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Instanton calculus in R-R background and the topological string

Marco Billó and Marialuisa Frau

Dipartimento di Fisica Teorica, Università di Torino
Istituto Nazionale di Fisica Nucleare - sezione di Torino
via P. Giuria 1, I-10125 Torino, Italy
E-mail: billo@to.infn.it, frau@to.infn.it

Francesco Fucito

Istituto Nazionale di Fisica Nucleare - sezione di Roma 2
Dipartimento di Fisica, Università di Roma Tor Vergata
Via della Ricerca Scientifica, I-00133 Roma, Italy
E-mail: Francesco.Fucito@roma2.infn.it

Alberto Lerda

Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale
I-15100 Alessandria, Italy, and
Istituto Nazionale di Fisica Nucleare - sezione di Torino
Italy
E-mail: lerda@to.infn.it

Abstract: We study a system of fractional D3 and D(–1) branes in a Ramond-Ramond closed string background and show that it describes the gauge instantons of $\mathcal{N} = 2$ super Yang-Mills theory and their interactions with the graviphoton of $\mathcal{N} = 2$ supergravity. In particular, we analyze the instanton moduli space using string theory methods and compute the prepotential of the effective gauge theory exploiting the localization methods of the instanton calculus showing that this leads to the same information given by the topological string. We also comment on the relation between our approach and the so-called Ω-background.

Keywords: Supersymmetric gauge theory, Solitons Monopoles and Instantons, D-branes, Topological Strings.

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1. Introduction

The relationship between string theory and perturbative field theories has been thoroughly investigated for many years. The study of the non-perturbative effects in string theory and their comparison with field theory is instead much more recent. In particular, only after the introduction of D branes it has been possible to significantly improve our knowledge of the non-perturbative aspects of string theory. From the open string point of view, the D branes are hyper-surfaces spanned by the string end points on which a supersymmetric gauge theory is defined. For instance, a stack of N D3 branes in flat space supports a four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory. Adding a set of k D(−1) branes (also called D-instantons) allows to describe instanton configurations of winding number k [1 – 5]. In fact, the excitations of the open strings stretching between two D-instantons or between a D3 brane and a D-instanton, are in one-to-one correspondence

with the ADHM moduli of the super Yang-Mills instantons, and their interactions correctly account for the measure on the moduli space [6] [1].

In a recent paper [8], to substantiate the above remarks, its has been shown how the computation of tree-level string scattering amplitudes on disks with mixed boundary conditions for a D3/D(–1) system leads, in the infinite tension limit $\alpha' \to 0$, to the effective action on the instanton moduli space of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Furthermore, it has been proved that the very same disk diagrams also yield the classical field profile of the instanton solution, and that these mixed disks effectively act as sources for the various components of the $\mathcal{N} = 4$ gauge multiplet. In this framework [9, 6] it is also possible to describe theories with a smaller number of supersymmetries and their instantons by suitably orbifolding the D3/D(–1) system. In particular, considering a configuration with fractional D3 branes and fractional D-instantons in the orbifold $\mathbb{R}^4 \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ one can describe a $\mathcal{N} = 2$ super Yang-Mills theory in four dimensions together with its instantons [3]. This is the system we study in this paper.

Our aim is to generalize the construction mentioned above to encompass the so-called multi-instanton equivariant calculus [10 – 16] and try to clarify some surprising properties that have been noticed in the literature. In the multi-instanton equivariant calculus, the instanton moduli action is deformed by means of certain $\mathrm{U}(1) \times \mathrm{U}(1)$ transformations which act only on a subset of the moduli and leave the ADHM constraints invariants [2]. This deformation turns out to be the crucial ingredient that allows the evaluation of the instanton partition function $Z^{(k)}$ for arbitrary winding numbers k. In fact, in the deformed theory the supersymmetry transformations on the moduli space have only a finite number of isolated fixed points so that it becomes possible to use the localization theorems and compute exactly the integrals over the instanton moduli space. The partition function obtained in this way turns out to be an even function of the deformation parameter ε and of the vacuum expectation value (v.e.v.) a of the chiral gauge superfield of the $\mathcal{N} = 2$ theory. Furthermore, by writing

$$Z(a;\varepsilon) = \sum_{k=1}^{\infty} Z^{(k)}(a;\varepsilon) = \exp\left(\frac{\mathcal{F}_{\mathrm{n.p.}}(a;\varepsilon)}{\varepsilon^2}\right) \ , \tag{1.1}$$

one finds that

$$\lim_{\varepsilon \to 0} \mathcal{F}_{\mathrm{n.p.}}(a;\varepsilon) = \mathcal{F}(a) \tag{1.2}$$

where $\mathcal{F}(a)$ is the non-perturbative Seiberg-Witten prepotential [17]. At this point an obvious question arises: what about the terms in $\mathcal{F}_{\mathrm{n.p.}}(a;\varepsilon)$ of higher order in ε and their physical interpretation? In ref. [10] (see also refs. [13, 16]) N. Nekrasov conjectured that the terms of order ε^{2h} describe gravitational corrections to the gauge prepotential coming from closed string amplitudes on Riemann surfaces of genus h and that they should correspond

[1] For an alternative string approach to gauge instantons based on tachyon condensation, see for example ref. [7]

[2] In this paper we consider the case in which the $\mathrm{U}(1) \times \mathrm{U}(1)$ transformations are represented by $\mathrm{e}^{+\mathrm{i}\,\varepsilon}$ and $\mathrm{e}^{-\mathrm{i}\varepsilon}$, where ε is the deformation parameter.

to F-term couplings in the $\mathcal{N} = 2$ effective action of the form

$$\int d^4x \ (R^+)^2 (\mathcal{F}^+)^{2h-2} \tag{1.3}$$

where R^+ is the self-dual part of the Riemann curvature tensor and $\mathcal{F}^+$ is the self-dual part of the graviphoton field strength of $\mathcal{N} = 2$ supergravity. It is a well-known result of string theory that the F-terms (1.3) are non-vanishing only on Riemann surfaces of genus h and that they can be also computed with the topological string [18, 19]. Using this information Nekrasov's conjecture was tested to be true in ref. [20] in the case of $\mathcal{N} = 2$ super Yang-Mills with SU(2) gauge group.

In this paper we try to confirm and make more evident the above interpretation by computing directly the couplings induced by a (self-dual) graviphoton background on the instanton moduli space, and by showing that they precisely match those that are induced by the ε deformation that fully localizes the instanton integrals. To do so we exploit the explicit string realization of the $\mathcal{N} = 2$ gauge theory and its instantons provided by a D3/D(–1) system of fractional branes, and use the description of the graviphoton of $\mathcal{N} = 2$ supergravity as a massless field the Ramond-Ramond closed string sector. Then we determine how the graviphoton modifies the instanton effective action by computing mixed open/closed string disk amplitudes. We do this using the RNS formalism and the methods already introduced in ref. [21] to study the non anti-commutative gauge theories [22]. Even if it is a common belief that the RNS formalism is not suited to deal with a R-R background, we show that this is not completely true, and that a lot of information can actually be extracted from this formalism in several cases, including the one studied in this paper. In fact, by computing the instanton partition function in a graviphoton background with a constant self-dual field strength proportional to ε, we can obtain through eq. (1.1) a non-perturbative prepotential $\mathcal{F}_{\text{n.p.}}(a;\varepsilon)$ which coincides with the one obtained with the multi-instanton equivariant calculus [10 – 15] but in which, by construction, the parameter ε represents the v.e.v. of the graviphoton field strength. Using standard superstring methods [5, 8], we promote ε to a fully dynamical graviphoton field strength $\mathcal{F}^+$ or even to the complete Weyl superfield [23] of which $\mathcal{F}^+$ is the lowest component. Then, expanding the instanton-induced prepotential in powers of this Weyl superfield, we obtain, among others, precisely the gravitational F-terms of eq. (1.3). It is worth noticing that in our approach these terms arise from disk diagrams, and specifically from $2h$ disks wich, even if apparently disconnected, must be effectively considered as connected because of the integration over the instanton moduli (see, for example, the discussion in section 6 of ref. [8]). Notice also that the Euler character of this topology is the same as that of the world-sheet with h handles that is used in the topological string derivation of eq. (1.3).

In refs. [13, 16] it has been argued that the ε-deformation on the instanton moduli space is due to a non-trivial metric, called Ω-background, on the gauge theory. At the linear order in the deformation this Ω-background is equivalent to a R-R background. In the present paper, we point out that, realizing the gauge theory and its instantons via a fractional D3/D(–1) brane system, the parameter ε is directly related to the graviphoton. In view of

the above considerations about the gravitational F-terms (1.3) and of the connection with the results of the topological string, we find this interpretation very natural.

This paper is organized as follows: in section 2 we review how to derive the $\mathcal{N}=2$ action (both in the gauge and in the instanton sectors) from tree-level open string scattering amplitudes in a D3/D(–1) system, and discuss also how to incorporate in the instanton effective action the v.e.v.'s of the scalar gauge fields. In section 3 we analyze the instanton moduli space in presence of a (constant) self-dual graviphoton background by computing mixed open/closed string disk amplitudes, and show that this R-R background induces precisely the same deformation of the ADHM moduli space which fully localizes the instanton integrals. In section 4 we compare our results with those obtained with the deformed ADHM construction, and show how to lift the graviphoton deformation to the gauge theory action in four dimensions. We also prove that the terms of this action that are linear in the graviphoton field strength coincide with those produced by the Ω-background considered in refs. [13, 16]. section 5 is devoted to show how the $\mathcal{N}=2$ effective action and the prepotential may be extracted from the deformed instanton partition function, and how this compares with the topological string approach. Finally, in section 6 we present our conclusions, and in appendix A we list our notations and collect some formulas that are useful for the explicit calculations.

2. $\mathcal{N}=2$ gauge instantons from D3/D(–1) systems

Instantons of charge k in $\mathcal{N}=2$ theories with gauge group SU(N) can be described within type IIB string theory by considering systems of N fractional D3 branes and k fractional D(–1) branes at the fixed point of the orbifold $\mathbb{R}^4\times\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$. In this section we recall this description, adapting to the $\mathcal{N}=2$ case the procedure discussed in refs. [8] and [21] for $\mathcal{N}=4$ and $\mathcal{N}=1$ models. Our notations and conventions, as well as the details of the $\mathbb{Z}_2$ orbifold projection, are explained in appendix A.

2.1 The gauge sector

Let us consider type IIB string theory in $\mathbb{R}^4\times\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ and place at the orbifold fixed point a stack of N fractional D3 branes that fill the four-dimensional (Euclidean) space $\mathbb{R}^4$. The massless excitations of open strings attached with both end points to these branes describe the $\mathcal{N}=2$ gauge vector multiplet in four dimensions, that comprises a gauge boson A_μ, two gauginos $\Lambda^{\alpha A}$ (with $\alpha, A=1,2$) and one complex scalar ϕ. This field content can be assembled in a $\mathcal{N}=2$ chiral superfield

$$\Phi(x,\theta)=\phi(x)+\theta\Lambda(x)+\frac{1}{2}\,\theta\sigma^{\mu\nu}\theta\,F^+_{\mu\nu}(x)+\cdots \tag{2.1}$$

where

$$\theta\Lambda(x)\equiv\theta^{\alpha A}\Lambda_\alpha{}^B(x)\epsilon_{AB}\,,\qquad \theta\sigma^{\mu\nu}\theta\equiv\theta^{\alpha A}\big(\sigma^{\mu\nu}\big)_{\alpha\beta}\theta^{\beta B}\epsilon_{AB}\,, \tag{2.2}$$

$F^+_{\mu\nu}$ is the self-dual part of the gauge field strength and the dots in (2.1) stand for terms containing auxiliary fields and derivatives. The various components of the chiral superfield

are represented by the following open string vertex operators

$$V_A(z) = \frac{A_\mu(p)}{\sqrt{2}}\,\psi^\mu(z)\,\mathrm{e}^{ip\cdot X(z)}\,\mathrm{e}^{-\varphi(z)}\ , \tag{2.3a}$$

$$V_\Lambda(z) = \Lambda^{\alpha A}(p)\,S_\alpha(z) S_A(z)\,\mathrm{e}^{ip\cdot X(z)}\,\mathrm{e}^{-\frac{1}{2}\varphi(z)}\ , \tag{2.3b}$$

$$V_\phi(z) = \frac{\phi(p)}{\sqrt{2}}\,\overline{\Psi}(z)\,\mathrm{e}^{ip\cdot X(z)}\,\mathrm{e}^{-\varphi(z)}\ , \tag{2.3c}$$

where z is a world-sheet point, p is the momentum along the D3 brane world-volume ($p^2 = 0$), and φ is the boson of the superghost fermionization formulas (for details, see appendix A). For completeness, we also write the vertex operators for the conjugate fields $\bar\Lambda_{\dot\alpha A}$ and $\bar\phi$, namely

$$V_{\bar\Lambda}(z) = \bar\Lambda_{\dot\alpha A}(p)\,S^{\dot\alpha}(z) S^A(z)\,\mathrm{e}^{ip\cdot X(z)}\,\mathrm{e}^{-\frac{1}{2}\varphi(z)}\ , \tag{2.4a}$$

$$V_{\bar\phi}(z) = \frac{\bar\phi(p)}{\sqrt{2}}\,\Psi(z)\,\mathrm{e}^{ip\cdot X(z)}\,\mathrm{e}^{-\varphi(z)}\,. \tag{2.4b}$$

In all these vertices, the polarizations have canonical dimensions [3] and are $[N]\times[N]$ matrices transforming in the adjoint representation of SU(N) (here we neglect an overall factor of U(1), associated to the center of mass of the N D3 branes, which decouples and does not play any rôle in our context).

By computing the field theory limit $\alpha' \to 0$ of all tree-level scattering amplitudes among the vertex operators (2.3a)-(2.4b), one obtains various couplings that lead to the $\mathcal{N}=2$ SU(N) super Yang-Mills action [4]

$$\begin{aligned} S_{\rm SYM} = \int d^4x\ \mathrm{Tr}\Big\{ &\frac{1}{2}\,F_{\mu\nu}^2 + 2\,D_\mu\bar\phi\,D^\mu\phi - 2\,\bar\Lambda_{\dot\alpha A}\bar{\not{D}}^{\dot\alpha\beta}\Lambda_\beta^A \\ &+ \mathrm{i}\sqrt{2}\,g\,\bar\Lambda_{\dot\alpha A}\epsilon^{AB}\big[\,\phi,\bar\Lambda_B^{\dot\alpha}\big] + \mathrm{i}\sqrt{2}\,g\,\Lambda^{\alpha A}\epsilon_{AB}\big[\,\bar\phi,\Lambda_\alpha^B\big] + g^2\big[\,\phi,\bar\phi\,\big]^2\ \Big\}\,. \end{aligned} \tag{2.5}$$

In the following we will study the non-perturbative structure of the $\mathcal{N}=2$ gauge effective action when the chiral superfield acquires a v.e.v.

$$\langle\Phi_{uv}\rangle \equiv \langle\phi_{uv}\rangle = a_{uv} = a_u\,\delta_{uv} \tag{2.6}$$

where $u,v = 1,\ldots,N$ and $\sum_u a_u = 0$, so that the gauge group SU(N) is broken to U(1)$^{N-1}$. In particular we will investigate non-perturbative instanton effects, also in presence of a non-trivial supergravity background.

2.2 The instanton sector

In this stringy set-up instanton effects can be introduced by adding k D(–1) branes (or D-instantons) which give rise to new types of excitations associated to open strings with

[3] Unless explicitly mentioned we will always understand the appropriate factors of $(2\pi\alpha')$, needed to have dimensionless string vertices.

[4] Compared to ref. [8], here we have rescaled all gauge fields with a factor of the Yang-Mills coupling constant g for later convenience.

at least one end point on the D-instantons. Due to the Dirichlet boundary conditions that prevent momentum in all directions, these new excitations describe moduli rather than dynamical fields and are in one-to-one correspondence with the ADHM moduli of gauge instantons (for a more detailed discussion see, for instance, ref. [6] and references therein).

Let us consider first the open strings with both end-points on the fractional D-instantons in the $\mathbb{Z}_2$ orbifold. In this case, the NS sector contains six physical bosonic excitations that can be conveniently organized in a vector a'_μ and a complex scalar χ, and also three auxiliary excitations D_c ($c = 1, 2, 3$). The corresponding vertex operators are

$$V_{a'}(z) = g_0\, a'_\mu\, \psi^\mu(z)\, \mathrm{e}^{-\varphi(z)} \ , \tag{2.7a}$$

$$V_\chi(z) = \frac{\chi}{\sqrt{2}}\, \overline{\Psi}(z)\, \mathrm{e}^{-\varphi(z)} \ , \tag{2.7b}$$

$$V_D(z) = \frac{D_c}{2}\, \bar\eta^c_{\mu\nu}\, \psi^\nu(z)\psi^\mu(z) \ , \tag{2.7c}$$

where $\bar\eta^c_{\mu\nu}$ are the three anti-self-dual 't Hooft symbols , and g_0 is the D-instanton coupling constant

$$g_0 = \frac{g}{4\pi^2\alpha'} \, . \tag{2.8}$$

The R sector of the D(–1)/D(–1) strings contains instead eight fermionic moduli, $M^{\alpha A}$ and $\lambda_{\dot\alpha A}$, described by the following vertices

$$V_M(z) = \frac{g_0}{\sqrt{2}}\, M^{\alpha A}\, S_\alpha(z) S_A(z)\, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \ , \tag{2.9a}$$

$$V_\lambda(z) = \lambda_{\dot\alpha A}\, S^{\dot\alpha}(z) S^A(z)\, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \, . \tag{2.9b}$$

All polarizations in the vertex operators (2.7) and (2.9) are $[k]\times[k]$ matrices and transform in the adjoint representation of $\mathrm{U}(k)$. In the following we will always understand the $\mathrm{U}(k)$ indices for simplicity, unless they are needed to avoid ambiguities. It is worth noticing that if the Yang-Mills coupling constant g is kept fixed when $\alpha' \to 0$ (as is appropriate to retrieve the gauge theory on the D3 branes), then the dimensionful coupling g_0 in (2.8) blows up. Thus, some of the vertex operators must be suitably rescaled with factors of g_0 (like in (2.7a) and (2.9a)) in order to yield non-trivial interactions when $\alpha' \to 0$ [8]. As a consequence some of the moduli acquire unconventional scaling dimensions which, however, are the right ones for their interpretation as parameters of an instanton solution [6, 8]. For instance, the a'_μ's in (2.7a) have dimensions of (length) and are related to the positions of the (multi)-centers of the instanton, while $M^{\alpha A}$ in (2.9a) have dimensions of $(\text{length})^{\frac{1}{2}}$ and are the fermionic partners of the instanton centers. Furthermore, if we write the $[k]\times[k]$ matrices a'^μ and $M^{\alpha A}$ as

$$\begin{aligned} a'^\mu &= x_0^\mu\, \mathbb{1}_{[k]\times[k]} + y_c^\mu\, T^c \ , \\ M^{\alpha A} &= \theta^{\alpha A}\, \mathbb{1}_{[k]\times[k]} + \zeta_c^{\alpha A}\, T^c \ , \end{aligned} \tag{2.10}$$

where T^c are the generators of $\mathrm{SU}(k)$, then the center of the instanton, x_0^μ, and its fermionic partner, $\theta^{\alpha A}$, can be identified respectively with the bosonic and fermionic coordinates of the $\mathcal{N} = 2$ superspace.

In the D3/D(–1) system there are also twisted string excitations corresponding to open strings with mixed boundary conditions that stretch between a D3 brane and a D-instanton, or vice versa. The twisted NS sectors contains the bosonic moduli $w_{\dot\alpha}$ and $\bar w_{\dot\alpha}$ which are associated to the following vertex operators

$$V_w(z) = \frac{g_0}{\sqrt{2}}\, w_{\dot\alpha}\, \Delta(z) S^{\dot\alpha}(z)\, \mathrm{e}^{-\varphi(z)} \quad \text{and} \quad V_{\bar w}(z) = \frac{g_0}{\sqrt{2}}\, \bar w_{\dot\alpha}\, \overline{\Delta}(z) S^{\dot\alpha}(z)\, \mathrm{e}^{-\varphi}(z)\,. \tag{2.11}$$

Here Δ and $\overline{\Delta}$ are the twist and anti-twist operators with conformal weight 1/4 which change the boundary conditions of the longitudinal coordinates X^μ from Neumann to Dirichlet and vice-versa by introducing a cut in the open-string world-sheet [24]. The moduli $w_{\dot\alpha}$ and $\bar w_{\dot\alpha}$ have dimension of (length) and are related to the instanton size. The twisted R sector contains instead the fermionic moduli μ^A and $\bar\mu^A$, with dimension of $(\text{length})^{1/2}$, described by the following vertex operators

$$V_\mu(z) = \frac{g_0}{\sqrt{2}}\, \mu^A\, \Delta(z) S_A(z)\, \mathrm{e}^{-\frac{1}{2}\varphi(z)} \quad \text{and} \quad V_{\bar\mu}(z) = \frac{g_0}{\sqrt{2}}\, \bar\mu^A\, \overline{\Delta}(z) S_A(z)\, \mathrm{e}^{-\frac{1}{2}\varphi(z)}\,. \tag{2.12}$$

In both (2.11) and (2.12) the polarizations transform in the bi-fundamental representations of the $\mathrm{U}(N) \times \mathrm{U}(k)$ group. Again, in most cases we will understand the corresponding indices for simplicity.

Following the procedure explained in ref. [8], by computing all tree-level interactions among the above vertex operators in the limit $\alpha' \to 0$ with g fixed (and hence with $g_0 \to \infty$) one can recover the ADHM action on the instanton moduli space for the $\mathcal{N} = 2$ theory, namely

$$S_{\text{moduli}} = S_k^{\text{bos}} + S_k^{\text{fer}} + S_k^{\text{c}} \tag{2.13}$$

with

$$S_k^{\text{bos}} = \mathrm{tr}_k\Big\{ -2\,[\chi^\dagger, a'_\mu][\chi, a'^\mu] + \chi^\dagger \bar w_{\dot\alpha} w^{\dot\alpha} \chi + \chi \bar w_{\dot\alpha} w^{\dot\alpha} \chi^\dagger \Big\} \tag{2.14a}$$

$$S_k^{\text{fer}} = \mathrm{tr}_k\Big\{ \mathrm{i}\,\frac{\sqrt{2}}{2}\, \bar\mu^A \epsilon_{AB} \mu^B \chi^\dagger - \mathrm{i}\,\frac{\sqrt{2}}{4}\, M^{\alpha A} \epsilon_{AB} [\chi^\dagger, M_\alpha^B] \Big\} \tag{2.14b}$$

$$\begin{aligned} S_k^{\text{c}} = \mathrm{tr}_k\Big\{ & -\mathrm{i} D_c \big(W^c + \mathrm{i} \bar\eta^c_{\mu\nu} [a'^\mu, a'^\nu] \big) \\ & - \mathrm{i} \lambda^{\dot\alpha}_A \big(\bar\mu^A w_{\dot\alpha} + \bar w_{\dot\alpha} \mu^A + [a'_{\alpha\dot\alpha}, M'^{\alpha A}] \big) \Big\} \end{aligned} \tag{2.14c}$$

where

$$(W^c)_{ij} = w_{ui\dot\alpha}\, (\tau^c)^{\dot\alpha}{}_{\dot\beta}\, \bar w^{\dot\beta}{}_{ju} \tag{2.15}$$

with $u = 1, ..., N$ and $i, j = 1, ..., k$. Since D_c and $\lambda^{\dot\alpha}_A$ act as Lagrange multipliers, the term (2.14c) yields the so-called bosonic and fermionic ADHM constraints

$$\begin{aligned} & W^c + \mathrm{i} \bar\eta^c_{\mu\nu} [a'^\mu, a'^\nu] = 0 \ , \\ & \bar\mu^A w_{\dot\alpha} + \bar w_{\dot\alpha} \mu^A + [a'_{\alpha\dot\alpha}, M'^{\alpha A}] = 0 \ , \end{aligned} \tag{2.16}$$

while by varying S_k^{bos} and S_k^{fer} with respect to $\chi^\dagger$ and χ we obtain the following equations

$$\frac{1}{2} \{ \bar w_{\dot\alpha} w^{\dot\alpha}, \chi \} + [a'_\mu, [a'^\mu, \chi]] + \mathrm{i}\,\frac{\sqrt{2}}{4}\, \epsilon_{AB} \left(\bar\mu^A \mu^B + M^{\alpha A} M_\alpha{}^B \right) = 0 \ , \tag{2.17a}$$

$$\frac{1}{2} \Big\{ \bar w_{\dot\alpha} w^{\dot\alpha}, \chi^\dagger \Big\} + \Big[a'_\mu, \Big[a'^\mu, \chi^\dagger \Big] \Big] = 0\,. \tag{2.17b}$$

It is interesting to observe the moduli action (2.13) does not depend on the superspace coordinates x_0^μ and $\theta^{\alpha A}$ defined in (2.10) and that the quartic interaction terms of the bosonic part (2.14a) can be completely disentangled by means of dimensionless auxiliary fields Y_μ, $X_{\dot\alpha}$ and $\bar X_{\dot\alpha}$ (plus their conjugate ones) which are associated to the following vertex operators [8]

$$V_Y(z) = \sqrt{2} g_0\, Y_\mu\, \overline{\Psi}(z)\, \psi^\mu(z) \quad , \qquad V_{Y^\dagger}(z) = \sqrt{2} g_0\, Y_\mu^\dagger\, \Psi(z)\, \psi^\mu(z) \ , \tag{2.18a}$$

$$V_X(z) = g_0\, X_{\dot\alpha}\, \Delta(z) S^{\dot\alpha}(z)\, \overline{\Psi}(z)\,, \qquad V_{X^\dagger}(z) = g_0\, X_{\dot\alpha}^\dagger\, \Delta(z) S^{\dot\alpha}(z)\, \Psi(z) \ , \tag{2.18b}$$

$$V_{\bar X}(z) = g_0\, \bar X_{\dot\alpha}\, \overline{\Delta}(z) S^{\dot\alpha}(z)\, \overline{\Psi}(z)\,, \qquad V_{\bar X^\dagger}(z) = g_0\, \bar X_{\dot\alpha}^\dagger(z)\, \overline{\Delta}(z) S^{\dot\alpha}(z)\, \Psi(z)\,. \tag{2.18c}$$

The operators (2.18a) describe excitations of the D(–1)/D(–1) strings, while the vertices (2.18b) and (2.18c) account for states of the D3/D(–1) and D(–1)/D3 sectors respectively. Like any vertex associated to an auxiliary field, also the vertices (2.18) can only be written in the 0 superghost picture and are not BRST invariant. Nevertheless, they can be safely used to compute scattering amplitudes in the field theory limit. For example, let us consider the 3-point amplitude corresponding to the disk diagram of figure 1*a*, namely

$$\Big\langle\!\Big\langle V_{\bar X^\dagger} V_w V_\chi \Big\rangle\!\Big\rangle \ \equiv\ C_0 \int \frac{\prod_i dz_i}{dV_{\rm CKG}} \ \times\ \langle V_{\bar X^\dagger}(z_1)\, V_w(z_2)\, V_\chi(z_3)\rangle \tag{2.19}$$

where $dV_{\rm CGKG}$ is the Conformal Killing Group volume element, and C_0 is the normalization of any D(–1) disk amplitude [8]

$$C_0 = \frac{2}{(2\pi\alpha')^2}\frac{1}{g_0^2} \ =\ \frac{8\pi^2}{g^2} \tag{2.20}$$

which is also the classical action of an instanton with charge $k = 1$. Computing the correlation function among the vertex operators using the OPE's reported in appendix A and reinstating in the polarizations the appropriate factors of $(2\pi\alpha')$ (which cancel against those of C_0), one easily finds that

$$\Big\langle\!\Big\langle V_{\bar X^\dagger} V_w V_\chi \Big\rangle\!\Big\rangle = -{\rm tr}_k\Big\{ \bar X_{\dot\alpha}^\dagger\, w^{\dot\alpha} \chi \Big\}\,. \tag{2.21}$$

Proceeding systematically in this way and computing all scattering amplitudes involving the auxiliary vertices, we obtain a bosonic moduli action with cubic interaction terms only, namely

$$\begin{aligned} {S'}_k^{\rm bos} = {\rm tr}_k\Big\{ & 2\, Y_\mu^\dagger Y^\mu + 2\, Y_\mu^\dagger \left[a'^\mu, \chi\right] + 2\, Y_\mu \left[a'^\mu, \chi^\dagger\right] \\ & + \bar X_{\dot\alpha}^\dagger X^{\dot\alpha} + \bar X_{\dot\alpha} X^{\dagger\,\dot\alpha} + \bar X_{\dot\alpha}^\dagger w^{\dot\alpha} \chi + \bar X_{\dot\alpha} w^{\dot\alpha} \chi^\dagger - \chi \bar w_{\dot\alpha} X^{\dagger\,\dot\alpha} - \chi^\dagger \bar w_{\dot\alpha} X^{\dot\alpha} \Big\} \end{aligned} \tag{2.22}$$

which is indeed equivalent to $S_k^{\rm bos}$ in (2.14a) after the auxiliary variables are integrated out.

The vertices (2.18) are also useful to discuss the supersymmetry transformation laws of the various moduli. In particular, for the supersymmetries which are preserved both

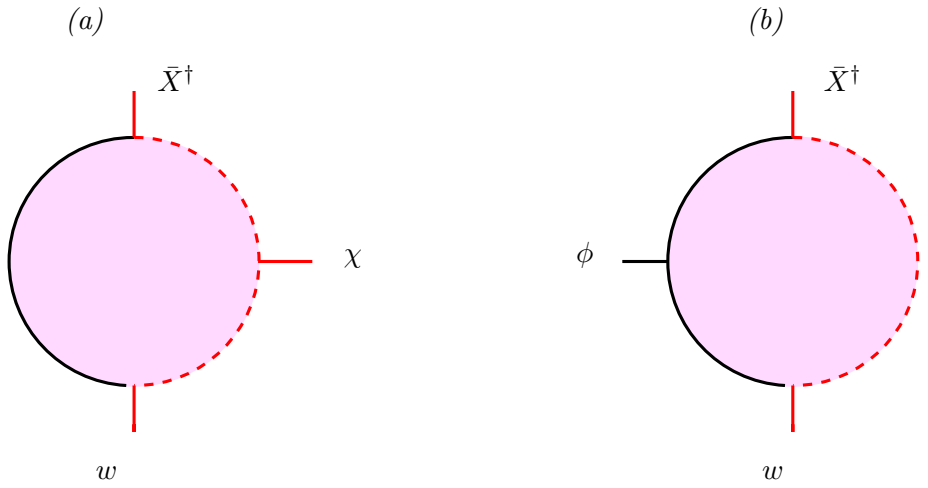

Figure 1: *(a)* A mixed diagram involving moduli and auxiliary moduli. *(b)* A mixed diagram involving also a gauge theory vertex. The solid part of the boundary is attached to a D3, the dotted part to a D-instanton.

on the D3 branes and on the D-instantons and which are generated by the following four supercharges

$$Q^{\dot\alpha A} = \oint \frac{dw}{2\pi \mathrm{i}}\, j^{\dot\alpha A}(w) \qquad \text{with} \qquad j^{\dot\alpha A}(w) = S^{\dot\alpha}(w) S^{A}(w)\, \mathrm{e}^{-\frac{1}{2}\varphi(w)} \ , \tag{2.23}$$

one can show that (see for example ref. [8])

$$\left[\xi_{\dot\alpha A}\, Q^{\dot\alpha A}, V_Y(z)\right] = \xi_{\dot\alpha A} \oint_z \frac{dw}{2\pi \mathrm{i}}\, j^{\dot\alpha A}(w)\, V_Y(z) = V_{\delta M}(z) \ , \tag{2.24}$$

where

$$\delta M^{\beta B} = -2\sqrt{2}\, \xi_{\dot\alpha A}\, \epsilon^{AB} (\bar\sigma_\mu)^{\dot\alpha\beta} Y^\mu \, . \tag{2.25}$$

After eliminating the auxiliary field Y via its equation of motion, one can rewrite the above transformation rule as

$$\delta M^{\beta B} = -2\sqrt{2}\, \xi_{\dot\alpha A}\, \epsilon^{AB} (\bar\sigma_\mu)^{\dot\alpha\beta} \left[\chi, a'^\mu\right] . \tag{2.26}$$

With similar calculations one can fully recover also the supersymmetry transformations of the other instanton moduli and find complete agreement with the standard results.

2.3 Introducing v.e.v.'s

The string formalism is well suited also to discuss the case in which the chiral superfield Φ has a v.e.v. like in (2.6). Indeed, to find how the constants a_{uv} enter in the instanton action one simply has to compute mixed disk diagrams with a constant scalar field ϕ emitted from the portion of the disk boundary that lies on the D3 branes. For example, one should consider the diagram of figure 1*b* which corresponds to the following amplitude

$$\left\langle\!\!\left\langle V_{\bar X^\dagger} V_\phi V_w \right\rangle\!\!\right\rangle \ \equiv \ C_0 \int \frac{\prod_i dz_i}{dV_{\mathrm{CKG}}} \ \times \ \langle V_{\bar X^\dagger}(z_1)\, V_\phi(z_2)\, V_w(z_3)\rangle \tag{2.27}$$

where the scalar vertex V_ϕ is taken at zero momentum to describe the emission of a constant field ϕ. The amplitude (2.27) is similar to that of eq. (2.19), the only difference being the presence of the D3/D3 vertex V_ϕ in place of the D(–1)/D(–1) vertex V_χ. Since these vertices differ only in their polarizations but not in their operator structure, the result can be simply inferred from (2.21), namely

$$\Big\langle\!\Big\langle V_{\bar{X}^\dagger} V_\phi V_w \Big\rangle\!\Big\rangle = \mathrm{tr}_k\Big\{ \bar{X}^\dagger_{\dot\alpha}\, a\, w^{\dot\alpha} \Big\}\,. \tag{2.28}$$

Proceeding systematically in this way, we can derive the modified moduli actions

$$\begin{aligned}\tilde{S}_k^{\rm bos} = \mathrm{tr}_k\Big\{ & 2\, Y_\mu^\dagger Y^\mu + 2\, Y_\mu^\dagger \,\big[a'^\mu, \chi\big] + 2\, Y_\mu \,\big[a'^\mu, \chi^\dagger\big] \\ & + \bar{X}^\dagger_{\dot\alpha} X^{\dot\alpha} + \bar{X}_{\dot\alpha} X^{\dagger\,\dot\alpha} + \bar{X}^\dagger_{\dot\alpha}\big(w^{\dot\alpha}\chi - a\, w^{\dot\alpha}\big) + \bar{X}_{\dot\alpha}\big(w^{\dot\alpha}\chi^\dagger - \bar{a}\, w^{\dot\alpha}\big) \\ & - \big(\chi \bar{w}_{\dot\alpha} - \bar{w}_{\dot\alpha}\, a\big) X^{\dagger\,\dot\alpha} - \big(\chi^\dagger \bar{w}_{\dot\alpha} - \bar{w}_{\dot\alpha}\, \bar{a}\big) X^{\dot\alpha} \Big\}\ ,\end{aligned} \tag{2.29}$$

and

$$\tilde{S}_k^{\rm fer} = \mathrm{tr}_k\Big\{ \mathrm{i}\,\frac{\sqrt{2}}{2}\,\bar{\mu}^A \epsilon_{AB}\big(\mu^B \chi^\dagger - \bar{a}\,\mu^B\big) - \mathrm{i}\,\frac{\sqrt{2}}{4}\, M^{\alpha A}\epsilon_{AB}[\chi^\dagger, M_\alpha^B] \Big\}\,. \tag{2.30}$$

Notice that these actions can be obtained from those in eqs. (2.22) and (2.14b) with the formal shifts

$$\chi_{ij}\,\delta_{uv} \;\to\; \chi_{ij}\,\delta_{uv} - \delta_{ij}\, a_{uv} \qquad \text{and} \qquad \chi^\dagger_{ij}\,\delta_{uv} \;\to\; \chi^\dagger_{ij}\,\delta_{uv} - \delta_{ij}\,\bar{a}_{uv} \tag{2.31}$$

where $i, j = 1, ...k$ and $u, v = 1, ..., N$. It is interesting to observe that the matrices a and $\bar{a}$ do not appear on equal footing; in particular a does not appear in the fermionic action $\tilde{S}_k^{\rm fer}$. This fact will have important consequences, like for example that the instanton partition function depends only on a and not on $\bar{a}$ (see also section 3.3).

3. Instantons in a graviphoton background

In this section we analyze the instanton moduli space of $\mathcal{N} = 2$ gauge theories in a non-trivial supergravity background. In particular we turn on a (self-dual) field strength for the graviphoton of the $\mathcal{N} = 2$ supergravity multiplet and see how it modifies the instanton moduli action. This graviphoton background breaks Lorentz invariance in space-time (leaving the metric flat) but it allows to explicitly perform instanton calculations and establish a direct correspondence with the localization techniques that have been recently discussed in the literature. Since our strategy is based on the use of string and D brane methods, we begin by reviewing how the graviphoton field is described in our stringy set-up.

3.1 Graviphoton in $\mathcal{N} = 2$ theories

The graviton multiplet of $\mathcal{N} = 2$ supergravity in four dimensions contains the metric $g_{\mu\nu}$, two gravitini $\psi_\mu^{\alpha A}$ and one vector C_μ called graviphoton. This field content can be organized in the chiral Weyl superfield [23]

$$W^+_{\mu\nu}(x,\theta) = \mathcal{F}^+_{\mu\nu}(x) + \theta\chi^+_{\mu\nu}(x) + \frac{1}{2}\,\theta\sigma^{\lambda\rho}\theta\, R^+_{\mu\nu\lambda\rho}(x) + \cdots \tag{3.1}$$

where $\mathcal{F}^+_{\mu\nu}$ is the self-dual part of the graviphoton field strength $\mathcal{F}_{\mu\nu} = \partial_{(\mu}C_{\nu)}$, $R^+_{\mu\nu\lambda\rho}$ is the self-dual Riemann curvature tensor and

$$\theta\chi_{\mu\nu} \equiv \theta^{\alpha A}\chi_{\mu\nu}{}^{\beta B}\,\epsilon_{\alpha\beta}\,\epsilon_{AB} \tag{3.2}$$

with $\chi_{\mu\nu}{}^{\alpha A}$ being the gravitino field strength, whose self-dual part appears in (3.1).

In our context these supergravity fields are associated to massless excitations of the type IIB closed string in $\mathbb{R}^4 \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$. Due to the presence of the fractional branes, the closed string world-sheet has boundaries and suitable identifications between left- and right-moving modes must be enforced. Therefore a closed string vertex operator, which is normally the product of two independent left and right components, *i.e.*

$$V_{\rm L}(z) \times V'_{\rm R}(\bar z)\ , \tag{3.3}$$

in the presence of D branes becomes of the form

$$V(z) \times V'(\bar z) \tag{3.4}$$

where both the holomorphic and the anti-holomorphic parts are written in terms of a single set of oscillators that describe the modes of a propagating open string attached to the D branes. Furthermore, due to these left/right identifications only eight of the sixteen bulk supercharges that exist in the $\mathbb{Z}_2$ orbifold of Type IIB string theory survive on the D brane world-volume.

Taking all this into account, we now write the vertex operators associated to the fields of the $\mathcal{N} = 2$ graviton multiplet in the open string formalism. The graviphoton vertex operator belongs to the R-R sector and in the $(-1/2, -1/2)$ superghost picture its properly normalized expression is

$$V_{\mathcal{F}}(z,\bar z) = \frac{1}{4\pi}\,\mathcal{F}^{\alpha\beta AB}(p)\Big[S_\alpha(z)S_A(z)\,{\rm e}^{-\frac{1}{2}\varphi(z)} \times S_\beta(\bar z)S_B(\bar z)\,{\rm e}^{-\frac{1}{2}\varphi(\bar z)}\Big]\,{\rm e}^{{\rm i}p\cdot X(z,\bar z)} \tag{3.5}$$

where the bi-spinor polarization is related to the self-dual part of the graviphoton field strength $\mathcal{F}^+_{\mu\nu}$ in the following manner

$$\mathcal{F}^{\alpha\beta AB} = \frac{\sqrt{2}}{4}\,\mathcal{F}^+_{\mu\nu}(\sigma^{\mu\nu})^{\alpha\beta}\,\epsilon^{AB}\ , \tag{3.6}$$

and

$$X^\mu(z,\bar z) = \frac{1}{2}\big[X^\mu(z) \pm X^\mu(\bar z)\big] \tag{3.7}$$

depending on whether X^μ is a longitudinal (+ sign) or transverse (− sign) direction [5].

The vertex operator for the gravitini $\psi^{\alpha A}_\mu$ belongs instead to the fermionic R-NS/NS-R sector and is given by

$$\begin{aligned} V_\psi(z,\bar z) = \frac{1}{4\pi}\,\psi^{\alpha A}_\mu(p)\,\Big[&S_\alpha(z)S_A(z)\,{\rm e}^{-\frac{1}{2}\varphi(z)} \times \psi^\mu(\bar z)\,{\rm e}^{-\varphi(\bar z)} \\ &+ \psi^\mu(z)\,{\rm e}^{-\varphi(z)} \times S_\alpha(\bar z)S_A(\bar z)\,{\rm e}^{-\frac{1}{2}\varphi(\bar z)}\Big]\,{\rm e}^{{\rm i}p\cdot X(z,\bar z)}\,. \end{aligned} \tag{3.8}$$

[5] To be very precise also the overall sign of $V_{\mathcal{F}}$ (and of other closed string vertices) depends on the type of boundary conditions; however, as we shall see in the following, this sign is irrelevant in our calculations.

Notice that in the first term the holomorphic part is of R type with half-integer superghost charge and the anti-holomorphic part is of NS type with integer superghost charge, while the roles are reversed in the second term. This "symmetrized" structure is a direct consequence of the left/right identifications we have mentioned above.

Finally, the vertex operator for the graviton $h_{\mu\nu}$ belongs to the NS-NS sector and in the $(-1,-1)$ picture it is

$$V_h(z,\bar{z}) = \frac{1}{4\pi}\, h_{\mu\nu}(p)\Big[\psi^\mu(z)\,\mathrm{e}^{-\varphi(z)} \times \psi^\nu(\bar{z})\,\mathrm{e}^{-\varphi(\bar{z})}\Big]\,\mathrm{e}^{\mathrm{i}p\cdot X(z,\bar{z})} \ . \tag{3.9}$$

The vertices (3.5), (3.8) and (3.9) are, as usual, dimensionless and their polarizations have canonical dimensions. In particular, since the graviphoton field strength $\mathcal{F}_{\mu\nu}$ has canonical dimensions of $(\text{length})^{-1}$, a factor of $(2\pi\alpha')^{1/2}$ should be understood in (3.5).

Using the explicit expression of the above vertex operators and the OPE's given in appendix A, it is possible to check various supersymmetry transformation rules. For example, taking the anti-chiral supercharges $Q^{\dot{\alpha}A}$ given in (2.23), one can show that

$$\big[\xi_{\dot{\alpha}A}\, Q^{\dot{\alpha}A}, V_{\mathcal{F}}(z,\bar{z})\big] = V_{\delta\psi}(z,\bar{z}) \tag{3.10}$$

where

$$\delta\psi_\mu^{\beta B} = \mathrm{i}\,\xi_{\dot{\alpha}A}\,(\bar{\sigma}_\nu)^{\dot{\alpha}\beta}\,\epsilon^{AB}\,\mathcal{F}^+_{\mu\nu}\,. \tag{3.11}$$

This is the correct graviphoton dependence of the anti-chiral supersymmetry transformations of the gravitini. Therefore, eq. (3.10) is also a confirmation for the vertex operators (3.5) and (3.8). With similar calculations one can check other pieces of the $\mathcal{N}=2$ supersymmetry transformation rules of the various supergravity fields.

In the graviphoton vertex operator (3.5) the holomorphic and anti-holomorphic components are both even under the $\mathbb{Z}_2$ orbifold projection. For reasons that will be clear in the following sections, it is convenient to consider also a R-R closed string vertex that is made up of two odd components, namely

$$V_{\bar{\mathcal{F}}}(z,\bar{z}) = \frac{1}{4\pi}\,\bar{\mathcal{F}}^{\alpha\beta\hat{A}\hat{B}}(p)\Big[S_\alpha(z)S_{\hat{A}}(z)\,\mathrm{e}^{-\frac{1}{2}\varphi(z)} \times S_\beta(\bar{z})S_{\hat{B}}(\bar{z})\,\mathrm{e}^{-\frac{1}{2}\varphi(\bar{z})}\Big]\,\mathrm{e}^{\mathrm{i}p\cdot X(z,\bar{z})} \tag{3.12}$$

where $\hat{A},\hat{B}=3,4$ in the notation of appendix A. This vertex operator clearly survives the orbifold projection since both $S_\alpha(z)S_{\hat{A}}(z)$ and $S_\beta(\bar{z})S_{\hat{B}}(\bar{z})$ are odd under $\mathbb{Z}_2$. In particular, we will consider the case in which the bi-spinor polarization of $V_{\bar{\mathcal{F}}}$ is

$$\bar{\mathcal{F}}^{\alpha\beta\hat{A}\hat{B}} = \frac{\sqrt{2}}{4}\,\bar{\mathcal{F}}^+_{\mu\nu}(\sigma^{\mu\nu})^{\alpha\beta}\,\epsilon^{\hat{A}\hat{B}} \tag{3.13}$$

where $\epsilon^{34} = -\epsilon^{43} = 1$. Notice that the antisymmetric tensor $\bar{\mathcal{F}}_{\mu\nu}$ cannot be interpreted as the graviphoton field strength, since the vertex operator (3.12) is not related to the gravitino vertex (3.8) as required by the rule (3.11) of $\mathcal{N}=2$ supersymmetry. In fact the tensor $\bar{\mathcal{F}}_{\mu\nu}$ corresponds to the field strength of some other vector in the $\mathcal{N}=2$ supergravity model, and as such it is independent of $\mathcal{F}_{\mu\nu}$. Despite their different meaning, the two vertices $V_{\mathcal{F}}$ and $V_{\bar{\mathcal{F}}}$ can be treated together in most of our calculations because of their very similar operator structure.

3.2 ADHM measure in graviphoton background

We now study how a graviphoton background modifies the instanton moduli action of the $\mathcal{N}=2$ gauge theory. To do so, we assume that the chiral graviphoton superfield (3.1) has a v.e.v.

$$\langle W^+_{\mu\nu}\rangle \equiv \langle \mathcal{F}^+_{\mu\nu}\rangle = f_{\mu\nu} \tag{3.14}$$

with $f_{\mu\nu}$ a constant self-dual tensor. This background can be described by a graviphoton vertex $V_{\mathcal{F}}$ at zero momentum with constant polarization $\mathcal{F}^+_{\mu\nu} = f_{\mu\nu}$, and the modified instanton action can be derived by computing all disk amplitudes among the various moduli with insertions of this closed string vertex. These are mixed open/closed string amplitudes which are very similar to the ones that have been previously studied in the context of non-anti-commutative theories [21].

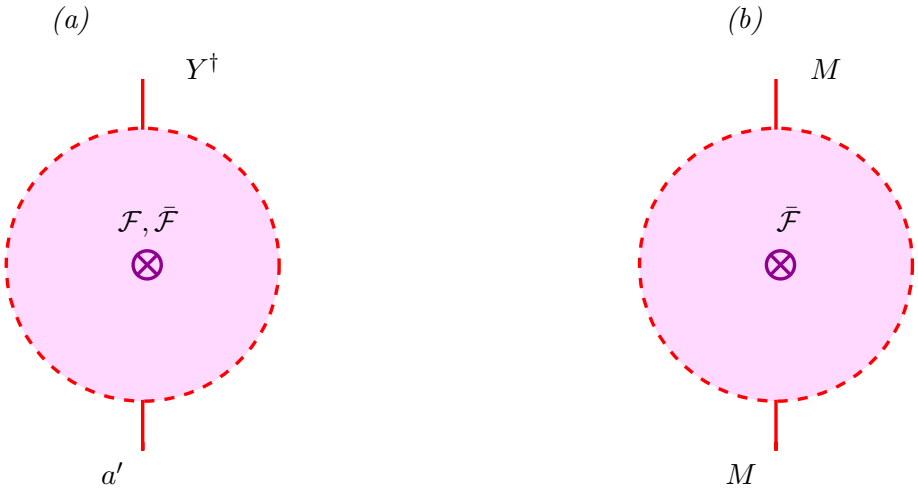

Figure 2: Disk diagrams encoding the coupling of the closed string RR vertices $\mathcal{F}_{\mu\nu}$ (the graviphoton) and $\bar{\mathcal{F}}_{\mu\nu}$ to bosonic *(a)* and fermionic *(b)* moduli. The boundary of the disks is entirely on a D-instanton.

Let us consider in detail the disk diagram represented in figure 2*a* which corresponds to the following amplitude:

$$\Big\langle\!\Big\langle V_{Y^\dagger} V_{a'} V_{\mathcal{F}} \Big\rangle\!\Big\rangle \;\equiv\; C_0 \int \frac{dz_1\, dz_2\, dw d\bar{w}}{dV_{\rm CKG}} \;\times\; \langle V_{Y^\dagger}(z_1)\, V_{a'}(z_2)\, V_{\mathcal{F}}(w,\bar{w})\rangle \tag{3.15}$$

where the open string punctures z_i are integrated along the real axis with $z_1 \geq z_2$ while the closed string puncture w is integrated on the upper half complex plane. More explicitly, we have

$$\begin{aligned}\Big\langle\!\Big\langle V_{Y^\dagger} V_{a'} V_{\mathcal{F}} \Big\rangle\!\Big\rangle = \frac{1}{4\pi}\,\mathrm{tr}_k\Big\{Y^\dagger_\mu\, a'_\nu\, f_{\lambda\rho}\Big\}\big(\sigma^{\lambda\rho}\big)^{\alpha\beta}\epsilon^{AB}\int \frac{dz_1\, dz_2\, dw d\bar{w}}{dV_{\rm CKG}}\,\times \\ \big\langle \mathrm{e}^{-\varphi(z_2)}\mathrm{e}^{-\frac{1}{2}\varphi(w)}\mathrm{e}^{-\frac{1}{2}\varphi(\bar{w})}\big\rangle\,\big\langle \Psi(z_1) S_A(w) S_B(\bar{w})\big\rangle\,\big\langle \psi^\mu(z_1)\psi^\nu(z_2) S_\alpha(w) S_\beta(\bar{w})\big\rangle\ .\end{aligned} \tag{3.16}$$

Using the correlation functions given in appendix A and exploiting the $\mathrm{Sl}(2,\mathbb{R})$ invariance to fix $z_1 \to \infty$ and $w \to \mathrm{i}$, we are left with the elementary integral

$$\int_{-\infty}^{\infty} dz_2\, \frac{1}{1+z_2^2} = \pi\ , \tag{3.17}$$

so that in the end we have

$$\Big\langle\!\Big\langle V_{Y^\dagger} V_{a'} V_{\mathcal{F}} \Big\rangle\!\Big\rangle = -4\mathrm{i}\,\mathrm{tr}_k\Big\{ Y_\mu^\dagger\, a'_\nu\, f^{\mu\nu}\Big\}\,. \tag{3.18}$$

A systematic analysis reveals that this is the only non-vanishing disk amplitude involving the graviphoton field strength $f_{\mu\nu}$ and the ADHM instanton moduli. Indeed, all other diagrams with insertions of $V_{\mathcal{F}}$ either vanish at the string theory level, or vanish in the field theory limit. However, there are a couple of non-vanishing amplitudes containing the vertex $V_{\bar{\mathcal{F}}}$ of eq. (3.12) at zero momentum (i.e. with a constant polarization $\bar{\mathcal{F}}^+_{\mu\nu} = \bar{f}_{\mu\nu}$). The first of these amplitudes, see figure 2*a*, is the strict analogue of the one we have presented above, namely

$$\Big\langle\!\Big\langle V_{Y} V_{a'} V_{\bar{\mathcal{F}}} \Big\rangle\!\Big\rangle = -4\mathrm{i}\,\mathrm{tr}_k\Big\{ Y_\mu\, a'_\nu\, \bar{f}^{\mu\nu}\Big\}\,. \tag{3.19}$$

The second is a fermionic amplitude involving the M moduli, see figure 2*b*, namely

$$\Big\langle\!\Big\langle V_{M} V_{M} V_{\bar{\mathcal{F}}} \Big\rangle\!\Big\rangle = \frac{1}{4\sqrt{2}}\,\mathrm{tr}_k\Big\{ M^{\alpha A}\, M^{\beta B}\, \bar{f}_{\mu\nu}\Big\}(\sigma^{\mu\nu})_{\alpha\beta}\epsilon_{AB}\,. \tag{3.20}$$

No other (irreducible) diagrams with $V_{\bar{\mathcal{F}}}$ insertions give a non-zero result. It is interesting to notice that the only non-vanishing amplitudes with insertions of the closed string vertices $V_{\mathcal{F}}$ and $V_{\bar{\mathcal{F}}}$ correspond to disks whose boundary lies entirely on the D-instantons and that there are no contributions to the instanton action due to graviphoton insertions on mixed disks [6].

By adding the contributions (3.18), (3.19) and (3.20) to the terms described in section 2 (in particular eqs. (2.29) and (2.30)), we can obtain the $\mathcal{N} = 2$ ADHM moduli action in the presence of a v.e.v. for the scalar field of the gauge multiplet, for the graviphoton field strength and for the anti-symmetric tensor $\bar{f}_{\mu\nu}$. Explicitly, after integrating out the auxiliary fields Y, X and $\bar{X}$, the resulting moduli action is

$$\begin{aligned} S_{\text{moduli}}(a,\bar{a};f,\bar{f}) = & -\mathrm{tr}_k\Big\{2\,\big([\chi^\dagger, a'_\mu] - 2\,\mathrm{i}\,\bar{f}_\mu{}^\nu a'_\nu\big)\big([\chi, a'^\mu] - 2\,\mathrm{i}\,f^{\mu\rho}a'_\rho\big) \\ & - \big(\chi^\dagger\bar{w}_{\dot\alpha} - \bar{w}_{\dot\alpha}\,\bar{a}\big)\big(w^{\dot\alpha}\chi - a\,w^{\dot\alpha}\big) - \big(\chi\bar{w}_{\dot\alpha} - \bar{w}_{\dot\alpha}\,a\big)\big(w^{\dot\alpha}\chi^\dagger - \bar{a}\,w^{\dot\alpha}\big)\Big\} \\ & + \mathrm{i}\,\frac{\sqrt{2}}{2}\,\mathrm{tr}_k\Big\{\bar{\mu}^A\epsilon_{AB}\big(\mu^B\chi^\dagger + \bar{a}\,\mu^B\big) \\ & - \frac{1}{2}\,M^{\alpha A}\epsilon_{AB}\big([\chi^\dagger, M^B_\alpha] - \frac{\mathrm{i}}{2}\,\bar{f}_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta}M^{\beta B}\big)\Big\} + S_k^{\mathrm{c}}\ , \end{aligned} \tag{3.21}$$

where S_k^{c} is the constraint part (2.14c) which is not affected by the background we have considered.

A few comments are in order at this point. First of all, if we write the two self-dual tensors f and $\bar{f}$ in terms of the three 't Hooft's symbols

$$f_{\mu\nu} = f_c\,\eta^c_{\mu\nu} \quad \text{and} \quad \bar{f}_{\mu\nu} = \bar{f}_c\,\eta^c_{\mu\nu}\ , \tag{3.22}$$

[6] Things would be different in an anti-self-dual graviphoton background.

after some standard manipulations the action (3.21) becomes

$$\begin{aligned} S_{\rm moduli}(a,\bar a;f,\bar f) = & -{\rm tr}_k\Big\{\big([\chi^\dagger,a'_{\alpha\dot\beta}]+2\,\bar f_c\,(\tau^c a')_{\alpha\dot\beta}\big)\big([\chi,a'^{\dot\beta\alpha}]+2\,f_c\,(a'\tau^c)^{\dot\beta\alpha}\big) \\ & -\big(\chi^\dagger\bar w_{\dot\alpha}-\bar w_{\dot\alpha}\,\bar a\big)\big(w^{\dot\alpha}\chi-a\,w^{\dot\alpha}\big)-\big(\chi\bar w_{\dot\alpha}-\bar w_{\dot\alpha}\,a\big)\big(w^{\dot\alpha}\chi^\dagger-\bar a\,w^{\dot\alpha}\big)\Big\} \\ & +{\rm i}\,\frac{\sqrt{2}}{2}\,{\rm tr}_k\Big\{\bar\mu^A\epsilon_{AB}\big(\mu^B\chi^\dagger-\bar a\,\mu^B\big) \\ & -\frac{1}{2}\,M^{\alpha A}\epsilon_{AB}\big([\chi^\dagger,M^B_\alpha]+2\,\bar f_c\,(\tau^c)_{\alpha\beta}M^{\beta B}\big)\Big\}+S^{\rm c}_k\ , \end{aligned} \tag{3.23}$$

where, as usual, $a'_{\alpha\dot\beta}=a'_\mu(\sigma^\mu)_{\alpha\dot\beta}$, $a'^{\dot\beta\alpha}=a'_\mu(\bar\sigma^\mu)^{\dot\beta\alpha}$ and τ^c are the three Pauli matrices. When we use this notation, it is clear that the effects on the instanton moduli of the gravitational backgrounds f and $\bar f$ can be formally introduced with the following shifts

$$[\chi,(\bullet)_\alpha]\ \to\ [\chi,(\bullet)_\alpha]+2\,f_c\,(\tau^c\,\bullet)_\alpha\quad\text{ and }\quad[\chi^\dagger,(\bullet)_\alpha]\ \to\ [\chi^\dagger,(\bullet)_\alpha]+2\,\bar f_c\,(\tau^c\,\bullet)_\alpha \tag{3.24}$$

where the notation $(\bullet)_\alpha$ stands for any field in the adjoint representation of $\mathrm{U}(k)$ that carries a chiral Lorentz index α. The shifts (3.24) are in some sense the gravitational counterparts of the ones in eq. (2.31), which account for the presence of a non-trivial v.e.v. for the gauge scalar fields, and appear as a rotation on the chiral indices. The rules (3.24) very much resemble the ones considered in refs. [10, 11, 13, 15] in the study of the localization properties of the instanton moduli space. Actually, we can be more precise in this respect. In fact, by choosing the independent parameters f_c and $\bar f_c$ as

$$f_c=\frac{\varepsilon}{2}\,\delta_{3c}\,,\qquad \bar f_c=\frac{\bar\varepsilon}{2}\,\delta_{3c} \tag{3.25}$$

with $\varepsilon=\bar\varepsilon$, the action (3.23) reduces exactly to the one of refs. [11, 15] with $\epsilon_1=-\epsilon_2=\varepsilon$. Thus, our derivation gives a direct gravitational meaning to the deformation parameter introduced in those references as a chiral rotation angle on some moduli; in fact, in our context this deformation naturally appears as due to a non-vanishing self-dual graviphoton field strength of $\mathcal{N}=2$ supergravity. In section 4 we will provide more details on this point and also comment on the relation of our results with the so-called Ω-background of ref. [13, 16].

3.3 Holomorphicity

Besides having a different meaning from the supergravity point of view, the parameters f_c and $\bar f_c$ are not on equal footing in the instanton moduli action (3.23). In particular the graviphoton parameters f_c do not appear in the fermionic part of $S_{\rm moduli}$. As noticed at the end of section 2.3, also the scalar v.e.v.'s a_{uv} and $\bar a_{uv}$ have a similar behaviour. This fact is not surprising since the effects of a_{uv} and f_c (or $\bar a_{uv}$ and $\bar f_c$) on the instanton action can be generated by shifting χ (or $\chi^\dagger$) according to the rules (2.31) and (3.24). This structure has a very important consequence, namely the instanton partition function

$$Z^{(k)}\equiv\int d\mathcal{M}_{(k)}\,{\rm e}^{-S_{\rm moduli}(a,\bar a;f,\bar f)} \tag{3.26}$$

where $\mathcal{M}_{(k)}$ collectively denotes the instanton moduli, depends only on a_{uv} and f_c and not on $\bar{a}_{uv}$ and $\bar{f}_c$. Such holomorphicity is a well-known property as far as the scalar v.e.v.'s are concerned [6], and here we extend it also to the gravitational parameters f_c and $\bar{f}_c$, with a straightforward generalization of the usual cohomology argument.

Let us give some details. In general, to prove holomorphicity it is rather convenient to rearrange everything by means of the so-called topological twist. This simply amounts to identify the index A of the internal $\mathrm{SU}(2)_{\mathrm{I}}$ symmetry group with the anti-chiral index $\dot{\alpha}$ of the $\mathrm{SU}(2)_{\mathrm{R}}$ factor of the Lorentz group. After this identification, the new Lorentz group becomes $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{d}}$ where $\mathrm{SU}(2)_{\mathrm{d}}$ is the diagonal subgroup of $\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{SU}(2)_{\mathrm{I}}$. The bosonic ADHM moduli $\{a'_\mu, \chi, \chi^\dagger, w_{\dot{\alpha}}, \bar{w}_{\dot{\alpha}}, D_c\}$ are not affected by the twist, while the fermionic ADHM moduli become $\{\mu^{\dot{\alpha}}, \bar{\mu}^{\dot{\alpha}}, \eta, \lambda_c, M^\mu\}$ where

$$\lambda_c = \frac{\mathrm{i}}{2}(\tau_c)^{\dot{\alpha}\dot{\beta}}\,\lambda_{\dot{\alpha}\dot{\beta}} \quad , \quad \eta = \frac{1}{2}\,\epsilon^{\dot{\alpha}\dot{\beta}}\,\lambda_{\dot{\alpha}\dot{\beta}} \quad , \quad M^\mu = (\sigma^\mu)_{\alpha\dot{\beta}}\,M^{\alpha\dot{\beta}}\,. \tag{3.27}$$

Similarly, the eight supersymmetry charges get reorganized as $\{Q, Q_c, Q^\mu\}$ where

$$Q = \frac{1}{2}\,\epsilon_{\dot{\alpha}\dot{\beta}}\,Q^{\dot{\alpha}\dot{\beta}} \quad , \quad Q_c = \frac{\mathrm{i}}{2}(\tau_c)_{\dot{\alpha}\dot{\beta}}\,Q^{\dot{\alpha}\dot{\beta}} \quad , \quad Q_\mu = (\sigma_\mu)^{\alpha\dot{\beta}}\,Q_{\alpha\dot{\beta}}\,. \tag{3.28}$$

The supercharge Q, which is the scalar component of the four supercharges that are preserved both by the D3 and the D(–1) branes (see eq. (2.23)), plays the role of a BRST charge in the topologically twisted version of the $\mathcal{N} = 2$ gauge theory. With some straightforward algebra, it is then possible to show that the moduli action (3.23) is Q-exact, namely

$$S_{\mathrm{moduli}}(a, \bar{a}; f, \bar{f}) = Q\,\Xi \tag{3.29}$$

where

$$\begin{aligned}\Xi \;=\; \mathrm{tr}_k\Big\{ & \frac{1}{\sqrt{2}}\Big(\big(\chi^\dagger\bar{w}_{\dot{\alpha}} - \bar{w}_{\dot{\alpha}}\,\bar{a}\big)\mu^{\dot{\alpha}} + \bar{\mu}^{\dot{\alpha}}\big(w^{\dot{\alpha}}\chi^\dagger - \bar{a}\,w^{\dot{\alpha}}\big)\Big) \\ & - \sqrt{2}\,M_\mu\big([\chi^\dagger, a'_\mu] - 2\,\mathrm{i}\,\bar{f}_\mu{}^\nu a'_\nu\big) - 2\,\mathrm{i}\,\lambda_c\big(\bar{w}_{\dot{\alpha}}(\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}}w^{\dot{\beta}} + \mathrm{i}\bar{\eta}^c_{\mu\nu}[a'^\mu, a'^\nu]\big)\Big\}\;,\end{aligned} \tag{3.30}$$

and the action of Q on the various moduli is

$$\begin{aligned} & Q\,a'^\mu = -\frac{\mathrm{i}}{2}\,M^\mu \;\;,\;\; Q\,\chi = 0 \;\;,\;\; Q\,\chi^\dagger = -\sqrt{2}\,\mathrm{i}\,\eta \;\;, \\ & Q\,D_c = -\,\mathrm{i}\sqrt{2}\,[\chi, \lambda_c] \;\;,\;\; Q\,M^\mu = \sqrt{2}\big([\chi, a'^\mu] - 2\,\mathrm{i}\,f_{\mu\rho}a'^\rho\big) \;\;, \\ & Q\,\eta = \frac{1}{2}\,[\chi, \chi\dagger] \;\;,\;\; Q\,\lambda_c = \frac{1}{2}\,D_c \;\;, \\ & Q\,w_{\dot{\alpha}} = -\frac{\mathrm{i}}{2}\,\mu_{\dot{\alpha}} \;\;,\;\; Q\,\bar{w}_{\dot{\alpha}} = -\frac{\mathrm{i}}{2}\,\bar{\mu}_{\dot{\alpha}} \;\;, \\ & Q\,\mu^{\dot{\alpha}} = \sqrt{2}\,\big(w^{\dot{\alpha}}\chi - a\,w^{\dot{\alpha}}\big) \;\;,\;\; Q\,\bar{\mu}^{\dot{\alpha}} = \sqrt{2}\,\big(\chi\bar{w}^{\dot{\alpha}} - w^{\dot{\alpha}}\,a\big)\,. \end{aligned} \tag{3.31}$$

It can be checked that Q is indeed nilpotent (up to gauge transformations and chiral rotations). Notice that $\bar{a}$ and $\bar{f}$ are present only in the fermion Ξ but not in the transformations (3.31). On the contrary a and f appear explicitly through the action of Q. Therefore, making a variation of the instanton partition function (3.26) with respect to $\bar{a}$ and $\bar{f}$ produces a Q-exact term and so $Z^{(k)}$ does not depend on $\bar{a}$ and $\bar{f}$. For this reason, $\bar{a}$ and $\bar{f}$ can be fixed to any convenient value.

4. Deformed ADHM construction and the Ω-background

As mentioned in the introduction, in supersymmetric models the computation of the instanton partition function $Z^{(k)}$ for arbitrary k is most easily performed if the gauge theory is suitably deformed by turning on the so-called Ω-background [13, 16]. On the instanton moduli space this deformation corresponds to enlarge the symmetries of the ADHM construction. In this section, following ref. [15], we briefly review this deformed ADHM construction and show that it is directly related to the instanton measure in a graviphoton background that we have described in the previous section.

To this aim, let us introduce the standard $[N+2k]\times[2k]$ ADHM matrix [7]

$$\Delta = \begin{pmatrix} w \\ a' - x \end{pmatrix} , \tag{4.1}$$

where w and a' are a shorthand for $w_{ui\dot\alpha}$ and $a'_{ij\alpha\dot\beta}$, and

$$x \equiv \mathbb{1}_{[k]\times[k]} \otimes \left(x_{\alpha\dot\beta}\right) = \mathbb{1}_{[k]\times[k]} \otimes \begin{pmatrix} z_1 & -\bar z_2 \\ z_2 & \bar z_1 \end{pmatrix} , \tag{4.2}$$

with z_1 and z_2 being the complex coordinates of the (Euclidean) space-time. In the $\mathcal{N}=2$ theory we introduce the fermionic partners of Δ, namely the $[N+2k]\times[k]$ ADHM matrices

$$\mathcal{M}^A = \begin{pmatrix} \mu^A \\ M^A \end{pmatrix} , \tag{4.3}$$

where μ^A and M^A stand for $\mu_{ui}{}^A$ and $M_{ij}{}^{\alpha A}$. In terms of these matrices, the bosonic and fermionic ADHM constraints (2.16) read

$$\bar\Delta\Delta = \ell^{-1} \otimes \mathbb{1}_{[2]\times[2]} \quad \text{and} \quad \bar\Delta\mathcal{M}^A + \bar{\mathcal{M}}^A\Delta = 0 \ , \tag{4.4}$$

where ℓ is the $[k]\times[k]$ matrix

$$\ell = \left(\bar w_{\dot\alpha} w^{\dot\alpha} + (a'-x)^2\right)^{-1} . \tag{4.5}$$

Introducing a $[N+2k]\times[N]$ matrix U such that $\bar\Delta U = \bar U\Delta = 0$, the $\mathcal{N}=2$ SU(N) super-instanton solution can be written as

$$A_\mu = \frac{1}{g}\,\bar U\partial_\mu U \ , \tag{4.6a}$$

$$\Lambda^{\alpha A} = \frac{1}{g^{1/2}}\,\bar U\left(\mathcal{M}^A\ell\,\bar b^\alpha - b^\alpha\ell\,\bar{\mathcal{M}}^A\right)U \ , \tag{4.6b}$$

$$\phi = \mathrm{i}\,\frac{\sqrt{2}}{4}\,\epsilon_{AB}\,\bar U\,\mathcal{M}^A\ell\,\bar{\mathcal{M}}^B U + \bar U\,\mathcal{J}\,U , \tag{4.6c}$$

[7] Not to be confused with the twist field Δ of section 2.

where $b^\alpha = \begin{pmatrix} 0 \\ \delta^\alpha_\beta\, \delta_{ij} \end{pmatrix}$ and $\mathcal{J}$ is the $[N+2k]\times[2k]$ matrix

$$\mathcal{J} = \begin{pmatrix} 0 & 0 \\ 0 & \chi \end{pmatrix} . \tag{4.7}$$

Here χ is a shorthand for $\chi \otimes \mathbb{1}_{[2]\times[2]}$, where χ is a hermitian $[k]\times[k]$ matrix such that

$$\mathbf{L}\,\chi = -\mathrm{i}\,\frac{\sqrt{2}}{4}\,\epsilon_{AB}\,\bar{\mathcal{M}}^A \mathcal{M}^B\,, \tag{4.8}$$

with the operator $\mathbf{L}$ defined by

$$\mathbf{L}\,\bullet = \frac{1}{2}\{\bar{w}_{\dot\alpha} w^{\dot\alpha}, \bullet\} + [a'_\mu, [a'^\mu, \bullet]]\,. \tag{4.9}$$

Notice that eq. (4.8) is the same equation (2.17a) that follows by varying the moduli action (2.13) with respect to $\chi^\dagger$. In showing that eq. (4.6c) is an instanton solution of the scalar field equation, a crucial role is played by the following zero-mode (see appendix C of ref. [6])

$$D^\mu\big(\bar{U}\,\mathcal{J}\,U\big) = 2\,\mathrm{Im}\,\big(\bar{U}\,\mathcal{A}\,\bar\sigma^\mu \ell \bar{b}\,U\big)\,, \tag{4.10}$$

where

$$\mathcal{A} = \begin{pmatrix} -w\chi \\ [\chi, a'] \end{pmatrix} . \tag{4.11}$$

This ADHM construction of the super-instanton solution can be generalized to include a v.e.v. for the scalar field ϕ. In this case the classical profile of ϕ at the leading order in the Yang-Mills coupling [8] is still given by eq. (4.6c), but with

$$\mathcal{J} \to \mathcal{J}(a) = \begin{pmatrix} a & 0 \\ 0 & \chi \end{pmatrix} \quad , \tag{4.12}$$

where a is the $[N]\times[N]$ v.e.v. matrix and χ now satisfies

$$\mathbf{L}\,\chi = -\mathrm{i}\,\frac{\sqrt{2}}{4}\,\epsilon_{AB}\,\bar{\mathcal{M}}^A \mathcal{M}^B + \bar{w}_{\dot\alpha}\, a\, w^{\dot\alpha}\,. \tag{4.13}$$

Note that (4.13) is precisely the equation of motion that follows varying w.r.t. to $\chi^\dagger$ the moduli action presented in section 2.3 (with $\bar{a} = 0$).

The presence of a non-zero v.e.v. for ϕ can be interpreted as a deformation of the ADHM construction. To appreciate this point, we can follow ref. [15] and show that the matrix $\mathcal{A}$ given in eq. (4.11) must be replaced by

$$\mathcal{A}(a) = \begin{pmatrix} aw - w\chi \\ [\chi, a'] \end{pmatrix} . \tag{4.14}$$

[8] We assume the following expansions for the various gauge fields: $A = g^{-1}A^{(0)} + gA^{(1)} + \ldots$; $\Lambda = g^{-1/2}\Lambda^{(0)} + g^{3/2}\Lambda^{(1)} + \ldots$; $\bar\Lambda = g^{1/2}\bar\Lambda^{(0)} + g^{5/2}\bar\Lambda^{(1)} + \ldots$; $\phi = \phi^{(0)} + g^2\phi^{(1)} + \ldots$; $\bar\phi = \bar\phi^{(0)} + g^2\bar\phi^{(1)} + \ldots$. At leading order for $g \to 0$, one can just work with the first terms in these expansions.

Notice that $\mathcal{A}(a)$ can be obtained from $\mathcal{A}$ by means of the same shift (2.31) that we have derived in section 2.3 from string amplitudes. The structure (4.14) has also a further interpretation if we note that the ADHM constraints are left invariant by the transformations $T_\chi = \mathrm{e}^{\mathrm{i}\,\chi} \in \mathrm{U}(k)$ and $T_a = \mathrm{e}^{\mathrm{i}\,a} \in \mathrm{SU}(N)$, which reflect the redundancy in the ADHM description and do not change the gauge connection (4.6a). Under these transformations the ADHM data change as

$$\Delta \;\to\; \begin{pmatrix} T_a\, w\, T_\chi^{-1} \\ T_\chi\, (a' - x)\, T_\chi^{-1} \end{pmatrix} = \Delta + \mathrm{i}\,\mathcal{A}(a) + \cdots \tag{4.15}$$

Thus, the matrix (4.14) is related to the first order variation of Δ under the symmetries of the ADHM constraints.

This construction can be further generalized [15] by including other symmetries of the ADHM constraints, in particular the chiral rotations in the four dimensional Euclidean space-time [9] which, as shown in refs. [10, 13, 11, 15], allow to fully localize the integral on the instanton moduli space on a discrete set of isolated fixed points. Thus, in place of (4.15) we consider

$$\Delta \;\to\; \begin{pmatrix} T_a\, w\, T_\chi^{-1} \\ T_\chi\, T_\varepsilon\, (a' - x)\, T_\chi^{-1} \end{pmatrix} \tag{4.16}$$

where $T_\varepsilon = \mathrm{e}^{\mathrm{i}\,\varepsilon_c \tau^c} \in \mathrm{SU}(2)_{\mathrm{L}}$ generates chiral rotations in the complex z_1 and z_2 planes with angles $\varepsilon_c^1 = -\varepsilon_c^2 = \varepsilon_c$. At first order we now have $\Delta \;\to\; \Delta + \mathrm{i}\,\mathcal{A}(a,\varepsilon)$, where

$$\mathcal{A}(a,\varepsilon) = \begin{pmatrix} aw - w\chi \\ [\chi, a'] + \varepsilon_c\, \tau^c a' \end{pmatrix} . \tag{4.17}$$

To find the instanton profile of the scalar field ϕ also in the presence of the ε-rotations, we take the Ansatz (4.10) with $\mathcal{A}$ replaced by $\mathcal{A}(a,\varepsilon)$. Since

$$\mathcal{A}(a,\varepsilon) = -\Delta\,\chi + \begin{pmatrix} a & 0 \\ 0 & \chi + \varepsilon_c\,\tau^c \end{pmatrix} \Delta + \varepsilon_c \begin{pmatrix} 0 \\ \tau^c x \end{pmatrix} \;\; , \tag{4.18}$$

we can show that [15]

$$2\,\mathrm{Im}\left(\bar{U}\mathcal{A}(a,\varepsilon)\,\bar{\sigma}^\mu \ell \bar{b}\, U\right) = D^\mu\left(\bar{U}\mathcal{J}(a,\varepsilon)U\right) - \mathrm{i}\,g\,\Omega_{\nu\rho}\,x^\rho F^{\mu\nu}\,, \tag{4.19}$$

where

$$\mathcal{J}(a,\varepsilon) = \begin{pmatrix} a & 0 \\ 0 & \chi + \varepsilon_c \tau^c \end{pmatrix} \;\; , \tag{4.20}$$

$$\Omega_{\mu\nu} = \varepsilon_c\, \eta^c_{\mu\nu} \;\; , \tag{4.21}$$

and $F_{\mu\nu}$ is the instanton gauge field strength which follows from (4.6a) and obeys $D^\mu F_{\mu\nu} = 0$. Furthermore, if χ satisfies the constraint

$$\mathbf{L}\,\chi = -\mathrm{i}\,\frac{\sqrt{2}}{4}\,\epsilon_{AB}\,\bar{\mathcal{M}}^A \mathcal{M}^B + \bar{w}_{\dot\alpha}\, a\, w^{\dot\alpha} - \mathrm{i}\,\Omega_{\mu\nu}\big[a'^\mu, a'^\nu\big] \;\; , \tag{4.22}$$

[9] Actually, as discussed in ref. [15], one could consider both chiral and anti-chiral rotations of the space-time. However, for our purposes it is enough to restrict our analysis to the chiral ones.

we can prove that eq. (4.6c) with $\mathcal{J} \to \mathcal{J}(a,\varepsilon)$ is a solution of the following field equation

$$D^2\phi = -\mathrm{i}\sqrt{2}\, g\, \epsilon_{AB}\Lambda^{\alpha A}\Lambda_\alpha{}^B - \mathrm{i}\, g\, \Omega_{\mu\nu}F^{\mu\nu}\,. \tag{4.23}$$

A few comments are in order. First of all, since the matrix $\mathcal{A}(a,\varepsilon)$ is not homogeneous in Δ but contains also a x-dependent piece proportional to ε_c, as is clear from (4.18), in computing $D^\mu\phi$ we produce also an extra piece proportional to Ω that in turns modifies the structure of the scalar field equation. Secondly, the constraint (4.22) is precisely the equation of motion for χ that follows by varying the moduli action (3.21) in a constant graviphoton background with field strength

$$f_{\mu\nu} = \frac{1}{2}\,\Omega_{\mu\nu} = \frac{1}{2}\,\varepsilon_c\,\eta^c_{\mu\nu} \tag{4.24}$$

(and with $\bar{a} = \bar{f}_{\mu\nu} = 0$). Thus, our analysis shows that the parameters ε_c of the deformed ADHM construction of ref. [15] have a direct interpretation in terms of a constant graviphoton background, as we already anticipated at the end of section 3.

At this point we can ask what is the meaning of ε_c at the level of the gauge theory action in four dimensions. In ref. [13, 16] it has been argued that these deformation parameters are related to a non-trivial metric in $\mathbb{R}^4 \times \mathbb{C}$, called Ω-background and characterized by a self-dual antisymmetric tensor $\Omega_{\mu\nu}$, which indeed leads to the deformed field equation (4.23) at the leading order in the Yang-Mills coupling constant. Here, however, we show that also on the gauge theory the ε-deformation can be directly related to the same graviphoton background (4.24) that modifies the action on the instanton moduli space.

To this aim, we first determine the deformed gauge theory action by computing the couplings among the various gauge fields and the graviphoton. This can be done by computing disk scattering amplitudes among the vertex operators for the open string massless excitations of the N fractional D3 branes given in (2.3) and (2.4), and the closed string vertex operator for the self-dual graviphoton field strength given in (3.5). For example, we have

$$\Big\langle\!\Big\langle V_A V_{\bar{\phi}} V_{\mathcal{F}} \Big\rangle\!\Big\rangle = 2\,\mathrm{i}\, g\,\mathrm{Tr}\Big\{\partial_{[\mu}A_{\nu]}\bar{\phi}f^{\mu\nu}\Big\}\,. \tag{4.25}$$

If we put V_ϕ instead of $V_{\bar{\phi}}$ we get a vanishing result due to an unbalanced internal charge. Actually, the amplitude (4.25) is the only non-zero 3-point function involving the graviphoton field strength.

To find higher order contributions it is convenient to follow the method described in detail in ref. [21] in the context of non-anti-commutative theories and introduce the auxiliary fields that disentangle the non-abelian quartic interactions among the gauge vector bosons. It turns out that these auxiliary fields have non-vanishing couplings also with the R-R graviphoton vertex $V_{\mathcal{F}}$ and, when they are integrated out, two effects are obtained: $\partial_{[\mu}A_{\nu]}$ in (4.25) is promoted to the full non-abelian field strength $F_{\mu\nu}$, and a quartic term $\sim g^2\left(\bar{\phi}f^{\mu\nu}\right)^2$ is produced.

Collecting all contributions, we find that the action for the gauge fields of N fractional D3 branes in a self-dual graviphoton background $f_{\mu\nu}$ is given by

$$S_{\mathrm{SYM}} + \int d^4x\ \mathrm{Tr}\Big\{ -2\,\mathrm{i}\, g\, F_{\mu\nu}\bar{\phi}f^{\mu\nu} - g^2\big(\bar{\phi}f^{\mu\nu}\big)^2\Big\} \tag{4.26}$$

where $S_{\rm SYM}$ is the super Yang-Mills action (2.5). We remark that the deformation terms in (4.26) are in perfect agreement with the general couplings between the Weyl and gauge vector multiplets required by $\mathcal{N}=2$ supergravity (see for instance the review [25] and references therein). It is now easy to see that at the leading order in the coupling constant g (using the standard expansions mentioned in footnote 8), the field equation for ϕ that follows from (4.26) is precisely eq. (4.23) once the relation (4.24) between $f_{\mu\nu}$ and $\Omega_{\mu\nu}$ is taken into account. Thus, the f-dependent terms in (4.26) correctly describe, at the gauge theory level, the same deformation which on the instanton moduli is realized as a chiral rotation. The action (4.26) coincides with the Ω-background action of refs. [13, 16] at the linear order in g, but differs at higher orders. However, since the instanton calculus is concerned with linearized actions, this difference is unimportant.

Our analysis shows that both on the ADHM moduli space and on the four-dimensional gauge theory the ε-rotations have the same supergravity interpretation since they are related to the components of the graviphoton field strength of the Weyl multiplet. Our method treats the ε-deformation in exactly the same way on the ADHM moduli and on the gauge fields. This is quite natural in the string realization of the instanton calculus by means of systems of D3 and D(–1) branes, in which gauge fields and ADHM moduli arise from different sectors of the same bound state of D branes.

5. Instanton contributions to the effective action

In this section we study the instanton partition function in a graviphoton background. Using the holomorphicity properties of section 3.3 we can set $\bar{a}=\bar{f}_{\mu\nu}=0$ with no loss of generality, and concentrate only on the a and $f_{\mu\nu}$ dependence. These parameters are actually the v.e.v.'s of the lowest components of chiral superfields but, implementing techniques and ideas of ref. [5] (see also ref. [8]), it is rather straightforward to derive the full dependence on the entire superfields $\Phi(x,\theta)$ (along the unbroken gauge directions of $\mathrm{U}(1)^{N-1}$) and $W^+_{\mu\nu}(x,\theta)$.

5.1 The field-dependent moduli action

Let us start by considering the gauge superfield Φ. The dependence of $S_{\rm moduli}$ on the lowest component ϕ can be derived by computing the same mixed disk diagrams which produce the a-dependent contributions, such as the one of figure 1*b*, but with a dynamical (i.e. momentum dependent) vertex V_ϕ. For example, the amplitude (2.28) becomes

$$\Big\langle\!\Big\langle V_{\bar{X}^\dagger} V_\phi V_w \Big\rangle\!\Big\rangle = \mathrm{tr}_k\Big\{\bar{X}^\dagger_{\dot\alpha} w^{\dot\alpha}\phi(p)\,\mathrm{e}^{\mathrm{i}p\cdot x_0}\Big\}\ , \tag{5.1}$$

where the dependence on the instanton center x_0 (defined in eq. (2.10)) originates from the world-sheet correlator $\langle\bar\Delta(z_1)\,\mathrm{e}^{\mathrm{i}p\cdot X(z_2)}\Delta(z_3)\rangle\propto\mathrm{e}^{\mathrm{i}p\cdot x_0}$. A similar dependence on x_0 arises in all other mixed diagrams involving the vertex operator V_ϕ. From these string correlators, after taking the Fourier transform with respect to p, we can extract a moduli action which is given by eq. (3.21) (at $\bar{a}=\bar{f}_{\mu\nu}=0$) with

$$a\to\phi(x_0)\,. \tag{5.2}$$

With this replacement, $S_{\rm moduli}$, which originally did not depend on the instanton center, acquires a non-trivial dependence on x_0 that, from now on, we will simply denote by x.

There are other non-vanishing disk diagrams that couple the various components of the gauge supermultiplet to the instanton moduli. These diagrams are related to the ones containing only ϕ by the Ward identities of the supersymmetries that are broken by the D(–1) branes and are generated by the chiral supercharges

$$Q_{\alpha A} = \oint \frac{dw}{2\pi \mathrm{i}}\, j_{\alpha A}(w) \qquad \text{with} \qquad j_{\alpha A} = S_\alpha S_A\, \mathrm{e}^{-\frac{1}{2}\varphi}\,. \tag{5.3}$$

Note that the supercurrents $j_{\alpha A}$ (which carry trivial Chan-Paton factors) coincide with the vertex for the moduli $\theta^{\alpha A}$ introduced in eqs. (2.9a) and (2.10). The transformations generated by $\theta^{\alpha A} Q_{\alpha A}$ are precisely the ones that connect the various components of the chiral superfield $\Phi(x,\theta)$. For instance, we have

$$\big[\theta^{\alpha A} Q_{\alpha A}, V_\Lambda(z)\big] = V_{\delta\phi}(z)\ , \tag{5.4}$$

where the vertices are given in eqs. (2.3b) and (2.3c), and the supersymmetry variation $\delta\phi = \theta^{\alpha A}\Lambda_\alpha{}^B \epsilon_{AB}$ is encoded in the superfield structure of $\Phi(x,\theta)$, see eq. (2.1). Thus, besides the amplitude (5.1) we also have a correlator in which the vertex V_ϕ is replaced by $V_{\delta\phi}$, *i.e.*

$$\Big\langle\!\Big\langle V_{\bar X^\dagger} V_{\delta\phi} V_w \Big\rangle\!\Big\rangle = \Big\langle\!\Big\langle V_{\bar X^\dagger} \big[\theta^{\alpha A} Q_{\alpha A}, V_\Lambda\big]\, V_w \Big\rangle\!\Big\rangle\,. \tag{5.5}$$

Deforming the integration contour for the supercharge and taking into account the fact that $Q_{\alpha A}$ commutes with $V_{\bar X^\dagger}$ and V_w, we can move it onto the D(–1) part of the boundary and get [10]

$$\Big\langle\!\Big\langle V_{\bar X^\dagger} \big[\theta^{\alpha A} Q_{\alpha A}, V_\Lambda\big]\, V_w \Big\rangle\!\Big\rangle = -\Big\langle\!\Big\langle V_{\bar X^\dagger} V_\Lambda V_w \int V_\theta \Big\rangle\!\Big\rangle\,. \tag{5.6}$$

In this way a 4-point amplitude containing one insertion of V_Λ on the D3 boundary and one (integrated) insertion of V_θ on the D(–1) boundary can be related to a 3-point amplitude with $V_{\delta\phi}$. Therefore, the corresponding result can be simply obtained from eq. (2.28) with the replacement

$$\phi \to \delta\phi = \theta\Lambda\,. \tag{5.7}$$

This analysis can be further iterated, revealing new couplings with the higher components of Φ and more θ-insertions. Altogether it turns out that these additional interactions in the moduli action can be summarized by extending the replacement (5.2) to

$$a \to \Phi(x,\theta)\,. \tag{5.8}$$

A similar pattern can be followed also with the Weyl superfield (3.1). Introducing the momentum dependence for the R-R vertex $V_{\mathcal F}$ in the amplitude (3.15), we obtain[11]

$$\Big\langle\!\Big\langle V_{Y^\dagger} V_{a'} V_{\mathcal F} \Big\rangle\!\Big\rangle = -4\mathrm{i}\,\mathrm{tr}_k\Big\{ Y_\mu^\dagger\, a'_\nu\, \mathcal{F}^{\mu\nu}(p)\, \mathrm{e}^{\mathrm{i}p\cdot x}\Big\}\,. \tag{5.9}$$

[10] We refer to ref. [5] and in particular to section 5.2 of the first paper in ref. [8] for more detailed explanations.

[11] Notice that the momentum dependence arises from the one-point function of the plane-wave term $\mathrm{e}^{\mathrm{i}p\cdot X(z,\bar z)}$ in the closed string vertex $V_{\mathcal F}$ on a disk with D(–1) boundary conditions.

The couplings with the other components of $W^+_{\mu\nu}$ are related to this one by the Ward identities for the supercharges $Q_{\alpha A}$ broken by the D-instantons, so that the dependence of $S_{\rm moduli}$ on $W^+_{\mu\nu}$ can be obtained by simply replacing in eq. (3.21)

$$f_{\mu\nu} \to W^+_{\mu\nu}(x,\theta)\,. \tag{5.10}$$

Altogether, performing the replacements (5.8) and (5.10) in $S_{\rm moduli}(a,0,f,0)$ given in eq. (3.21), we obtain

$$S_{\rm moduli}(a,0;f,0) \;\to\; S_{\rm moduli}\left(\Phi(x,\theta),0;W^+(x,\theta),0\right) \equiv S\big(\Phi;W^+;\widehat{\mathcal{M}}\big)\ , \tag{5.11}$$

which describes the couplings of the abelian superfield Φ in the unbroken gauge directions and of the Weyl superfield $W^+_{\mu\nu}$ to the centered instanton moduli $\widehat{\mathcal{M}}$, i.e. all moduli except x and θ.

5.2 The low-energy effective action and the prepotential

Let us neglect for the moment the Weyl multiplet and concentrate on the usual $\mathcal{N}=2$ SYM theory for which the low-energy effective action is a functional of the chiral superfield Φ in the unbroken gauge directions (and of its conjugate $\bar\Phi$). For simplicity, but without loss of generality, we focus on the SU(2) gauge theory; in this case, from eq. (2.6) we see that the unbroken U(1) chiral superfield is $\Phi = \Phi_3\,\tau^3$. In the following Φ_3 will be simply denoted by Φ without any ambiguity.

Up to two-derivative terms, $\mathcal{N}=2$ supersymmetry constrains the effective action for Φ to be of the form

$$S_{\rm eff}[\Phi] = \int d^4x\, d^4\theta\, \mathcal{F}(\Phi) + \text{c.c}\ , \tag{5.12}$$

where $\mathcal{F}$ is the prepotential. In the semi-classical limit $\mathcal{F}$ displays a 1-loop perturbative contribution plus instanton corrections [26], namely

$$\mathcal{F}(\Phi) = \frac{\mathrm{i}}{2\pi}\Phi^2 \log\frac{\Phi^2}{\Lambda^2} + \sum_{k=1}^{\infty}\mathcal{F}^{(k)}(\Phi)\ , \tag{5.13}$$

where Λ is the dynamically generated scale and k is the instanton number.

Focusing on a given sector with positive k, the instanton induced effective action for Φ is given by [12]

$$S^{(k)}_{\rm eff}[\Phi] = \int d^4x\, d^4\theta\, d\widehat{\mathcal{M}}_{(k)}\ \mathrm{e}^{-\frac{8\pi k}{g^2} - S\left(\Phi;\widehat{\mathcal{M}}_{(k)}\right)}\,, \tag{5.14}$$

where in the exponent we have added also the classical part of the instanton action $S^{\rm cl}_k = 8\pi k/g^2$. Upon comparison with eq. (5.12), we obtain

$$\mathcal{F}^{(k)}(\Phi) = \int d\widehat{\mathcal{M}}_{(k)}\ \mathrm{e}^{-\frac{8\pi k}{g^2} - S\left(\Phi;\widehat{\mathcal{M}}_{(k)}\right)}\,. \tag{5.15}$$

[12] Since we are dealing with $\mathcal{N}=2$ theories, we do not distinguish between effective actions a la Wilson and effective actions a la Coleman-Weinberg.

Thus, the k-instanton contribution to the prepotential is given by the centered k-instanton partition function [27, 28]. Since the superfield $\Phi(x,\theta)$ is a constant with respect to the integration variables $\widehat{\mathcal{M}}_{(k)}$, one can compute $\mathcal{F}^{(k)}$ by fixing $\Phi(x,\theta)$ to its v.e.v., use the existing results of the literature (see for example ref. [6]) and finally replace the v.e.v. with the complete superfield $\Phi(x,\theta)$. In this way one finds

$$\mathcal{F}^{(k)}(\Phi) = c_k\,\Phi^2\left(\frac{\Lambda}{\Phi}\right)^{4k} . \tag{5.16}$$

where the factor Λ^{4k} originates from the classical action term $\exp(-8\pi k/g^2)$, upon taking into account the β-function of the $\mathcal{N}=2$ SU(2) theory. The numerical coefficients c_k have been explicitly computed for $k=1$ and $k=2$ by evaluating the integral over the instanton moduli space [29, 27] and checked against the predictions of the Seiberg-Witten theory [17], finding perfect agreement. More recently, using the localization formulas of the instanton integrals, the coefficients c_k have been computed also for arbitrary k [10, 11, 20, 15].

Let us now consider the gravitational corrections to the $\mathcal{N}=2$ effective theory by introducing also the dependence on the Weyl superfield $W^+_{\mu\nu}$ induced by the D-instantons. In perfect analogy with eqs. (5.14) and (5.15), we have to construct connected diagrams that describe also the couplings of the instanton moduli to $W^+_{\mu\nu}$, and thus write

$$S^{(k)}_{\text{eff}}[\Phi;W^+] = \int d^4x\,d^4\theta\,\mathcal{F}^{(k)}(\Phi;W^+) \tag{5.17}$$

where the prepotential is

$$\mathcal{F}^{(k)}(\Phi;W^+) = \int d\widehat{\mathcal{M}}_{(k)}\,\mathrm{e}^{-\frac{8\pi k}{g^2}-S\left(\Phi;W^+;\widehat{\mathcal{M}}_{(k)}\right)} . \tag{5.18}$$

Since both $\Phi(x,\theta)$ and $W^+_{\mu\nu}(x,\theta)$ are constant with respect to the integration variables, we can simply compute $\mathcal{F}^{(k)}(a;f)$ and then replace the v.e.v.'s with the corresponding superfields in the result. By examining the explicit form of the moduli action $S_{\text{moduli}}(a,0;f,0)$ given in eq. (3.21), we see that it is invariant under the simultaneous sign reversal of a and f, if at the same time also the signs of χ, w and of the SU(k) part of a' (named y_c in eq. (2.10)) are reversed. This is a change of integration variables in eq. (5.18) with unit Jacobian, so that we can conclude that $\mathcal{F}^{(k)}(a;f)$ is invariant under the exchange

$$a, f_{\mu\nu} \to -a, -f_{\mu\nu} . \tag{5.19}$$

The prepotential $\mathcal{F}^{(k)}(a;f)$ has a regular expansion for $f\to 0$, where it reduces to the super Yang-Mills expression $\mathcal{F}^{(k)}(a)$ of eq. (5.16). Moreover, it cannot contain odd powers of $(af_{\mu\nu})$, that would be compatible with the symmetry (5.19) but would have necessarily some uncontracted indices and therefore are unacceptable because of their tensorial nature. As a consequence, the expansion of $\mathcal{F}^{(k)}(a;f)$ must contain even powers of both a and $f_{\mu\nu}$, the latter suitably contracted. Replacing the v.e.v.'s with the corresponding superfields, and remembering that the prepotential has dimensions of (length)$^{-2}$, from the previous arguments we can deduce that

$$\mathcal{F}^{(k)}(\Phi;W^+) = \sum_{h=0}^{\infty} c_{k,h}\,\Phi^2\left(\frac{\Lambda}{\Phi}\right)^{4k}\left(\frac{W^+}{\Phi}\right)^{2h} , \tag{5.20}$$

where $c_{k,0} = c_k$ so as to reproduce eq. (5.16) at $W^+ = 0$. The problem of finding the non-perturbative gravitational contributions to the $\mathcal{N} = 2$ superpotential is then reduced to that of finding the numerical coefficients $c_{k,h}$.

The series (5.20) is obtained from a perturbative expansion of the linear couplings to the graviphoton multiplet which appear in $\mathrm{e}^{-S(\Phi;W^+;\mathcal{M}_{(k)})}$. As discussed in detail in section 3.2, these couplings originate from disk diagrams whose boundary lies on the D(–1) branes, each of which emits a graviphoton. Thus, the term of order $(W^+)^{2h}$ in eq. (5.20) comes from $2h$ disks, which correspond to a single (degenerate) Riemann surface with $2h$ boundaries and Euler characteristic

$$\chi_{\mathrm{Euler}} = 2h - 2\,. \tag{5.21}$$

This world-sheet is seemingly disconnected, but in the construction of the effective action it plays the rôle of a *connected* diagram because of the integration over the instanton moduli (see the related discussion in section 6 of ref. [8]).

Let us consider now the entire set of non-perturbative contributions to the prepotential

$$\mathcal{F}_{\mathrm{n.p.}}(\Phi;W^+) = \sum_{k=1}^{\infty} \mathcal{F}^{(k)}(\Phi;W^+) = \sum_{h=0}^{\infty} C_h(\Lambda,\Phi)(W^+)^{2h}\ , \tag{5.22}$$

where

$$C_h(\Lambda,\Phi) = \sum_{k=1}^{\infty} c_{k,h}\,\frac{\Lambda^{4k}}{\Phi^{4k+2h-2}}\,. \tag{5.23}$$

The effective action corresponding to this prepotential contains many different terms, connected to each other by supersymmetry. For instance, if we saturate the θ-integral with four θ's all coming from the Φ superfield, we obtain, among others, four-fermion contributions proportional to

$$\Lambda^{4k}\int d^4x\ \phi^{-4k-2h-2}\big(\Lambda^{\alpha A}\Lambda_{\alpha}{}^{B}\epsilon_{AB}\big)^2\left(\mathcal{F}^+\right)^{2h}\ , \tag{5.24}$$

which for $h \neq 0$ represent the gravitational corrections to the four-gaugino interaction induced by an instanton of charge k.

If instead the θ-integral is saturated with four θ's all coming from the W^+ superfields, we obtain, among others, a contribution proportional to

$$\int d^4x\ C_h(\Lambda,\phi)\,(R^+)^2(\mathcal{F}^+)^{2h-2}\,. \tag{5.25}$$

When the scalar field ϕ is frozen to its expectation value, this describes a purely gravitational F-term of the $\mathcal{N} = 2$ effective action. As we remarked above, in our approach based on the instanton calculus, the contribution (5.25) arises from mixed open/closed string amplitudes on $2h$ disks. Originally, this structure was discovered by computing pure closed string amplitudes on Riemann surfaces of genus h and through them a precise connection with the topological string was established. We will comment more on this point in the following subsection.

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5.3 Relation with topological amplitudes

The gravitational F-terms (5.25) can be computed also in the context of Type II strings compactified on a Calabi-Yau (CY) manifold, which is another well-known string setting from which one can obtain a $\mathcal{N} = 2$ low-energy effective theory. In this case, the coefficients C_h are functions of the moduli of the CY manifold [18, 19] which can be computed by topological h-loop string amplitudes arising from closed world-sheets of genus h, whose Euler characteristic is given again by eq. (5.21). When the two settings correspond to the same effective theory, the topological string computation of C_h should be in agreement with the gauge theory instanton calculations presented here, and indeed this is the case.

As shown by Seiberg and Witten (SW) [17], the $\mathcal{N} = 2$ low-energy effective action for a super Yang-Mills theory with a gauge group G can be described in terms of an auxiliary Riemann surface. The so-called geometrical engineering constructions [30, 31] embed the $\mathcal{N} = 2$ theory in a consistent Type IIB string context, thus accounting for the "physical" emergence of the SW Riemann surface. For this one has to consider Type IIB strings on a "local" CY manifold $\mathcal{M}_G^{(B)}$ whose geometrical moduli are related to the quantities characterizing the gauge theory, namely the dynamically generated scale Λ and the gauge invariant composites $\mathrm{Tr}\,\Phi^k$ of the scalars [13]. The dependence of the IIB prepotential on the moduli of $\mathcal{M}_G^{(B)}$ has a geometric expression in terms of periods of suitable forms. Mapping the geometrical moduli to gauge theory quantities, the prepotential matches the field-theoretic SW expression [30, 31]. Also the higher genus topological [18, 19] amplitudes C_h on $\mathcal{M}_G^{(B)}$ can be computed as functions of the moduli [33, 20], and hence of the gauge theory parameters. In this way one can get, for instance, explicit expressions for the couplings $C_h(\Lambda, \Phi)$ of eq. (5.23) in the SU(2) case.

As we argued above, the non-perturbative superpotential $\mathcal{F}_{\mathrm{n.p.}}(\Phi; W^+)$ can be computed also on the gauge theory side, where it is given by the multi-instanton centered partition functions (5.18), by freezing Φ and $W^+_{\mu\nu}$ to their v.e.v.'s a and $f_{\mu\nu}$. We have shown in section 3.2 (see in particular eqs. (3.22) and (3.25)), that the choice $f_{\mu\nu} = \frac{1}{2}\,\varepsilon\,\eta^3_{\mu\nu}$, corresponds to the deformation of the instanton moduli space with parameter ε that has been introduced in refs. [10, 11, 15] as a tool for the explicit evaluation of multi-instanton partition functions. Therefore, the expansion in powers of the deformation parameter ε and of the v.e.v. a of the centered multi-instanton partition functions determines the coefficient $c_{k,h}$ of eq. (5.23).

That these coefficients must agree with those derived from the topological amplitudes on $\mathcal{M}_G^{(B)}$ is a conjecture put forward in ref. [10] and checked in ref. [20] for the SU(2) case. We think that in the present paper we have made this conjecture extremely natural and self-evident by recognizing that the deformation parameter is nothing else that the graviphoton itself. Furthermore, our analysis puts the evidence found in ref. [20] in a broader perspective.

[13] The manifold $\mathcal{M}_G^{(B)}$ is usually determined via "local mirror symmetry" [31] from a type IIA CY manifold $\mathcal{M}_G^{(A)}$, whose singularity structure reproduces the $\mathcal{N} = 2$ effective theory for G, in the low-energy limit and upon decoupling gravity. The form of the IIB local CY space $\mathcal{M}_G^{(B)}$ can also be inferred directly, independently of the mirror construction, as explained for example in ref. [32].

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5.4 Consequences of the ε-deformation for the prepotential

There is however an important subtlety to be considered in checking the agreement between the instanton calculations and the topological string results. To reproduce the deformations (3.25), besides the graviphoton background we have to turn on also a vacuum expectation value $\bar{f}_{\mu\nu} = \frac{1}{2}\,\bar{\varepsilon}\,\eta^3_{\mu\nu}$ for a different R-R field strength (setting moreover $\bar{\varepsilon} = \varepsilon$ as discussed in section 3.2). The holomorphicity properties of the instanton moduli action discussed in section 3.3 ensure that the instanton partition function does not smoothly depend of $\bar{\varepsilon}$; however, the case $\bar{\varepsilon} = 0$ is a limiting one and some care is needed.

To determine the coefficients $c_{k,h}$ of the prepotential expansion, it is enough to consider constant background values for the scalar and Weyl multiplets. In this case, the instanton contributions to the prepotential $\mathcal{F}^{(k)}(a; f)$ of eq. (5.18) are well defined, but of course the corresponding contributions to the effective action $S^{(k)}[a; f]$ diverge because of the (super)volume integral $\int d^4x\, d^4\theta$. However, in presence of the complete deformations (3.25) (i.e. when also $\bar{\varepsilon}$ is present), the superspace integral is regularized by a gaussian term, and thus it becomes possible to work at the level of the effective action, that is at the level of the *full* instanton partition function (as opposed to the centered one). To be concrete, let us consider the simplest case $k = 1$. The moduli action (3.23) (with $\bar{a} = 0$) reduces simply to

$$S^{(k=1)}_{\text{moduli}} = -2\bar{\varepsilon}\varepsilon\, x^2 - \frac{\bar{\varepsilon}}{2}\,\theta^{\alpha A}\epsilon_{AB}(\tau_3)_{\alpha\beta}\theta^{\beta B} + \widehat{S}^{(k=1)}_{\text{moduli}}\ , \tag{5.26}$$

where the last term, containing only the centered moduli $\widehat{\mathcal{M}}^{(k=1)} = \{w, \mu,\, \chi, D_c, \lambda\}$, is

$$\widehat{S}^{(k=1)}_{\text{moduli}} = -2\bar{w}_{\dot\alpha} w^{\dot\alpha}\chi^\dagger\,(\chi + a) + \mathrm{i}\frac{\sqrt{2}}{2}\chi^\dagger\mu^A\epsilon_{AB}\mu^B - \mathrm{i}D_c W^c - \mathrm{i}\lambda^{\dot\alpha}_A\left(w_{\dot\alpha}\bar\mu^A + \mu^A\bar{w}_{\dot\alpha}\right) \tag{5.27}$$

and does not depend on the deformation parameters ε and $\bar{\varepsilon}$ [14]. The corresponding partition function is therefore given by

$$Z^{(k=1)}(a, \varepsilon) = \int d^4x\, d^4\theta\ \mathrm{e}^{-2\bar{\varepsilon}\varepsilon x^2 - \frac{1}{2}\,\bar{\varepsilon}\,\theta\cdot\theta}\,\mathcal{F}^{(k=1)}(a) = \frac{1}{\varepsilon^2}\,\mathcal{F}^{(k=1)}(a)\,. \tag{5.28}$$

In the last step we have trivially performed the gaussian integration over x and θ and produced a factor of $1/\varepsilon^2$, since the centered partition function

$$\mathcal{F}^{(k=1)}(a) = \int d\widehat{\mathcal{M}}^{(k=1)}\mathrm{e}^{-\frac{8\pi^2}{g^2} - \hat{S}^{(k=1)}_{\text{moduli}}} \tag{5.29}$$

is $\varepsilon, \bar{\varepsilon}$-independent. Effectively, in the presence of the full deformation we have the rule

$$\int d^4x\, d^4\theta \to \frac{1}{\varepsilon^2}\,. \tag{5.30}$$

It is interesting to remark that the same effective rule appears also in other contexts related to topological string amplitudes, like in their relation to black hole free energy recently proposed by Ooguri-Strominger-Vafa (see in particular section 3.2 of ref. [34]).

[14] Notice that all the D(–1)/D(–1) moduli, which in general are $k \times k$ matrices, reduce just to numbers for $k = 1$, and that a'_μ and $M^{\alpha A}$ contain only their components along the identity, namely the center coordinate x_μ, and its super-partner $\theta^{\alpha A}$ (cf. eq. (2.10)).

Turning on the deformations, we can therefore compute the full partition function $Z(a;\varepsilon)$ by integrating over all the moduli and convert the would-be (super)-volume divergences into ε-singularities. In this respect, a further important point has to be taken into account. The combined effect of the scalar v.e.v.'s a_u and of the $\varepsilon, \bar{\varepsilon}$-deformations localizes completely the integral over the moduli space, in the sense that only point-like solutions contribute. Thus, a trivial superposition of two instantons of charge k_1 and k_2 contributes to the sector of charge $k_1 + k_2$. This cluster decomposition implies [10, 16] that the localized integral over the deformed instanton moduli space of a fixed charge k contains both *connected* and *disconnected* contributions. For $\bar{\varepsilon} = 0$ the disconnected configurations, consisting of separated instantons of charges k_i such that $\sum_i k_i = k$, do not contribute to the sector of charge k, but instead they do in the fully localized case when $\bar{\varepsilon} \neq 0$. Therefore, the partition function computed via the localization techniques corresponds to the exponential of the non-perturbative prepotential, namely

$$\begin{aligned} Z(a;\varepsilon) &= \exp\left(\frac{\mathcal{F}_{\text{n.p.}}(a;\varepsilon)}{\varepsilon^2}\right) = \exp\left(\sum_{k=1}^{\infty} \frac{\mathcal{F}^{(k)}(a;\varepsilon)}{\varepsilon^2}\right) \\ &= \exp\left(\sum_{h=0}^{\infty}\sum_{k=1}^{\infty} c_{k,h} \frac{\varepsilon^{2h-2}}{a^{2h}} \left(\frac{\Lambda}{a}\right)^{4k}\right) \end{aligned} \tag{5.31}$$

where $\mathcal{F}_{\text{n.p.}}(a;\varepsilon)$ and $\mathcal{F}^{(k)}(a;\varepsilon)$ are the expressions given in eqs. (5.22), (5.18) and (5.20) evaluated for constant values of the scalar and Weyl multiplets. Notice that the factor of $1/\varepsilon^2$ in the exponent of eq. (5.31) effectively represents the (super)-volume integral according to eq. (5.30), while the disconnected contributions now cancel.

The relation (5.31) allows to check successfully [20] the expression of the coefficients $c_{k,h}$ obtained from the multi-instanton deformed calculus against the results from topological string amplitudes, as originally conjectured in ref. [10].

6. Conclusions

Realizing supersymmetric gauge theories by means of fractional D3 branes allows to compute instanton effects by considering the inclusion of D(–1) branes and offers a natural way to study the effect of turning on closed string "gravitational" backgrounds. In this paper we have considered, in particular, the effect of including a self-dual graviphoton field-strength coming from the R-R closed string sector in a $\mathcal{N} = 2$ gauge theory.

We have shown that a constant graviphoton field-strength proportional to ε exactly produces those modifications of the instanton sectors which have been advocated in the literature to fully localize the integration over the moduli. This localization allows to perform explicitly calculation of the instanton partition functions $Z_k(a, \varepsilon)$, where a is the scalar v.e.v., for arbitrary value of the topological charge k. Moreover, we have shown that extending the computation to a dynamical graviphoton determines a prepotential for the resulting $\mathcal{N} = 2$ low-energy effective theory which includes gravitational F-terms.

These F-terms can be alternatively computed in a different setting, where the low-energy $\mathcal{N} = 2$ effective action is engineered by considering closed strings on a suitable CY

manifold; in this case such couplings are encoded in topological string amplitudes on the same manifold. The two different roads to determine these F-couplings must lead to the same result. This is a very natural way to state the conjecture by N. Nekrasov [10] that the coefficients arising in the ε-expansion of multi-instanton partition functions match those appearing in higher genus topological string amplitudes on CY manifolds. It would be very nice[15] to be able to follow the fate of the constant RR background that we turn on in the fractional brane set-up through a series of geometrical operations (including the blow-up of the orbifold) and string dualities connecting this set-up to the local CY set-up. This does not seem to be a completely trivial task and this point deserves further investigation.

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A. Useful formulae

$\mathbb{Z}_2$ **orbifold:** our notation and conventions are as follows: we label the four longitudinal directions of the D3 branes with indices $\mu, \nu, ... = 1, 2, 3, 4$, and the six transverse directions with indices $a, b, ... = 5, ..., 10$. On the complexified internal string coordinates

$$\begin{aligned} Z &\equiv \frac{X^5 + \mathrm{i}X^6}{\sqrt{2}} \quad , \quad Z^1 \equiv \frac{X^7 + \mathrm{i}X^8}{\sqrt{2}} \quad , \quad Z^2 \equiv \frac{X^9 + \mathrm{i}X^{10}}{\sqrt{2}} \ , \\ \Psi &\equiv \frac{\psi^5 + \mathrm{i}\psi^6}{\sqrt{2}} \quad , \quad \Psi^1 \equiv \frac{\psi^7 + \mathrm{i}\psi^8}{\sqrt{2}} \quad , \quad \Psi^2 \equiv \frac{\psi^9 + \mathrm{i}\psi^{10}}{\sqrt{2}} \ , \end{aligned} \tag{A.1}$$

the $\mathbb{Z}_2$ orbifold generator h acts as

$$h \ : \ \begin{cases} (Z, Z^1, Z^2) \ \to \ (Z, -Z^1, -Z^2) \\ (\Psi, \Psi^1, \Psi^2) \ \to \ (\Psi, -\Psi^1, -\Psi^2) \end{cases} . \tag{A.2}$$

Under the $\mathrm{SO}(10) \to \mathrm{SO}(4) \times \mathrm{SO}(6)$ decomposition induced by the presence of the D3 branes, the ten-dimensional (anti-chiral) spin fields $S^{\dot{\mathcal{A}}}$ ($\dot{\mathcal{A}} = 1, \ldots, 16$) of the RNS formalism become products of four- and six-dimensional spin fields according to

$$S^{\dot{\mathcal{A}}} \ \to \ (S_\alpha S_{A'} \, , \, S^{\dot{\alpha}} S^{A'}) \tag{A.3}$$

where the index α (or $\dot{\alpha}$) denotes positive (or negative) chirality in four dimensions, and the upper (or lower) index A' labels the chiral (or anti-chiral) spinor representation of SO(6).

[15] We thank N. Nekrasov and M. Vonk for having pointed out to us this issue.

Under the further SO(6) $\to$ SO(2) $\times$ SO(4) breaking induced by the orbifold projection, a chiral SO(6) spinor $S^{A'}$ splits into $\left(+\frac{1}{2};(\mathbf{2},\mathbf{1})\right)+\left(-\frac{1}{2};(\mathbf{1},\mathbf{2})\right)$, labeled respectively by an upper index $A=1,2$ and $\hat{A}=3,4$, while an anti-chiral SO(6) spinor $S_{A'}$ splits into $\left(+\frac{1}{2};(\mathbf{1},\mathbf{2})\right)+\left(-\frac{1}{2};(\mathbf{2},\mathbf{1})\right)$, labeled respectively by a lower index $A=1,2$ and $\hat{A}=3,4$.

On the internal spinor indices the orbifold generator acts as a SO(4) chirality operator [16] as follows

$S_{A'}$	$S^{A'}$	h
$S_1 = S_{---}$	$S^1 = S^{+++}$	$+1$
$S_2 = S_{-++}$	$S^2 = S^{+--}$	$+1$
$S_3 = S_{+-+}$	$S^3 = S^{-+-}$	-1
$S_4 = S_{++-}$	$S^4 = S^{--+}$	-1

(A.4)

Thus, the spinors belonging to the $\left(\pm\frac{1}{2};(\mathbf{2},\mathbf{1})\right)$ representation are even under the orbifold projection while the ones transforming in the $\left(\pm\frac{1}{2};(\mathbf{1},\mathbf{2})\right)$ representation are odd.

d = 4 Clifford algebra: the matrices $(\sigma^\mu)_{\alpha\dot\beta}$ and $(\bar\sigma^\mu)^{\dot\alpha\beta}$ which generate the Clifford algebra in four dimensions are defined as

$$\sigma^\mu = (\mathbb{1}, -\mathrm{i}\vec\tau)\,, \qquad \bar\sigma^\mu = \sigma^\dagger_\mu = (\mathbb{1}, \mathrm{i}\vec\tau)\ , \tag{A.5}$$

where τ^c are the ordinary Pauli matrices.

Out of these matrices, the SO(4) generators are defined by

$$\sigma_{\mu\nu} = \frac{1}{2}(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu)\,, \qquad \bar\sigma_{\mu\nu} = \frac{1}{2}(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu)\ ; \tag{A.6}$$

the matrices $\sigma_{\mu\nu}$ are self-dual and thus generate the $\mathrm{SU(2)_L}$ factor of SO(4); the anti-self-dual matrices $\bar\sigma_{\mu\nu}$ generate instead the $\mathrm{SU(2)_R}$ factor. The explicit mapping of a self-dual SO(4) tensor into the adjoint representation of the $\mathrm{SU(2)_L}$ factor is realized by the 't Hooft symbols $\eta^c_{\mu\nu}$; the analogous mapping of an anti-self dual tensor into the adjoint of the $\mathrm{SU(2)_R}$ subgroup is realized by $\bar\eta^c_{\mu\nu}$. One has

$$(\sigma_{\mu\nu})_\alpha{}^\beta = \mathrm{i}\,\eta^c_{\mu\nu}\,(\tau^c)_\alpha{}^\beta, \qquad (\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \mathrm{i}\,\bar\eta^c_{\mu\nu}\,(\tau^c)^{\dot\alpha}{}_{\dot\beta}\,. \tag{A.7}$$

String field correlators: the non-trivial OPE's of the world-sheet fields with space-time indices that are used in the main text are

$$\begin{aligned} S^{\dot\alpha}(z)\,S_\beta(w) &\sim \frac{1}{\sqrt{2}}\,(\bar\sigma^\mu)^{\dot\alpha}{}_\beta\,\psi_\mu(w) \quad , \quad S^{\dot\alpha}(z)\,S^{\dot\beta}(w) \sim -\frac{\epsilon^{\dot\alpha\,\dot\beta}}{(z-w)^{1/2}} \\ S_\alpha(z)\,S_\beta(w) &\sim \frac{\epsilon_{\alpha\beta}}{(z-w)^{1/2}} \quad , \quad \psi^\mu(z)\,S^{\dot\alpha}(w) \sim \frac{1}{\sqrt{2}}\,\frac{(\bar\sigma^\mu)^{\dot\alpha\beta}\,S_\beta(w)}{(z-w)^{1/2}} \end{aligned} \tag{A.8}$$

Other relations can be obtained from eq. (A.8) by suitable changes of chirality.

[16] The indices $\pm$ appearing in the table (A.4) denote the charge $\pm 1/2$ carried by the spin field under the three bosons that bosonize the three world-sheet spinors Ψ, Ψ^1 and Ψ^2.

The relevant OPE's between fields with internal indices are instead

$$
\begin{aligned}
&S^A(z)\,S_B(w) \sim \frac{\mathrm{i}\,\delta^A_{\ B}}{(z-w)^{3/4}}\ , && S^A(z)\,S^B(w) \sim -\frac{\mathrm{i}\,\epsilon^{AB}\Psi(w)}{(z-w)^{1/4}}\ ,\\
&S^{\hat{A}}(z)\,S^{\hat{B}}(w) \sim -\frac{\mathrm{i}\,\epsilon^{\hat{A}\hat{B}}\bar{\Psi}(w)}{(z-w)^{1/4}}\ , && S_A(z)\,S_B(w) \sim \frac{\mathrm{i}\,\epsilon_{AB}\bar{\Psi}(w)}{(z-w)^{1/4}}\ ,\\
&S_{\hat{A}}(z)\,S_{\hat{B}}(w) \sim \frac{\mathrm{i}\,\epsilon_{\hat{A}\hat{B}}\Psi(w)}{(z-w)^{1/4}}\ , && \Psi(z)\,S_A(w) \sim \frac{\epsilon_{AB}S^B(w)}{(z-w)^{1/2}}\ ,\\
&\bar{\Psi}(z)\,S_{\hat{A}}(w) \sim \frac{\epsilon_{\hat{A}\hat{B}}S^{\hat{B}}(w)}{(z-w)^{1/2}}\ , && \Psi(z)\,S^{\hat{A}}(w) \sim \frac{\epsilon_{\hat{A}\hat{B}}S^{\hat{B}}(w)}{(z-w)^{1/2}}\ ,\\
&\bar{\Psi}(z)\,S^A(w) \sim \frac{\epsilon_{AB}S^B(w)}{(z-w)^{1/2}}\ , && \Psi(z)\,\bar{\Psi}(w) \sim \frac{1}{z-w}\,.
\end{aligned}
\tag{A.9}
$$

From these OPE's we can derive the following 3- and 4-point correlators which are needed for the calculation of the scattering amplitudes presented in the main text

$$
\begin{aligned}
\big\langle \Psi(z_1)\,S_A(z_2)\,S_B(z_3)\big\rangle &= \frac{\mathrm{i}\,\epsilon_{AB}}{(z_1-z_2)^{1/2}(z_1-z_3)^{-1/2}(z_2-z_3)^{1/4}}\ ,\\
\big\langle \bar{\Psi}(z_1)\,S_{\hat{A}}(z_2)\,S_{\hat{B}}(z_3)\big\rangle &= \frac{\mathrm{i}\,\epsilon_{\hat{A}\hat{B}}}{(z_1-z_2)^{1/2}(z_1-z_3)^{1/2}(z_2-z_3)^{1/4}}\ ,
\end{aligned}
\tag{A.10}
$$

and

$$
\big\langle \psi^\mu(z_1)\,\psi^\nu(z_2)\,S_\alpha(w)\,S_\beta(\bar{w})\big\rangle = A\,\delta^{\mu\nu}\,\epsilon_{\alpha\beta} + B(\sigma^{\mu\nu})_{\alpha\beta}\ , \tag{A.11}
$$

where

$$
A = \frac{1}{2}\,\frac{(z_1-w)(z_2-\bar{w})+(z_2-w)(z_1-\bar{w})}{(z_1-z_2)\big[(z_1-w)(z_1-\bar{w})(z_2-w)(z_2-\bar{w})(w-\bar{w})\big]^{1/2}} \tag{A.12}
$$

and

$$
B = -\frac{1}{2}\,\frac{(w-\bar{w})^{1/2}}{\big[(z_1-w)(z_1-\bar{w})(z_2-w)(z_2-\bar{w})\big]^{1/2}}\,. \tag{A.13}
$$

Bosonic twist fields: for the open strings that stretch between a D3 and a D(–1) brane, the string fields X^μ along the D3 brane world-volume have mixed Neumann-Dirichlet boundary conditions, which can be seen as due to twist and anti-twist fields $\Delta(z)$ and $\bar{\Delta}(z)$. These fields change the boundary conditions from Neumann to Dirichlet and vice-versa by introducing a cut in the world-sheet (see for example ref. [24]). The twist fields $\Delta(z)$ and $\bar{\Delta}(z)$ are bosonic operators with conformal dimension 1/4 and their OPE's are

$$
\Delta(z_1)\,\bar{\Delta}(z_2) \sim (z_1-z_2)^{1/2}\quad ,\quad \bar{\Delta}(z_1)\,\Delta(z_2) \sim -\,(z_1-z_2)^{1/2}\ , \tag{A.14}
$$

where the minus sign in the second correlator is an "effective" rule to correctly account for the space-time statistics in correlation functions.

Superghost correlators:

$$
\big\langle \mathrm{e}^{-\varphi(z_1)}\,\mathrm{e}^{-\frac{1}{2}\varphi(z_2)}\,\mathrm{e}^{-\frac{1}{2}\varphi(z_3)}\big\rangle = (z_1-z_2)^{-1/2}\,(z_1-z_3)^{-1/2}\,(z_2-z_3)^{-1/4}\ , \tag{A.15}
$$

$$
\begin{aligned}
&\big\langle \mathrm{e}^{-\frac{1}{2}\varphi(z_1)}\,\mathrm{e}^{-\frac{1}{2}\varphi(z_2)}\,\mathrm{e}^{-\frac{1}{2}\varphi(z_3)}\,\mathrm{e}^{-\frac{1}{2}\varphi(z_4)}\big\rangle\\
&\quad = \Big[(z_1-z_2)\,(z_1-z_3)\,(z_1-z_4)\,(z_2-z_3)\,(z_2-z_4)\,(z_3-z_4)\Big]^{-1/4}\,.
\end{aligned}
\tag{A.16}
$$

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