The exact 8d chiral ring from 4d recursion relations

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Abstract: We consider the local F-theory set-up corresponding to four D7 branes in type I′ theory, in which the exact axio-dilaton background $\tau(z)$ is identified with the low-energy effective coupling of the four-dimensional $\mathcal{N} = 2$ super Yang-Mills theory with gauge group SU(2) and $N_f = 4$ flavours living on a probe D3 brane placed at position z. Recently, an intriguing relation has been found between the correlators forming the chiral ring of the eight-dimensional theory on the D7 branes and the large-z expansion of the τ profile. Here we apply to the SU(2) $N_f = 4$ theory some recursion techniques that allow to derive the coefficients of the large-z expansion of τ in terms of modular functions of the UV coupling τ_0. In this way we obtain exact expressions for the elements of the eight-dimensional chiral ring that resum their instanton expansions, previously known only up to the first few orders by means of localization techniques.

Keywords: D-branes, F-Theory, Supersymmetric Effective Theories

Contents

1   Introduction

F-theory is a very interesting framework for building string models that may be potentially relevant for phenomenology (for reviews see, for instance, refs. [1, 2]). It also represents an intriguing arena from the more formal point of view, as it is supposed to incorporate the non-perturbative corrections of type IIB string compactifications by geometrizing them in a very non-trivial way.

By considering local type I$'$ models containing D7 branes and an O7 plane, a remarkable relation has been recently pointed out [3–5] between the profile of the axio-dilaton field τ in F-theory and certain correlators in the eight-dimensional gauge theory living on the D7 branes that provide its microscopic description. This relation reads

$$\tau(a) = \tau_0 + \frac{1}{2\pi\mathrm{i}} \Big\langle \log \det \Big(1 - \frac{m}{a} \Big) \Big\rangle = \tau_0 - \frac{1}{2\pi\mathrm{i}} \sum_{\ell=1}^{\infty} \frac{1}{2\ell} \frac{\langle \operatorname{Tr} m^{2\ell} \rangle}{a^{2\ell}} \,. \tag{1.1}$$

Here $a = z/(2\pi\alpha')$, where z is the complex coordinate transverse to the D7 world-volume. Moreover, m is a complex scalar field belonging to the $8d$ chiral supermultiplet that contains the massless degrees of freedom of the open strings attached to the D7 branes. This multiplet transforms in the adjoint representation of SO($2N_f$) if there are N_f D7 branes, so it is an antisymmetric matrix and the traces of odd powers in the expansion of the logarithm above vanish. The vacuum expectation values in eq. (1.1) are taken with respect

to the D7 brane world-volume theory. Finally, we remark that τ depends on the vacuum expectation values m_i of the adjoint field m:

$$\langle\, m\,\rangle = \frac{1}{\sqrt{2}}\,\mathrm{diag}\,(m_1, m_2, \ldots, -m_1, -m_2, \ldots)\,. \tag{1.2}$$

The parameters $m_i = z_i/(2\pi\alpha')$ correspond to the locations z_i of the D7 branes when they are displaced from the orientifold plane.

In this paper we will focus on the case in which there are $N_f = 4$ D7 branes, supporting an SO(8) gauge theory in eight dimensions. This is the local limit of F-theory considered long ago by Sen in ref. [6], where he proposed that the exact profile of τ is given by the effective low-energy coupling of the $\mathcal{N} = 2$ super Yang-Mills theory in four dimensions with gauge group SU(2), $N_f = 4$ fundamental flavours and τ_0 as UV coupling. This non-trivial relation can be understood [7] by considering a D3 brane probe of the geometry created by the D7 branes and the O7 plane: indeed, the D3 branes supports an Sp(1) $\sim$ SU(2) gauge theory with four flavours (plus a decoupled hypermultiplet in the antisymmetric representation), and its gauge kinetic term couples to the axio-dilaton field. From this perspective, the parameters m_i are the flavour masses. When the complex scalar field ϕ that belongs to the gauge multiplet on the D3 takes a vacuum expectation value

$$\langle\, \phi\,\rangle = (a, -a)\,, \tag{1.3}$$

i.e. when the D3 brane and its orientifold image are placed at $z = \pm 2\pi\alpha' a$, the exact axio-dilaton $\tau(a)$ represents the effective coupling of this gauge theory. As such, it is encoded in the corresponding Seiberg-Witten (SW) curve [8, 9] of which it describes the complex structure parameter.

Eq. (1.1) was put forward in ref. [3] based on the computation of the first few D-instanton corrections to the D3 coupling in the D3/D7 system as a series in the non-perturbative parameter $q = \mathrm{e}^{\mathrm{i}\pi\tau_0}$. These corrections were found to match the first few terms in the q-expansion of the $8d$ chiral ring elements obtained in ref. [10] via localization techniques. In [4] the same relation has been understood entirely from the D7 brane point of view by showing how the D-instantons that correct the chiral ring correlators also modify the source terms for τ, hence its profile. Adopting this point of view, in ref. [5] the relation has been proven at all instanton orders and extended to any number of D7 branes in presence of an O7 plane, both in a flat background and in a $\mathbb{R}^4/\mathbb{Z}_2$ orbifold.

The exact expression of τ encoded in the SW curve *implicitly* contains, via eq. (1.1), all information about the $8d$ chiral ring correlators. However, to extract the exact expression of a given correlator, we must be able to single out a specific term in the $1/a$ expansion of τ. This is not trivial, but it can be done systematically by using recursive techniques, akin to the Matone relation [11]. Here we will rediscuss this type of recursions, gathering an understanding that allows us to apply them also to the $N_f = 4$ case, where the structure of the SW curve is complicated by the presence of several different invariants constructed with the flavour masses. In this way we are able to obtain exact expressions for the $8d$ chiral ring elements that resum their instanton expansions, previously known only up to the first few orders. We find this a remarkable by-product of the deep relation between the

$4d$ effective physics on the D3 brane and the $8d$ theory on the "flavour" D7 branes encoded in eq. (1.1). Let us note that for conformal theories there exists also another recursive approach, based on the modular anomaly equation [12, 13], which allows to partially fix the coefficients of the large-a expansion of the effective coupling τ; this approach, which has been used in ref. [14] for the so-called $\mathcal{N} = 2^*$ theory (also known as mass deformed $\mathcal{N} = 4$ theory) where there is a single mass invariant, could also be applied to the $N_f = 4$ model, but it is less efficient than the one we are going to discuss.

The structure of this paper is as follows: in section 2 we discuss in general the recursion relations for rank one $\mathcal{N} = 2$ theories, and describe the procedure to follow when the SW curve is not in a factorized form. In section 3 we discuss the SU(2) $N_f = 4$ theory and show how to obtain from its SW curve a recursion relation that yields the various correlators of the "flavour" theory, whose properties are presented in section 4 together with comments and concluding remarks. Finally, in the appendices we give some more technical details and discuss the recursion relation in the $\mathcal{N} = 2^*$ model seen as a particular case of the $N_f = 4$ theory.

2 Recursion relations for rank one $\mathcal{N} = 2$ theories

The SW curve for $\mathcal{N} = 2$ super Yang-Mills theories with gauge group SU(2) is a torus and can be thus described as an algebraic surface via a cubic equation of the form

$$y^2 = \big(x - \mathcal{E}_1(z)\big)\big(x - \mathcal{E}_2(z)\big)\big(x - \mathcal{E}_3(z)\big)\,, \tag{2.1}$$

where the precise expression for the roots $\mathcal{E}_\ell$ depends on the gauge-invariant coordinate on the Coulomb moduli space

$$u = \langle\, \mathrm{Tr}\, \phi^2 \,\rangle\,, \tag{2.2}$$

on the masses (if there is matter) and on the dynamically generated scale Λ (or the bare UV coupling τ_0 in the conformal cases). All these dependencies are here summarized by the variable z. The complex structure parameter τ of the torus describes the complexified IR coupling of the gauge theory according to

$$\tau = \frac{\theta_{\mathrm{eff}}}{\pi} + \frac{8\pi\mathrm{i}}{g_{\mathrm{eff}}^2} \tag{2.3}$$

where θ_{eff} and g_{eff} are the θ-angle and the Yang-Mills coupling constant at low-energy, and is related to the anharmonic ratio κ of the roots as follows

$$\kappa = \frac{\mathcal{E}_3(z) - \mathcal{E}_2(z)}{\mathcal{E}_1(z) - \mathcal{E}_2(z)} = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\,, \tag{2.4}$$

where the θ's are the Jacobi θ-functions (see appendix A for our conventions). In the semiclassical regime, i.e. when u is large, we have

$$u \sim \mathrm{Tr}\, \langle\phi\rangle^2 = 2a^2\,, \tag{2.5}$$

where the second equality follows from eq. (1.3). At a generic point u on the moduli space we have a symplectic section $(a(u), a_D(u))$ given by the periods of the SW differential, such that

$$\frac{\partial a}{\partial u} = \omega_1 \,, \qquad \frac{\partial a_D}{\partial u} = \omega_2 \,, \tag{2.6}$$

where ω_1 and ω_2 are the periods of the torus with

$$\tau = \frac{\omega_2}{\omega_1} = \frac{\partial a_D}{\partial a} \,. \tag{2.7}$$

The low-energy physics can be described by an effective theory for an abelian multiplet with lowest component a and a prepotential $\mathcal{F}(a)$ such that

$$a_D = \frac{1}{2\pi\mathrm{i}} \frac{\partial \mathcal{F}(a)}{\partial a} \,, \tag{2.8}$$

which implies that

$$\tau = \frac{1}{2\pi\mathrm{i}} \frac{\partial^2 \mathcal{F}(a)}{\partial a^2} \,. \tag{2.9}$$

The SW curve (2.1) encodes the exact expressions for the physical quantities in the corresponding gauge theory, including its effective coupling τ. These quantities admit a large-a expansion which exhibits, beside the tree-level and perturbative terms, also non-perturbative contributions from all instanton sectors which can be computed directly using localization techniques in the multi-instanton calculus [15]. To extract the instanton expansion from the SW curve one has to determine $\tau(a)$. This can be done by first obtaining the expression of $\tau(u)$ by inverting eq. (2.4), and then by determining u as a function of a by inverting the first relation in eq. (2.6). This procedure is straightforward but may become rather cumbersome in practice.

A more efficient way to proceed is to write the prepotential $\mathcal{F}(a)$, and hence $\tau(a)$, as an expansion for large a with unknown coefficients and to obtain a recursion relation for the latter. This can be done by expanding the right hand side of eq. (2.4) around a specific value of τ corresponding to the semiclassical limit, and the left hand side of eq. (2.4) around particular values of the roots $\mathcal{E}_\ell$ of the SW curve that correspond to this limit. This is basically the idea behind the recursion relation originally devised in refs. [11, 16] for the pure SU(2) theory. In this case the SW curve is [8]

$$y^2 = \big(x - \hat{u}\big)\big(x - \hat{\Lambda}^2\big)\big(x + \hat{\Lambda}^2\big) \,, \tag{2.10}$$

so that the anharmonic ratio of the roots defined in eq. (2.4) is

$$\hat{\kappa} = \frac{-2\hat{\Lambda}^2}{\hat{u} - \hat{\Lambda}^2} \,. \tag{2.11}$$

In these expressions we have introduced a "hat" sign to take into account the fact that the complex structure of the SW curve (2.10) turns out to be related to the gauge theory parameters by [8]

$$\hat{\tau} = \frac{\theta_{\mathrm{eff}}}{2\pi} + \frac{4\pi\mathrm{i}}{g_{\mathrm{eff}}^2} \,, \tag{2.12}$$

as opposed to eq. (2.3); moreover, the parameter $\hat{u}$ is related to the vacuum expectation value a in the semi-classical regime by $\hat{u} \sim a^2/2$, to be contrasted with eq. (2.5).[1]

In ref. [11] it was shown that that the quantity $\hat{u}/\hat{\Lambda}^2$ satisfies a differential equation whose solution is

$$\frac{\hat{u}}{\hat{\Lambda}^2} = 1 - 2\,\frac{\theta_3^4(\hat{\tau})}{\theta_2^4(\hat{\tau})}\,; \tag{2.13}$$

from this result the identification $\hat{\kappa} = \theta_2^4(\hat{\tau})/\theta_3^4(\hat{\tau})$ immediately follows.

One extra ingredient that is needed to proceed is the relation between u (or $\hat{u}$) and a, which generalizes the classical one given in eq. (2.5). Such a relation is provided [11] through the prepotential $\mathcal{F}(a)$ by means of

$$u(a) = 2\Lambda\frac{\partial\mathcal{F}(a)}{\partial\Lambda}\,. \tag{2.14}$$

Inserting this in the left hand side of eq. (2.13) and using eq. (2.9) in the right hand side, one obtains a non-trivial equation for the prepotential from which, by expanding in inverse powers of a, one can derive a recursion relation for the coefficients of this expansion. For dimensional reasons these coefficients correspond to different powers of Λ^4, i.e. to different instantonic sectors,[2] and hence this recursion relation allows to reconstruct the higher instanton contributions starting from the lower ones.

Things are a bit different in conformal theories. In this case the relation (2.14) is replaced by

$$u(a) = 2q\frac{\partial\mathcal{F}(a)}{\partial q} \tag{2.15}$$

where

$$q = \mathrm{e}^{\pi\mathrm{i}\tau_0} \tag{2.16}$$

with τ_0 being the bare UV coupling. Inserting eq. (2.15) in the left hand side of eq. (2.4) and replacing in the right hand side τ via eq. (2.9), one generates a recursion relation for the coefficients of the large-a expansion of the prepotential in which the q dependence is *exact*.

However, not always the SW curve is given or known in the factorized form (2.1) considered so far. For example, for the SU(2) theories with $N_f \leq 4$ massive flavours the SW curves are written as cubic polynomials in a non-factorized form [9] for which it is not easy or practical[3] to find the three roots $\mathcal{E}_\ell$. Thus, in these cases the recursion relation

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[1]To avoid this change of conventions and normalizations, in place of the curve (2.10) one could use for the pure SU(2) theory the isogenic SW curve

$$y^2 = \left(x - u + \sqrt{u^2 - \Lambda^4}\right) x \left(x - u - \sqrt{u^2 - \Lambda^4}\right)$$

with complex structure $\tau = 2\hat{\tau}$ in agreement with eq. (2.3). The anharmonic ratio of the roots would then read

$$\kappa = \frac{u - \sqrt{u^2 - \Lambda^4}}{u + \sqrt{u^2 - \Lambda^4}}\,.$$

[2]Recall that corrections from the sector with instanton number k are weighted by $\Lambda^{b_1 k}$, where b_1 is the 1-loop coefficient of the β-function. For the pure SU(2) theory we have $b_1 = 4$.

[3]Even if, in principle, it is always possible via the Cardano formula.

cannot be directly obtained by applying the above procedure. Nevertheless, a simple generalization exists and a recursion relation can be implemented also in these cases.[4] In fact, by shifting the variable x if needed, it is always possible to put a cubic polynomial in a Weierstraß form:

$$y^2 = x^3 - \frac{G_2(z)}{4} x - \frac{G_3(z)}{4} \,. \tag{2.17}$$

In this description the complex structure τ can be directly related to the coefficients G_2 and G_3 by forming the combination

$$J = \frac{G_2^3(z)}{G_2^3(z) - 27 \, G_3^2(z)} \,, \tag{2.18}$$

and identifying it with the "absolute modular invariant" by writing

$$J = \frac{E_4^3(\tau)}{E_4^3(\tau) - E_6^2(\tau)} \tag{2.19}$$

where E_4 and E_6 are the Eisenstein series of weight 4 and 6, respectively. By equating the right hand sides of eqs. (2.18) and (2.19) we obtain the relation between τ and u, and then we can proceed as described above and establish a recursion relation by exploiting either eq. (2.15) or eq. (2.14) depending on whether or not the theory is conformal.

In the next section we will apply this method to the SU(2) theory with $N_f = 4$ massive flavours whose SW curve is known in a non-factorized form [9], and explicitly derive a large-a expansion of its effective coupling τ in which each coefficient is determined *exactly* as a function of q by means of a recursion relation. As explained in the Introduction, via eq. (1.1) this is tantamount to finding the exact expression of the elements of the $8d$ chiral ring on the "flavour" D7 branes.

3 The SU(2) $N_f = 4$ theory

The SU(2) $N_f = 4$ theory for vanishing masses is conformal and the corresponding SW curve is just a torus of complex structure τ_0, representing the UV coupling [9]. Such a torus can be given a simple description as the locus of a factorized cubic equation in Weierstraß form

$$y^2 = x^3 - \frac{g_2}{4}x - \frac{g_3}{4} = (x - e_1)(x - e_2)(x - e_3) \,, \tag{3.1}$$

where the three roots e_ℓ, satisfying $e_1 + e_2 + e_3 = 0$, are the following functions of τ_0:

$$e_1 = \frac{1}{3}(\theta_3^4 + \theta_4^4) \,, \qquad e_2 = -\frac{1}{3}(\theta_3^4 + \theta_2^4) \,, \qquad e_3 = -\frac{1}{3}(\theta_4^4 - \theta_2^4) \,, \tag{3.2}$$

with θ_a being the Jacobi θ-functions.[5] The coefficients g_2 and g_3 can then be expressed in terms of the Eisenstein series E_i as

$$g_2 = \frac{4}{3} E_4 \,, \qquad g_3 = \frac{8}{27} E_6 \,, \tag{3.3}$$

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[4]The technique we describe here is similar to the one used in the first part of ref. [12] to find the instanton expansion of toroidally compactified non-critical strings.

[5]For brevity, here and in the following, when no modular variable is indicated and no confusion is possible, we always understand that the modular functions are evaluated at τ_0; for example $\theta_a \equiv \theta_a(\tau_0)$. We refer to appendix A for our conventions and useful relations.

so that the absolute modular invariant $J_0 \equiv J(\tau_0)$, in accordance with eqs. (2.18) and (2.19), becomes

$$\frac{1}{J_0} = 1 - 27\,\frac{g_3^2}{g_2^3} = 1 - \frac{E_6^2}{E_4^3}\,. \tag{3.4}$$

Notice that all the functions of τ_0 involved in the above definitions are expressible as power series in q defined in eq. (2.16).

When masses are turned on, the equation of the curve is modified as described in ref. [9]. The result is still a cubic polynomial which can be written as

$$y^2 = W_1 W_2 W_3 + A\big[W_1 T_1(e_2 - e_3) + W_2 T_2(e_3 - e_1) + W_3 T_3(e_1 - e_2)\big] - A^2 N, \tag{3.5}$$

where[6]

$$A = (e_1 - e_2)(e_2 - e_3)(e_3 - e_1) = 16\,\eta^{12}, \tag{3.6}$$

with η being the Dedekind η-function and, for $\ell = 1, 2, 3$,

$$W_\ell = x - e_\ell\,\tilde{u} - e_\ell^2\,R\,, \tag{3.7}$$

with

$$\tilde{u} = u - \frac{e_1}{2}\,R\,. \tag{3.8}$$

Here R, T_ℓ and N are invariants of the flavour group SO(8) that are, respectively, quadratic, quartic and sextic in the masses m_i:

$$
\begin{aligned}
R &= \frac{1}{2}\sum_i m_i^2\,, \\
T_1 &= \frac{1}{12}\sum_{i<j} m_i^2 m_j^2 - \frac{1}{24}\sum_i m_i^4\,, \\
T_2 &= -\frac{1}{24}\sum_{i<j} m_i^2 m_j^2 + \frac{1}{48}\sum_i m_i^4 - \frac{1}{2}\,\mathrm{Pf}m\,, \\
N &= \frac{3}{16}\sum_{i<j<k} m_i^2 m_j^2 m_k^2 - \frac{1}{96}\sum_{i\neq j} m_i^4 m_j^2 + \frac{1}{96}\sum_i m_i^6
\end{aligned}
\tag{3.9}
$$

with $\mathrm{Pf}m = m_1 m_2 m_3 m_4$. The third quartic invariant T_3 is not independent, rather it is defined through the relation $T_1 + T_2 + T_3 = 0$.

When the masses are set to zero, it is straightforward to see that eq. (3.5) reduces to the equation[7] of a torus of complex parameter τ_0. For non-zero masses, by a suitable shift[8]

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[6]Note that A^2 is proportional to the discriminant of the cubic equation and can be written as

$$A^2 = \frac{1}{16}(g_2^3 - 27 g_3^2) = \frac{4}{27}(E_4^3 - E_6^2)\,.$$

[7]With respect to eq. (3.1), the roots are rescaled by u; this does not affect the complex structure and the absolute modular invariant.

[8]Explicitly, $x \to x + \frac{2}{9}E_4 R$.

in x, the curve can still be cast in the Weierstraß form (2.17) with coefficients G_2 and G_3 depending on u, on the flavour invariants, and on q. Their explicit expressions are

$$G_2 = \frac{4}{3} E_4 \tilde{u}^2 + \frac{8}{9} E_6 R \tilde{u} + \frac{4}{27} E_4^2 R^2 + 12 A(e_1 T_2 - e_2 T_1),$$

$$G_3 = \frac{8}{27} E_6 \tilde{u}^3 + \frac{8}{27} E_4^2 R \tilde{u}^2 + \frac{8}{81} E_4 E_6 R^2 \tilde{u} - \frac{8}{729}(E_4^3 - 2E_6^2)R^3 + 4 A^2 N \qquad (3.10)$$

$$+ \frac{4}{3} A E_4 (e_1 T_2 - e_2 T_1) R - \frac{4}{3} A\big(E_4(T_1 - T_2) + 9 e_3(e_1 T_1 - e_2 T_2)\big)\tilde{u}.$$

It is easy to check that in the massless limit we have

$$G_2 \to G_2^{(0)} = g_2 u^2, \qquad G_3 \to G_3^{(0)} = g_3 u^3 \qquad (3.11)$$

with g_2 and g_3 given in eq. (3.3).

The modular invariant J can then be explicitly determined in terms of u, of the flavour invariants and of q by eq. (2.18), which we rewrite as

$$\frac{1}{J} = 1 - 27 \frac{G_3^2}{G_2^3}. \qquad (3.12)$$

The complex structure τ, namely the exact IR complexified gauge coupling, is in turn implicitly determined by the modular invariant, to which it is related by eq. (2.19), that we rewrite as

$$\frac{1}{J} = 1 - \frac{E_6^2(\tau)}{E_4^3(\tau)}. \qquad (3.13)$$

Comparing these two expressions for J we can establish a relation between u and τ. However, in order to make contact with the standard field theory results, we have to write everything in terms of a. The ingredients that are needed for this purpose, namely the functions $u(a)$ and $\tau(a)$, are provided by the prepotential $\mathcal{F}(a)$ through eqs. (2.15) and (2.9), respectively. At this point, when both eqs. (3.12) and (3.13) give J as a function of a, we can proceed in two distinct ways. On the one hand, we can expand eq. (3.13) in powers of

$$Q = e^{\pi i \tau}, \qquad (3.14)$$

and eq. (3.12) in powers of q, and then compare the two expansions, thus finding order by order a relation between Q and q in which the dependence on a (and the mass invariants) is exact [3]. On the other hand, we can expand both eqs. (3.12) and (3.13) in (inverse) powers of a and, by comparing the two expansions, obtain a recursion relation for their coefficients which allows to determine exactly the full dependence on q.

We now show that using the first approach we can easily reconstruct the 1-loop corrections to the gauge coupling constant and the prepotential of the $N_f = 4$ theory. Later, we will exploit the second approach and study the recursion relation.

3.1 Tree-level and 1-loop terms

Expanding eq. (3.13) for small Q, we obtain

$$\frac{1}{J} = 1728 Q^2 + \mathcal{O}(Q^4); \qquad (3.15)$$

likewise, using eq. (3.10) and then expanding eq. (3.12) for small q, we get

$$\frac{1}{J} = 1728\,q^2 \left(1 - \frac{2R}{u} + \frac{R^2 + 6T_1}{u^2} - \frac{4N + 2RT_1}{u^3} + \frac{(T_1 + 2T_2)^2}{u^4}\right) + \mathcal{O}(q^4)\,. \qquad (3.16)$$

Equating these two expressions and using the classical approximation (2.5) to replace u with $2a^2$, we deduce that

$$\begin{aligned}
Q &= q\,\sqrt{1 - \frac{R}{a^2} + \frac{R^2 + 6T_1}{4a^4} - \frac{4N + 2RT_1}{8a^6} + \frac{(T_1 + 2T_2)^2}{16a^8} + \mathcal{O}(q^2)} \\
&= q\,\sqrt{1 - \frac{\sum_i m_i^2}{a^2} + \frac{\sum_{i<j} m_i^2 m_j^2}{4a^4} - \frac{\sum_{i<j<k} m_i^2 m_j^2 m_k^2}{8a^6} + \frac{(\mathrm{Pf}m)^2}{16a^8} + \mathcal{O}(q^2)}
\end{aligned} \qquad (3.17)$$

where the second line follows from the definitions (3.9) of the mass invariants. The square root in eq. (3.17) represents the complete 1-loop correction to the UV coupling as a function of the mass deformations and of the classical vacuum expectation value a. By taking the logarithm of both sides and using eq. (1.2), after some simple algebra, we can rewrite eq. (3.17) as follows

$$\tau = \tau_0 - \frac{1}{2\pi\mathrm{i}} \sum_{\ell=1}^{\infty} \frac{1}{2\ell} \frac{\mathrm{Tr}\,\langle m\rangle^{2\ell}}{a^{2\ell}} + \mathcal{O}(q)\,, \qquad (3.18)$$

which indeed is the correct expression for the gauge coupling constant of the massive SU(2) $N_f = 4$ theory in the 1-loop approximation. We can also write this result in terms of the prepotential $\mathcal{F}(a)$ in accordance with eq. (2.9). Introducing for later convenience the quantities

$$h_\ell^{(0)} = \frac{2^\ell}{(2\ell+1)(2\ell+2)}\,\mathrm{Tr}\,\langle m\rangle^{2\ell+2}\,, \qquad (3.19)$$

for $\ell \geq 0$, the prepotential that follows from eq. (3.18) is then

$$\mathcal{F}(a) = \pi\mathrm{i}\tau_0\,a^2 + \log\left(\frac{a}{\Lambda}\right)h_0^{(0)} - \sum_{\ell=1}^{\infty} \frac{1}{2\ell\,2^\ell}\frac{h_\ell^{(0)}}{a^{2\ell}} + \mathcal{O}(q)\,, \qquad (3.20)$$

up to possible a-independent terms. One can easily check that this agrees with the perturbative expression obtained with standard field theory methods (see, for example, ref. [17]) up to constant terms which can always be absorbed into a redefinition of the (arbitrary) scale Λ.

To obtain the q-dependent terms in the prepotential, one can go to higher order in the q-expansion, as discussed in ref. [3]. Alternatively, one can expand the modular invariant J given by eqs. (3.12) and (3.13) in inverse powers of a and by comparing the two expansions establish a recursion relation for their coefficients which fixes the complete q-dependence. This is the approach we are going to discuss in the following.

3.2 Initial condition

In order to successfully implement a recursion relation, we need to know, as an initial condition, the exact expression in q of the first sub-leading term of $u(a)$ for large a. In other words, writing

$$u(a) = 2a^2 + 2\lambda(q)\,R + \mathcal{O}(a^{-2})\,, \qquad (3.21)$$

we need to determine the function $\lambda(q)$. To do this, we start by inverting the above relation, obtaining

$$a = \frac{u^{1/2}}{\sqrt{2}} - \frac{\lambda(q)\,R}{\sqrt{2}\,u^{1/2}} + \mathcal{O}(u^{-3/2})\,. \tag{3.22}$$

From this it readily follows that the first period ω_1 of the torus has the expansion

$$\omega_1 = \frac{\partial a}{\partial u} = \frac{1}{2\sqrt{2}\,u^{1/2}} + \frac{\lambda(q)\,R}{2\sqrt{2}\,u^{3/2}} + \mathcal{O}(u^{-5/2})\,. \tag{3.23}$$

On the other hand, given the Weierstraß form (2.17), the period ω_1 can be expressed as [18]

$$\omega_1 = (48\,G_2)^{-1/4}\,F\left(\frac{1}{12}, \frac{5}{12}, 1; \frac{1}{J}\right), \tag{3.24}$$

where $F(a, b, c; z)$ is the hypergeometric function and the normalization has been chosen so that the leading behaviour $1/(2\sqrt{2}\,u^{1/2})$ is correctly reproduced. From this expression we can obtain the term $\delta\omega_1$ linear in R as a perturbation around the massless case; then, by writing it as

$$\delta\omega_1 = \frac{\lambda(q)\,R}{2\sqrt{2}\,u^{3/2}} \tag{3.25}$$

according to eq. (3.23), we can read off $\lambda(q)$. Let us now give some details. From eq. (3.10) we easily find

$$\begin{aligned}
\delta G_2 &\equiv G_2 - G_2^{(0)} = G_2^{(0)}\left(\frac{2E_6}{3E_4} - e_1\right)\frac{R}{u} + \cdots, \\
\delta G_3 &\equiv G_3 - G_3^{(0)} = G_3^{(0)}\left(\frac{E_4^2}{E_6} - \frac{3e_1}{2}\right)\frac{R}{u} + \cdots,
\end{aligned} \tag{3.26}$$

where the dots stand for terms of higher order in the mass deformations which are not relevant for our present purposes. Moreover, from eq. (3.12) we get

$$\delta\left(\frac{1}{J}\right) \equiv \frac{1}{J} - \frac{1}{J_0} = \frac{1 - J_0}{J_0}\left(\frac{2\,\delta G_3}{G_3^{(0)}} - \frac{3\,\delta G_2}{G_2^{(0)}}\right) = \frac{2E_6(E_6^2 - E_4^3)}{E_4^4}\frac{R}{u} + \cdots. \tag{3.27}$$

On the other hand, by varying eq. (3.24) we obtain

$$\delta\omega_1 = \omega_1^{(0)}\left[-\frac{\delta G_2}{4G_2^{(0)}} + \partial_z \log F\left(\frac{1}{12}, \frac{5}{12}, 1; z\right)\bigg|_{z=\frac{1}{J_0}} \delta\left(\frac{1}{J}\right)\right]. \tag{3.28}$$

Taking into account that $F\left(\frac{1}{12}, \frac{5}{12}, 1; \frac{1}{J_0}\right) = E_4^{1/4}$, and using eqs. (3.26) and (3.27), after some algebra involving the properties of the Eisenstein series and the θ-functions collected in appendix A, we get

$$\delta\omega_1 = \frac{R}{2\sqrt{2}\,u^{3/2}}\left(\frac{e_1}{4} - \frac{E_2}{6}\right) + \cdots \tag{3.29}$$

Upon comparison with eq. (3.25), we thus obtain

$$\lambda(q) = \frac{e_1}{4} - \frac{E_2}{6} = \frac{1}{12}(\theta_3^4 + \theta_4^4 - 2E_2) = -q\frac{\partial}{\partial_q} \log(\theta_3\theta_4)\,. \tag{3.30}$$

Inserting this into eq. (3.21) and using the resulting expression for $u(a)$ in eq. (2.15), we can easily derive the leading terms of the semiclassical expansion of the prepotential $\mathcal{F}(a)$, namely

$$\mathcal{F}(a) = \pi \mathrm{i}\tau_0\, a^2 - \log(\theta_3\theta_4)\, R + \mathcal{O}(a^{-2})\,. \tag{3.31}$$

It is interesting to remark that this structure has also been obtained in ref. [19] using the AGT conjecture [20] and the Zamolodchikov formula for the 4-point conformal blocks in the two-dimensional Liouville theory.[9]

3.3 Recursion relation

By comparing the two expressions (3.20) and (3.31), we are immediately led to write the following expansion for the complete prepotential

$$\mathcal{F}(a) = \pi \mathrm{i}\tau_0\, a^2 - \log(\theta_3\theta_4)\, R + \log\left(\frac{a}{\Lambda}\right) h_0 - \sum_{\ell=1}^{\infty} \frac{1}{2\ell\, 2^\ell}\, \frac{h_\ell}{a^{2\ell}}\,. \tag{3.32}$$

The coefficients h_ℓ, to be determined, are the generalizations of those defined in eq. (3.19) when non-perturbative instanton corrections are taken into account. Via eq. (2.9), the expansion (3.32) corresponds to writing the IR coupling τ as

$$\tau - \tau_0 = -\frac{1}{2\pi \mathrm{i}} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2^\ell}\, \frac{h_\ell}{a^{2\ell+2}}\,. \tag{3.33}$$

Comparing this expression with eq. (1.1), we see that the coefficients h_ℓ are related to the elements of the chiral ring of the eight-dimensional SO(8) theory by[10]

$$h_\ell = \frac{2^\ell}{(2\ell+2)(2\ell+1)} \langle\, \mathrm{Tr}\, m^{2\ell+2} \,\rangle\,. \tag{3.34}$$

The absence in the effective prepotential of any dependence on the scale Λ other than that arising at 1-loop implies that

$$h_0 = h_0^{(0)} = R\,, \tag{3.35}$$

or, equivalently, that $\langle\, \mathrm{Tr}\, m^2 \,\rangle = \mathrm{Tr}\, \langle m^2 \rangle$. This relation, which is explicitly confirmed by instanton calculations [10], can be understood also as a simple consequence of the scaling dimensions of the instanton moduli space which do not allow to generate any non-perturbative contribution to $\langle\, \mathrm{Tr}\, m^2 \,\rangle$.

---

[9]We observe that the result reported in eq. (33) of ref. [19] does not respect the SO(8) flavour symmetry due the presence of the structure $(\sum_i m_i)^2$ which is not SO(8) invariant. However, as remarked in ref. [3], this fact can be easily corrected by choosing a different "dressing factor" in the AGT relation. With this choice the resulting SO(8) invariant expression for the prepotential fully agrees with eq. (3.31).

[10]We have written the expansion (3.32) in terms of the coefficient h_ℓ and not of $\langle\, \mathrm{Tr}\, m^{2l+2} \,\rangle$ because the former turn out to be more convenient to exhibit the results in a compact way, thanks to the fact that the "modular anomaly" equation they satisfy (to be discussed in the next section) takes a particularly simple form with this choice.

The next ingredient is obtained from eq. (2.15) which, together with the expansion (3.32) and eq. (3.35), implies that

$$u(a) = 2a^2 + 2\lambda(q)\, h_0 - \sum_{\ell=1}^{\infty} \frac{1}{\ell\, 2^\ell} \frac{q\partial_q h_\ell}{a^{2\ell}} \tag{3.36}$$

with $\lambda(q)$ given in eq. (3.30).

Now, we are ready to establish the desired recursion relation. On the one hand, we expand $1/J$ around τ_0, getting

$$\delta\left(\frac{1}{J}\right) = \frac{1}{J} - \frac{1}{J_0} = \sum_{n=1}^{\infty} \frac{1}{n!} \left.\frac{\partial^n(1/J)}{\partial\tau^n}\right|_{\tau_0} (\tau - \tau_0)^n. \tag{3.37}$$

The difference $(\tau - \tau_0)$ is parametrized as in eq. (3.33), while the derivatives of $1/J$ can be straightforwardly computed from eq. (3.13) in terms of the derivatives of the Eisenstein series given in appendix A, taking into account that eq. (2.16) implies that $\partial_\tau\big|_{\tau_0} = \mathrm{i}\pi\, q\partial_q$. For instance, the first two derivatives are

$$\begin{aligned}
\left.\frac{\partial(1/J)}{\partial\tau}\right|_{\tau_0} &= \frac{\mathrm{i}\pi}{J_0} \frac{2E_6}{E_4}\,, \\
\left.\frac{\partial^2(1/J)}{\partial\tau^2}\right|_{\tau_0} &= \frac{(\mathrm{i}\pi)^2}{J_0} \frac{2(8E_6^2 + E_6 E_4 E_2 - 3E_4^3)}{3E_4^2}\,.
\end{aligned} \tag{3.38}$$

On the other hand, we can compute the difference $\delta(1/J)$ via eq. (3.12), by expanding the coefficients G_2 and G_3 given in eq. (3.10) with respect to the values they assume in the massless case, writing

$$G_2 = g_2\, (2a^2)^2 (1 + \eta_2)\,, \qquad G_3 = g_3\, (2a^2)^3 (1 + \eta_3)\,. \tag{3.39}$$

It is not particularly useful here to spell η_2 and η_3 in detail, but we just remark that they depend on u and on the flavour invariants. With easy manipulations we then find

$$\delta\left(\frac{1}{J}\right) = -\frac{E_6^2}{E_4^3}\left(\frac{(1 + \eta_3)^2}{(1 + \eta_2)^3} - 1\right). \tag{3.40}$$

We can now equate the two different expressions of $\delta\left(\frac{1}{J}\right)$, given in eqs. (3.37) and (3.40), order by order in the large-a expansion that is obtained substituting eqs. (3.33) and (3.36) in the first and in the second one, respectively.

At the first non trivial order, namely $1/a^2$, eq. (3.37) gives, through eq. (3.38),

$$\left.\delta\left(\frac{1}{J}\right)\right|_{\frac{1}{a^2}} = -\frac{1}{J_0} \frac{E_6}{E_4} \frac{h_0}{a^2}\,. \tag{3.41}$$

On the other hand, eq. (3.40) reduces to

$$\left.\delta\left(\frac{1}{J}\right)\right|_{\frac{1}{a^2}} = \frac{E_6(E_6^2 - E_4^3)}{E_4^4} \frac{R}{a^2} = -\frac{1}{J_0} \frac{E_6}{E_4} \frac{R}{a^2}\,; \tag{3.42}$$

this is basically the same computation that leads to eq. (3.27). The two expressions (3.41) and (3.42) are identical, since we have already set $h_0 = R$.

At the next order, $1/a^4$, we start getting non-trivial information. From eq. (3.37) we get

$$\delta\left(\frac{1}{J}\right)\bigg|_{\frac{1}{a^4}} = -\frac{1}{J_0}\left(\frac{3E_6}{2E_4}\,h_1 - \frac{8E_6^2 + E_6E_4E_2 - 3E_4^3}{12E_4^2}\,h_0^2\right)\frac{1}{a^4}\,. \tag{3.43}$$

With a bit of algebra, from eq. (3.40) we get instead

$$\delta\left(\frac{1}{J}\right)\bigg|_{\frac{1}{a^4}} = -\frac{1}{J_0}\left(\frac{3E_4^3 - 8E_6^2 + 3E_6E_4\big(e_1 - 4\lambda(q)\big)}{12E_4^2}\,R^2 - \frac{3E_6}{2E_4}(\theta_4^4 T_1 - \theta_2^4 T_2)\right)\frac{1}{a^4}\,. \tag{3.44}$$

Comparing these two expressions, and taking into account eq. (3.30), we find

$$h_1 = \frac{E_2}{6}\,R^2 - \theta_4^4\,T_1 + \theta_2^4\,T_2\,. \tag{3.45}$$

At the next-to-next order, we are able to determine the coefficient h_2, which appears in the $1/a^6$ term of eq. (3.37), since the corresponding term in the expansion of eq. (3.40) contains only quantities already determined, namely h_0, λ and $q\partial_q h_1$. This pattern, which is easily implemented on a symbolic computation program like *Mathematica*, continues at all orders, and allows us to determine recursively the coefficients h_ℓ. The explicit results up to h_4 are

$$\begin{aligned}
h_2 &= \frac{1}{90}(E_4 + 5E_2^2)\,R^3 + \frac{2}{5}\,E_4\,N - \frac{1}{3}\,\theta_4^4(2E_2 + 2\theta_2^4 + \theta_4^4)\,RT_1 \\
&\quad + \frac{1}{3}\,\theta_2^4(2E_2 - \theta_2^4 - 2\theta_4^4)\,RT_2\,, \tag{3.46} \\
h_3 &= \frac{1}{7560}(11E_6 + 84E_4E_2 + 175E_2^3)\,R^4 + \frac{2}{35}(3E_6 + 7E_4E_2)\,RN \\
&\quad - \frac{1}{12}\,\theta_4^4(3E_4 + 5E_2^2 + 8E_2\theta_2^4 + 4E_2\theta_4^4)\,R^2T_1 \\
&\quad + \frac{1}{12}\,\theta_2^4(3E_4 + 5E_2^2 - 4E_2\theta_2^4 - 8E_2\theta_4^4)\,R^2T_2 \\
&\quad - \frac{1}{14}(4E_6 - 7E_2\theta_4^8 - 14\,\theta_4^{12} - 28\,\theta_2^4\theta_4^8)\,T_1^2 \\
&\quad - \frac{1}{14}(4E_6 - 7E_2\theta_2^8 + 14\,\theta_2^{12} + 28\,\theta_2^8\theta_4^4)\,T_2^2 \\
&\quad - \frac{1}{7}(2E_6 + 7E_2\theta_2^4\theta_4^4 + 7\,\theta_2^8\theta_4^4 - 7\,\theta_2^4\theta_4^8)\,T_1T_2\,, \tag{3.47}
\end{aligned}$$

$$h_4 = \frac{1}{22680}\left(44E_6E_2 + 19E_4^2 + 196E_4E_2^2 + 245E_2^4\right)R^5$$
$$+ \frac{2}{315}\left(36E_6E_2 + 20E_4^2 + 49E_4E_2^2\right)R^2N$$
$$+ \frac{1}{135}\theta_4^4\left(12E_6 - 47E_4E_2 - 35E_2^3 - (35E_2^2 + 30\theta_4^8)(2\theta_2^4 + \theta_4^4)\right)R^3T_1$$
$$- \frac{1}{135}\theta_2^4\left(12E_6 - 47E_4E_2 - 35E_2^3 + (35E_2^2 + 30\theta_2^8)(\theta_2^4 + 2\theta_4^4)\right)R^3T_2$$
$$+ \frac{4}{15}\theta_4^4\left(2E_6 - 2E_4E_2 - 5\theta_4^8(2\theta_2^4 + \theta_4^4)\right)NT_1$$
$$- \frac{4}{15}\theta_2^4\left(2E_6 - 2E_4E_2 + 5\theta_2^8(\theta_2^4 + 2\theta_4^4)\right)NT_2$$
$$- \frac{1}{63}\Big(E_6(24E_2 - 8\theta_4^4) - \theta_4^8(133E_4 + 49E_2^2) - 112E_2\theta_4^8(2\theta_2^4 + \theta_4^4)$$
$$+ 75\theta_4^{16} + 20\theta_4^{12}\theta_2^4 + 4\theta_2^{16}\Big)RT_1^2$$
$$- \frac{1}{63}\Big(E_6(24E_2 + 8\theta_2^4) - \theta_2^8(133E_4 + 49E_2^2) + 112E_2\theta_2^8(\theta_2^4 + 2\theta_4^4)$$
$$+ 75\theta_2^{16} + 20\theta_2^{12}\theta_4^4 + 4\theta_4^{16}\Big)RT_2^2$$
$$+ \frac{2}{189}\Big(76E_6E_2 - 147E_2^2(E_4 - \theta_2^8 - \theta_4^8) + 112E_2(\theta_2^{12} - \theta_4^{12})$$
$$- 6\theta_2^{16} - 33\theta_2^{12}\theta_4^4 + 150\theta_2^8\theta_4^8 - 33\theta_2^4\theta_4^{12} - 6\theta_4^{16}\Big)RT_1T_2 . \tag{3.48}$$

From the modular transformation properties of the Eisenstein series and the θ-functions, we see that the coefficients h_ℓ are almost modular forms of degree 2ℓ; the failure to be exact modular forms is due to the appearance of the Eisenstein series E_2 whose modular transformations are anomalous. Note that the recursive method we have described fixes completely the coefficients h_ℓ, differently from the recursion relation based on the modular anomaly equation [14] which only determines the E_2 dependence. In models with many different structures like the $N_f = 4$ theory, this is a big computational advantage. In the next section we are going to analyze these results and comment on their properties.

4 Discussion of the results and comments

The explicit formulas for the coefficients h_ℓ obtained in the previous section allow us to read, via eq. (3.34), the exact expression for the elements of the SO(8) chiral ring of the eight-dimensional theory on the D7 branes. For example, from eq. (3.45) we have

$$\langle \operatorname{Tr} m^4 \rangle = E_2 R^2 - 6\theta_4^4 T_1 + 6\theta_2^4 T_2 . \tag{4.1}$$

Expanding the modular functions in powers of q, we can obtain the various instanton contributions. Explicitly, we have

$$\langle \operatorname{Tr} m^4 \rangle = (1 - 24q^2 - 72q^4)R^2 - (6 - 48q + 144q^2 - 192q^3 + 144q^4 - 288q^5)T_1$$
$$+ (96q + 384q^3 + 576q^5)T_2 + \mathcal{O}(q^6)$$
$$= \operatorname{Tr}\langle m \rangle^4 - 48\operatorname{Pf}m\, q - 24\sum_{i<j}m_i^2m_j^2\, q^2 - 192\operatorname{Pf}m\, q^3 \tag{4.2}$$
$$- \left(12\sum_i m_i^4 + 48\sum_{i<j}m_i^2m_j^2\right)q^4 - 288\operatorname{Pf}m\, q^5 + \mathcal{O}(q^6),$$

where in the final step we used the definitions (3.9) of the mass invariants. One can check that this result completely agrees with the one obtained via localization techniques in refs. [10, 21] from direct multi-instanton calculations performed in the D7/D(-1) brane system of type I′. Furthermore, the non-perturbative part of the quartic correlator (4.1) matches precisely against the exact results for the BPS-saturated quartic coupling in the dual Heterotic string (see, for instance, the discussion in ref. [21]).

This analysis can be extended to the higher elements of the SO(8) chiral ring, and again we find perfect agreement with all results existing in the literature on this matter. The details for the chiral correlators up to $\langle \operatorname{Tr} m^{10} \rangle$ are given in appendix B, together with their expansions up to the first few instantons.

We observe that by setting $T_1 = T_2 = N = 0$ and retaining only the dependence on the quadratic invariant R, our results reduce to those found in ref. [14] with a different method; indeed with these positions the SU(2) $N_f = 4$ theory reduces to the so-called $\mathcal{N} = 2^*$ model (also known as mass deformed $\mathcal{N} = 4$ SU(2) theory) that was studied in that reference. In appendix C we present an alternative derivation of the recursion relation for the $\mathcal{N} = 2^*$ model and also discuss the decoupling limits to the pure SU(2) theory.

Another interesting remark is that the coefficients h_ℓ given in eqs. (3.45)–(3.48) satisfy

$$\frac{\partial h_\ell}{\partial E_2} = \frac{\ell}{6} \sum_{m=1}^{\ell} h_{m-1}\, h_{\ell-m} \quad \text{for } \ell \geq 1 \,, \tag{4.3}$$

with the initial condition $\partial h_0 / \partial E_2 = 0$. This recursion relation fixes the E_2 dependence of all coefficients h_ℓ and could be used to reconstruct them in analogy to what has been done for the $\mathcal{N} = 2^*$ SU(2) model in ref. [14].

Eq. (4.3) can be given a nice interpretation in terms of the modular anomaly equation [12, 13]. Let us consider the combination $(a_D - \tau_0\, a)$, which, using eqs. (2.8) and (3.32), can be written as

$$a_D - \tau_0\, a = \frac{1}{2\pi \mathrm{i}} \sum_{\ell=0}^{\infty} \frac{1}{2^\ell}\, \frac{h_\ell}{a^{2\ell+1}} \,, \tag{4.4}$$

and study its modular transformation properties. We recall that under an S modular transformation we have

$$\tau_0 \to -\frac{1}{\tau_0} \,, \qquad a_D \to -a \,, \qquad a \to a_D = \tau_0\, a \left(1 + \frac{1}{2\pi \mathrm{i} \tau_0} \sum_{\ell=0}^{\infty} \frac{1}{2^\ell}\, \frac{h_\ell}{a^{2\ell+2}} \right); \tag{4.5}$$

moreover we assume that the coefficients h_ℓ are almost modular forms of weight 2ℓ transforming under S as follows

$$h_\ell \to h_\ell' = \tau_0^{2\ell} (h_\ell + \delta h_\ell) \,, \tag{4.6}$$

and that the "anomalous" term δh_ℓ arises only through the fact that h_ℓ depends on E_2, i.e. it is of the form

$$h_\ell = \alpha_0\, E_2^\ell + \alpha_1\, E_2^{\ell-1} + \ldots + \alpha_{\ell-1}\, E_2 + \alpha_\ell \,, \tag{4.7}$$

with α_ℓ modular forms of weight 2ℓ. Taking into account the modular properties of E_2 (see eq. (A.10)), this implies that

$$\delta h_\ell = \frac{6}{\mathrm{i}\pi\tau_0}\frac{\partial h_\ell}{\partial E_2} + \mathcal{O}(\tau_0^{-2})\,. \tag{4.8}$$

Applying these rules to eq. (4.4), on the one hand we have

$$a_D - \tau_0\,a \to -a + \frac{1}{\tau_0}\,a_D = \frac{1}{2\pi\mathrm{i}\,\tau_0}\sum_{\ell=0}^{\infty}\frac{h_\ell}{2^\ell\,a^{2\ell+1}}\,, \tag{4.9}$$

while on the other hand we have

$$a_D - \tau_0\,a \to \frac{1}{2\pi\mathrm{i}}\sum_{\ell=0}^{\infty}\frac{h_\ell'}{2^\ell(\tau_0\,a)^{2\ell+1}}\left(1 + \frac{1}{2\pi\mathrm{i}\,\tau_0}\sum_{m=0}^{\infty}\frac{h_m}{2^m a^{2m+2}}\right)^{-2\ell-1}. \tag{4.10}$$

Comparing the right hand sides of these equations, and using the form given in eq. (4.8) for h_ℓ', we find

$$\sum_{\ell=0}^{\infty}\frac{6}{2^\ell}\frac{\partial h_\ell}{\partial E_2}\frac{1}{a^{2\ell}} = \sum_{m,n=0}^{\infty}\frac{(2m+1)}{2^{m+n+1}}\frac{h_m h_n}{a^{2(m+n+1)}} \tag{4.11}$$

from which, after a suitable relabeling of the indices, the recursion relation (4.3) and its initial condition easily follow. It is interesting to notice that eq. (4.11) is nothing but the mode expansion of the following partial differential equation

$$\frac{\partial}{\partial E_2}(a_D - \tau_0\,a) = \frac{\pi}{6\mathrm{i}}(a_D - \tau_0\,a)\frac{\partial}{\partial a}(a_D - \tau_0\,a)\,, \tag{4.12}$$

which is a type of inviscid Burgers' equation.

We conclude by observing that the recursive methods we have described in this paper could be generalized in several ways. In particular it would be very intersting to apply them to models with gauge groups of higher rank corresponding to SW curves of higher genus and use them to find further connections with their F-theory interpretation. Another interesting possibility would be to apply these techniques to the gravitational corrections of the 4d Yang-Mills theories and establish a connection with the topological amplitudes at higher genus. Finally, it would be nice to study the relation between the elements of the SO(8) chiral ring we have found and the corresponding amplitudes in the dual Heterotic string, generalizing the connection already established for the first element $\langle\,\mathrm{Tr}\,m^4\,\rangle$ in refs. [10, 21] using the D-instanton interpretation. In particular, it would be interesting to see how the modular anomaly we have found in our expressions for the chiral ring elements might be related to the holomorphic anomaly for the dual Heterotic amplitudes. We hope to return to some of these issues in the near future.

A Useful formulae

Modular functions. All the functions we are going to discuss depend on a modulus τ and admit a Fourier expansion in terms of $q = \exp(\mathrm{i}\pi\tau)$. To keep the formulae short, we do not indicate this dependence explicitly, except when some confusion is possible, and we write θ_a for $\theta_a(0|\tau)$, E_2 for $E_2(\tau)$ and so on.

The Jacobi θ-functions are defined as

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(v|\tau) = \sum_{n \in \mathbb{Z}} q^{(n-\frac{a}{2})^2}\, \mathrm{e}^{2\pi\mathrm{i}\left(n-\frac{a}{2}\right)\left(v-\frac{b}{2}\right)}\,, \tag{A.1}$$

for $a, b = 0, 1$. We simplify the notation by writing, as usual, $\theta_1 \equiv \theta\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\theta_2 \equiv \theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\theta_3 \equiv \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\theta_4 \equiv \theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The functions θ_a, $a = 2, 3, 4$, satisfy the "*aequatio identica satis abstrusa*"

$$\theta_3^4 - \theta_2^4 - \theta_4^4 = 0\,. \tag{A.2}$$

The Dedekind η-function is defined by

$$\eta(q) = q^{\frac{1}{12}} \prod_{n=1}^{\infty}\left(1 - q^{2n}\right)\,. \tag{A.3}$$

The first Eisenstein series can be expressed as follows:

$$
\begin{aligned}
E_2 &= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)\, q^{2n} = 1 - 24q^2 - 72q^4 - 96q^6 + \dots\,, \\
E_4 &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, q^{2n} = 1 + 240q^2 + 2160q^4 + 6720q^6 + \dots\,, \\
E_6 &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, q^{2n} = 1 - 504q^2 - 16632q^4 - 122976q^6 + \dots\,,
\end{aligned}
\tag{A.4}
$$

where $\sigma_k(n)$ is the sum of the k-th power of the divisors of n, i.e., $\sigma_k(n) = \sum_{d|n} d^k$. The series E_4 and E_6 are expressible as polynomials in the θ-functions according to

$$
\begin{aligned}
E_4 &= \frac{1}{2}(\theta_2^8 + \theta_3^8 + \theta_4^8)\,, \\
E_6 &= \frac{1}{2}(\theta_3^4 + \theta_4^4)(\theta_2^4 + \theta_3^4)(\theta_4^4 - \theta_2^4)\,.
\end{aligned}
\tag{A.5}
$$

The series E_2, E_4 and E_6 are connected among themselves by logarithmic q-derivatives and form a sort of a "ring":

$$
\begin{aligned}
q\partial_q E_2 &= \frac{1}{6}(E_2^2 - E_4)\,, \\
q\partial_q E_4 &= \frac{2}{3}(E_4 E_2 - E_6)\,, \\
q\partial_q E_6 &= E_6 E_2 - E_4^2\,.
\end{aligned}
\tag{A.6}
$$

Also the derivatives of the functions θ_a^4 have simple expressions:

$$q\partial_q\theta_2^4 = \frac{\theta_2^4}{3}(E_2 + \theta_3^4 + \theta_4^4)\,,$$
$$q\partial_q\theta_3^4 = \frac{\theta_3^4}{3}(E_2 + \theta_2^4 - \theta_4^4)\,, \tag{A.7}$$
$$q\partial_q\theta_4^4 = \frac{\theta_4^4}{3}(E_2 - \theta_2^4 - \theta_3^4)\,.$$

Modular transformations. Under a $\mathrm{Sl}(2,\mathbb{Z})$ modular transformation

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}\,, \tag{A.8}$$

the Eisenstein series E_4 and E_6 are modular forms of weight 4 and 6, respectively:

$$E_4(\tau') = (c\tau + d)^4\, E_4(\tau)\,, \qquad E_6(\tau') = (c\tau + d)^6\, E_6(\tau)\,. \tag{A.9}$$

The series E_2, instead, is an almost modular form of degree 2:

$$E_2(\tau') = (c\tau + d)^2\, E_2(\tau) - \frac{6\mathrm{i}}{\pi}\, c\,(c\tau + d)\,. \tag{A.10}$$

The behaviour of the relevant θ-functions and the Dedekind function under the generators T and S of the modular group is given by

$$\begin{aligned} T &: & \theta_3^4 &\leftrightarrow \theta_4^4\,, & \theta_2^4 &\to \theta_2^4\,, & \eta &\to \mathrm{e}^{\frac{\mathrm{i}\pi}{12}}\,\eta\,, \\ S &: & \theta_2^4 &\to \tau^2\,\theta_4^4\,, & \theta_3^4 &\to \tau^2\,\theta_3^4\,, & \theta_4^4 &\to \tau^2\,\theta_2^4\,, & \eta &\to \sqrt{-\mathrm{i}\tau}\,\eta\,. \end{aligned} \tag{A.11}$$

B   Chiral ring elements and their instanton expansion

Here we write the exact expressions for the first few elements of the SO(8) chiral ring (beyond $\langle\,\mathrm{Tr}\,m^4\,\rangle$ that we already discussed in section 4), together with their expansions up to the first few instantons. The formulas we are going to write follow from eqs. (3.46)–(3.48) and eq. (3.34)), as well as the definitions (3.9) of the mass invariants. We have

$$\begin{aligned} \langle\,\mathrm{Tr}\,m^6\,\rangle &= \frac{1}{12}(E_4 + 5E_2^2)\,R^3 + 3\,E_4\,N - \frac{5}{2}\,\theta_4^4(2E_2 + 2\theta_2^4 + \theta_4^4)\,RT_1 \\ &\quad + \frac{5}{2}\,\theta_2^4(2E_2 - \theta_2^4 - 2\theta_4^4)\,RT_2 \\ &= \mathrm{Tr}\,\langle\,m\,\rangle^6 + 180 \sum_{i<j<k} m_i^2 m_j^2 m_k^2\, q^2 + 960\,\mathrm{Pf}m \sum_i m_i^2\, q^3 \\ &\quad + \left(180 \sum_{i\neq j} m_i^4 m_j^2 + 2160 \sum_{i<j<k} m_i^2 m_j^2 m_k^2\right) q^4 + 5760\,\mathrm{Pf}m \sum_i m_i^2\, q^5 + \ldots\,, \end{aligned} \tag{B.1}$$

$$\langle \operatorname{Tr} m^8 \rangle = \frac{1}{1080}(11E_6 + 84E_4E_2 + 175E_2^3)\,R^4 + \frac{2}{5}(3E_6 + 7E_4E_2)\,RN$$
$$- \frac{7}{12}\,\theta_4^4(3E_4 + 5E_2^2 + 8E_2\theta_2^4 + 4E_2\theta_4^4)\,R^2T_1$$
$$+ \frac{7}{12}\,\theta_2^4(3E_4 + 5E_2^2 - 4E_2\theta_2^4 - 8E_2\theta_4^4)\,R^2T_2$$
$$- \frac{1}{2}(4E_6 - 7E_2\theta_4^8 - 14\,\theta_4^{12} - 28\,\theta_2^4\theta_4^8)\,T_1^2$$
$$- \frac{1}{2}(4E_6 - 7E_2\theta_2^8 + 14\,\theta_2^{12} + 28\,\theta_2^8\theta_4^4)\,T_2^2$$
$$- (2E_6 + 7E_2\theta_2^4\theta_4^4 + 7\,\theta_2^8\theta_4^4 - 7\,\theta_2^4\theta_4^8)\,T_1T_2 \tag{B.2}$$
$$= \operatorname{Tr}\langle m \rangle^8 - 420\,(\operatorname{Pf}m)^2\,q^2 - 2240\operatorname{Pf}m \sum_{i<j} m_i^2 m_j^2\, q^3$$
$$- \left(210 \sum_{i<j} m_i^4 m_j^4 + 4200 \sum_i m_i^4 \sum_{j<k\neq i} m_j^2 m_k^2 + 40320\,(\operatorname{Pf}m)^2\right) q^4$$
$$- \operatorname{Pf}m\left(9408 \sum_i m_i^4 + 53760 \sum_{i<j} m_i^2 m_j^2\right) q^5 + \dots,$$

$$\langle \operatorname{Tr} m^{10} \rangle = \frac{1}{4032}(44E_6E_2 + 19E_4^2 + 196E_4E_2^2 + 245E_2^4)\,R^5$$
$$+ \frac{1}{28}(36E_6E_2 + 20E_4^2 + 49E_4E_2^2)\,R^2N$$
$$+ \frac{1}{24}\,\theta_4^4\big(12E_6 - 47E_4E_2 - 35E_2^3 - (35E_2^2 + 30\theta_4^8)(2\theta_2^4 + \theta_4^4)\big)\,R^3T_1$$
$$- \frac{1}{24}\,\theta_2^4\big(12E_6 - 47E_4E_2 - 35E_2^3 + (35E_2^2 + 30\theta_2^8)(\theta_2^4 + 2\theta_4^4)\big)\,R^3T_2$$
$$+ \frac{3}{2}\,\theta_4^4\big(2E_6 - 2E_4E_2 - 5\theta_4^8(2\theta_2^4 + \theta_4^4)\big)\,NT_1$$
$$- \frac{3}{2}\,\theta_2^4\big(2E_6 - 2E_4E_2 + 5\theta_2^8(\theta_2^4 + 2\theta_4^4)\big)\,NT_2$$
$$- \frac{5}{56}\big(E_6(24E_2 - 8\theta_4^4) - \theta_4^8(133E_4 + 49E_2^2) - 112E_2\theta_4^8(2\theta_2^4 + \theta_4^4)$$
$$+ 75\theta_4^{16} + 20\theta_4^{12}\theta_2^4 + 4\theta_2^{16}\big)\,RT_1^2$$
$$- \frac{5}{56}\big(E_6(24E_2 + 8\theta_2^4) - \theta_2^8(133E_4 + 49E_2^2) + 112E_2\theta_2^8(\theta_2^4 + 2\theta_4^4)$$
$$+ 75\theta_2^{16} + 20\theta_2^{12}\theta_4^4 + 4\theta_4^{16}\big)\,RT_2^2$$
$$+ \frac{5}{84}\big(76E_6E_2 - 147E_2^2(E_4 - \theta_2^8 - \theta_4^8) + 112E_2(\theta_2^{12} - \theta_4^{12})$$
$$- 6\theta_2^{16} - 33\theta_2^{12}\theta_4^4 + 150\theta_2^8\theta_4^8 - 33\theta_2^4\theta_4^{12} - 6\theta_4^{16}\big)\,RT_1T_2 \tag{B.3}$$
$$= \operatorname{Tr}\langle m \rangle^{10} + 3360\operatorname{Pf}m \sum_{i<j<k} m_i^2 m_j^2 m_k^2\, q^3$$
$$+ \left(5670 \sum_i m_i^2 \sum_{j<k\neq i} m_j^4 m_k^4 + 52920\,(\operatorname{Pf}m)^2 \sum_i m_i^2\right) q^4$$
$$+ \operatorname{Pf}m\left(60480 \sum_{i\neq j} m_i^4 m_j^2 + 362880 \sum_{i<j<k} m_i^2 m_j^2 m_k^2\right) q^5 + \dots. \tag{B.4}$$

These results agree with those found with explicit multi-instanton calculations in refs. [3, 10, 21].

C Recursion relation for the $\mathcal{N} = 2^*$ theory

The SW curve for the $\mathcal{N} = 2^*$ theory with gauge group SU(2) corresponds to a particular case of the $N_f = 4$ theory where the invariants T_i and N vanish. In this case eq. (3.5) gets the simple factorized form

$$y^2 = (x - W_1)(x - W_2)(x - W_3) = (x - e_1 \tilde{u} - e_1^2 R)(x - e_2 \tilde{u} - e_2^2 R)(x - e_3 \tilde{u} - e_3^2 R), \quad \text{(C.1)}$$

so that one can establish a recursion relation based on eq. (2.4). Indeed the anharmonic ratio of the roots is found to be

$$\kappa = \frac{W_3 - W_2}{W_1 - W_2} = \kappa_0 \left(1 - \frac{\theta_4^4 R}{u - \theta_2^4 R/2} \right), \quad \text{(C.2)}$$

where

$$\kappa_0 = \frac{e_3 - e_2}{e_1 - e_2} = \frac{\theta_2^4}{\theta_3^4} \quad \text{(C.3)}$$

is the corresponding ratio of the roots in the massless case.

On the one hand, we can use eq. (C.2) to express the difference $(\kappa - \kappa_0)$ in terms of u, and then plug in the large-a expansion of the latter in the form of eq. (3.36). On the other hand, we can Taylor expand κ, seen as the function of τ given by eq. (2.4), around κ_0:

$$\kappa - \kappa_0 = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n \kappa}{\partial \tau^n} \bigg|_{\tau_0} (\tau - \tau_0)^n, \quad \text{(C.4)}$$

and then insert the large-a expansion of $(\tau - \tau_0)$ given in eq. (3.33).

Comparing order by order the two different expansions of $(\kappa - \kappa_0)$ we can recursively determine the unknown coefficients h_ℓ of the expansion exactly in q and in R, finding in the end that

$$h_0 = R, \qquad h_1 = \frac{E_2}{6} R^2, \qquad h_2 = \frac{E_4 + 5E_2^2}{90} R^3, \qquad h_3 = \frac{11E_6 + 84E_4 E_2 + 175E_2^3}{7560} R^4,$$

$$h_4 = \frac{44E_6 E_2 + 19E_4^2 + 196E_4 E_2^2 + 245E_2^4}{22680} R^5, \dots . \quad \text{(C.5)}$$

These expressions correspond to setting $T_i = N = 0$ in the $N_f = 4$ results given in eqs. (3.45)–(3.48), and coincide with what was found in ref. [14].

Decoupling limits to the pure SU(2) theory. As already discussed in ref. [9], it is possible to recover the pure SU(2) theory from the $\mathcal{N} = 2^*$ model by sending the mass invariant R to infinity and at the same time q to zero, so as to keep the combination

$$\hat{\Lambda}^2 = 2Rq \quad \text{(C.6)}$$

finite. Indeed, expanding the θ-functions, in this limit from eq. (C.2) we find

$$\kappa \to -\frac{16qR}{u - 8qR} = \frac{-2\hat{\Lambda}^2}{\hat{u} - \hat{\Lambda}^2} \tag{C.7}$$

which coincides with the result (2.11) derived from the SW curve (2.10) of the pure SU(2) theory. Notice that above we have taken into account the fact that $u = 4\hat{u}$ according to what we have explained after eq. (2.12).

Let us now consider the complex structure τ given in eq. (3.33); using the h_ℓ's of eq. (C.5), it is not difficult to check that in this limit one gets

$$\tau \to \hat{\tau} = \frac{i}{\pi} \log \frac{4a^2}{\hat{\Lambda}^2} + \frac{1}{2\pi i} \left\{ \frac{3}{2} \frac{\hat{\Lambda}^4}{a^4} + \frac{105}{64} \frac{\hat{\Lambda}^8}{a^8} + \cdots \right\}, \tag{C.8}$$

which is the correct expression of the effective SU(2) coupling in the normalization (2.12) appropriate for the form eq. (2.11) of the Matone relation.

Starting from the generic $N_f = 4$ theory one can decouple some masses and recover the asymptotically free theories with $N_f = 3, 2, 1, 0$. In particular, one can reach the pure SU(2) case by sending $q \to 0$ while keeping

$$\Lambda^4 = 64 \, q \, \mathrm{Pf} m \tag{C.9}$$

fixed. Such a limit can be taken starting from a particular form of the $N_f = 4$ curve (3.5) in which $R = N = T_1 = 0$ and only $T_2 = -T_3 = -\mathrm{Pf}m/2$ is kept; in this situation, the curve becomes

$$y^2 = W_1 W_2 W_3 + \frac{3\mathrm{Pf}m \, A}{2} \left[e_1 x + \frac{u}{3}(e_1^2 + 2e_2 e_3) \right]. \tag{C.10}$$

In the decoupling limit described above, this curve factorizes and the anharmonic ratio of the roots becomes

$$\kappa = \frac{u - \sqrt{u^2 - 64q\mathrm{Pf}m}}{u + \sqrt{u^2 - 64q\mathrm{Pf}m}} = \frac{u - \sqrt{u^2 - \Lambda^4}}{u + \sqrt{u^2 - \Lambda^4}}, \tag{C.11}$$

which agrees with the pure SU(2) result mentioned in footnote 1. In this case, for the complex structure τ we find

$$\tau \to \frac{2i}{\pi} \log \frac{16a^2}{\Lambda^2} + \frac{1}{\pi i} \left\{ \frac{3}{32} \frac{\Lambda^4}{a^4} + \frac{105}{16384} \frac{\Lambda^8}{a^8} + \cdots \right\}, \tag{C.12}$$

which, with the position $\hat{\Lambda}^4 = \Lambda^4/16$, corresponds to *twice* the effective coupling $\hat{\tau}$ of eq. (C.8), as appropriate for this case.

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