Statistical aspects of fuzzy monotone set–valued stochastic processes. Application to birth–and–growth processes.

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Abstract

The paper considers a particular family of fuzzy monotone set–valued stochastic processes. The proposed setting allows us to investigate suitable α-level sets of such processes, modeling birth–and–growth processes. A decomposition theorem is established to characterize the nucleation and the growth. As a consequence, different consistent set–valued estimators are studied for growth process. Moreover, the nucleation process is studied via the hitting function, and a consistent estimator of the nucleation hitting function is derived.

Key words: Random closed sets, Stochastic geometry, Birth–and–growth processes, Set–valued processes, Non–additive measures, Fuzzy random sets, Fuzzy set–valued stochastic processes

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Introduction

A birth–and–growth crystal process may be studied by means of a positive time– and space–dependent stochastic function representing a concentration process as in [3]. In particular, concentration in the crystal phase takes a constant value, namely c_s (obtained from physical evidences), and outside the crystal it is represented by a sufficiently regular function c_{ex} such that $c_{ex} < c_s$; i.e., the crystal phase is more dense than the mother phase, and a jump in the concentration always occurs on the crystal boundary (see Figure 1a). Figure 1b can

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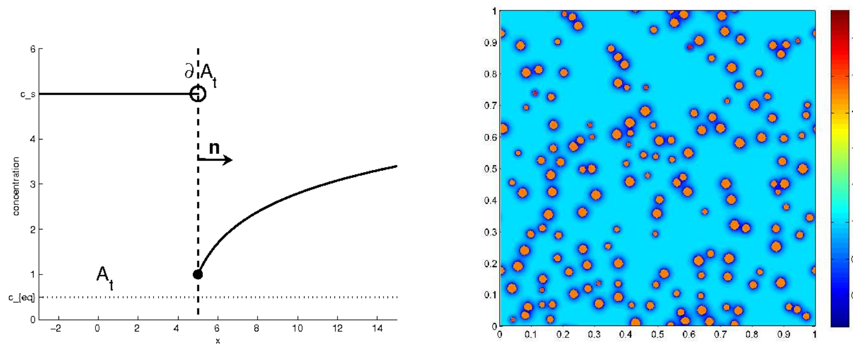
be interpreted as a sequence of membership functions, and so, crystal growth can be seen as a fuzzy monotone set–valued stochastic process; where "monotone" means that every α–level set at each time is included in the α–level set at successive times.



(a) (b)

Figure 1: (Credits to [3]). (a) is a 1D sketch of the concentration for an analytical growth model. (b) is a 2D simulation of a crystallization process on a square grid where the color scale represents the concentration. The figures may be interpreted also from a fuzzy point of view.

In order to study some statistical aspects of the fuzzy monotone set–valued stochastic process, we notice that the α–level process is a closed set–valued stochastic process, that can be modeled as a birth–and–growth process. In this paper, we underline some geometrical properties and statistical aspects of birth–and–growth processes.
The importance of nucleation and growth processes is well known, since they arise in several natural and technological applications (cf. [7, 6] and the references therein) such as, for example, solidification and phase–transition of materials, semiconductor crystal growth, biomineralization, and DNA replication (cf., e.g., [18]). During the years, several authors studied stochastic spatial processes (cf. [11, 32, 24] and references therein) nevertheless they essentially consider static approaches modeling real phenomenons. For what concerns the dynamical point of view, a parametric *birth–and–growth process* was studied in [26, 27]. A birth–and–growth process is a random closed sets (RaCS) family given by $\Theta_t = \bigcup_{n:T_n \leq t} \Theta^t_{T_n}(X_n)$, for $t \in \mathbb{R}_+$, where $\Theta^t_{T_n}(X_n)$ is the RaCS obtained as the evolution up to time $t > T_n$ of the germ born at (random) time T_n in (random) location X_n, according to some growth model. An analytical approach is often used to model birth–and–growth process, in particular it is assumed that the growth of a spherical nucleus of infinitesimal radius is driven according to a non–negative normal velocity, i.e. for every instant t, a border point of the crystal $x \in \partial\Theta_t$ "grows" along the outwards normal unit (e.g. [17, 5, 4, 9, 3]). In view of the chosen framework, different parametric and non–parametric estimations are proposed over the years (cf. [28, 25, 13, 6, 8, 2, 10]

and references therein). Note that the existence of the outwards normal vector imposes a regularity condition on $\partial\Theta_t$ (and also on the nucleation process: it cannot be a point process).
On the other hand, it is well known that sets are particular cases of fuzzy sets. Now, in the class of all convex fuzzy sets stochastic process having compact support, Doob–type decomposition for sub- and super–martingales was studied (e.g. [14, 16, 15, 33]). Nevertheless, a more general case (than the convex one) has not yet been considered; surely, in order to do this, the first easiest step is to consider decomposition for random set–valued processes. The work in progress aims to generalize results of this paper to birth–and–growth fuzzy set–valued stochastic processes.
This paper is an attempt to offer an original approach based on a purely geometric stochastic point of view in order to avoid regularity assumptions describing birth–and–growth processes. The pioneer work [22] studies a growth model for a single convex crystal based on Minkowski sum, whilst in [1], the authors derive a computationally tractable mathematical model of such processes that emphasizes the geometric growth of objects without regularity assumptions on the boundary of crystals. Here, in view of this approach, we introduce different set–valued parametric estimators of the rate of growth of the process. They arise naturally from a decomposition via Minkowski sum and they are consistent as the observation window expands to the whole space. On the other hand, keeping in mind that distributions of random closed sets are determined by Choquet capacity functionals and that the nucleation process cannot be observed directly, the paper provides an estimation procedures of the hitting function of the nucleation process.
The article is organized as follows. Section 1 contains preliminary properties. Section 2 introduces a birth–and–growth model for random closed sets as the combination of two set–valued processes (nucleation and growth respectively). Further, a decomposition theorem is established to characterize the nucleation and the growth. Section 3 studies different estimators of the growth process and correspondent consistent properties are proved. In Section 4, the nucleation process is studied via the hitting function, and a consistent estimator of the nucleation hitting function is derived. Section 5 concludes the paper with some brief discussions.

1. Preliminary results

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$ be the sets of all non–negative integer, integer, real and non–negative real numbers respectively, and let $\mathfrak{X} = \mathbb{R}^d$. Let $\mathbb{F}$ be the family of all closed subsets of $\mathfrak{X}$ and $\mathbb{F}' = \mathbb{F} \setminus \{\emptyset\}$. The subscripts b, k and c denote boundedness, compactness and convexity properties respectively (e.g. $\mathbb{F}_{kc}$ denotes the family of all compact convex subsets of $\mathfrak{X}$).

For all $A, B \subseteq \mathfrak{X}$ and $\alpha \in \mathbb{R}_+$, let us define

$$\begin{aligned}
A + B &= \{a + b : a \in A,\ b \in B\} = \textstyle\bigcup_{b \in B} b + A, && \text{(Minkowski Sum)},\\
\alpha \cdot A &= \alpha A = \{\alpha a : a \in A\}, && \text{(Dilation by Scalars)},\\
A \ominus B &= \left(A^C + B\right)^C = \textstyle\bigcap_{b \in B} b + A, && \text{(Minkowski Subtraction)},\\
\check{A} &= \{-a : a \in A\}, && \text{(Symmetric Set)},
\end{aligned}$$

where $A^C = \{x \in \mathfrak{X} : x \notin A\}$ is the complement to A, $x + A$ means $\{x\} + A$ (i.e. A translate by vector x), and, by definition, $\emptyset + A = \emptyset = \alpha\emptyset$. It is well known that $+$ is a commutative and associative operation with a neutral element but, in general, $A \subseteq \mathfrak{X}$ does not admit inverse (cf. [30, 19]) and $\ominus$ is not the inverse operation of $+$. The following relations are useful in the sequel (see [31]): for every $A, B, C \subseteq \mathfrak{X}$

$$\begin{gathered}
(A \cup B) + C = (A + C) \cup (B + C),\\
\text{if } B \subseteq C, \quad A + B \subseteq A + C,\\
(A \ominus B) + \check{B} \subseteq A \quad \text{and} \quad (A + B) \ominus \check{B} \supseteq A,\\
(A \cup B) \ominus C \supseteq (A \ominus C) \cup (B \ominus C).
\end{gathered}$$

In the following, we shall work with closed sets. In general, if $A, B \in \mathbb{F}$ then $A + B$ does not belong to $\mathbb{F}$ (e.g., in $\mathfrak{X} = \mathbb{R}$ let $A = \{n + 1/n : n > 1\}$ and $B = \mathbb{Z}$, then $\{1/n = (n + 1/n) + (-n)\} \subset A + B$ and $1/n \downarrow 0$, but $0 \notin A + B$). In view of this fact, we define $A \oplus B = \overline{A + B}$ where $\overline{(\cdot)}$ denotes the closure in $\mathfrak{X}$. It can be proved that, if $A \in \mathbb{F}$ and $B \in \mathbb{F}_k$ then $A + B \in \mathbb{F}$ (see [31]).
For any $A, B \in \mathbb{F}'$ the *Hausdorff distance* (or *metric*) is defined by

$$\delta_H(A, B) = \max\left\{\sup_{a \in A} \inf_{b \in B} \|a - b\|_{\mathfrak{X}}, \sup_{b \in B} \inf_{a \in A} \|a - b\|_{\mathfrak{X}}\right\}.$$

A random closed set (RaCS) is a map X defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with values in $\mathbb{F}$ such that $\{\omega \in \Omega : X(\omega) \cap K \neq \emptyset\}$ is measurable for each compact set K in $\mathfrak{X}$. It can be proved (see [20]) that, if X, X_1, X_2 are RaCS and if ξ is a random variable, then $X_1 \oplus X_2$, $X_1 \ominus X_2$, ξX and $(\text{Int } X)^C$ are RaCS. Moreover, if $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of RaCS then $X = \overline{\bigcup_{n \in \mathbb{N}} X_n}$ is so.
Let X be a RaCS, then $T_X(K) = \mathbb{P}(X \cap K \neq \emptyset)$, for all $K \in \mathbb{F}_k$, is its *hitting function* (or *Choquet capacity functional*). The well known Choquet–Kendall–Matheron Theorem states that, the probability law $\mathbb{P}_X$ of any RaCS X is uniquely determined by its hitting function (see [21]) and hence by $Q_X(K) = 1 - T_X(K)$.

Remark 1.1. *(See [23].)* If both X and Y are RaCS, then, for every $K \in \mathbb{F}_k$,

$$T_{X \oplus Y}(K) = \mathbb{E}\left[\mathbb{E}\left[T_X\left(K \oplus \check{Y}\right)\middle| Y\right]\right].$$

Moreover, if X, Y are independent, then, for every $K \in \mathbb{F}_k$,

$$T_{X \cup Y}(K) = T_X(K) + T_Y(K) - T_X(K)\, T_Y(K).$$

A RaCS X is *stationary* if the probability laws of X and $X+v$ coincide for every $v \in \mathfrak{X}$. Thus, the hitting function of a stationary RaCS clearly is invariant up to translation $T_X(K) = T_X(K+v)$ for each $K \in \mathbb{F}_k$ and any $v \in \mathfrak{X}$.
A stationary RaCS X is *ergodic*, if, for all $K_1, K_2 \in \mathbb{F}$,

$$\frac{1}{|W_n|}\int_{W_n} Q_X((K_1+v)\cup K_2)dv \to Q_X(K_1)Q_X(K_2), \qquad \text{as} \qquad n\to\infty,$$

where $\{W_n\}_{n\in\mathbb{N}}$ is a *convex averaging sequence of sets* in $\mathfrak{X}$ (see [12]), i.e. each $\{W_n\}$ is convex and compact, $W_n \subset W_{n+1}$ for all $n \in \mathbb{N}$ and

$$\sup\{r \geq 0 : B(x,r) \subset W_n \text{ for some } x \in W_n\} \uparrow \infty, \qquad \text{as} \qquad n \to \infty.$$

Proposition 1.2. Let X, Y be RaCS with $Y \in \mathbb{F}'_k$ a.s. and X stationary, then $X+Y$ is a stationary RaCS. Moreover, if X is ergodic, then $X+Y$ is so.

Proof. Let $Z = X+Y$, it is a RaCS. Note that

$$T_Z(K) = \mathbb{E}\left[\mathbb{E}\left[T_X\left(K+\check{Y}\right)\middle| Y\right]\right] = \mathbb{E}\left[\mathbb{E}\left[T_X\left(K+\check{Y}+v\right)\middle| Y\right]\right] = T_Z(K+v),$$

for every $K \in \mathbb{F}_k$ and $v \in \mathfrak{X}$, then $Z = X+Y$ is stationary. Further, let us suppose that X is ergodic, then, by Fubini–Tonelli's Theorem and by dominated convergence theorem, we obtain

$$\begin{aligned}\int_{W_n}\frac{Q_Z((K_1+v)\cup K_2)}{|W_n|}dv &= \mathbb{E}\left[\mathbb{E}\left[\frac{1}{|W_n|}\int_{W_n} Q_X(((K_1+v)\cup K_2)+\check{Y})dv\middle| Y\right]\right]\\ &\to \mathbb{E}\left[\mathbb{E}\left[Q_X(K_1+\check{Y})Q_X(K_2+\check{Y})\middle| Y\right]\right]\\ &= Q_Z(K_1)Q_Z(K_2),\end{aligned}$$

for every $K_1, K_2 \in \mathbb{F}_k$. Hence $X+Y$ is ergodic. ∎

2. A Birth–and–Growth process

Here, $\mathcal{F}$ denotes the family of all fuzzy sets $\nu : \mathfrak{X} \to [0,1]$. We recall that a fuzzy random set is defined as a measurable map $X : \Omega \to \mathcal{F}$, where Ω and $\mathcal{F}$ are endowed with the relevant σ–algebra's (see [20]). A fuzzy set–valued stochastic process is a measurable map $X : \Omega \times \mathbb{N} \to \mathcal{F}$. For $\beta \in (0,1]$, we call *β–fuzzy monotone set–valued stochastic process* a fuzzy set–valued stochastic process X such that, for every $\omega \in \Omega$ and $t_1, t_2 \in \mathbb{N}$ with $t_1 \leq t_2$,

$$X_\alpha(\omega,t_1) \subseteq X_\alpha(\omega,t_2), \text{ for each } \alpha \in (0,1] \text{ with } \beta \leq \alpha$$

where $X_\alpha(\omega,t) = \{x \in \mathfrak{X} : X(\omega,t)(x) \geq \alpha\}$ is the α–level set of the fuzzy set $X(\omega,t)$. In other words, a β–fuzzy monotone set–valued stochastic process is a time dependent fuzzy random set for which every α–level processes are not–decreasing RaCS processes, for any $\beta \leq \alpha$. Clearly, the associated α–level set stochastic processes X_α are useful in order to study a fuzzy monotone set–valued stochastic process X. In the following, we deal with 1–fuzzy monotone

set–valued stochastic process. A set–valued stochastic process is modeled to describe $\Theta = X_1$ process and analyzed from a statistical point of view.
Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space with the usual properties. Let $\{B_n : n \geq 0\}$ and $\{G_n : n \geq 1\}$ be two families of RaCS such that B_n is $\mathfrak{F}_n$–measurable and G_n is $\mathfrak{F}_{n-1}$–measurable. These processes represent the *birth* (or *nucleation*) *process* and the *growth process* respectively. Thus, let us define recursively a birth–and–growth process $\Theta = \{\Theta_n : n \geq 0\}$ by

$$\Theta_n = \begin{cases} (\Theta_{n-1} \oplus G_n) \cup B_n, & n \geq 1, \\ B_0, & n = 0. \end{cases} \tag{1}$$

Roughly speaking, Equation (1) means that Θ_n is the enlargement of Θ_{n-1} due to a Minkowski *growth* G_n while *nucleation* B_n occurs. Without loss of generality let us consider the following assumption.
(A-1) For every $n \geq 1$, $0 \in G_n$.
Note that, Assumption (A-1) is equivalent to $\Theta_{n-1} \subseteq \Theta_n$.
In [1], the authors derive (1) from a continuous time birth–and–growth process; here, in order to make inference, the discrete time case is sufficient. Indeed, a sample of a birth–and–growth process is usually a time sequence of pictures that represent process Θ at different temporal step; namely Θ_{n-1}, Θ_n. Thus, in view of (1), it is interesting to investigate $\{G_n\}$ and $\{B_n\}$; in particular, we shall estimate the maximal growth G_n and the capacity functional of B_n. For the sake of simplicity, Y, X, G and B will denote RaCS Θ_n, Θ_{n-1}, G_n and B_n respectively (then $X \subseteq Y$). Thus, let us consider the following general definition.

Definition 2.1. Let Y, X be RaCS with $X \subseteq Y$. A *X–decomposition of Y* is a couple of RaCS (G, B) for which

$$Y = (X \oplus G) \cup B. \tag{2}$$

Note that, since we can consider $(G, B) = (\{0\}, Y)$, there always exists a X–decomposition of Y. It can happen that G and B in (2) are not unique. As example, let $Y = [0, 1]$ and $X = \{0\}$, then both $(G_1, B_1) = (Y, Y)$ and $(G_2, B_2) = (X, Y)$ satisfy (2). As a consequence, since we can not distinguish between two different decompositions, we shall choose a maximal one according to the following proposition.

Proposition 2.2. *(See [31])* Let Y, X be RaCS with $X \subseteq Y$. Then

$$G = Y \ominus \check{X} = \{g \in \mathfrak{X} : g + X \subseteq Y\}. \tag{3}$$

is the greatest RaCS, with respect to set inclusion, such that $(X \oplus G) \subseteq Y$.

Corollary 2.3. The couple $(G = Y \ominus \check{X}, B = Y \cap \overline{(X \oplus G)^C})$ is the max-min X–decomposition of Y. As a consequence, (G, B) is a X–decomposition of Y and for any other X–decomposition of Y, say (G', B'), then $G' \subseteq G$ and $B' \supseteq B$.

In other words, if X, G', B' are RaCS and $Y = (X \oplus G') \cup B'$, then $G = Y \ominus \check{X} \supseteq G'$ and $Y = (X \oplus G) \cup B'$.
Let Θ be as in (1). From now on, G_n denotes $\Theta_n \ominus \check{\Theta}_{n-1}$ that, as a consequence of Assumption (A-1), contains the origin. Moreover, we shall suppose
(A-2) There exists $K \in \mathbb{F}'_b$ such that $G_n = \Theta_n \ominus \check{\Theta}_{n-1} \subseteq K$ for every $n \in \mathbb{N}$.
(A-3) For every $n \geq 1$, $\left(B_n \ominus \check{\Theta}_{n-1}\right) = \emptyset$ almost surely.
Roughly speaking, Assumption (A-2) means that process Θ does not grow too "fast", whilst Assumption (A-3) means that it cannot born something that, up to a translation, is larger (or equal) than what there already exists.
Let us remark that Assumption (A-2) implies $\{G_n\} \subset \mathbb{F}'_k$ and $X \oplus G_n = X + G_n$, for any RaCS X.

3. Estimators of G

On the one hand Proposition 2.2 gives a theoretical formula for G, but, on the other hand, in practical cases, data are bounded by some observation window and edge effects may cause problems. Hence, as the standard statistical scheme for spatial processes (e.g. [24]) suggests, we wonder if there exists a consistent estimator of G as the observation window expands to the whole space $\mathfrak{X}$.

Proposition 3.1. If $\{W_i\}_{i \in \mathbb{N}} \subset \mathbb{F}'_{ck}$ is a convex averaging sequence of sets, then, for any $K \in \mathbb{F}'_k$, $\mathfrak{X} = \bigcup_{i \in \mathbb{N}} W_i \ominus \check{K}$. In this case, we say that $\{W_i\}_{i \in \mathbb{N}}$ expands to $\mathfrak{X}$ and we shall write $W_i \uparrow \mathfrak{X}$.

Proof. At first note that $\mathfrak{X} = \bigcup_{i \in \mathbb{N}} \text{Int } W_i$ and for any $i \in \mathbb{N}$, $W_i \subseteq W_{i+1}$.
Let $x \in \mathfrak{X}$ and $K \in \mathbb{F}'_k$. Note that, $x + K \in \mathbb{F}'_k$ is a compact set. Then there exists a finite family of indices $I \subset \mathbb{N}$ such that, if $N = \max I$, then

$$x + K \subseteq \bigcup_{j \in I} \text{Int } W_j = \text{Int } W_N.$$

Hence, we have that $x \in \text{Int } W_N \ominus \check{K} \subseteq W_N \ominus \check{K}$, i.e., for any $x \in \mathfrak{X}$, there exists $n_0 \in \mathbb{N}$ such that $x \in W_{n_0} \ominus \check{K}$. ■
Let $W \in \{W_i\}_{i \in \mathbb{N}}$ be an observation window and let us denote by Y_W and X_W, the (random) observation of Y and X through W, i.e. $Y \cap W$ and $X \cap W$ respectively. Let us consider the estimator of G given by the maximal X_W–decomposition of Y_W:

$$\widehat{G}_W = \left(Y_W \ominus \check{X}_W\right) \tag{4}$$

so that $X_W \oplus \widehat{G}_W \subseteq Y_W \subseteq W$. Notice that, whenever Y and X are bounded, then there exists $W_j \in \{W_i\}_{i \in \mathbb{N}}$ such that $Y \subseteq W_j$ and $\check{X} \subseteq W_j$, hence $\widehat{G}_{W_j} = Y \ominus \check{X} = G$. In other words, on the set $\{\omega \in \Omega : X(\omega), Y(\omega) \text{ bounded}\}$, the estimator (4) is consistent

$$\widehat{G}_{W_i}(Y, X | Y, X \text{ bounded}) \to G, \qquad \text{as } W_i \uparrow \mathfrak{X};$$

otherwise, as we already said, if Y and X are unbounded, edge effects may cause problems and the estimator (4) is, in general, not consistent as we discussed in the following example.

Example 3.2. Let $\mathfrak{X} = \mathbb{R}^2$, let us consider $X = (\{x = 0\} \cup \{y = 0\})$ and $Y = X + B(0,1)$ where $B(0,1)$ is the closed unit ball centered in the origin. Surely $X \subset Y$, and they are unbounded. Note that $Y = (X + G)$ for any G such that $(\{0\} \times [-1,1] \cup [-1,1] \times \{0\}) \subseteq G \subseteq B(0,1)$. On the other hand, by Proposition 2.2, there exists a unique G that is the greatest set, with respect to set inclusion; in this case $G = [-1,1] \times [-1,1]$.
Let us suppose $0 \in W_0$ and let $W \in \{W_i\}_{i\in\mathbb{N}}$, then, by Equation (4), the estimator of G is $\widehat{G}_W = \{0\} \neq G$. This is an edge effect due to the fact that, for every G' with $\{0\} \subset G' \subseteq G$, it holds $(X_W + G') \cap W^C \neq \emptyset$ and then $X_W + G' \nsubseteq Y_W$ that does not agree with Proposition 2.2.

Edge effects can be reduced by considering the following estimators of G

$$\widehat{G}^1_W = \left(Y_W \ominus \check{X}_{W\ominus\check{K}}\right) \cap K, \tag{5}$$
$$\widehat{G}^2_W = \left(\left[Y_W \cup \left(\partial^{+K}_W X_W\right)\right] \ominus \check{X}_W\right) \cap K; \tag{6}$$

where K is given in Assumption (A-2) and where $\left(\partial^{+K}_W X_W\right) = \overline{(X_W + K) \setminus W}$. The role of K will be clarified in Proposition 3.3 where it guarantees the monotonicity of $\widehat{G}^1_W$. Note that, estimators (5) (6) are bounded (i.e. compact) RaCS, moreover, if Y and X are bounded, then $\widehat{G}^1_{W_j}$, $\widehat{G}^2_{W_j}$ eventually coincide with the estimator (4); i.e. there exists n_0 such that for all $j \geq n_0$, $\widehat{G}_{W_j} = \widehat{G}^1_{W_j} = \widehat{G}^2_{W_j} = G$.
Let us explain how $\widehat{G}^1_W$ and $\widehat{G}^2_W$ work. Estimator $\widehat{G}^1_W$ is obtained by reducing the information given by X to the smaller window $W \ominus \check{K}$, whilst Y is observed in W. Then $\widehat{G}^1_W$ is the greatest subset of K, with respect to set inclusion, such that $X_{W\ominus\check{K}} + \widehat{G}^1_W \subseteq Y_W$ (see Proposition 2.2). Estimator $\widehat{G}^2_W$ is obtained by observing X in W (and not $W \ominus \check{K}$), whilst Y is increased (at least) by $\left(\partial^{+K}_W X_W\right)$, that is the greatest possible set of growth for X outside of the observed window W. Then $\widehat{G}^2_W$ is the greatest subset of K, with respect to set inclusion, such that $(X_W + \widehat{G}^2_W) \cap W \subseteq Y_W$, or, alternatively, $X_W + \widehat{G}^2_W \subseteq Y_{W'}$, where $Y_{W'} = Y_W \cup \left(\partial^{+K}_W X_W\right)$ (see Proposition 2.2).
Note that by definition of Minkowski Subtraction

$$\widehat{G}^1_W = \bigcap_{x\in X_{W\ominus\check{K}}} x + ((-x+K) \cap Y_W),$$
$$\widehat{G}^2_W = \bigcap_{x\in X_W} x + ((-x+K) \cap Y_{W'});$$

i.e. every $x \in X_{W\ominus\check{K}}$ (resp. $x \in X_W$) "grows" at most as $(-x+K) \cap Y_W$ (resp. $(-x+K) \cap Y_{W'}$).
Now, we are ready to show the consistency property of $\widehat{G}^1_{W_i}$ and $\widehat{G}^2_W$. In particular, Proposition 3.3 proves that $\widehat{G}^1_{W_i}$ decreases, with respect to set inclusion, to the theoretical G, whenever W_i expands to the whole space ($W_i \uparrow \mathfrak{X}$). Proposition 3.4 proves that, for every $W \in \mathbb{F}'$, $\widehat{G}^2_W$ is a better estimator than $\widehat{G}^1_W$ and hence it is a consistent estimator of G.

Proposition 3.3. Let Y, X be RaCS, let $0 \in G = Y \ominus \check{X} \subseteq K$. The following statements hold for $\widehat{G}^1_W$:
(1) $G \subseteq \widehat{G}^1_W$ for every W;
(2) $\widehat{G}^1_{W_2} \subseteq \widehat{G}^1_{W_1}$ if $W_2 \supseteq W_1$;
(3) If $W_i \uparrow \mathfrak{X}$, then $\bigcap_{i\in\mathbb{N}} \widehat{G}^1_{W_i} = G$. Moreover,

$$\lim_{i\to\infty} \delta_H(\widehat{G}^1_{W_i}, G) = 0. \tag{7}$$

Proof.
(1) Since $0 \in K$, $\bigcap_{k\in K} -k + W = W \ominus \check{K} \subseteq W$ and then $X_{W\ominus\check{K}} \subseteq W$. Let $g \in G$, then $g + X \subseteq Y$. Since $g \in K$, last inclusion still holds when X and Y are substituted by $X_{W\ominus\check{K}}$ and Y_W respectively: $g + X_{W\ominus\check{K}} \subseteq Y_W$. Thus $g \in \widehat{G}^1_W$ follows by Equation (5) and Proposition 2.2.
(2) In order to obtain $\widehat{G}^1_{W_2} \subseteq \widehat{G}^1_{W_1}$, it is sufficient to prove that

$$X_{W_1\ominus\check{K}} + \widehat{G}^1_{W_2} \subseteq Y_{W_1} \tag{8}$$

since $\widehat{G}^1_{W_1}$ is the greatest set, with respect to set inclusion, for which the inclusion (8) holds. In fact, $W_1 \ominus \check{K} \subseteq \big(W_1 \ominus \check{K}\big) + K \subseteq W_1 \subseteq W_2$, then $X_{W_1\ominus\check{K}} \subseteq X_{W_2}$. Let $x \in X_{W_1\ominus\check{K}} = X \cap \big(W_1 \ominus \check{K}\big)$, then $x \in X_{W_2}$. By definition of $\widehat{G}^1_{W_2}$, we have

$$x + \widehat{G}^1_{W_2} \subseteq Y_{W_2} \subseteq Y.$$

On the other hand, since $x \in W_1 \ominus \check{K}$ and $\widehat{G}^1_{W_2} \subseteq K$, we have

$$x + \widehat{G}^1_{W_2} \subseteq \big(W_1 \ominus \check{K}\big) + K \subseteq W_1;$$

i.e. $x + \widehat{G}^1_{W_2}$ is included both in Y and in W_1.
(3) Since $G \subseteq \bigcap_{i\in\mathbb{N}} \widehat{G}^1_{W_i}$, it remains to prove that

$$\bigcap_{i\in\mathbb{N}} \widehat{G}^1_{W_i} \subseteq G;$$

i.e. if $g \in \widehat{G}^1_{W_i}$ for each $i \in \mathbb{N}$, then $g \in G$. Take $g \in \bigcap_{i\in\mathbb{N}} \widehat{G}^1_{W_i}$. By definition of $\widehat{G}^1_{W_1}$, we have

$$g + x \in Y \text{ for all } x \in X_{W_i\ominus\check{K}} \text{ and } \forall i \in \mathbb{N}. \tag{9}$$

By contradiction, assume $g \notin G$. Then $g + X \nsubseteq Y$, i.e. there exists $\overline{x} \in X$ such that $(g + \overline{x}) \notin Y$. On the one hand, Proposition 3.1 implies that there exists $j \in \mathbb{N}$ such that $\overline{x} \in W_j \ominus \check{K}$. On the other hand, Equation (9) implies $g + \overline{x} \in Y$ which is a contradiction. Thus Theorem 1.1.18 in [20] implies (7). ■

Proposition 3.4. For every $W \in \mathbb{F}'$, $G \subseteq \widehat{G}^2_W \subseteq \widehat{G}^1_W$.

Proof. Let us divide the proof in two parts; in the first one we prove that $\widehat{G}^2_W \subseteq \widehat{G}^1_W$, in the second one that $G \subseteq \widehat{G}^2_W$. Let $g \in \widehat{G}^2_W$ and $x \in X_{W \ominus \check{K}}$. Since $\widehat{G}^2_W \subseteq K$, we have

$$x + g \in \left(W \ominus \check{K}\right) + \widehat{G}^2_W \subseteq \left(W \ominus \check{K}\right) + K \subseteq W; \tag{10}$$

where we use properties of monotonicity of the Minkwoski Subtraction and Sum. Moreover, by definition of $\widehat{G}^2_W$,

$$x + g \in Y_W, \qquad \text{or} \qquad x + g \in \left(\partial^{+K}_W X_W\right) \subseteq W^C.$$

By (10), $x + g \in Y_W$. The arbitrary choice of $x \in X_{W \ominus \check{K}}$ completes the first part of the proof. For the second part, let $g \in G$ and $x \in X_W$. By definition of G, $x + g \in Y$. We have two cases:
- $x + g \in W$, and therefore $x + g \in Y_W$,
- $x + g \notin W$. Since $x \in X_W$, $x + g \in (X_W + G) \setminus W \subseteq \left(\partial^{+K}_W X_W\right)$. ■

Corollary 3.5. $\widehat{G}^2_W$ is consistent (i.e. $\widehat{G}^2_W \downarrow G$ whenever $W \uparrow \mathfrak{X}$).

In Figure 2, we consider two pictures of a simulated birth–and–growth process, at two different time instants, that in our notations are X and Y. In the same figure, emphasizing the differences, we report here the magnified pictures of the true growth used for the simulation, the computed $\widehat{G}^2_W$, $\widehat{G}^1_W$ and $\widehat{G}^1_{W\ominus\check{K}}$. Note that they agree with Proposition 3.3 and Proposition 3.4 since $\widehat{G}^1_{W\ominus\check{K}} \supseteq \widehat{G}^1_W \supseteq \widehat{G}^2_W$.

A General Definition of $\widehat{G}^2_W$. The following proposition shows that the estimator in (6) can be defined in an equivalent way by

$$\widehat{G}^2_W(Z) = \left\{\left[Y_W \cup \left(\partial^{+K}_W Z\right)\right] \ominus \check{X}_W\right\}_K;$$

where $\left(\partial^{+K}_W X\right)$ in (6) is substituted by $\left(\partial^{+K}_W Z\right)$ with

$$X_{W\setminus\left(W\ominus\check{K}\right)} \subseteq Z \subseteq W. \tag{11}$$

In other words, we are saying that, under condition (11), $\widehat{G}^2_W(Z)$ does not depend on Z. From a computational point of view, this means that Z can be chosen in a way that reduces the computational costs. On the one hand, the best choice of Z seems to be the smallest possible set, i.e. $Z = X_{W\setminus\left(W\ominus\check{K}\right)}$. On the other hand, in order to get $X_{W\setminus\left(W\ominus\check{K}\right)}$, we have to compute $\left(W \ominus \check{K}\right)$ that may be costly if at least one between W and K has a "bad shape" (for instance it is not a rectangular one).

Proposition 3.6. If $Z_1, Z_2 \in \mathfrak{P}'$ both satisfy condition (11), then $\widehat{G}^2_W(Z_1) = \widehat{G}^2_W(Z_2)$.

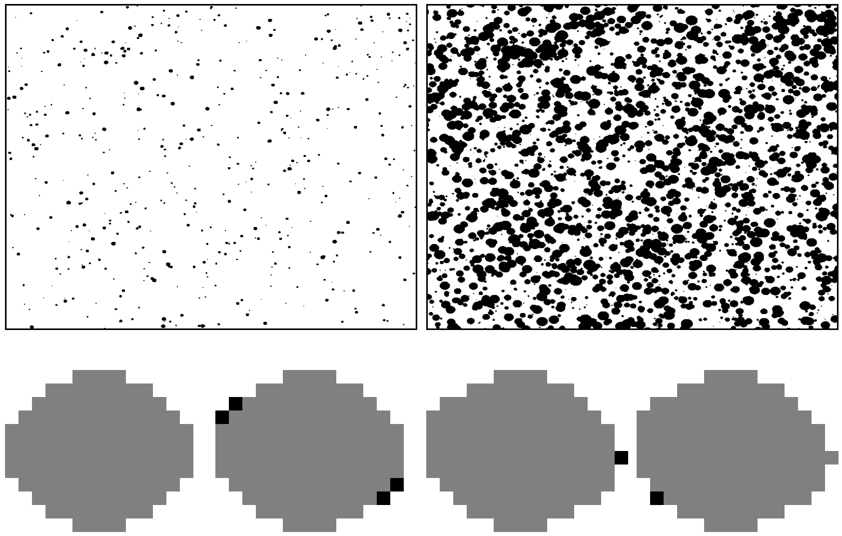


Figure 2: Two pictures of a simulated birth–and–growth process at two different time instants (X and Y). The magnified pictures of the true growth used for the simulation, the computed $\widehat{G}^2_W$, $\widehat{G}^1_W$ and $\widehat{G}^1_{W\ominus\check{K}}$.

Proof. It is sufficient to prove:
(1) $Z_1 \subseteq Z_2$ implies $\widehat{G}^2_W(Z_1) \subseteq \widehat{G}^2_W(Z_2)$;
(2) $\widehat{G}^2_W(W) \subseteq \widehat{G}^2_W\left(X_{W\setminus(W\ominus\check{K})}\right)$.
In fact, (1) and (2) imply that $\widehat{G}^2_W(W) = \widehat{G}^2_W\left(X_{W\setminus(W\ominus\check{K})}\right)$. At the same time they imply $\widehat{G}^2_W(Z) = \widehat{G}^2_W\left(X_{W\setminus(W\ominus\check{K})}\right)$ holds for every Z that satisfies (11); that is the thesis.
STEP (1) is a consequence of the following implications

$$\begin{aligned} Z_1 \subseteq Z_2 \quad &\Rightarrow \quad Z_1 + K \subseteq Z_2 + K, \\ &\Rightarrow \quad Y_W \cup [(Z_1+K)\setminus W] \subseteq Y_W \cup [(Z_2+K)\setminus W], \\ &\Rightarrow \quad \widehat{G}^2_W(Z_1) \subseteq \widehat{G}^2_W(Z_2); \end{aligned}$$

where the last one holds since $X_1 \ominus Y \subseteq X_2 \ominus Y$ if $X_1 \subseteq X_2$ (see [31]).
Before proving the second step, we show that $\widehat{G}^2_W(Z) = \widehat{G}^2_W\left(Z_{W\setminus(W\ominus\check{K})}\right)$ for all Z that satisfies (11). This statement is true if $\left(Z_{W\setminus(W\ominus\check{K})} + K\right)\setminus W$ and $(Z+K)\setminus W$ are the same set. Since Minkowski sum is distributive with respect to union, we get

$$\begin{aligned} (Z+K)\setminus W \quad &= \quad \left[\left(Z_{W\setminus(W\ominus\check{K})} \cup Z_{W\ominus\check{K}}\right) + K\right]\setminus W \\ &= \quad \left[\left(Z_{W\setminus(W\ominus\check{K})} + K\right)\setminus W\right] \cup \left[(Z_{W\ominus\check{K}} + K)\setminus W\right]. \end{aligned}$$

Then we have to prove that $\left[\left(Z_{W\ominus\check{K}}+K\right)\setminus W\right]=\emptyset$:

$$\begin{aligned}\left(Z_{W\ominus\check{K}}+K\right)\setminus W &= \left\{\left[Z\cap\left(W\ominus\check{K}\right)\right]+K\right\}\setminus W\\ &\subseteq \left\{(Z+K)\cap\left[\left(W\ominus\check{K}\right)+K\right]\right\}\setminus W\\ &\subseteq [(Z+K)\cap W]\setminus W=\emptyset.\end{aligned}$$

STEP (2). Since $\widehat{G}^2_W\left(X_W\right)=\widehat{G}^2_W\left(X_{W\setminus(W\ominus\check{K})}\right)$, thesis becomes $\widehat{G}^2_W(W)\subseteq\widehat{G}^2_W(X_W)$. Let $g\in\widehat{G}^2_W(W)$. We must prove $g\in\widehat{G}^2_W(X_W)$, i.e. for every $x\in X_W$

$$g+x\in Y_W,\qquad\text{or}\qquad g+x\in(X_W+K)\setminus W.$$

Since $g\in\widehat{G}^2_W(W)$, for any $x\in X_W$ we can have two possibilities
(a) $g+x\in Y_W$,
(b) $g+x\in(W+K)\setminus W$.
It remains to prove that (b) implies $g+x\in(X_W+K)\setminus W$. In particular, (b) implies $g+x\in W^C$. At the same time $g+x\in X_W+K$, i.e. $g+x\in(X_W+K)\setminus W$. ■

4. Hitting Function Associated to B

In many practical cases, an observer, through a window W and at two different instants, observes the nucleation and growth processes namely X and Y. According to Section 3 we can estimate G via the consistent estimator $\widehat{G}^2_W$ or $\widehat{G}^1_W$ (in the following we shall write $\widehat{G}_W$ meaning one of them). From the birth–and–growth process point of view, it is also interesting to test whenever the nucleation process $B=\{B_n\}_{n\in\mathbb{N}}$ is a specific RaCS (for example a Boolean model or a point process). In general, we cannot directly observe the n–th nucleation B_n since it can be overlapped by other nuclei or by their evolutions. Nevertheless, we shall infer on the hitting function associated to the nucleation process $T_{B_n}(\cdot)$. Let us consider the decomposition given by (2) $Y=(X+G)\cup B$ then the following proposition is a consequence of Remark 1.1.

Proposition 4.1. If (G,B) is a X–decomposition of Y such that B is independent on X and on G, then, for each $K\in\mathbb{F}_k$,

$$T_Y\left(K\right)=T_{X+G}\left(K\right)+T_B\left(K\right)-T_{X+G}\left(K\right)T_B\left(K\right),$$

that, in terms of $Q_\cdot(K)=(1-T_\cdot\left(K\right))$, is equivalent to

$$Q_Y(K)=Q_B(K)Q_{X+G}(K).$$

In other words, the probability for the exploring set K to miss Y is the probability for K to miss B multiplied by the probability for K to miss $X+G$.

Remark 4.2. Working with data we shall consider two estimators of the hitting function (we refer to [24, p. 57–63] and references therein). In particular, if X is a stationary ergodic RaCS, then $T_X(\cdot)$ can be estimated by a single realization of X and two empirical estimators are given by

$$\widehat{T}_{X,W}(K) = \frac{\mu_\lambda\left((X + \check{K}) \cap (W \ominus K_0)\right)}{\mu_\lambda\left(W \ominus K_0\right)}, \quad K \in \mathbb{F}_k;$$

where μ_λ is the Lebesgue measure on $\mathfrak{X} = \mathbb{R}^d$ and K_0 is a compact set such that $K \subset K_0$ for all $K \in \mathbb{F}_k$ of interest.

A *regular closed* set in $\mathfrak{X}$ is a closed set $G \in \mathbb{F}'$ for which $G = \overline{\text{Int } G}$; i.e. G is the closure (in $\mathfrak{X}$) of its interior.

Proposition 4.3. Let $G \in \mathbb{F}'_k$ be a regular closed subset in $\mathfrak{X}$. Then, for every $X \in \mathbb{F}'$, $X + G$ is a regular closed set.

Proof. Since $X + G$ is a closed set, $\overline{\text{Int }(X + G)} \subseteq X + G$. It remains to prove that $X + G \subseteq \overline{\text{Int }(X + G)}$. Let $y \in X + G$, then there exists $x \in X$ and $g \in G$ such that $y = x + g$. If $g \in \text{Int } G$, then there exists an open neighborhood of g for which $U(g) \subseteq \text{Int } G$ and $x + U(g)$ is an open neighborhood of $x + g$ included in $X + G$; i.e. $x + g \in \text{Int }(X + G)$. On the other hand, let $g \in \partial G = G \setminus \text{Int } G$, then there exists $\{g_n\}_{n\in\mathbb{N}} \subset G$ such that $g_n \to g$ and $g_n \in \text{Int } G$, for all $n \in \mathbb{N}$. Thus, for every $n \in \mathbb{N}$, $x + g_n$ is an interior point of $X + G$ and $x + g_n \to x + g \in \overline{\text{Int }(X + G)}$. ■

Proposition 4.4. *(See [24, Theorem 4.5 p. 61] and references therein)* Let X be an ergodic stationary random closed set. If the random set X is almost surely regular closed

$$\sup_{\substack{K \in \mathbb{F}_k \\ K \subseteq K_0}} \left|\widehat{T}_{X,W}(K) - T_X(K)\right| \to 0, \quad \text{a.s.} \tag{12}$$

as $W \uparrow \mathfrak{X}$ and for every $K_0 \in \mathbb{F}'$.

Remark 4.5. Proposition 4.3, together to Equation (1) means that, if $\{G_n\}_{n\in\mathbb{N}}$ is a sequence of almost surely regular closed sets, then $\{\Theta_n\}_{n\in\mathbb{N}}$ is so.

The following Theorem shows that the hitting functional Q_B of the hidden nucleation process can be exstimated by the observable quantity $\widetilde{Q}_{B,W}$, where for every $K \in \mathbb{F}_k$,

$$\widetilde{Q}_{B,W}(K) := \frac{\widehat{Q}_{Y,W}(K)}{\widehat{Q}_{X+\widehat{G}_W,W}(K)}, \tag{13}$$

and $\widehat{G}_W$ is given by (5) or (6).

Theorem 4.6. Let X, Y be two RaCS a.s. regular closed. Let (G, B) be a $X-$ decomposition of Y with B a stationary ergodic RaCS independent on G and X. Assume that G is an a.s. regular closed set and $\widetilde{Q}_{B,W}$ defined in Equation (13). Then, for any $K \in \mathbb{F}_k$,

$$\left|\widetilde{Q}_{B,W}(K) - Q_B(K)\right| \underset{W\uparrow\mathfrak{X}}{\longrightarrow} 0, \quad \text{a.s.}$$

Proof. Let $K \in \mathbb{F}_k$ be fixed. For the sake of simplicity, $Q_\cdot$, $\widetilde{Q}_\cdot$ and $\widehat{Q}_\cdot$ denote $Q_\cdot(K)$, $\widetilde{Q}_{\cdot,W}(K)$ and $\widehat{Q}_{\cdot,W}(K)$ respectively. Thus,

$$\left|\widetilde{Q}_B - Q_B\right| = \left|\frac{\widehat{Q}_Y}{\widehat{Q}_{X+\widehat{G}_W}} - \frac{Q_Y}{Q_{X+G}}\right| = \left|\frac{\widehat{Q}_Y Q_{X+G} - Q_Y \widehat{Q}_{X+\widehat{G}_W}}{\widehat{Q}_{X+\widehat{G}_W} Q_{X+G}}\right|.$$

Since $Y \supseteq X + \widehat{G}_W$, $\widehat{Q}_{X+\widehat{G}_W} > \widehat{Q}_Y$. Accordingly to (12), $\widehat{Q}_Y$ converges to Q_Y that is a positive quantity. Thus, thesis is equivalent to prove that

$$\left|\widehat{Q}_Y Q_{X+G} - Q_Y \widehat{Q}_{X+\widehat{G}_W}\right| \to 0, \quad \text{a.s.}$$

as $W \uparrow \mathfrak{X}$. The following inequalities hold

$$\begin{aligned}
\left|\widehat{Q}_Y Q_{X+G} - Q_Y \widehat{Q}_{X+\widehat{G}_W}\right| &\leq Q_{X+G}\left|\widehat{Q}_Y - Q_Y\right| + Q_Y\left|Q_{X+G} - \widehat{Q}_{X+\widehat{G}_W}\right| \\
&\leq Q_{X+G}\left|\widehat{Q}_Y - Q_Y\right| + Q_Y\left|Q_{X+G} - Q_{X+\widehat{G}_W}\right| \\
&\quad + Q_Y\left|Q_{X+\widehat{G}_W} - \widehat{Q}_{X+\widehat{G}_W}\right|.
\end{aligned}$$

Proposition 1.2 and Proposition 4.3 guarantee that $X+G$ is a stationary ergodic RaCS and a.s. regular closed, then we can apply (12) to the first and the third addends. It remains to prove that

$$\left|Q_{X+G} - Q_{X+\widehat{G}_W}\right| \to 0 \quad \text{as } W \uparrow \mathfrak{X}. \tag{14}$$

Since Minkowski sum is a continuous map from $\mathbb{F} \times \mathbb{F}_k$ to $\mathbb{F}$ (see [31]), $\widehat{G}_W \downarrow G$ a.s. implies $X+\widehat{G}_W \downarrow X+G$ a.s. As a consequence, we get that $X+\widehat{G}_W \downarrow X+G$ in distribution [29, p. 182], which is Equation (14). ■

5. Conclusions

Fuzzy monotone set–valued stochastic processes can be used to describe crystal growth processes. In this framework, α–level sets, modeled as birth–and–growth process, are considered to analyze statistical aspects of crystal process.
In this paper, statistical aspects of α–level sets have been considered; in particular, consistent estimators have been provided for a general birth–and–growth stochastic process. A pure geometrical approach reduces the estimation of a growth process to simple operations among sets. At the same time, consistent estimators for the hitting function of nucleation process have been also provided.
Finally, we want to suggest some possible future developments. It may be interesting to define a continuous time set–valued stochastic process modeling birth–and–growth process, as a natural extension of the discrete time process in Section 2. Moreover, it may be interesting to define new mathematical models for fuzzy monotone set–valued stochastic process, in order to study distributions of estimators and to construct confidence intervals for the model parameters.

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