inverse of (4.2). These expressions agree with those that can be found for instance in [18] (modulo the different sign conventions and the different number of variables we are using). The perturbative corrections W_g have a diagrammatic interpretation in which the propagator is $1/\partial^2 F_0$ and the vertices are given by multiple derivatives of the F_g, all evaluated at a_0; in the expression of W_g only vertices constructed out of derivatives of F_k with $k < g$ appear; this is again in full similarity to what is described in [18].

In the NS limit $g_s^2 \to 0$, (4.3) reduces to

$$\widetilde{F}_0(\tilde{a}) = F_0(a_0) - \partial F_0(a_0)\, a_0 = F_0(a_0) - 2\pi \mathrm{i}\tilde{a}\, a_0 \tag{4.5}$$

which is the standard and well-known Legendre transform relation between the NS prepotential and its S-dual. From this it is straightforward to obtain the S-duality transformations of the derivatives of F_0. For example one has

$$\mathcal{S}[\partial F_0] = \frac{\partial \mathcal{S}[F_0]}{\partial \tilde{a}} = \frac{\partial a_0}{\partial \tilde{a}}\, \frac{\partial \mathcal{S}[F_0]}{\partial a_0} = -2\pi \mathrm{i}a_0\,, \tag{4.6}$$

$$\mathcal{S}[\partial^2 F_0] = -\frac{(2\pi \mathrm{i})^2}{\partial^2 F_0}\,, \tag{4.7}$$

and so on and so forth. Then, exploiting (4.6) and (4.2), it immediately follows that

$$\mathcal{S}^2[a_0] = \mathcal{S}[\tilde{a}] = -a_0\,, \tag{4.8}$$

and, using (4.5), also that

$$\mathcal{S}^2[F_0] = \mathcal{S}[\widetilde{F}_0] = F_0\,. \tag{4.9}$$

We now analyze what happens when the corrections in g_s^2 are taken into account. We will show that, with suitable redefinitions, it is possible to write the relation between the prepotential and its S-dual in the form of a Legendre transform and to generalize the relations (4.8) and (4.9) also for finite values of g_s^2. The procedure we follow is similar to the one that is used to derive the effective action from the generating functional of the connected Green functions in quantum field theory, but with some important differences which will point out later. Let us consider the first-order corrections in g_s^2, namely F_1 inside F and W_1 in the S-duality formula. If we want that the right hand side of (4.3) becomes a Legendre transform, we have to redefine the classical saddle-point a_0 and the prepotential F according to

$$a = a_0 + \sum_{g=1}^{\infty} g_s^{2g}\, \delta a_g\,, \tag{4.10}$$

$$\widehat{F} = F + \sum_{g=1}^{\infty} g_s^{2g}\, \Delta_g = F_0 + \sum_{g=1}^{\infty} g_s^{2g}\big(F_g + \Delta_g\big)\,. \tag{4.11}$$

Inserting these expressions into (4.3), and keeping consistently only the terms up to order g_s^2, we can rewrite the S-duality relation as

$$\begin{aligned}
\mathcal{S}[\widehat{F}](\tilde{a}) = \widehat{F}(a) - \partial\widehat{F}(a)\, a &+ g_s^2\Big[W_1(a) - \Delta_1(a) + \mathcal{S}[\Delta_1](a)\Big] \\
&+ g_s^2\Big[\partial^2 F_0(a)\, \delta a_1 + \partial F_1(a) + \partial\Delta_1(a)\Big]a + \mathcal{O}(g_s^4)\,.
\end{aligned} \tag{4.12}$$

This reduces to a Legendre transform if the square brackets vanish, i.e. if

$$\Delta_1 - \mathcal{S}[\Delta_1] = W_1 \,, \tag{4.13}$$

$$\delta a_1 = -\frac{1}{\partial^2 F_0}\big(\partial F_1 + \partial \Delta_1\big)\,. \tag{4.14}$$

It is easy to verify that a solution of these equations is given by

$$\Delta_1 = \frac{1}{2}\,W_1 = \frac{1}{4}\log\frac{\partial^2 F_0}{2\pi\mathrm{i}\tau_0}\,, \tag{4.15}$$

$$\delta a_1 = -\frac{\partial F_1}{\partial^2 F_0} - \frac{1}{4}\frac{\partial^3 F_0}{(\partial^2 F_0)^2}\,. \tag{4.16}$$

Indeed, from (4.7) and $\mathcal{S}[\tau_0] = -1/\tau_0$, one has

$$\mathcal{S}\left[\frac{\partial^2 F_0}{2\pi\mathrm{i}\tau_0}\right] = \frac{2\pi\mathrm{i}\tau_0}{\partial^2 F_0}\,, \tag{4.17}$$

which leads to $\mathcal{S}[W_1] = -W_1$ and hence to (4.15) and, in turn, to (4.16).

This procedure can be extended to higher orders in g_s^2 without any difficulty; in appendix D we provide some details for the computation of the corrections at order g_s^4, as well as an alternative (and computationally more efficient) method to determine the higher order terms in the saddle-point expansion. For example, the next-to-leading order correction of the prepotential turns out to be given by (see (D.9))

$$\Delta_2 = \frac{1}{4}\frac{\partial^2 F_1}{\partial^2 F_0} + \frac{1}{16}\frac{\partial^4 F_0}{(\partial^2 F_0)^2} - \frac{11}{192}\frac{(\partial^3 F_0)^2}{(\partial^2 F_0)^3}\,. \tag{4.18}$$

This analysis shows that the S-duality relation (4.3) can be written as a Legendre transform of a redefined prepotential $\widehat{F}$, namely

$$\mathcal{S}[\widehat{F}](\tilde{a}) = \widehat{F}(a) - 2\pi\mathrm{i}\tilde{a}\,a \tag{4.19}$$

with

$$\begin{aligned}\widehat{F} &= F + \frac{g_s^2}{4}\log\frac{\partial^2 F_0}{2\pi\mathrm{i}\tau_0} + g_s^4\left[\frac{1}{4}\frac{\partial^2 F_1}{\partial^2 F_0} + \frac{1}{16}\frac{\partial^4 F_0}{(\partial^2 F_0)^2} - \frac{11}{192}\frac{(\partial^3 F_0)^2}{(\partial^2 F_0)^3}\right] + \mathcal{O}(g_s^6) \\ &= F + \frac{g_s^2}{4}\log\frac{\partial^2 F}{2\pi\mathrm{i}\tau_0} + \frac{g_s^4}{16}\left[\frac{\partial^4 F}{(\partial^2 F)^2} - \frac{11}{12}\frac{(\partial^3 F)^2}{(\partial^2 F)^3}\right] + \mathcal{O}(g_s^6)\,,\end{aligned} \tag{4.20}$$

and

$$2\pi\mathrm{i}\tilde{a} = \partial\widehat{F} = \partial F + \frac{g_s^2}{4}\frac{\partial^3 F}{\partial^2 F} + \frac{g_s^4}{16}\left[\frac{\partial^5 F}{(\partial^2 F)^2} - \frac{23}{6}\frac{\partial^3 F\,\partial^4 F}{(\partial^2 F)^3} + \frac{11}{4}\frac{(\partial^3 F)^3}{(\partial^2 F)^4}\right] + \mathcal{O}(g_s^6)\,. \tag{4.21}$$

Using the formulas derived in appendix D (see in particular (D.4)–(D.6)), it is easy to check that

$$\mathcal{S}^2[a] = \mathcal{S}[\tilde{a}] = -a \quad\text{and}\quad \mathcal{S}^2[\widehat{F}] = \widehat{F}\,, \tag{4.22}$$

which generalize the undeformed S-duality relations (4.8) and (4.9) to the case of a non-vanishing deformation $g_s^2 = \epsilon_1\epsilon_2$. Finally, we observe that the expressions for $\widehat{F}$ and $\tilde{a}$ we have obtained here completely agree with the results of [46] that were derived by explicitly enforcing, order by order in $\epsilon_1\epsilon_2$, the requirement that $\mathcal{S}^2[a] = -a$ (see in particular eq. (5.3) of [46] which exactly matches (4.20)).

We end with a few comments. The saddle-point evaluation of the Fourier transform of the deformed prepotential leading to (4.3) is a standard procedure, corresponding to the usual perturbative treatment of quantum field theories. The correspondence is as follows: the topological string coupling $g_s^2 = \epsilon_1\epsilon_2$ represents $\hbar$; a single degree of freedom x plays the rôle of the field ϕ, while $-2\pi\mathrm{i}\tilde{a}$ corresponds to the current j; the prepotential $F(x)$ represents the tree-level action and its S-dual $\widetilde{F}(\tilde{a})$ is like the generating functional $W[j]$ of the connected diagrams. In the diagrammatic interpretation of the perturbative corrections W_g, like those given in (4.4), F_0 is seen as the "free" action and the higher genus terms F_g as interactions. Note however that these interactions are not weighted by independent couplings, as is usually the case in field theory, but by powers of g_s^2, i.e. of $\hbar$ itself; in other words, one is quantizing a theory whose action already contains $\hbar$ corrections. This makes the choice of F_0 as the "free action" somewhat ambiguous, and in fact in appendix D we show that it is possible to organize the expansion around a shifted saddle-point by using as free action a different function, which differs from F_0 by $\hbar$ corrections, obtaining exactly the same results.

In the standard field theory procedure one defines the quantum effective action $\Gamma[\Phi]$, which is related to $W[j]$ by a Legendre transform, so that $\Phi = \partial W/\partial j = \langle\phi\rangle_j$. In our situation this would amount to express the S-dual prepotential as the (inverse) Legendre transform of an "effective prepotential" $F_{\mathrm{eff}}(a)$, namely as

$$\mathcal{S}[F](\tilde{a}) = F_{\mathrm{eff}}(a) - 2\pi\mathrm{i}\tilde{a}\,a\,, \tag{4.23}$$

where a can be obtained by inverting[11] the relation $2\pi\mathrm{i}\tilde{a} = \partial F_{\mathrm{eff}}(a)/\partial a$.

This of course can be done, but it is not what we have done! Indeed, we have modified the procedure by introducing a shifted prepotential $\widehat{F}$ such that its S-dual is the Legendre transform of itself and not of a different function, as one can realize by comparing (4.19) with (4.23). This result has been obtained by defining the modified prepotential $\widehat{F}$ to differ from F by *half* the perturbative quantum corrections appearing in the computation of $\widetilde{F}(\tilde{a})$ (see in particular (D.23)–(D.25)). In other words we have divided the quantum corrections democratically between the prepotential $\widehat{F}$ and its S-dual $\mathcal{S}[\widehat{F}]$, giving these quantities a symmetric rôle with respect to S-duality. In the analogous field theory computation, redefining the action by half the quantum corrections would not make much sense, given that one starts from a purely classical action and the goal is precisely to understand how

---

[11]This is tantamount to writing

$$a = -\frac{1}{2\pi\mathrm{i}}\frac{\partial\widetilde{F}}{\partial\tilde{a}} = \frac{\int dx\, x\, \mathrm{e}^{\frac{2\pi\mathrm{i}\tilde{a}x - F(x)}{g_s^2}}}{\int dx\, \mathrm{e}^{\frac{2\pi\mathrm{i}\tilde{a}x - F(x)}{g_s^2}}} = \langle x\rangle_{\tilde{a}}$$

where we used (3.20).

the quantum corrections modify it into an effective action. In our case, however, the prepotential F is expressed as a series in $\hbar$, *i.e.* it already contains "quantum" terms before the saddle-point evaluation of the Fourier transform. It is therefore not particularly disturbing to shift it by some $\hbar$ contributions, in order to obtain a better behaved quantity and a more symmetric formulation of the S-duality relation.

5 Conclusions

The fact that the deformed partition function of $\mathcal{N} = 2$ superconformal SU(2) theories satisfies a modular anomaly equation, equivalent to the heat equation, was already known in the literature but not all its implications were exploited.

In this paper we have shown that writing the deformed prepotential of $\mathcal{N} = 2^*$ and $N_f = 4$ theories with gauge group SU(2) as a solution of the corresponding heat equation is an efficient way to compute the exact dependence on the bare coupling τ_0 of the coefficients of its large-a expansion. With this formalism, quite high orders in the expansion can be achieved with a limited effort, using as input data the explicit knowledge of the perturbative (and the very first few instanton) terms of the prepotential.

From a more conceptual point of view, starting from the solution of the modular anomaly equation via the heat kernel, we have proved that $\mathcal{S}$-duality is realized on the deformed partition function as a Fourier transform also for generic ϵ-deformations and masses. This fact is perfectly consistent with the interpretation of $\exp\left(-\frac{F(a)}{\epsilon_1 \epsilon_2}\right)$ as a wave function and of a and $\tilde{a} = \mathcal{S}[a]$ as a pair of canonically conjugate variables on which S-duality act as a canonical transformation. Thus there is a complete analogy with what has been observed in topological string models and local CY compactifications of Type II string theories [17–19]. The Fourier transform can be evaluated in the saddle-point approximation, yielding the perturbative expansion (4.3) of the dual prepotential. As discussed at the end of section 4, this procedure is very similar to the usual perturbative expansion of field theories, and in particular it implies that the S-dual variable $\tilde{a}$ coincides with the derivative of the prepotential F only at the leading order in g_s^2.

In the last part of our work, we have shown that it is possible to introduce a modified prepotential $\widehat{F}$, differing from F by a series of g_s^2 corrections determined order by order, which exhibits a "classical" behavior under S-duality: its S-dual is simply given by its Legendre transform, see (4.19), so that we also have that $\tilde{a} = \partial \hat{F}(a)/\partial a$. We believe $\widehat{F}$ should have a direct "physical" meaning and uncovering it represents an interesting open problem left to future investigation.

Acknowledgments

A Modular functions

In this appendix we collect some useful formulas involving the modular functions we used in the main text.

θ-functions. The Jacobi θ-functions are defined as

$$\theta \begin{bmatrix} a \\ b \end{bmatrix}(v|\tau) = \sum_{n \in \mathbb{Z}} q^{(n-\frac{a}{2})^2} \, e^{2\pi i (n-\frac{a}{2})(v-\frac{b}{2})}, \tag{A.1}$$

for $a, b = 0, 1$ and $q = e^{\pi i \tau}$. We simplify the notation by writing, as usual, $\theta_1 \equiv \theta\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\theta_2 \equiv \theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\theta_3 \equiv \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\theta_4 \equiv \theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The functions θ_a, $a = 2, 3, 4$, satisfy the "*aequatio identica satis abstrusa*"

$$\theta_3^4 - \theta_2^4 - \theta_4^4 = 0, \tag{A.2}$$

and admit the following series expansions

$$\begin{aligned}
\theta_2(0|\tau) &= 2q^{\frac{1}{4}}\left(1 + q^2 + q^6 + q^{12} + \cdots\right), \\
\theta_3(0|\tau) &= 1 + 2q + 2q^4 + 2q^9 + \cdots, \\
\theta_4(0|\tau) &= 1 - 2q + 2q^4 - 2q^9 + \cdots.
\end{aligned} \tag{A.3}$$

Eisenstein series. The first Eisenstein series can be expressed as follows:

$$\begin{aligned}
E_2 &= 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)\, q^{2n} = 1 - 24q^2 - 72q^4 - 96q^6 + \dots, \\
E_4 &= 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)\, q^{2n} = 1 + 240q^2 + 2160q^4 + 6720q^6 + \dots, \\
E_6 &= 1 - 504\sum_{n=1}^{\infty} \sigma_5(n)\, q^{2n} = 1 - 504q^2 - 16632q^4 - 122976q^6 + \dots,
\end{aligned} \tag{A.4}$$

where $\sigma_k(n)$ is the sum of the k-th power of the divisors of n, i.e. $\sigma_k(n) = \sum_{d|n} d^k$. The series E_4 and E_6 are related to the θ-functions in the following way

$$E_4 = \frac{1}{2}\left(\theta_2^8 + \theta_3^8 + \theta_4^8\right), \quad E_6 = \frac{1}{2}\left(\theta_3^4 + \theta_4^4\right)\left(\theta_2^4 + \theta_3^4\right)\left(\theta_4^4 - \theta_2^4\right). \tag{A.5}$$

The series E_2, E_4 and E_6 are connected among themselves by logarithmic q-derivatives according to

$$q\partial_q E_2 = \frac{1}{6}\left(E_2^2 - E_4\right), \quad q\partial_q E_4 = \frac{2}{3}\left(E_4 E_2 - E_6\right), \quad q\partial_q E_6 = E_6 E_2 - E_4^2. \tag{A.6}$$

Also the logarithmic q-derivatives of the functions θ_a^4 have simple expressions:

$$q\partial_q \theta_2^4 = \frac{\theta_2^4}{3}\left(E_2 + \theta_3^4 + \theta_4^4\right), \quad q\partial_q \theta_3^4 = \frac{\theta_3^4}{3}\left(E_2 + \theta_2^4 - \theta_4^4\right), \quad q\partial_q \theta_4^4 = \frac{\theta_4^4}{3}\left(E_2 - \theta_2^4 - \theta_3^4\right). \tag{A.7}$$

Modular transformations. Under the $\mathrm{Sl}(2, \mathbb{Z})$ modular transformation

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \qquad \text{with} \quad a, b, c, d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1, \tag{A.8}$$

the Eisenstein series E_4 and E_6 behave as modular forms of weight 4 and 6, respectively:

$$E_4(\tau') = (c\tau + d)^4 E_4(\tau), \quad E_6(\tau') = (c\tau + d)^6 E_6(\tau). \tag{A.9}$$

The series E_2, instead, is a quasi-modular form of degree 2:

$$E_2(\tau') = (c\tau + d)^2 E_2(\tau) + \frac{6}{\mathrm{i}\pi} c(c\tau + d). \tag{A.10}$$

Under the generators $\mathcal{T}$ and $\mathcal{S}$ of the modular group the θ-functions transform as follows

$$\begin{aligned}
\mathcal{T} &: \quad \theta_3^4 \leftrightarrow \theta_4^4, \quad \theta_2^4 \to -\theta_2^4, \\
\mathcal{S} &: \quad \theta_2^4 \to -\tau^2 \theta_4^4, \quad \theta_3^4 \to -\tau^2 \theta_3^4, \quad \theta_4^4 \to -\tau^2 \theta_2^4.
\end{aligned} \tag{A.11}$$

B Higher order coefficients in the generalized prepotential

In this appendix we provide some technical details for the calculation of higher order coefficients h_ℓ in the expansion of the generalized prepotential for large values of the vacuum expectation value a. In particular we will compute the coefficient h_4 in the $\mathcal{N} = 2^*$ theory and the coefficient h_3 in the $N_f = 4$ theory.

B.1 The coefficient h_4 of the $\mathcal{N} = 2^*$ prepotential

The coefficient $h_4^{(0)}$ of the perturbative part of the generalized prepotential (2.17) for the $\mathcal{N} = 2^*$ SU(2) theory is given by

$$\begin{aligned}
h_4^{(0)} = \frac{1}{1440} h_0 \big(h_0 + \epsilon_1 \epsilon_2 \big) \big(2h_0^3 + 13h_0^2 \epsilon_1 \epsilon_2 - 5h_0^2 s^2 - 29 h_0 \epsilon_1 \epsilon_2 s^2 + 29 h_0 (\epsilon_1 \epsilon_2)^2 + 6 h_0 s^4 \\
+ 21(\epsilon_1 \epsilon_2)^3 - 42(\epsilon_1 \epsilon_2)^2 s^2 + 21 \epsilon_1 \epsilon_2 s^4 - 3s^6 \big).
\end{aligned} \tag{B.1}$$

On the other hand, taking as initial condition for the heat kernel equation the expression given in (2.36), we obtain a contribution to the prepotential proportional to $1/a^8$ with a coefficient

$$\begin{aligned}
h_4 \simeq \frac{1}{20736} h_0 \big(h_0 + \epsilon_1 \epsilon_2 \big) \Big[\big(14h_0^3 + 79h_0^2 \epsilon_1 \epsilon_2 + 155 h_0 (\epsilon_1 \epsilon_2)^2 + 105(\epsilon_1 \epsilon_2)^3 \big) E_2^4 \\
+ \frac{2}{5} \big(2h_0 + 3\epsilon_1 \epsilon_2 - 6s^2 \big) \big(14h_0^2 + 74 h_0 \epsilon_1 \epsilon_2 + 105(\epsilon_1 \epsilon_2)^2 \big) E_2^2 E_4 \\
+ \frac{4}{35} \big(2h_0 + 7\epsilon_1 \epsilon_2 \big) \big(11h_0^2 + 59h_0 \epsilon_1 \epsilon_2 + 60(\epsilon_1 \epsilon_2)^2 - 108 h_0 s^2 - 270 s^2 \epsilon_1 \epsilon_2 + 180 s^4 \big) E_2 E_6 \Big].
\end{aligned} \tag{B.2}$$

Notice that this expression does not contain the structure E_4^2, which instead should be expected on the basis of the modularity properties since h_4 is a (quasi) modular form of

weight 8. Furthermore, in the perturbative limit when $E_2, E_4, E_6 \to 1$, (B.2) does not reproduce the 1-loop result (B.1); indeed we have

$$
\begin{aligned}
\Theta_4 \equiv h_4^{(0)} &- h_4\big|_{E_2, E_4, E_6 \to 1} \\
&= \frac{1}{725760} h_0 \big(h_0 + \epsilon_1\epsilon_2\big)\big(38h_0^3 + 347h_0^2\epsilon_1\epsilon_2 - 480h_0^2 s^2 + 1011h_0\epsilon_1\epsilon_2 s^2 \\
&\quad + 1584h_0 s^4 + 819(\epsilon_1\epsilon_2)^3 - 4789(\epsilon_1\epsilon_2)^2 s^2 + 5544\epsilon_1\epsilon_2 s^4 - 1512 s^6\big).
\end{aligned}
\tag{B.3}
$$

This mismatch is easily cured by changing the initial condition in the heat kernel equation (2.13), and by using

$$
\varphi_0(y, 0) \simeq -\frac{1}{2}\, h_0 \log \frac{4y^2}{\Lambda^2} + \frac{\Theta_2\, E_4}{16\, y^4} + \frac{\Theta_3\, E_6}{48\, y^6} + \frac{\Theta_4\, E_4^2}{128\, y^8}
\tag{B.4}
$$

instead of (2.36). In this way, in fact, we obtain the same expressions as before for h_1, h_2 and h_3, given respectively in (2.27), (2.33) and (2.37), and also the exact expression for h_4, namely

$$
\begin{aligned}
h_4 = \frac{1}{20736}\, h_0 \big(h_0 + \epsilon_1\epsilon_2\big)\Big[& \big(14h_0^3 + 79h_0^2\epsilon_1\epsilon_2 + 155h_0(\epsilon_1\epsilon_2)^2 + 105(\epsilon_1\epsilon_2)^3\big) E_2^4 \\
&+ \frac{2}{5}\big(2h_0 + 3\epsilon_1\epsilon_2 - 6s^2\big)\big(14h_0^2 + 74h_0\epsilon_1\epsilon_2 + 105(\epsilon_1\epsilon_2)^2\big) E_2^2 E_4 \\
&+ \frac{4}{35}\big(2h_0 + 7\epsilon_1\epsilon_2\big)\big(11h_0^2 + 59h_0\epsilon_1\epsilon_2 + 60(\epsilon_1\epsilon_2)^2 - 108h_0 s^2 - 270s^2\epsilon_1\epsilon_2 + 180s^4\big) E_2 E_6 \\
&+ \frac{1}{35}\big(38h_0^3 + 347h_0^2\epsilon_1\epsilon_2 - 480h_0^2 s^2 + 1011h_0\epsilon_1\epsilon_2 s^2 \\
&\quad + 1584h_0 s^4 + 819(\epsilon_1\epsilon_2)^3 - 4789(\epsilon_1\epsilon_2)^2 s^2 + 5544\epsilon_1\epsilon_2 s^4 - 1512s^6\big)E_4^2\Big].
\end{aligned}
\tag{B.5}
$$

By expanding the Eisenstein series, one may retrieve the various instanton contributions and one can check that for the first few instanton numbers these contributions exactly agree with the explicit results obtained in the multi-instanton calculus from the localization formulas [46].

B.2 The coefficient h_3 of the $N_f = 4$ prepotential

In the $N_f = 4$ SYM theory the coefficient $h_3^{(0)}$ of the 1-loop prepotential (2.42) is [46]

$$
\begin{aligned}
h_3^{(0)} = \frac{1}{672}\Big[& 24h_0^4 + 176h_0^3\epsilon_1\epsilon_2 - 64h_0^3 s^2 + 372h_0^2(\epsilon_1\epsilon_2)^2 - 352h_0^2 s^2\epsilon_1\epsilon_2 \\
&+ 64h_0^2 s^4 - 2688h_0^2 T_1 + 124h_0(\epsilon_1\epsilon_2)^3 - 240h_0 s^2(\epsilon_1\epsilon_2)^2 + 64h_0 s^4\epsilon_1\epsilon_2 \\
&+ 3072h_0 N - 10752h_0 T_1\epsilon_1\epsilon_2 + 4032h_0 T_1 s^2 + 9216N\epsilon_1\epsilon_2 - 3840N s^2 \\
&+ 15s^4(\epsilon_1\epsilon_2)^2 - 36s^2(\epsilon_1\epsilon_2)^3 + 10080 T_1\epsilon_1\epsilon_2(s^2 - \epsilon_1\epsilon_2) - 2016 T_1 s^4 \\
&+ 13056 T_1^2 - 3072 T_1 T_2 - 3072 T_2^2\Big].
\end{aligned}
\tag{B.6}
$$

On the other hand, using the initial condition (2.59) in the heat-kernel formula (2.40) we obtain the following expression for the coefficient of the $1/a^6$ term

$$
\begin{aligned}
h_3 \simeq & \frac{1}{216} h_0 \big(h_0 + \epsilon_1\epsilon_2\big)\big(5h_0^2 + 17h_0\epsilon_1\epsilon_2 + 15(\epsilon_1\epsilon_2)^2\big) E_3^2 \\
& - \frac{1}{3}\big(5h_0^2 + 17h_0\epsilon_1\epsilon_2 + 15(\epsilon_1\epsilon_2)^2\big)\big(T_1\,\theta_4^4 - T_2\,\theta_2^4\big) E_2^2 \\
& + \frac{1}{1440}\big(2h_0 + 5\epsilon_1\epsilon_2\big)\Big[8h_0^3 + 68h_0^2\epsilon_1\epsilon_2 - 48h_0^2 s^2 + 24h_0(\epsilon_1\epsilon_2)^2 \\
& \hspace{4.5cm} - 48h_0 s^2\epsilon_1\epsilon_2 - 9s^2(\epsilon_1\epsilon_2)^2 + 2304N\Big]E_4 E_2 \\
& - \frac{1}{3}\big(2h_0 + 5\epsilon_1\epsilon_2\big)\big(2h_0 - 3s^2 + 3\epsilon_1\epsilon_2\big)\big(T_1\theta_4^8 + 2(T_1 + T_2)\theta_2^4\theta_4^4 + T_2\theta_2^8\big)E_2 \\
& + 8(T_1\theta_4^4 - T_2\theta_2^4)^2 E_2 \ .
\end{aligned}
\tag{B.7}
$$

It is not difficult to realize that in the perturbative limit where $E_2, E_4, \theta_4 \to 1$ and $\theta_2 \to 0$ the above expression does not reproduce the 1-loop result (B.6); indeed

$$
h_3^{(0)} - h_3\big|_{E_4, E_2 \to 1, \theta_4 \to 1, \theta_2 \to 0} = \Theta_3 + \Theta_3' T_1 + 16T_1^2
\tag{B.8}
$$

where

$$
\begin{aligned}
\Theta_3 = & \frac{1}{30240}\Big[44h_0^4 + 1144h_0^3\epsilon_1\epsilon_2 - 864h_0^3 s^2 + 4112h_0^2(\epsilon_1\epsilon_2)^2 - 8784h_0^2 s^2\epsilon_1\epsilon_2 \\
& \quad + 2880h_0^2 s^4 + 41472h_0 N + 960h_0(\epsilon_1\epsilon_2)^3 - 5382h_0 s^2(\epsilon_1\epsilon_2)^2\epsilon_1\epsilon_2 \\
& \quad + 2880h_0 s^4 - 172800N(s^2 - \epsilon_1\epsilon_2) + 675s^2(\epsilon_1\epsilon_2)^2(s^2 - \epsilon_1\epsilon_2) \\
& \quad - 138240(T_1^2 + T_1 T_2 + T_2^2)\Big],
\end{aligned}
\tag{B.9}
$$

$$
\Theta_3' = -h_0^2 - 5h_0\epsilon_1\epsilon_2 + 4h_0 s^2 - 3s^4 + 10s^2\epsilon_1\epsilon_2 - 5(\epsilon_1\epsilon_2)^2 \,.
\tag{B.10}
$$

The mismatch (B.8) signals the fact that some structures are missing in our analysis. In fact, we have three modular forms of weight 6 which have not been considered so far. They are: the Eisenstein series E_6, which can be multiplied by any modular invariant combination of the SO(8) mass invariants (2.39), and the following two structures:

$$
E_4(T_1\theta_4^4 - T_2\theta_2^4) \quad \text{and} \quad T_1^2(\theta_4^4 + 2\theta_2^4)\theta_4^8 - T_2^2(\theta_2^4 + 2\theta_4^4)\theta_2^8 - T_1 T_2\,\theta_2^4\theta_4^4(\theta_2^4 - \theta_4^4) \tag{B.11}
$$

whose modular transformation properties can be found using (2.50) and (2.51), and whose perturbative limits are, respectively, T_1 and T_1^2. Exploiting this fact, we can cure the mismatch (B.8) by changing the initial condition in the heat-kernel formula (2.40) and use

$$
\begin{aligned}
\varphi_0(y; 0) \simeq & -\frac{1}{2} h_0 \log\frac{y^2}{\Lambda^2} - \frac{T_1\,\theta_4^4 - T_2\,\theta_2^4}{y^2} - \frac{\Theta_2\,E_4 + \Theta_2'\big(T_1\theta_4^8 + 2(T_1 + T_2)\theta_2^4\theta_4^4 + T_2\theta_2^8\big)}{16y^4} \\
& - \frac{\Theta_3\,E_6 + \Theta_3' E_4\big(T_1\,\theta_4^4 - T_2\,\theta_2^4\big)}{48y^6} \\
& - \frac{16\big(T_1^2(\theta_4^4 + 2\theta_2^4)\theta_4^8 - T_2^2(\theta_2^4 + 2\theta_4^4)\theta_2^8 - T_1 T_2\,\theta_2^4\theta_4^4(\theta_2^4 - \theta_4^4)\big)}{48y^6} \,.
\end{aligned}
\tag{B.12}
$$

In this way one obtains the exact expression for h_3 which is the sum of (B.7) and

$$\Theta_3 E_6 + \Theta_3' E_4 \big(T_1 \theta_4^4 - T_2 \theta_2^4 \big) + 16 \big(T_1^2 (\theta_4^4 + 2\theta_2^4)\theta_4^8 - T_2^2 (\theta_2^4 + 2\theta_4^4)\theta_2^8 - T_1 T_2 \theta_2^4 \theta_4^4 (\theta_2^4 - \theta_4^4) \big) \,. \tag{B.13}$$

One can check that this result exactly reproduces in all details eq. (B.2) of [46], which was obtained from the multi-instanton calculus and the localization method.

C The parabolic cylinder functions

The parabolic cylinder functions $D_q(z)$ have the following integral representation (see for example 9.241 of [61])

$$D_q(z) = \frac{1}{\sqrt{\pi}} \, 2^{q + \frac{1}{2}} \, \mathrm{e}^{-\frac{q\pi \mathrm{i}}{2}} \, \mathrm{e}^{\frac{z^2}{4}} \int_{-\infty}^{+\infty} du \; u^q \, \mathrm{e}^{-2z^2 + 2\mathrm{i}uz} \tag{C.1}$$

for $\mathrm{Re}\, q > -1$, and the following asymptotic expansion for large z (see for example 9.246 of [61])

$$D_q(z) \simeq \mathrm{e}^{-\frac{z^2}{4}} \, z^q \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \, (q)_{2\ell}}{2^\ell \, \ell!} \, \frac{1}{z^{2\ell}} \tag{C.2}$$

where $(q)_{2\ell}$ is the Pochhammer symbol

$$(q)_{2\ell} = q(q - 1) \cdots (q - 2\ell + 1) \,. \tag{C.3}$$

Making use of these expressions and their analytic extensions, we can obtain the useful formula

$$\int_{-\infty}^{+\infty} dy \; y^{q-2k} \, \exp\left(-\frac{(x - y)^2}{2\epsilon_1 \epsilon_2 t} \right) = \sqrt{2\pi \epsilon_1 \epsilon_2 t} \; x^{q-2k} \sum_{\ell=0}^{\infty} \frac{(q - 2k)_{2\ell}}{2^\ell \, \ell!} \, \frac{(\epsilon_1 \epsilon_2 t)^\ell}{x^{2\ell}} \,. \tag{C.4}$$

The functions $D_q(z)$ satisfy the following recursion relations:

$$\begin{aligned}
D_{q+1}(z) - z D_q(z) + q D_{q-1}(z) &= 0 \,, \\
\partial_z D_q(z) + \frac{z}{2} D_q(z) - q D_{q-1}(z) &= 0 \,, \\
\partial_z D_q(z) - \frac{z}{2} D_q(z) + D_{q+1}(z) &= 0 \,,
\end{aligned} \tag{C.5}$$

that can be easily rewritten also for the asymptotic series $\widetilde{D}_q(z)$ defined as

$$\widetilde{D}_q(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \, (q)_{2\ell}}{2^\ell \, \ell!} \, \frac{1}{z^{2\ell}} = \mathrm{e}^{\frac{z^2}{4}} \, z^{-q} \, D_q(z) \,. \tag{C.6}$$

The functions $\widetilde{D}_q$ are useful since they appear explicitly in the expressions of the various terms of the prepotential $\varphi_0(a, t)$ obtained in the main text. For example, as shown in section 2 for the $\mathcal{N} = 2^*$ SU(2) theory, the terms of the prepotential which depend only on

the Eisenstein series E_2 or which are linear in E_4 and E_6 are reconstructed from the initial condition (2.36), from which one obtains

$$\varphi_0(a,t) \simeq -\frac{1}{2} h_0 \log \frac{4a^2}{\Lambda^2} + \log \widetilde{D}_q(z) + \frac{\Theta_2 E_4}{16 a^4} \frac{\widetilde{D}_{q-4}(z)}{\widetilde{D}_q(z)} + \frac{\Theta_3 E_6}{48 a^6} \frac{\widetilde{D}_{q-6}(z)}{\widetilde{D}_q(z)} + \cdots \quad (C.7)$$

where $t = E_2/24$ and $z = ia/\sqrt{\epsilon_1\epsilon_2 t}$, while Θ_2 and Θ_3, which are respectively polynomials of order 2 and 3 in h_0, are given in (2.31) and (2.35).

The interesting point is that, while the coefficient of $a^{-2\ell}$ in the expansion of $\exp\left(\frac{\varphi_0(a;t)}{\epsilon_1\epsilon_2}\right)$ is a polynomial of degree 2ℓ in h_0 (see (2.24)), the coefficients h_ℓ in φ_0 have to be polynomials in h_0 of degree $\ell+1$ for dimensional reasons. This translates into the fact that while the coefficient of $z^{-2\ell}$ in $\widetilde{D}_q(z)$ is a polynomial of degree 2ℓ in q, the coefficients of $z^{-2\ell}$ in expressions like $\frac{\widetilde{D}_{q-4}(z)}{\widetilde{D}_q(z)}$ or $\frac{\widetilde{D}_{q-6}(z)}{\widetilde{D}_q(z)}$ (respectively $\log \widetilde{D}_q(z)$) are polynomials of degree ℓ (respectively $(\ell+1)$) only.

We now show that the cancellations needed for this to occur are a simple consequence of the recursion relations (C.5). We first note that these relations imply

$$z\partial_z \log \widetilde{D}_q(z) = \frac{q(q-1)}{z^2} \frac{\widetilde{D}_{q-2}(z)}{\widetilde{D}_q(z)} , \quad (C.8)$$

so that to prove our statement it is enough to demonstrate that for any integer s

$$\frac{\widetilde{D}_{q-s}(z)}{\widetilde{D}_q(z)} = 1 + \sum_{n \geq 1} \frac{R[q;s;n]}{z^{2n}} \quad (C.9)$$

where $R[q;s;n]$ is a polynomial in q of order n.

As a lemma, we first demonstrate by induction that, for any integer s, we have

$$\frac{\widetilde{D}_{q-s}(z)}{\widetilde{D}_q(z)} = 1 + \sum_{n \geq 1} \frac{P[q;s;n]}{z^{2n}} \frac{\widetilde{D}_{q-s-n}(z)}{\widetilde{D}_q(z)} \quad (C.10)$$

where $P[q;s;n]$ is a polynomial in q of order n. One can easily see that this is true for $s = 1$ since the recursion relations (C.5) imply

$$\frac{\widetilde{D}_{q-1}(z)}{\widetilde{D}_q(z)} = 1 + \frac{q-1}{z^2} \frac{\widetilde{D}_{q-2}(z)}{\widetilde{D}_q(z)} . \quad (C.11)$$

Then, using the relation (C.11) with $q \to q - s$, we get

$$\frac{\widetilde{D}_{q-s-1}(z)}{\widetilde{D}_q(z)} = 1 + \sum_{n \geq 1} \frac{P[q;s;n]}{z^{2n}} \frac{\widetilde{D}_{q-s-n}(z)}{\widetilde{D}_q(z)} + \frac{q-s-1}{z^2} \frac{\widetilde{D}_{q-s-2}(z)}{\widetilde{D}_q(z)} , \quad (C.12)$$

which has the form we seek, except that the index range in the sum is not the correct one. However, using (C.11) with $q \to q - s - (n+1)$, we can easily put the right hand side of (C.12) in the desired form and thus prove by induction that (C.10) holds true

for any integer s. Finally, (C.9) follows from (C.10) by induction on n and by using the recursion (C.11) for an appropriate value of q.

The functions $\widetilde{D}_q$ are useful also for the computation of the prepotential for the $N_f = 4$ theory, which indeed goes essentially along the same lines as above, apart from the existence of an order one term proportional to $\left(T_1\,\theta_4^4 - T_2\,\theta_2^4\right)$ in the initial condition (2.59). When calculating h_3, this term is responsible for the contribution $8(T_1\,\theta_4^4 - T_2\,\theta_2^4)E_2$ in the last line of (B.7) which does not seem to have a direct precursor in the initial condition (2.59). Actually, such a term originates from the fact that the heat equation is satisfied by the partition function rather than the prepotential. An explicit computation shows that it is the first term in $\varphi_0(a,t)$ of a series taking the form

$$\varphi_0(a,t) \simeq \frac{(T_1\,\theta_4^4 - T_2\,\theta_2^4)^2}{2\epsilon_1\epsilon_2\,a^4} \left(\frac{\widetilde{D}_{q-4}(z)}{\widetilde{D}_q(z)} - \frac{\widetilde{D}_{q-2}^2(z)}{\widetilde{D}_q^2(z)} \right) + \cdots, \tag{C.13}$$

which has indeed the features discussed above.

D   Higher orders in the saddle-point approximation

In this appendix we provide some technical details on the calculation of the next-to-leading order correction in the S-duality transformation of the prepotential. Before doing this, however, we make explicit some properties of the leading order results described in section 4 which will be useful later on.

At order $g_s^2 = \epsilon_1\epsilon_2$, we have found that

$$2\pi\mathrm{i}\tilde{a} = \partial F + \frac{g_s^2}{4}\frac{\partial^3 F}{\partial^2 F} + \mathcal{O}(g_s^4) = \partial F_0 + g_s^2\Big(\partial F_1 + \frac{1}{4}\frac{\partial^3 F_0}{\partial^2 F_0}\Big) + \mathcal{O}(g_s^4), \tag{D.1}$$

and

$$\widetilde{F}(\tilde{a}) = F(a) - 2\pi\mathrm{i}\,\tilde{a}\,a + \frac{g_s^2}{2}\,\log\frac{\partial^2 F}{2\pi\mathrm{i}\tau_0} + \mathcal{O}(g_s^4). \tag{D.2}$$

From these relations it is simple to obtain

$$\mathcal{S}[\partial F] = \frac{\partial a}{\partial \tilde{a}}\frac{\partial\mathcal{S}[F]}{\partial a} = -2\pi\mathrm{i}\,a + \frac{2\pi\mathrm{i}\,g_s^2}{4}\frac{\partial^3 F}{(\partial^2 F)^2} + \mathcal{O}(g_s^4), \tag{D.3}$$

$$\mathcal{S}[\partial^2 F] = \frac{\partial a}{\partial \tilde{a}}\frac{\partial\mathcal{S}[\partial F]}{\partial a} = -\frac{(2\pi\mathrm{i})^2}{\partial^2 F} + \frac{(2\pi\mathrm{i})^2 g_s^2}{2}\left[\frac{\partial^4 F}{(\partial^2 F)^3} - \frac{3}{2}\frac{(\partial^3 F)^2}{(\partial^2 F)^4}\right] + \mathcal{O}(g_s^4), \tag{D.4}$$

and so on and so forth. These relations generalize (4.6) and (4.7) to include the first-order corrections in g_s^2. Expanding the prepotential as in (4.1) and reading the coefficients of g_s^0 and g_s^2, from (D.4) we get

$$\mathcal{S}[\partial^2 F_0] = -\frac{(2\pi\mathrm{i})^2}{\partial^2 F_0}\,,$$
$$\mathcal{S}[\partial^2 F_1] = (2\pi\mathrm{i})^2\left[\frac{\partial^2 F_1}{(\partial^2 F_0)^2} + \frac{1}{2}\frac{\partial^4 F_0}{(\partial^2 F_0)^3} - \frac{3}{4}\frac{(\partial^3 F_0)^2}{(\partial^2 F_0)^4}\right]. \tag{D.5}$$

Proceeding similarly for the higher derivatives, we find

$$\mathcal{S}[\partial^3 F_0] = (2\pi\mathrm{i})^3 \frac{\partial^3 F_0}{(\partial^2 F_0)^3} \,,$$
$$\mathcal{S}[\partial^4 F_0] = (2\pi\mathrm{i})^4 \left[\frac{\partial^4 F_0}{(\partial^2 F_0)^4} - 3 \frac{(\partial^3 F_0)^2}{(\partial^2 F_0)^5} \right] \,. \tag{D.6}$$

All these formulas will be useful for the next-to-leading order calculation.

In order to find the shifted prepotential $\widehat{F}$ at order g_s^4, we proceed as described in section 4 and rewrite the saddle-point result (4.3) in terms of a and $\widehat{F}$ as given in (4.10) and (4.11), keeping all terms up to order g_s^4. The new structures that in this way are generated in the right hand side of (4.12) are

$$g_s^4 \left[W_2(a) - \partial W_1(a)\, \delta a_1 - \partial F_1(a)\, \delta a_1 - \frac{1}{2}\, \partial^2 F_0(a)\, \delta a_1^2 - \Delta_2(a) + \mathcal{S}[\Delta_2](a) \right]$$
$$+ g_s^4 \left[\partial^2 F_0(a)\, \delta a_2 + \partial F_2(a) + \partial\Delta_2(a) - \frac{1}{2}\, \partial^3 F_0(a)\, \delta a_1^2 \right] a \tag{D.7}$$

Thus, in order to have a Legendre transform relation between $\widehat{F}$ and its S-dual we must require that the above square brackets vanish. From the second line we fix the form of δa_2, while from the first line we obtain

$$\Delta_2 - \mathcal{S}[\Delta_2] = W_2 - \partial W_1\, \delta a_1 - \partial F_1\, \delta a_1 - \frac{1}{2}\, \partial^2 F_0\, \delta a_1^2$$
$$= \frac{1}{2} \frac{\partial^2 F_1}{\partial^2 F_0} + \frac{1}{8} \frac{\partial^4 F_0}{(\partial^2 F_0)^2} - \frac{11}{96} \frac{(\partial^3 F_0)^2}{(\partial^2 F_0)^3} \tag{D.8}$$

where in the second step we have inserted the expressions of W_1 and W_2 given in (4.4) and of δa_1 given in (4.16). Using (D.5) and (D.6), one can easily check that the combination in the right hand side of (D.8) changes sign under S-duality; thus a solution to this equation is given by

$$\Delta_2 = -\mathcal{S}[\Delta_2] = \frac{1}{4} \frac{\partial^2 F_1}{\partial^2 F_0} + \frac{1}{16} \frac{\partial^4 F_0}{(\partial^2 F_0)^2} - \frac{11}{192} \frac{(\partial^3 F_0)^2}{(\partial^2 F_0)^3} \,, \tag{D.9}$$

as reported in the main text.

This procedure can be easily extended to compute higher-order corrections. However, there is a simpler (and computationally more efficient) way to perform these calculations. Since we aim at writing the S-duality relation as a Legendre transform for a shifted prepotential $\widehat{F}$ and at having

$$2\pi\mathrm{i}\tilde{a} = \partial\widehat{F}(a)\,, \tag{D.10}$$

we can organize the calculation in such a way that the identification (D.10) arises as the saddle-point condition. Thus, we rewrite (3.20) as

$$\exp\left(-\frac{\widetilde{F}(\tilde{a})}{g_s^2} \right) = \sqrt{\frac{\mathrm{i}\tau_0}{g_s^2}} \int_{-\infty}^{+\infty} dx \, \exp\left(\frac{2\pi\mathrm{i}\,\tilde{a}\,x - \widehat{F}(x)}{g_s^2} \right) \exp\left(\frac{\widehat{F}(x) - F(x)}{g_s^2} \right)$$
$$= \sqrt{\frac{\mathrm{i}\tau_0}{g_s^2}} \int_{-\infty}^{+\infty} dx \, \exp\left(\frac{2\pi\mathrm{i}\,\tilde{a}\,x - \widehat{F}(x)}{g_s^2} \right) \exp\left(\sum_{g=1}^{\infty} g_s^{2g-2}\, X_g(x) \right) \tag{D.11}$$

where in the second step we have used

$$\widehat{F} = F + \sum_{g=1}^{\infty} g_s^{2g} \, X_g \, . \tag{D.12}$$

For small values of g_s^2, we can evaluate (D.11) in the saddle-point approximation by setting

$$x = a + g_s y \tag{D.13}$$

with a given by (D.10), so that

$$\exp\left(-\frac{\widetilde{F}(\tilde{a})}{g_s^2}\right) = \sqrt{i\tau_0} \, \exp\left(\frac{2\pi i\tilde{a}\, a - F(a)}{g_s^2}\right) \int_{-\infty}^{+\infty} dy \, \exp\left(-\frac{1}{2}\partial^2 F(a)\, y^2\right) \tag{D.14}$$

$$\times \exp\left(-\sum_{k=3}^{\infty} g_s^{k-2} \, \partial^k F(a) \, \frac{y^k}{k!} + y \sum_{g=1}^{\infty} g_s^{2g-1} \, \partial X_g(a)\right) .$$

We can now expand in powers of g_s and carry out the gaussian integrations, obtaining

$$\exp\left(-\frac{\widetilde{F}(\tilde{a})}{g_s^2}\right) = \sqrt{\frac{2\pi i\tau_0}{\partial^2 F(a)}} \, \exp\left(\frac{2\pi i\tilde{a}\, a - F(a)}{g_s^2}\right)\left(1 + \sum_{g=1}^{\infty} g_s^{2g} \, C_g\right), \tag{D.15}$$

where the coefficients C_g depend on the quantities indicated below:

$$C_g \equiv C_g\big(\partial X_1, \ldots, \partial X_{g-1}; \partial^2 F, \ldots, \partial^{2g+2} F\big) \, . \tag{D.16}$$

Taking the logarithm of (D.15), we finally obtain

$$\widetilde{F}(\tilde{a}) = F(a) - 2\pi i\tilde{a}\, a + \sum_{g=1}^{\infty} g_s^{2g} \, \Gamma_g \, , \tag{D.17}$$

where the coefficients Γ_g depend on the same type of quantities as the coefficients C_g. The explicit computation of the first few terms yields

$$\Gamma_1 = \frac{1}{2} \log \frac{\partial^2 F}{2\pi i\tau_0} \, , \tag{D.18}$$

$$\Gamma_2 = -\frac{1}{2}\frac{(\partial X_1)^2}{\partial^2 F} + \frac{1}{2}\frac{\partial X_1 \, \partial^3 F}{(\partial^2 F)^2} + \frac{1}{8}\frac{\partial^4 F}{(\partial^2 F)^2} - \frac{5}{24}\frac{(\partial^3 F)^2}{(\partial^2 F)^3} \, , \tag{D.19}$$

$$\Gamma_3 = -\frac{\partial X_1 \partial X_2}{\partial^2 F} + \frac{1}{2}\frac{\partial X_2 \, \partial^3 F}{(\partial^2 F)^2} + \frac{1}{6}\frac{(\partial X_1)^3 \, \partial^3 F}{(\partial^2 F)^3} + \frac{1}{4}\frac{(\partial X_1)^2 \partial^4 F}{(\partial^2 F)^3} + \frac{1}{8}\frac{\partial X_1 \, \partial^5 F}{(\partial^2 F)^3}$$

$$+ \frac{1}{48}\frac{\partial^6 F}{(\partial^2 F)^3} - \frac{1}{2}\frac{(\partial X_1)^2 \, (\partial^3 F)^2}{(\partial^2 F)^4} - \frac{2}{3}\frac{\partial X_1 \, \partial^3 F \, \partial^4 F}{(\partial^2 F)^4} - \frac{7}{48}\frac{\partial^3 F \, \partial^5 F}{(\partial^2 F)^4} - \frac{1}{12}\frac{(\partial^4 F)^2}{(\partial^2 F)^4}$$

$$+ \frac{5}{8}\frac{\partial X_1 \, (\partial^3 F)^3}{(\partial^2 F)^5} + \frac{25}{48}\frac{(\partial^3 F)^2 \, \partial^4 F}{(\partial^2 F)^5} - \frac{5}{16}\frac{(\partial^3 F)^4}{(\partial^2 F)^6} \, . \tag{D.20}$$

Obtaining higher Γ_g's is quite straightforward.

Using (D.12), we can rewrite (D.17) as

$$\mathcal{S}[\widehat{F}](\tilde{a}) = \widehat{F}(a) - 2\pi i\tilde{a}\, a + \sum_{g=1}^{\infty} g_s^{2g} \left[\mathcal{S}(X_g) - X_g + \Gamma_g\right], \tag{D.21}$$

and hence we have the desired Legendre transform relation if

$$\mathcal{S}[X_g] - X_g + \Gamma_g = 0 \,. \tag{D.22}$$

These equations can be easily solved iteratively. For example, we have

$$X_1 = -\mathcal{S}[X_1] = \frac{1}{2}\,\Gamma_1 = \frac{1}{4}\log\frac{\partial^2 F}{2\pi\mathrm{i}\tau_0}\ ; \tag{D.23}$$

plugging this expression into (D.19), we then obtain

$$X_2 = -\mathcal{S}[X_2] = \frac{1}{2}\,\Gamma_2 = \frac{1}{16}\frac{\partial^4 F}{(\partial^2 F)^2} - \frac{11}{192}\frac{(\partial^3 F)^2}{(\partial^2 F)^3}\ ; \tag{D.24}$$

in turn, using these results into (D.20), we find

$$\begin{aligned}
X_3 = -\mathcal{S}[X_3] = \frac{1}{2}\,\Gamma_3 = {} & \frac{1}{96}\frac{\partial^6 F}{(\partial^2 F)^3} - \frac{19}{384}\frac{\partial^3 F\,\partial^5 F}{(\partial^2 F)^4} - \frac{1}{24}\frac{(\partial^4 F)^2}{(\partial^2 F)^4} \\
& + \frac{119}{768}\frac{\partial^4 F\,(\partial^3 F)^2}{(\partial^2 F)^5} + \frac{109}{1536}\frac{(\partial^3 F)^4}{(\partial^2 F)^6}\,,
\end{aligned} \tag{D.25}$$

and we can continue iteratively this way. The prepotential $\widehat{F} = F + g_s^2\,\Gamma_1 + g_s^4\,\Gamma_2 + \cdots$ perfectly agrees with the one derived with the more "conservative" saddle-point method as described in the main text and at the beginning of this appendix (see in particular (4.20)).

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