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Haezendonck-Goovaerts capital allocation rules

Gabriele Canna [a], Francesca Centrone [b], Emanuela Rosazza Gianin [a,*]

[a] Dipartimento di Statistica e Metodi Quantitativi, Università di Milano Bicocca, Via Bicocca degli Arcimboldi 8, 20126 Milano, Italy
[b] Dipartimento di Studi per l'Economia e l'Impresa, Università del Piemonte Orientale, Via Perrone 18, 28100 Novara, Italy

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ABSTRACT

This paper deals with the problem of capital allocation for a peculiar class of risk measures, namely the Haezendonck-Goovaerts (HG) ones (Bellini and Rosazza Gianin, 2008; Goovaerts et al., 2004). To this aim, we generalize the capital allocation rule (CAR) introduced by Xun et al. (2019) for Orlicz risk premia (Haezendonck and Goovaerts, 1982) as well as for HG risk measures, using an approach based on Orlicz quantiles (Bellini and Rosazza Gianin, 2012). We therefore study the properties of different CARs for HG risk measures in the quantile-based setting. Finally, we provide robust versions of the introduced CARs, considering ambiguity both over the probabilistic model and over the Young function, following the scheme of Bellini et al. (2018).

1. Introduction

In this paper, we focus on the problem of capital allocation in the context of a well known class of risk measures, namely the Haezendonck-Goovaerts (HG) ones. Roughly speaking, a capital allocation problem consists in, given a risk measure ρ and a set of financial positions $\mathcal{X}$, finding a suitable way (that is, satisfying some desirable properties) of sharing the risk capital $\rho(X)$ (interpreted as a buffer against default) among the sub-units of X, for each $X \in \mathcal{X}$.

The capital allocation problem has been investigated for general risk measures with different approaches and broad scopes (see, among others, Denault, 2001; Kalkbrener, 2005; Delbaen, 2000; Dhaene et al., 2012; Centrone and Rosazza Gianin, 2018; Tsanakas, 2009). Formally, following (Delbaen, 2000; Denault, 2001; Kalkbrener, 2005), a capital allocation rule (CAR) for a risk measure $\rho: L^\infty \to \mathbb{R}$ is a map $\Lambda: L^\infty \times L^\infty \to \mathbb{R}$ such that $\Lambda(X, X) = \rho(X)$ for every $X \in L^\infty$, where $\Lambda(X, Y)$ is interpreted as the risk contribution of a sub-position X to the risk of the whole position Y.

At the same time, Haezendonck-Goovaerts risk measures have been studied in the last decades both from a mathematical point of view and from an actuarial one (see, among others, Bellini et al., 2018; Bellini and Rosazza Gianin, 2008, 2012; Bellini et al., 2014; Goovaerts et al., 2004; Haezendonck and Goovaerts, 1982). This class of risk measures, based on the so called Orlicz premium introduced by Haezendonck and Goovaerts (1982), has become popular also because it generalizes the well-known Conditional Value at Risk (CVaR).

Among the different methods of capital allocations, the quantile-based approach is the more natural when focusing on HG risk measures since this family of risk measures intrinsically depends on generalized quantiles. Many works on quantile-based capital allocation are present in the literature and face the problem both from a theoretical and from an empirical standpoint. For instance, Kalkbrener (2005) and Tasche (2004) study and derive explicit formulas, by using VaR and CVaR as underlying risk measures, for the popular gradient allocation. In this regard, the reader is also referred to the recent works of Asimit et al. (2019) and Gómez et al. (2021), among others.

A specific capital allocation method, which is "tailored" for Orlicz premia and works beyond the special case of CVaR, has been recently introduced by Xun et al. (2019), generalizing the contribution to shortfall, provided by Overbeck (2000) for CVaR. However, despite its desirable properties, the method works only for

* Corresponding author.
E-mail addresses: gabriele.canna@unimib.it (G. Canna), francesca.centrone@uniupo.it (F. Centrone), emanuela.rosazza1@unimib.it (E. Rosazza Gianin).

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$X \in L^\infty_+$ and depends on the quantile (or VaR) of the aggregated risk $Y \in L^\infty_+$. Therefore, it is still somehow connected to CVaR and excludes the possibility of allocating capital to positions not representing just a loss, which is instead financially meaningful e. g. for internal purposes, when assessing the performances of various business lines.

Starting from the approach of Xun et al. (2019) and motivated by the fact that HG risk measures are meaningful beyond the case of non-negative random variables, our main goal is to introduce a capital allocation method for HG risk measures which is defined for any pair $(X, Y) \in L^\infty \times L^\infty$ (not only positive) and overcomes the special case of CVaR, by maintaining some of the properties required for a significant capital allocation rule.

The aim of the paper is, therefore, to generalize the CAR proposed by Xun et al. (2019) for Orlicz premia in two directions. First, inspired by Bellini and Rosazza Gianin (2012), we extend the work of Xun et al. (2019) by providing capital allocation rules both for Orlicz risk premia and for HG risk measures, not only in terms of VaR but also of Orlicz quantiles (defined in Bellini and Rosazza Gianin, 2012) that are more appropriate when the involved Young function is not necessarily linear. We show indeed that such CARs satisfy most of the usually required properties and are also reasonable from a financial point of view. A comparison among the approaches here introduced and two popular capital allocation rules, that is, the gradient method and the Aumann-Shapley one (Centrone and Rosazza Gianin, 2018; Kalkbrener, 2005), is also provided. Since a deep analysis on the gradient approach has been recently provided by Gómez et al. (2021) for higher moment risk measures, corresponding to HG risk measures for power Young functions, we also extend one of their results on Orlicz quantiles to the case of general Young functions.

Second, inspired by robust Orlicz premia and robust HG risk measures recently introduced by Bellini et al. (2018), we provide some extensions of the proposed methods of capital allocation to cover ambiguity over the probabilistic model and over the risk perception of the decision-maker. In particular, we first introduce robust Orlicz quantiles and study their properties, obtaining results similar to the non-robust case. By using robust Orlicz quantiles, we then provide robust versions of the presented methods to account for ambiguity over the probability measure and for ambiguity over the utility/loss function. We find out that the robust versions work well for the quantile-based methods, providing results very close to the non-robust case.

The paper is organized as follows: in Section 2 we briefly recall some known facts about HG risk measures and capital allocation rules; in Section 3 we present the capital allocation methods based on Orlicz quantiles and study their properties. Section 4 is instead devoted to the robust versions.

2. Preliminaries

In this section, we fix the notation used in the paper and recall some well-known definitions and results.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the space of all $\mathbb{P}$-essentially bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and let L^∞_+ be the space of non-negative elements of L^∞. Equalities and inequalities must be understood to hold $\mathbb{P}$-almost surely.

Let $\Phi\colon [0, +\infty) \to [0, +\infty)$ be a normalized Young function, that is, a convex and strictly increasing function satisfying $\Phi(0) = 0$ and $\Phi(1) = 1$. It follows that Φ is continuous and satisfies $\lim_{x\to+\infty} \Phi(x) = +\infty$. Given a Young function Φ, the Orlicz space L^Φ is defined as

$$L^\Phi := \left\{ X \in L^0 \,\middle|\, \mathbb{E}\left[\Phi\left(\frac{|X|}{a}\right)\right] < +\infty \text{ for some } a > 0 \right\}, \tag{2.1}$$

where $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$ is the space of all random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. For a detailed treatment on Orlicz spaces we refer to Rao and Ren (1991).

Throughout the work, we will adopt the actuarial notation about signs, that is, positive values have to be interpreted as losses while negative as gains. In this setting, the Value at Risk (VaR) of X at level $\alpha \in (0, 1)$ is defined by

$$\mathrm{VaR}_\alpha(X) := \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \le x) > \alpha\} = q^+_\alpha(X),$$

where $q^+_\alpha(X)$ denotes the upper α-quantile of X.

We remind that a risk measure $\rho\colon L^\infty \to \mathbb{R}$ is coherent when it satisfies monotonicity, cash-additivity, positive homogeneity and subadditivity (see Artzner et al., 1999 and Delbaen, 2002 for the precise definition and further details), where – for the sign convention above – monotonicity means increasing monotonicity, i. e. $Y \ge X$ implies $\rho(Y) \ge \rho(X)$, while cash-additivity corresponds to $\rho(X + m) = \rho(X) + m$ for any $m \in \mathbb{R}$ and $X \in L^\infty$.

2.1. Orlicz premia and Haezendonck-Goovaerts risk measures

We recall now the definition and some basic results on Orlicz risk premia and Haezendonck-Goovaerts (HG) risk measures, well-known in the actuarial theory. In particular, Orlicz risk premia can be interpreted as a multiplicative version of the certainty equivalent. See, for further details, Haezendonck and Goovaerts (1982), Goovaerts et al. (2004) and Bellini and Rosazza Gianin (2008).

Definition 2.1 *(see Haezendonck and Goovaerts, 1982).* Let a Young function Φ be given and let $\alpha \in [0, 1)$ be fixed. The *Orlicz risk premium* of $X \in L^\infty_+$, with $X \neq 0$, is the unique solution $H^\Phi_\alpha(X)$ of

$$\mathbb{E}\left[\Phi\left(\frac{X}{H^\Phi_\alpha(X)}\right)\right] = 1 - \alpha, \tag{2.2}$$

while, by convention, for $X = 0$, $H^\Phi_\alpha(0) := 0$.

For simplicity of notation, the dependence on Φ is usually omitted in $H^\Phi_\alpha(X)$, i.e. $H_\alpha := H^\Phi_\alpha$, and $H(X) := H_0(X)$.

Notice that a more general definition of Orlicz premia on Orlicz spaces L^Φ has been given in Haezendonck and Goovaerts (1982) by means of the Luxemburg norm $\|\cdot\|_\Phi$ (see Rao and Ren, 1991 for more details on this norm), that is,

$$H(X) = \|X\|_\Phi := \inf\left\{ k > 0 \,\middle|\, \mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] \le 1 \right\}. \tag{2.3}$$

It is clear that H_α is simply given by (2.3) with $\Phi_\alpha := \frac{\Phi}{1-\alpha}$ instead of Φ.

Haezendonck and Goovaerts (1982) proved also that $H_\alpha(X)$ satisfies the following properties:

Monotonicity: if $X \ge Y$, $X, Y \in L^\infty_+$, then $H_\alpha(X) \ge H_\alpha(Y)$.

Subadditivity: $H_\alpha(X + Y) \le H_\alpha(X) + H_\alpha(Y)$ for every $X, Y \in L^\infty_+$.

Positive homogeneity: $H_\alpha(\lambda X) = \lambda H_\alpha(X)$ for every $X \in L^\infty_+$, $\lambda \ge 0$.

Since Orlicz risk premia are defined only for $X \in L^\infty_+$ and fail, in general, to be cash-additive, Haezendonck-Goovaerts risk measures were introduced by Haezendonck and Goovaerts (1982) (see also Goovaerts et al., 2004, Bellini and Rosazza Gianin, 2008) to extend Orlicz risk premia so to obtain cash-additive risk measures defined on the whole L^∞.

Definition 2.2 *(see Bellini and Rosazza Gianin, 2008).* Let $\alpha \in [0, 1)$. The *Haezendonck-Goovaerts risk measure* of $X \in L^\infty$ is defined by

$$\pi_\alpha(X) := \inf_{x\in\mathbb{R}} \pi_\alpha(X,x) \tag{2.4}$$

where

$$\pi_\alpha(X,x) := x + H_\alpha\left((X-x)^+\right). \tag{2.5}$$

Bellini and Rosazza Gianin (2008) proved that π_α defines a coherent risk measure in our setting. Moreover, the following result holds.

Proposition 2.1 *(see Bellini and Rosazza Gianin, 2012). For any $\alpha \in (0,1)$ and $X \in L^\infty$, the infimum in (2.4) is attained at some x^*_X. Moreover, π_α admits the following representation*

$$\pi_\alpha(X) = \max_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X],$$

where $\mathcal{Q}$ is a subset of $\mathcal{D}^\Psi := \left\{\eta \in L^\Psi_+ \,\middle|\, \mathbb{E}[\eta]=1\right\}$ and Ψ is the convex conjugate of Φ, that is, $\Psi(y) := \sup_{x\geq 0}\left\{xy - \Phi(x)\right\}$ for $y \geq 0$.

If Φ is also differentiable and $H_\alpha\left((X-\cdot)^+\right)$ is differentiable at x^*_X, then

$$\pi_\alpha(X) = \mathbb{E}_{\mathbb{Q}_X}[X] \tag{2.6}$$

where

$$\frac{d\mathbb{Q}_X}{d\mathbb{P}} = \frac{\Phi'\left(\frac{(X-x^*_X)^+}{\|(X-x^*_X)^+\|_{\Phi_\alpha}}\right)\mathbb{1}_{\{X>x^*_X\}}}{\mathbb{E}\left[\Phi'\left(\frac{(X-x^*_X)^+}{\|(X-x^*_X)^+\|_{\Phi_\alpha}}\right)\mathbb{1}_{\{X>x^*_X\}}\right]}. \tag{2.7}$$

2.2. Capital allocation

We conclude this section by recalling the notion of a capital allocation rule and some related properties (see Denault, 2001, Kalkbrener, 2005, Delbaen, 2000, Dhaene et al., 2012 and Centrone and Rosazza Gianin, 2018 for a detailed discussion).

Definition 2.3 *(see Centrone and Rosazza Gianin, 2018; Denault, 2001; Dhaene et al., 2012; Kalkbrener, 2005).* Given a risk measure ρ on a linear space $\mathcal{X}$, a *capital allocation rule (CAR)*[1] for ρ is a map $\Lambda : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ such that

$$\Lambda(X,X) = \rho(X) \quad \text{for all } X \in \mathcal{X}. \tag{2.8}$$

As pointed out by Brunnermeier and Cheridito (2019), equality in (2.8) might be not indispensable in some cases, for example when capital is collected for monitoring purpose. For that reason, (2.8) can be replaced by $\Lambda(X,X) \leq \rho(X)$ for any $X \in \mathcal{X}$ (in that case, Λ will be called audacious CAR) or by $\Lambda(X,X) \geq \rho(X)$ for any $X \in \mathcal{X}$ (prudential CAR). See also Centrone and Rosazza Gianin (2018) for a further discussion.

Given a non-empty set $\mathcal{X}$ of random variables, we say that $X \in \mathcal{X}$ is a sub-portfolio (or sub-unit) of $Y \in \mathcal{X}$ if there exists $Z \in \mathcal{X}$ such that $Y = X + Z$. Note that, since $\mathcal{X} := L^\infty$ throughout the work, every random variable is a sub-portfolio of any other.

In this framework, $\Lambda(Y,Y)$ defines the capital allocated to Y considered as a stand-alone portfolio, that is, as a sub-portfolio of itself. Furthermore, $\Lambda(X,Y)$ defines the capital allocated to X considered as a sub-portfolio of Y, which can be interpreted as the contribution of X to the aggregated risk capital $\rho(Y)$ of Y.

[1] Notice that, throughout the work and with an abuse of notation, we still call CAR a map defined on a restricted domain D of $L^\infty \times L^\infty$ (e.g. on $L^\infty_+ \times L^\infty$) and satisfying (2.8) for the corresponding ρ restricted to D.

For a CAR to be economically sound, it is customary to require the following properties (see Denault, 2001; Kalkbrener, 2005; Dhaene et al., 2012 and the references therein for further details and interpretations):

No-undercut: $\Lambda(X,Y) \leq \rho(X)$ for any $X, Y \in L^\infty$.

Monotonicity: if $X \geq Z$, $X, Z \in L^\infty$, then $\Lambda(X,Y) \geq \Lambda(Z,Y)$ for any $Y \in L^\infty$.

Riskless: $\Lambda(a,Y) = a$ for any $a \in \mathbb{R}$, $Y \in L^\infty$.

No-undercut is a well-known property in standard capital allocation problems (see Centrone and Rosazza Gianin, 2018, Denault, 2001, Kalkbrener, 2005), including game-theoretical features of stability as it implies that the sub-portfolio X has no incentive to be split from the whole portfolio Y, since staying alone would be more costly. Monotonicity (in the first variable) means that the capital allocated to a position with a higher loss has to be greater or equal than the capital allocated to another position with a lower loss. Riskless requires that the capital allocated to a fixed monetary amount is exactly such amount.

Another property often required for a CAR is:

Full allocation: if $\sum_{i=1}^n X_i = X$, then $\sum_{i=1}^n \Lambda(X_i, X) = \rho(X)$;

meaning that when a portfolio X is decomposed into sub-units $X_1, X_2, \ldots, X_n$ such that $\sum_{i=1}^n X_i = X$, then the whole capital should be shared among the different sub-units. Although full allocation seems to be a desirable property, in a general framework and for general risk measures, it is not compatible with other properties (e.g. with no-undercut, as shown in Kalkbrener (2005) – see also Canna et al. (2020a) for a further discussion). However, notice that it is always possible to modify a capital allocation rule so to guarantee the full allocation property (potentially by losing other properties), for instance, by normalizing the risk contribution of each X_i as

$$\widehat{\Lambda}(X_i, X) := \frac{\rho(X)}{\sum_{j=1}^n \Lambda(X_j, X)}\Lambda(X_i, X),$$

or by shifting by a suitable exogenous amount (see Brunnermeier and Cheridito, 2019; Dhaene et al., 2012; Kromer et al., 2016).

Some among the following further properties on Λ can be also required (see Canna et al., 2020a for details):

1-cash-additivity: $\Lambda(X+c, Y) = \Lambda(X,Y) + c$ for any $c \in \mathbb{R}$, $X, Y \in L^\infty$.

1-law invariance: if $X \sim Z$, $X, Z \in L^\infty$, then $\Lambda(X,Y) = \Lambda(Z,Y)$ for any $Y \in L^\infty$.

1-positive homogeneity: $\Lambda(\lambda X, Y) = \lambda\Lambda(X,Y)$ for any $\lambda \geq 0$, $X, Y \in L^\infty$.

2-translation-invariance: $\Lambda(X, Y+c) = \Lambda(X,Y)$ for any $c \in \mathbb{R}$, X, $Y \in L^\infty$.

1-cash-additivity requires that, whenever we add any cash amount to the sub-portfolio X, the capital allocated to such a pair is exactly that allocated to the pair (X,Y) plus the cash amount; while 1-positive homogeneity imposes that the capital allocated to a pair of sub-portfolio and portfolio formed by λ shares of X is exactly λ times the capital allocated to (X,Y). 1-law invariance requires that the capital allocated to any couple of sub-portfolios with the same distribution is equal. Finally, 2-translation-invariance means that the capital allocated to the sub-portfolios does not change if we add or remove any cash amount to the portfolio Y. Notice that 1-cash-additivity and 2-translation invariance imply cash-additivity, that is, $\Lambda(X+c, Y+c) = \Lambda(X,Y) + c$ for any $c \in \mathbb{R}$, $X, Y \in L^\infty$.

We finally recall two well-known capital allocation methods, which we are going to use as a comparison with those proposed in the following. In particular, the first one is very popular in the literature for its natural interpretation as marginal contribution of each sub-unit to the overall risk and as it allows to get closed-form formulas for some of the most popular quantile-based risk measures (see, for example, Kalkbrener, 2005, Section 5.2 and Tasche, 2004).

Given a coherent Gateaux differentiable risk measure ρ, the *gradient (or Euler) allocation* is given by

$$\Lambda^{\rho}_{\nabla}(X,Y) := \mathbb{E}_{\mathbb{Q}_Y}[X], \tag{2.9}$$

where $\frac{d\mathbb{Q}_Y}{d\mathbb{P}}$ is the gradient of ρ at Y (see Kalkbrener, 2005 for more details). The gradient allocation is linear (which implies full allocation) and satisfies no-undercut and riskless.

Based on the gradient approach, the Aumann-Shapley method has been proposed by Tsanakas (2009) and later extended by Centrone and Rosazza Gianin (2018) to risk measures that are not necessarily differentiable. In particular, for a coherent risk measure $\rho(Y) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[Y]$ where the supremum is attained at some $\mathbb{Q}_Y$, the *Aumann-Shapley* CAR is defined as

$$\Lambda^{\rho}_{AS}(X,Y) := \int_0^1 \mathbb{E}_{\mathbb{Q}_{\gamma Y}}[X]\, d\gamma. \tag{2.10}$$

3. Capital allocation via Orlicz quantiles

Quite recently, Xun et al. (2019) introduced the following capital allocation rule for Orlicz risk premia. In particular, they defined the risk contribution $H_{Y,\alpha}(X)$ of X as a sub-portfolio of Y as the solution of[2]

$$\mathbb{E}\left[\Phi\left(\frac{X\mathbb{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}}}{H_{Y,\alpha}(X)}\right)\right] = 1-\alpha \tag{3.1}$$

where $\alpha \in [0,1)$. Such a definition reduces to the so-called "contribution to shortfall", proposed by Overbeck (2000) and given by

$$\mathrm{CS}_\alpha(X,Y) := \mathbb{E}\left[X \mid Y > \mathrm{VaR}_\alpha(Y)\right], \tag{3.2}$$

for $\Phi(x) = x$ and for continuous random variables Y.

The above capital allocation is appealing for different reasons: firstly, because – once normalized – it is a coherent allocation (that is, it satisfies suitable properties such as full allocation, no-undercut and riskless, see Denault, 2001); secondly, because it is a generalization of the contribution to shortfall for nonlinear Φ; finally, because it is the first CAR defined specifically for the family of Orlicz premia.

In definition (3.1), the fact of considering the loss Y only in the event $\{Y > \mathrm{VaR}_\alpha(Y)\}$ is justified by the capital allocation approaches based on the insurer's default option and it is motivated by the fact that the shareholders of a company have limited liability and therefore, in the event of default, they are not obliged to pay when the loss exceeds such fixed threshold; see, for more details, Dhaene et al. (2012), Myers and Read (2001).

Nevertheless, while the use of VaR_α in the CAR (3.1) can be justified for $\Phi(x) = x$ by the arguments above, this is no more the case for a general Φ. Roughly speaking, VaR_α can be seen as the "right quantile" for $\Phi(x) = x$ while it is not for a general Φ where the use of Orlicz quantiles seems to be more appropriate. Motivated by this, we aim at generalizing the definition above in two directions: first, by replacing VaR_α with an Orlicz quantile in (3.1); second, by defining a CAR for HG risk measures π_α, starting from that for Orlicz risk premia H_α, so to obtain a CAR defined for any pair $(X,Y) \in L^\infty \times L^\infty$ (and not only for X in L^∞_+). Hence, in this section, we are going to provide some capital allocation methods, both for the Orlicz risk premium and for the HG risk measure, by means of the so called Orlicz quantiles, introduced by Bellini and Rosazza Gianin (2008) (see also Bellini et al., 2014).

Before proceeding with the definition of our new CARs, a preliminary discussion on the attainment of the infimum in (2.4) is in order. Indeed, by Proposition 2.1, for any $\alpha \in (0,1)$ the infimum in (2.4) of π_α is realized at some point x^* (see Bellini and Rosazza Gianin, 2008). Thus, π_α can be written as

$$\pi_\alpha(X) = x^*_\alpha(X) + H_\alpha\left((X - x^*_\alpha(X))^+\right) \tag{3.3}$$

where

$$x^*_\alpha(X) \in \arg\min_{x\in\mathbb{R}} \left\{x + H_\alpha\left((X-x)^+\right)\right\} \tag{3.4}$$

is called an Orlicz quantile (see Bellini and Rosazza Gianin, 2012 for details).

Furthermore, in Bellini and Rosazza Gianin (2012) the authors claimed the uniqueness of an Orlicz quantile under the hypothesis of Φ being strictly convex. Unfortunately, we have recently realized that such a result does not hold without some additional hypotheses (see also the example below). Therefore, we correct and replace Proposition 3(c-d) of Bellini and Rosazza Gianin (2012) with Proposition 3.1, whose proof is similar to those of (Bellini and Rosazza Gianin, 2012, Propp. 3, 11) and of (Bellini et al., 2014, Propp. 1, 5).

Example 3.1. Consider the Young function $\Phi(x) = x^2$, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\{\omega_i\}) = \frac{1}{2}$, $i = 1,2$; and the random variable $X = \begin{cases} 4, & \text{on } \omega_1 \\ 8, & \text{on } \omega_2 \end{cases}$. Hence,

$$\pi_\alpha(X,x) = x + \sqrt{\frac{\mathbb{E}[((X-x)^+)^2]}{1-\alpha}}$$
$$= \begin{cases} x + \sqrt{\frac{(4-x)^2+(8-x)^2}{2(1-\alpha)}}, & \text{if } x \le 4, \\ x + \frac{8-x}{\sqrt{2(1-\alpha)}}, & \text{if } 4 < x \le 8, \\ x, & \text{if } x > 8. \end{cases}$$

It follows that, in the interval $(4,8]$, $\pi_\alpha(X,x)$ is linear (thus not strictly convex) and even constant for $\alpha = \frac{1}{2}$. Furthermore, for $\alpha = \frac{1}{2}$ the minimizer is not unique but attained at any point of the interval $[4,8]$, where $\pi_\alpha(X,x) \equiv 8$.

Note that the non-uniqueness of the minimizers as well as their formulations is not surprising in view of the recent paper of Gómez et al. (2021). In their Lemma 2.1, indeed, the aforementioned authors proved that, for power functions Φ and for a slightly different formulation of $\pi_\alpha(X,x)$, the (upper) Orlicz quantiles coincide with the $\operatorname{ess\,sup}(X)$ for some α and for X assuming $\operatorname{ess\,sup}(X)$ with positive probability (as in the present case).

The following result investigates the properties of Orlicz quantiles in general (with no assumptions on the distribution of X and on α), in line with Proposition 3 of Bellini and Rosazza Gianin (2012) and using the following notations

$$x^{*,-}_\alpha(X) := \inf \arg\min_{x\in\mathbb{R}} \pi_\alpha(X,x),$$
$$x^{*,+}_\alpha(X) := \sup \arg\min_{x\in\mathbb{R}} \pi_\alpha(X,x).$$

[2] Differently from the present paper, in Xun et al. (2019) VaR_α is defined as the lower α-quantile. This different definition, however, is irrelevant for the study.

Proposition 3.1. *For any $\alpha \in (0,1)$ and $X \in L^\infty$, the set of minimizers is a closed interval, that is,*

$$\arg\min_{x\in\mathbb{R}} \pi_\alpha(X,x) = \left[x_\alpha^{*,-}(X),\, x_\alpha^{*,+}(X)\right].$$

Moreover, it satisfies the following properties:

(a) Cash-additivity: for any $h \in \mathbb{R}$, $X \in L^\infty$

$$\left[x_\alpha^{*,-}(X+h),\, x_\alpha^{*,+}(X+h)\right] = \left[x_\alpha^{*,-}(X)+h,\, x_\alpha^{*,+}(X)+h\right].$$

(b) Positive homogeneity: for any $\lambda \geq 0$, $X \in L^\infty$,

$$\left[x_\alpha^{*,-}(\lambda X),\, x_\alpha^{*,+}(\lambda X)\right] = \left[\lambda x_\alpha^{*,-}(X),\, \lambda x_\alpha^{*,+}(X)\right].$$

(c) Riskless: if $X = a \in \mathbb{R}$ then $x_\alpha^{*,-}(X) = x_\alpha^{*,+}(X) = a$.
(d) Boundedness from above: $x_\alpha^{*,+}(X) \leq \operatorname{ess\,sup}(X)$ for any $X \in L^\infty$.

Proof. Since, for any $\alpha \in (0,1)$ and $X \in L^\infty$, $\pi_\alpha(X,x)$ is finite, convex and $\lim_{x\to\pm\infty}\pi_\alpha(X,x) = +\infty$, from (Bellini and Rosazza Gianin, 2012, Prop. 3(a-b)) it follows that the set of minimizers is a closed interval.

(a) For any $h \in \mathbb{R}$ and $X \in L^\infty$ it holds that

$$\begin{aligned}\left[x_\alpha^{*,-}(X+h),\, x_\alpha^{*,+}(X+h)\right] &= \arg\min_{x\in\mathbb{R}} \pi_\alpha(X+h,x)\\ &= \arg\min_{x\in\mathbb{R}} \{\pi_\alpha(X,x-h)\}\\ &= \left[x_\alpha^{*,-}(X)+h,\, x_\alpha^{*,+}(X)+h\right].\end{aligned}$$

(b) The case $\lambda = 0$ follows by riskless (proved below). By positive homogeneity of H_α, for any $\lambda > 0$ and $X \in L^\infty$ it follows that

$$\begin{aligned}\left[x_\alpha^{*,-}(\lambda X),\, x_\alpha^{*,+}(\lambda X)\right] &= \arg\min_{x\in\mathbb{R}} \pi_\alpha(\lambda X,x)\\ &= \arg\min_{x\in\mathbb{R}} \left\{\lambda\pi_\alpha\left(X,\frac{x}{\lambda}\right)\right\}\\ &= \arg\min_{x\in\mathbb{R}} \left\{\pi_\alpha\left(X,\frac{x}{\lambda}\right)\right\}\\ &= \left[\lambda x_\alpha^{*,-}(X),\, \lambda x_\alpha^{*,+}(X)\right].\end{aligned}$$

(c) Since

$$\pi_\alpha(a) = \inf_{x\in\mathbb{R}}\left\{x + H_\alpha\left((a-x)^+\right)\right\} = \inf_{x\in\mathbb{R}}\left\{x + \frac{(a-x)^+}{\Phi^{-1}(1-\alpha)}\right\}$$

for any $a \in \mathbb{R}$ and $\Phi^{-1}(1-\alpha) < 1$ for any $\alpha \neq 0$, it follows that the unique minimizer is $x_\alpha^*(a) = a$.

(d) Notice that $Y := X - \operatorname{ess\,sup}(X) \leq 0$, hence

$$0 \geq \pi_\alpha(Y) = x_\alpha^{*,+}(Y) + H_\alpha\left(\left(Y - x_\alpha^{*,+}(Y)\right)^+\right) \geq x_\alpha^{*,+}(Y),$$

by monotonicity of π_α. Therefore, $x_\alpha^{*,+}(X) \leq \operatorname{ess\,sup}(X)$ follows by cash-additivity. □

Notice that Orlicz quantiles fail to be monotone, as they fail to be bounded from below; furthermore, for $\Phi(x) = x$, the upper Orlicz quantile $x_\alpha^{*,+}$ is exactly the Value at Risk at level α (see Bellini and Rosazza Gianin, 2012 for details).

The following result better describes Orlicz quantiles under suitable assumptions on α and on the distribution of X, by extending Lemma 2.1 of Gómez et al. (2021) to general (not necessarily power) functions Φ. Differently from the aforementioned result, however, we are not able to cover the case of α belonging to the whole interval $(0,1)$ but only to a subset.

Proposition 3.2. *Let Φ be a Young function and $\alpha \in (0,1)$. Assume that $X \in L^\infty$ satisfies $\hat{\pi}_X = \mathbb{P}(X = \operatorname{ess\,sup}(X)) < 1$.*

(i) If $\alpha \in (0,C)$ for some suitable $C \in (0, 1-\hat{\pi}_X)$, then any $x_\alpha^*(X) < \operatorname{ess\,sup}(X)$ and $\pi_\alpha(X) < \operatorname{ess\,sup}(X)$.
(ii) If $\alpha \in (1-\Phi(\hat{\pi}_X), 1)$, then $x_\alpha^*(X) \equiv \operatorname{ess\,sup}(X)$ and $\pi_\alpha(X) = \operatorname{ess\,sup}(X)$.
(iii) If $\alpha = 1-\Phi(\hat{\pi}_X)$, then $x_\alpha^*(X) \in [x_\alpha^{*,-}(X), \operatorname{ess\,sup}(X)]$ and $\pi_\alpha(X) = \operatorname{ess\,sup}(X)$.

Proof. From Bellini and Rosazza Gianin (2012) it follows that any $x_\alpha^*(X) \leq \operatorname{ess\,sup}(X)$ and that $\pi_\alpha(X) \leq \operatorname{ess\,sup}(X)$.
(ii), (iii) For any $x \leq \operatorname{ess\,sup}(X)$ it holds that

$$\begin{aligned} x + H_\alpha\left((X-x)^+\right) &\geq x + \frac{\mathbb{E}[(X-x)^+]}{\Phi^{-1}(1-\alpha)}\\ &= x + \frac{\mathbb{E}\left[(X-x)^+\mathbb{1}_{\{X=\operatorname{ess\,sup}(X)\}} + (X-x)^+\mathbb{1}_{\{X<\operatorname{ess\,sup}(X)\}}\right]}{\Phi^{-1}(1-\alpha)}\\ &\geq x + \frac{(\operatorname{ess\,sup}(X)-x)\hat{\pi}_X}{\Phi^{-1}(1-\alpha)}\\ &= \operatorname{ess\,sup}(X) + (\operatorname{ess\,sup}(X)-x)\left(\frac{\hat{\pi}_X}{\Phi^{-1}(1-\alpha)} - 1\right)\\ &\geq \operatorname{ess\,sup}(X), \end{aligned} \tag{3.5}$$

where the first inequality is due to Goovaerts et al. (2004) and the last inequality to the hypothesis $\alpha > 1-\Phi(\hat{\pi}_X)$. Moreover, the last inequality is strict for $x < \operatorname{ess\,sup}(X)$. It then follows that $x_\alpha^* = \operatorname{ess\,sup}(X)$ and $\pi_\alpha(X) = \operatorname{ess\,sup}(X)$.

If $\alpha = 1-\Phi(\hat{\pi}_X)$, instead, the inequality (3.5) becomes an equality and $x_\alpha^*(X) \in [x_\alpha^{*,-}(X), \operatorname{ess\,sup}(X)]$.

(i) By definition of H_α, it follows that

$$\begin{aligned} 1-\alpha &= \mathbb{E}\left[\Phi\left(\frac{(X-x)^+}{H_\alpha((X-x)^+)}\right)\right]\\ &\leq \Phi\left(\frac{\operatorname{ess\,sup}(X)-x}{H_\alpha((X-x)^+)}\right)\mathbb{P}(X \geq x)\end{aligned}$$

for any $x < \operatorname{ess\,sup}(X)$. Hence, for any $x < \operatorname{ess\,sup}(X)$

$$x + H_\alpha\left((X-x)^+\right) \leq x + \frac{\operatorname{ess\,sup}(X)-x}{\Phi^{-1}\left(\frac{1-\alpha}{\mathbb{P}(X\geq x)}\right)} < \operatorname{ess\,sup}(X)$$

if $\Phi^{-1}\left(\frac{1-\alpha}{\mathbb{P}(X\geq x)}\right) > 1$ (or, equivalently, $\alpha < 1-\mathbb{P}(X \geq x)$). This implies that $x_\alpha^*(X) < \operatorname{ess\,sup}(X)$ and $\pi_\alpha(X) < \operatorname{ess\,sup}(X)$ for $\alpha < 1-\mathbb{P}(X \geq x_\alpha^*) = C \leq 1-\hat{\pi}_X$. □

From the previous result it follows that, for random variables X that are not continuous and for suitable values of α, the (upper) Orlicz quantile reduces to $\operatorname{ess\,sup}(X)$.

3.1. Capital allocation rules for H_α

With the previous section at hand, we now generalize the approach of Xun et al. (2019) at the level of H_α by means of Orlicz quantiles. The idea of defining the capital to be allocated to a sub-portfolio X of the whole portfolio Y by focusing only on the scenario where Y exceeds the threshold $x_\alpha^*(Y)$ comes from similar arguments as for the contribution to shortfall (see also Section 2.2) and it is in line with the capital allocation approaches based on the insurer's default option (see Dhaene et al., 2012; Myers and Read, 2001). In particular, such an approach is motivated by the fact that an insurer is mainly concerned about losses exceeding a fixed threshold, hence it seems reasonable to restrict ourselves on that scenario also when computing $\Lambda(X, Y)$. The previous motivation is even more convincing once $x^*_\alpha(Y)$ is interpreted as a deductible. In that case, indeed, for an insurer it is relevant to consider only the case where the losses of the whole portfolio Y exceed the deductible and to evaluate the capital to be allocated to the sub-portfolio X accordingly.

Definition 3.1. Given the Orlicz risk premium H_α, we define the map $\Lambda^H : L^\infty_+ \times L^\infty \to \mathbb{R}^+$ as

$$\Lambda^H(X, Y) := H_\alpha\left(X\mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\right) \tag{3.6}$$

where $x^*_\alpha(Y)$ is an Orlicz quantile at level α of Y.

As pointed out by a Referee, the definition above seems to be financially reasonable once Λ^H is defined on $L^\infty_+ \times L^\infty_+$ instead of $L^\infty_+ \times L^\infty$. If all the sub-portfolios X are positive, indeed, it is reasonable to assume that the whole portfolio is positive a fortiori. Nevertheless, we have chosen to define $\Lambda^H(X, Y)$ for any $Y \in L^\infty$ in view of the definition of Λ^π in (3.8) where Y is not necessarily positive. The results below, however, hold as well also for the restriction of Λ^H to $L^\infty_+ \times L^\infty_+$.

Notice that, for $X \neq 0$ and $\mathbb{P}(Y \geq x^*_\alpha(Y)) > 0$, $\Lambda^H(X, Y)$ is the unique solution of

$$\mathbb{E}\left[\Phi\left(\frac{X\mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}}{\Lambda^H(X, Y)}\right)\right] = 1 - \alpha. \tag{3.7}$$

Condition $\mathbb{P}(Y \geq x^*_\alpha(Y)) > 0$ is quite commonly satisfied since, by the properties of Orlicz quantiles, $x^*_\alpha(Y) \leq \operatorname{ess\,sup}(Y)$ (see Proposition 3.1). When $\Phi(x) = x$ and Y is a continuous random variable, for instance, $\mathbb{P}(Y \geq \mathrm{VaR}_\alpha(Y)) > 0$ is guaranteed by definition of VaR_α.

It is worth mentioning that, differently from (3.1), in our definition we consider the loss Y on the event $\left\{Y \geq x^*_\alpha(Y)\right\}$ (instead of $\{Y > \mathrm{VaR}_\alpha(Y)\}$). In addition to the use of Orlicz quantiles instead of VaR_α, the choice of considering $\left\{Y \geq x^*_\alpha(Y)\right\}$ (instead of a strict inequality) has been done to avoid that, for constant Y, $\Lambda^H(X, Y) = 0$ for any $X \in L^\infty_+$. For $\Phi(x) = x$ and for Y with $\mathbb{P}(Y = x^*_\alpha(Y)) = 0$, however, Λ^H reduces to the CAR proposed by Xun et al. (2019) since $x^*_\alpha = x^{*,+}_\alpha = \mathrm{VaR}_\alpha$.

Moreover, we point out that the definition of Λ^H depends on the choice of the Orlicz quantile, thus, roughly speaking, Λ^H can be seen as a family of CARs "parameterized" by the Orlicz quantile chosen. Thanks to Proposition 3.2, however, under suitable assumptions on α and on the distribution of X, the Orlicz quantile is uniquely given by the essential supremum. In that case, therefore, Λ^H is uniquely determined. In general, in order to define Λ^H in a unique way, in the following $x^*_\alpha(Y)$ (used in (3.6)) will be fixed as the upper Orlicz quantile; that is, we set $x^*_\alpha(Y) := x^{*,+}_\alpha(Y)$. Similar arguments would hold if the lower Orlicz quantile was fixed.

Since, from Proposition 3.2, $x^{*,+}_\alpha(Y) = \operatorname{ess\,sup}(Y)$ for $\alpha \in [1 - \Phi(\hat{\pi}_Y), 1)$ with $\hat{\pi}_Y = \mathbb{P}(Y = \operatorname{ess\,sup}(Y)) > 0$, for non-continuous Y the definition of $\Lambda^H(X, Y)$ would be more reasonable for $\alpha < 1 - \Phi(\hat{\pi}_Y)$ (or, possibly, with respect to lower Orlicz quantiles). On the contrary, indeed, it solves

$$\mathbb{E}\left[\Phi\left(\frac{X\mathbb{1}_{\{Y = \operatorname{ess\,sup}(Y)\}}}{\Lambda^H(X, Y)}\right)\right] = 1 - \alpha,$$

meaning that it would depend on the scenario where Y has a maximal loss. For $\Phi(x) = x$, for instance, $\Lambda^H(X, Y) = \frac{\hat{\pi}_Y}{1-\alpha}\mathbb{E}[X \mid Y = \operatorname{ess\,sup}(Y)]$, corresponding, up to the factor $\frac{\hat{\pi}_Y}{1-\alpha}$, to the contribution to shortfall of Overbeck (2000).

In the following result we list the main properties satisfied by Λ^H.

Proposition 3.3. *The map Λ^H is an audacious CAR for H_α satisfying: no-undercut with respect to H_α (that is, $\Lambda^H(X, Y) \leq H_\alpha(X)$ for any $X \in L^\infty_+, Y \in L^\infty$), monotonicity, 1-law invariance, 1-positive homogeneity and 2-translation-invariance. Moreover, the following holds:*

$$\Lambda^H(a, Y) = a\left(\Phi^{-1}\left(\frac{1-\alpha}{\mathbb{P}\left(Y \geq x^*_\alpha(Y)\right)}\right)\right)^{-1} \quad \text{for any } a \geq 0,\ Y \in L^\infty.$$

Proof. Audacious CAR, no-undercut and monotonicity follow easily by monotonicity of H_α, while 1-law invariance and 1-positive homogeneity follow from the corresponding properties of H_α.

2-translation-invariance follows because, by cash-additivity of the Orlicz quantile, $\left\{Y + c \geq x^*_\alpha(Y + c)\right\} = \left\{Y \geq x^*_\alpha(Y)\right\}$ for any $c \in \mathbb{R}$.

As regards the last statement, notice that

$$\Lambda^H(a, Y) = H_\alpha\left(a\mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\right) = aH_\alpha\left(\mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\right)$$

holds for any $a \geq 0$ and $Y \in L^\infty$ by positive homogeneity of H_α. By taking $A := \left\{Y \geq x^*_\alpha(Y)\right\}$, it follows that

$$\begin{aligned} H_\alpha(\mathbb{1}_A) &= \inf\left\{k > 0 \,\middle|\, \mathbb{E}\left[\Phi\left(\frac{\mathbb{1}_A}{k}\right)\right] \leq 1 - \alpha\right\} \\ &= \inf\left\{k > 0 \,\middle|\, \mathbb{E}\left[\Phi\left(\frac{1}{k}\right)\mathbb{1}_A\right] \leq 1 - \alpha\right\} \\ &= \inf\left\{k > 0 \,\middle|\, \frac{1}{k} \leq \Phi^{-1}\left(\frac{1-\alpha}{\mathbb{P}(A)}\right)\right\} \\ &= \left(\Phi^{-1}\left(\frac{1-\alpha}{\mathbb{P}(A)}\right)\right)^{-1}, \end{aligned}$$

where the second equality holds because Φ is a Young function while the third one by strict monotonicity of Φ. □

3.2. Different capital allocation rules for π_α

So far, we have generalized the CAR given by Xun et al. (2019) for H_α, by using Orlicz quantiles. The introduction of the CAR Λ^H, besides having an independent economic meaning, will serve as a step for the construction of CARs with no restrictions on the sign of the involved portfolios and sub-portfolios. In fact, in the following, we will propose different CARs for HG risk measures π_α and not only for Orlicz risk premia H_α, again by means of Orlicz quantiles. A comparison among the different CARs here proposed and the classical ones will be also provided.

Starting from the CAR proposed for H_α in (3.6) and from (3.3), we introduce a CAR for π_α whose construction is inspired by that of HG risk measures. In particular, $\pi_\alpha(X) = \min_{x \in \mathbb{R}}\{x + H_\alpha((X - x)^+)\}$ can be interpreted in an insurance-reinsurance view (because it is the inf-convolution of two risk measures, see Bellini and Rosazza Gianin, 2008) and can be seen as the minimal riskiness of a position X in the presence of an insurance with franchise/deductible and where minimality is with respect to the deductible x. Indeed, $H_\alpha((X - x)^+)$ can be interpreted as the premium to be paid for an insurance with a deductible x (or as the riskiness of the corresponding position). The main idea is therefore to define a CAR Λ^π by mimicking the behavior of π_α with respect to H_α and by focusing only on the situation where Y exceeds a given threshold (or deductible). In other words, the CAR below will measure the risk contribution of X by taking into account both the threshold of the whole portfolio Y and the minimal deductible for X.

Definition 3.2. Given π_α, we define the map $\Lambda^\pi : L^\infty \times L^\infty \to \mathbb{R}$ as

$$\Lambda^\pi(X,Y) := x_\alpha^*(X) + \Lambda^H\left(\left(X - x_\alpha^*(X)\right)^+, Y\right) = x_\alpha^*(X) + H_\alpha\left(\left(X - x_\alpha^*(X)\right)^+ \mathbb{1}_{\{Y \geq x_\alpha^*(Y)\}}\right) \quad (3.8)$$

where $x_\alpha^*(X)$ (resp. $x_\alpha^*(Y)$) is an Orlicz quantile at level α of X (resp. of Y).

The above definition extends Λ^H to the whole L^∞, similarly to π_α which extends H_α. From an economic point of view, we assume that each sub-portfolio has a different liability threshold and focus on allocating the risk capital over such threshold, taking into account that the aggregate loss also exceeds its liability threshold.

As for Λ^H, the definition of Λ^π also depends on the choice of the two Orlicz quantiles $x_\alpha^*(X)$ and $x_\alpha^*(Y)$, hence Λ^π can be seen as a family of CARs. Furthermore, by Proposition 3.2 it follows that Orlicz quantiles of X and Y may reduce to essential supremum of these random variables under suitable assumptions on α and on the distributions of X and Y.

In the following, we will again fix x_α^* (used in (3.8)) to be the upper Orlicz quantile. Similar arguments would hold if the lower Orlicz quantile was fixed.

Here below, we provide an example of Λ^π and then we list its general properties.

Example 3.2. Consider the case where $\Phi(x) = x$. Then Λ^π reduces to

$$\Lambda^\pi(X,Y) = q_\alpha^+(X) + \frac{\mathbb{E}\left[\left(X - q_\alpha^+(X)\right)^+ \mathbb{1}_{\{Y \geq q_\alpha^+(Y)\}}\right]}{1-\alpha} = q_\alpha^+(X) + \frac{\mathbb{P}(A_{X,Y})}{1-\alpha}\mathbb{E}\left[X - q_\alpha^+(X) \mid A_{X,Y}\right]$$

where $A_{X,Y} := \{X \geq q_\alpha^+(X), Y \geq q_\alpha^+(Y)\}$. For continuous X and Y, Λ^π becomes

$$\begin{aligned}\Lambda^\pi(X,Y) &= \mathrm{VaR}_\alpha(X) + \frac{\mathbb{P}(A_{X,Y})}{1-\alpha}\mathbb{E}\left[X - \mathrm{VaR}_\alpha(X) \mid A_{X,Y}\right] \\ &= \left(1 - \frac{\mathbb{P}(A_{X,Y})}{1-\alpha}\right)\mathrm{VaR}_\alpha(X) + \frac{\mathbb{P}(A_{X,Y})}{1-\alpha}\mathbb{E}\left[X \,\middle|\, A_{X,Y}\right] \\ &= (1-\beta)\,\mathrm{VaR}_\alpha(X) + \beta\,\mathbb{E}\left[X \mid X \geq \mathrm{VaR}_\alpha(X), Y \geq \mathrm{VaR}_\alpha(Y)\right]\end{aligned}$$

where $\beta = \beta_{X,Y,\alpha} := \frac{\mathbb{P}(A_{X,Y})}{1-\alpha} \in [0,1]$. In other words, Λ^π is a convex combination of $\mathrm{VaR}_\alpha(X)$ and of a term that is somehow related to the contribution to shortfall (3.2) but taking into account also $\mathrm{VaR}_\alpha(X)$.

Proposition 3.4. *The map Λ^π is a CAR for π_α satisfying: no-undercut, riskless, 1-cash-additivity, 1-law invariance, 1-positive homogeneity, 2-translation-invariance and cash-additivity.*

Proof. It is easy to check that Λ^π is a CAR with respect to π_α.
No-undercut: by monotonicity of H_α, it follows that for any $X, Y \in L^\infty$

$$\begin{aligned}\Lambda^\pi(X,Y) &= x_\alpha^*(X) + H_\alpha\left(\left(X - x_\alpha^*(X)\right)^+ \mathbb{1}_{\{Y \geq x_\alpha^*(Y)\}}\right) \\ &\leq x_\alpha^*(X) + H_\alpha\left(\left(X - x_\alpha^*(X)\right)^+\right) \\ &= \pi_\alpha(X).\end{aligned}$$

Riskless follows immediately by riskless of Orlicz quantiles, while *1-law invariance* follows from the law invariance of the Orlicz quantile and of H_α.

1-cash-additivity and *2-translation invariance:* by cash-additivity of Orlicz quantiles, it holds that for any $c \in \mathbb{R}$ and $X, Y \in L^\infty$

$$\begin{aligned}&\Lambda^\pi(X+c,Y) \\ &= x_\alpha^*(X+c) + H_\alpha\left(\left(X + c - x_\alpha^*(X+c)\right)^+ \mathbb{1}_{\{Y \geq x_\alpha^*(Y)\}}\right) \\ &= x_\alpha^*(X) + c + H_\alpha\left(\left(X - x_\alpha^*(X)\right)^+ \mathbb{1}_{\{Y \geq x_\alpha^*(Y)\}}\right) \\ &= \Lambda^\pi(X,Y) + c\end{aligned}$$

and

$$\begin{aligned}\Lambda^\pi(X,Y+c) &= x_\alpha^*(X) + H_\alpha\left(\left(X - x_\alpha^*(X)\right)^+ \mathbb{1}_{\{Y+c \geq x_\alpha^*(Y+c)\}}\right) \\ &= x_\alpha^*(X) + H_\alpha\left(\left(X - x_\alpha^*(X)\right)^+ \mathbb{1}_{\{Y \geq x_\alpha^*(Y)\}}\right) \\ &= \Lambda^\pi(X,Y).\end{aligned}$$

1-positive homogeneity: by positive homogeneity of Orlicz quantiles and of H_α, it follows that, for any $\lambda \geq 0$ and any $X, Y \in L^\infty$,

$$\begin{aligned}\Lambda^\pi(\lambda X,Y) &= x_\alpha^*(\lambda X) + H_\alpha\left(\left(\lambda X - x_\alpha^*(\lambda X)\right)^+ \mathbb{1}_{\{Y \geq x_\alpha^*(Y)\}}\right) \\ &= \lambda\left(x_\alpha^*(X) + H_\alpha\left(\left(X - x_\alpha^*(X)\right)^+ \mathbb{1}_{\{Y \geq x_\alpha^*(Y)\}}\right)\right) \\ &= \lambda\Lambda^\pi(X,Y).\end{aligned}$$

Cash-additivity follows by 2-translation-invariance and 1-cash-additivity. □

Notice that Λ^π fails to be monotone, as a consequence of the same failure of the Orlicz quantiles.

To remain in streamline of the definition of π_α as an infimum, here below we propose an alternative capital allocation rule, roughly speaking, on a "common quantile" whenever the infimum is attained. Thus, we consider a common liability threshold both for the sub-portfolio and for the aggregated loss Y, focusing on allocating the risk capital over such threshold, as in the previous definitions. The main reason is to investigate what happens when a threshold is not fixed a priori for X and Y but it is chosen so to minimize the capital to be allocated. As pointed out by an anonymous Referee, it could seem not reasonable to consider the same level both for X and for Y since they may have a different scale. Nevertheless, although X and Y are absolutely arbitrary in general, one should take into account their financial interpretation where X is a sub-unit (that is, roughly speaking, an "ingredient") of the global position Y. Therefore, in our opinion it seems financially justifiable the idea of choosing a common level (to be minimized) for the deductible of X and for the threshold of Y.

Definition 3.3. Given the Orlicz risk premium H_α, we define the map $\bar{\Lambda}^\pi : L^\infty \times L^\infty \to \mathbb{R}$ as

$$\bar{\Lambda}^\pi(X,Y) := \inf_{x \in \mathbb{R}}\left\{x + H_\alpha\left((X - x)^+ \mathbb{1}_{\{Y \geq x\}}\right)\right\}. \quad (3.9)$$

We first establish some properties satisfied by $\bar{\Lambda}^\pi$, then investigate whether the infimum in (3.9) is attained or not.

Proposition 3.5. *The map $\bar{\Lambda}^\pi$ is a CAR with respect to π_α satisfying: no-undercut, monotonicity, 1-law invariance, cash-additivity and positive homogeneity.*

Proof. It is immediate to check that $\bar{\Lambda}^{\pi}$ is a CAR with respect to π_α. No-undercut and monotonicity follow easily by monotonicity of H_α, while 1-law invariance is a straightforward consequence of law invariance of H_α.

Cash-additivity: for any $c \in \mathbb{R}$ and $X, Y \in L^\infty$ it holds that

$$\begin{aligned}\bar{\Lambda}^{\pi}(X+c, Y+c) &= \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\left((X+c-x)^+ \mathbb{1}_{\{Y+c\geq x\}}\right)\right\} \\ &= \inf_{y\in\mathbb{R}} \left\{y + c + H_\alpha\left((X-y)^+ \mathbb{1}_{\{Y\geq y\}}\right)\right\} \\ &= \bar{\Lambda}^{\pi}(X, Y) + c,\end{aligned}$$

hence cash-additivity.

Positive homogeneity: the case of $\lambda = 0$ is immediate. For any $\lambda > 0$ and $X, Y \in L^\infty$, instead,

$$\begin{aligned}\bar{\Lambda}^{\pi}(\lambda X, \lambda Y) &= \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\left((\lambda X - x)^+ \mathbb{1}_{\{\lambda Y\geq x\}}\right)\right\} \\ &= \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\left(\lambda\left(X - \frac{x}{\lambda}\right)^+ \mathbb{1}_{\{Y\geq \frac{x}{\lambda}\}}\right)\right\} \\ &= \inf_{y\in\mathbb{R}} \left\{\lambda y + \lambda H_\alpha\left((X-y)^+ \mathbb{1}_{\{Y\geq y\}}\right)\right\} \\ &= \lambda \bar{\Lambda}^{\pi}(X, Y)\end{aligned}$$

where the third equality holds by positive homogeneity of H_α. □

Notice that the infimum is clearly attained in (3.9) whenever $X \leq Y$ because in such case $\bar{\Lambda}^{\pi}$ coincides with π_α. As shown in the following example, however, the infimum in (3.9) may be not attained in general.

To simplify the notation, for $\Phi(x) = x$, we set

$$L_{X,Y}(x) := x + \frac{\mathbb{E}\left[(X-x)^+ \mathbb{1}_{\{Y\geq x\}}\right]}{1-\alpha}. \tag{3.10}$$

Hence (3.9) becomes

$$\bar{\Lambda}^{\pi}(X, Y) = \inf_{x\in\mathbb{R}} \left\{x + \frac{\mathbb{E}[(X-x)^+ \mathbb{1}_{\{Y\geq x\}}]}{1-\alpha}\right\} = \inf_{x\in\mathbb{R}} L_{X,Y}(x).$$

Example 3.3. Take $\Phi(x) = x$ and two random variables X, Y with the following joint distribution $\mathbb{P}(X = k, Y = j) = \frac{1}{9}$ for any $k, j = -1, 0, 1$.

It is easy to check that

$$L_{X,Y}(x) = \begin{cases} -x\frac{\alpha}{1-\alpha}, & x \leq -1, \\ x\left(1 - \frac{4}{9(1-\alpha)}\right) + \frac{2}{9(1-\alpha)}, & -1 < x \leq 0, \\ x\left(1 - \frac{1}{9(1-\alpha)}\right) + \frac{1}{9(1-\alpha)}, & 0 < x \leq 1, \\ x, & x > 1. \end{cases}$$

For $\alpha = \frac{1}{9}$, it can be easily seen that $L_{X,Y}$ is not convex in x and that $\inf_{x\in\mathbb{R}} L_{X,Y}(x) = -\frac{1}{4}$ is not attained. A similar result holds also for $\alpha = \frac{5}{9}$, corresponding to $\mathbb{P}(X \geq 0, Y \geq 0) = 1 - \alpha = \frac{4}{9}$, that is, "more or less" the α-quantile of $\min\{X, Y\}$.

For $\alpha = \frac{8}{9}$ or, equivalently, $1-\alpha = \frac{1}{9} = \mathbb{P}(X \geq 1, Y \geq 1)$, it holds instead that $\inf_{x\in\mathbb{R}} L_{X,Y}(x) = 1$ is attained at any point of the interval $(0, 1]$ but $L_{X,Y}$ is still not convex in x.

Also, due to non-convexity of $L_{X,Y}$ in x, it is quite hard to obtain a general result for the existence of a minimum. However, as shown in the following result, the existence is guaranteed (at least) for continuous X and Y.

Proposition 3.6. *If $X, Y \in L^\infty$ are two continuous random variables in L^∞ and $\Phi(x) = x$, then the infimum in (3.9) is attained at some $x^* \in [\operatorname{ess\,inf}(\min\{X, Y\}), \operatorname{ess\,sup}(\max\{X, Y\})]$.*

Proof. Assume that X and Y have joint density function $f_{X,Y}$ and that $\Phi(x) = x$. Then, it is immediate to check that $L_{X,Y}$ is continuous in $x \in \mathbb{R}$ and

$$\begin{aligned}L_{X,Y}(x) &= x + \frac{\mathbb{E}[X-x]}{1-\alpha} \\ &= -\frac{\alpha}{1-\alpha}x + \frac{\mathbb{E}[X]}{1-\alpha}, \text{ for } x \leq \operatorname{ess\,inf}(\min\{X, Y\})\end{aligned}$$

while $L_{X,Y}(x) = x$ for $x > \operatorname{ess\,sup}(\max\{X, Y\})$. Hence $L_{X,Y}$ is decreasing on $(-\infty, \operatorname{ess\,inf}(\min\{X, Y\}))$ and increasing on $(\operatorname{ess\,sup}(\max\{X, Y\}), +\infty)$. By continuity of $L_{X,Y}$ in x, it follows that there exists (at least) a minimum point belonging to the interval $[\operatorname{ess\,inf}(\min\{X, Y\}), \operatorname{ess\,sup}(\max\{X, Y\})]$. □

Remember now that, in the literature, a very popular approach to capital allocation is the one based on the sub-differential of a risk measure ρ (see Delbaen, 2000); namely, in this case, the CAR is defined by $\Lambda_\partial(X, Y) = \mathbb{E}_{\mathbb{Q}_Y}[X]$ with $\mathbb{Q}_Y$ being an optimal scenario[3] in the dual representation of ρ. By analogy, when dealing with capital allocation in the context of HG measures, it is meaningful to replace, in the formulation of $\pi_\alpha(X)$, the Orlicz quantile $x^*_\alpha(Y)$ realizing the infimum in the definition of $\pi_\alpha(Y)$. This gives rise to a new CAR where the liability threshold is the same for each sub-portfolio and it is given by the Orlicz quantile of the aggregated loss Y. As in the case of the sub-differential approach, also in this case there can be different optimal scenarios among which to choose, according to different financial criteria.

Definition 3.4. Given π_α as in Definition 2.2, we define the map $\widetilde{\Lambda}^{\pi}: L^\infty \times L^\infty \to \mathbb{R}$ as

$$\widetilde{\Lambda}^{\pi}(X, Y) := x^*_\alpha(Y) + H_\alpha\left(\left(X - x^*_\alpha(Y)\right)^+\right)$$

where $x^*_\alpha(Y)$ is an Orlicz quantile at level α of Y.

In the following, we fix x^*_α to be the upper Orlicz quantile. As previously, similar results would hold if the lower Orlicz quantiles were considered.

Proposition 3.7. *The map $\widetilde{\Lambda}^{\pi}$ is a CAR with respect to π_α satisfying monotonicity, 1-law invariance and undercut (that is, $\widetilde{\Lambda}^{\pi}(X, Y) \geq \widetilde{\Lambda}^{\pi}(X, X) = \pi_\alpha(X)$ for any $X, Y \in L^\infty$). Furthermore,*

$$\widetilde{\Lambda}^{\pi}(a, Y) = x^*_\alpha(Y) + \frac{(a - x^*_\alpha(Y))^+}{\Phi^{-1}(1-\alpha)} \quad \text{for any } a \in \mathbb{R},\ Y \in L^\infty. \tag{3.11}$$

Proof. By definition of $x^*_\alpha(X)$, it is straightforward to check that $\widetilde{\Lambda}^{\pi}$ is a CAR with respect to π_α. Furthermore, monotonicity and 1-law invariance of $\widetilde{\Lambda}^{\pi}$ follow by monotonicity and law invariance of H_α, while (3.11) by riskless of H_α.

Undercut: for any $X, Y \in L^\infty$

$$\begin{aligned}\pi_\alpha(X) &= x^*_\alpha(X) + H_\alpha\left(\left(X - x^*_\alpha(X)\right)^+\right) \\ &\leq x^*_\alpha(Y) + H_\alpha\left(\left(X - x^*_\alpha(Y)\right)^+\right) \\ &= \widetilde{\Lambda}^{\pi}(X, Y)\end{aligned}$$

[3] Notice that the sub-differential approach coincides with the gradient CAR whenever there is a unique optimal scenario.

where the inequality holds by $x^*_\alpha(X) \in \arg\min_{x\in\mathbb{R}}\{x + H_\alpha((X - x)^+)\}$. □

Remark 3.1. In this section, we have introduced a generalization of the work of Xun et al. (2019) which shares with it the idea of linking X and Y in the argument of the Orlicz risk premium H_α, with a specific functional form.

This approach can be further generalized by considering a general function $f : L^\infty_+ \times L^\infty \to L^\infty_+$ "linking" the sub-portfolio X and the portfolio Y to yield the aggregated position $f(X, Y) \in L^\infty_+$. Thus, the latter goes beyond the above allocation methods which necessarily focus on default events, allowing us to aggregate X and Y by means of a general function. The particular case of $f(X, Y) := X\mathbb{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}}$ corresponds then to the approach in Xun et al. (2019).

A CAR for the Orlicz risk premium and its extension for the HG risk measure can be provided by using the same procedure as for the corresponding π_α and H_α. We refer to the working paper (Canna et al., 2020b) for a full treatment. A key role will play the assumptions on f, with special attention to the hypothesis $f(X, X) = X$ (called linking in Canna et al., 2020b). It becomes therefore evident that the risk contribution defined by Xun et al. (2019) is not a capital allocation rule with respect to the Orlicz risk premium because the function $f(X, Y) = X\mathbb{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}}$ is not linking.

3.3. Comparison among different approaches and full allocation

A comparison among the three approaches introduced above and some well known capital allocation rules will be provided here below.

Proposition 3.8. *The following relations hold for any $X, Y \in L^\infty$:*

$$\widetilde{\Lambda}^\pi(X, Y) \geq \pi_\alpha(X) \geq \Lambda^\pi(X, Y) \quad \text{and} \quad \widetilde{\Lambda}^\pi(X, Y) \geq \bar{\Lambda}^\pi(X, Y).$$

Proof. First of all, for any $X, Y \in L^\infty$, $\widetilde{\Lambda}^\pi(X, Y) \geq \pi_\alpha(X)$ follows by undercut of $\widetilde{\Lambda}^\pi$ (or by definition of x^*_α) while $\pi_\alpha(X) \geq \Lambda^\pi(X, Y)$ by no-undercut of Λ^π. Concerning the last inequality, instead, it holds that

$$\begin{aligned}\widetilde{\Lambda}^\pi(X, Y) &\geq x^*_\alpha(Y) + H_\alpha\left(\left(X - x^*_\alpha(Y)\right)^+ \mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}\right)\\ &\geq \inf_{x\in\mathbb{R}}\{x + H_\alpha\left((X - x)\right)^+ \mathbb{1}_{\{Y\geq x\}}\right) = \bar{\Lambda}^\pi(X, Y). \quad \Box\end{aligned}$$

As could be expected, $\widetilde{\Lambda}^\pi$ dominates both Λ^π and $\bar{\Lambda}^\pi$: indeed, $\widetilde{\Lambda}^\pi$ depends only on the Orlicz quantile of Y, while Λ^π depends on both the Orlicz quantiles of X and Y, and $\bar{\Lambda}^\pi$ on a common quantile. So, in a certain sense, $\bar{\Lambda}^\pi$ does not take into account the possibility that the risks of X and Y can compensate each other and hence assigns to X a higher "cost".

Notice that, by Proposition 2.1 (see also Bellini and Rosazza Gianin, 2008), for HG risk measures the gradient capital allocation (2.9) becomes

$$\begin{aligned}\Lambda^\pi_\nabla(X, Y) &= \mathbb{E}_{\mathbb{Q}_Y}[X]\\ &= \mathbb{E}\left[\frac{\Phi'\left(\frac{(Y-x^*_\alpha(Y))^+}{\|(Y-x^*_\alpha(Y))^+\|_{\Phi_\alpha}}\right)\mathbb{1}_{\{Y>x^*_\alpha(Y)\}}}{\mathbb{E}\left[\Phi'\left(\frac{(Y-x^*_\alpha(Y))^+}{\|(Y-x^*_\alpha(Y))^+\|_{\Phi_\alpha}}\right)\mathbb{1}_{\{Y>x^*_\alpha(Y)\}}\right]}X\right]\end{aligned} \tag{3.12}$$

since

$$\frac{d\mathbb{Q}_Y}{d\mathbb{P}} = \frac{\Phi'\left(\frac{(Y-x^*_\alpha(Y))^+}{\|(Y-x^*_\alpha(Y))^+\|_{\Phi_\alpha}}\right)\mathbb{1}_{\{Y>x^*_\alpha(Y)\}}}{\mathbb{E}\left[\Phi'\left(\frac{(Y-x^*_\alpha(Y))^+}{\|(Y-x^*_\alpha(Y))^+\|_{\Phi_\alpha}}\right)\mathbb{1}_{\{Y>x^*_\alpha(Y)\}}\right]}. \tag{3.13}$$

This allows us to compare Λ^π and $\widetilde{\Lambda}^\pi$ with the gradient approach and with the Aumann-Shapley method.

Proposition 3.9. *(a) For any $X, Y \in L^\infty$ it holds that*

$$\Lambda^\pi(X, Y) \geq \mathbb{E}\left[X\frac{\mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}}{\Phi^{-1}(1-\alpha)}\right] + x^*_\alpha(X)\left(1 - \frac{\mathbb{P}(Y \geq x^*_\alpha(Y))}{\Phi^{-1}(1-\alpha)}\right). \tag{3.14}$$

Moreover, if $\Phi(x) = x$ and Y is a continuous random variable, then
$\Lambda^\pi(X, Y) \geq \Lambda^\pi_\nabla(X, Y)$ and $\Lambda^\pi(X, Y) \geq \Lambda^\pi_{AS}(X, Y)$ for any $X \in L^\infty$.
(b) $\widetilde{\Lambda}^\pi \geq \Lambda^\pi_\nabla$ and $\widetilde{\Lambda}^\pi \geq \Lambda^\pi_{AS}$.

Proof. (a) For any $X, Y \in L^\infty$ it holds that

$$\begin{aligned}\Lambda^\pi(X, Y) &= x^*_\alpha(X) + H_\alpha\left((X - x^*_\alpha(X))^+\mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}\right)\\ &\geq x^*_\alpha(X) + \frac{\mathbb{E}\left[(X - x^*_\alpha(X))^+\mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}\right]}{\Phi^{-1}(1-\alpha)}\\ &\geq \mathbb{E}\left[X\frac{\mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}}{\Phi^{-1}(1-\alpha)}\right] + x^*_\alpha(X)\left(1 - \frac{\mathbb{P}(Y \geq x^*_\alpha(Y))}{\Phi^{-1}(1-\alpha)}\right),\end{aligned} \tag{3.15}$$

where the first inequality is due to $H_\alpha(Z) \geq \frac{\mathbb{E}[Z]}{\Phi^{-1}(1-\alpha)}$ for $Z \in L^\infty_+$ (see Haezendonck and Goovaerts, 1982 and Goovaerts et al., 2004).

Since, for $\Phi(x) = x$ and for a continuous Y, (3.13) becomes

$$\frac{d\mathbb{Q}_Y}{d\mathbb{P}} = \frac{\mathbb{1}_{\{Y>q^+_\alpha(Y)\}}}{\mathbb{P}(Y > q^+_\alpha(Y))} = \frac{\mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}}{1-\alpha},$$

(3.15) implies $\Lambda^\pi(X, Y) \geq \Lambda^\pi_\nabla(X, Y)$ and, for any $\gamma \in (0, 1)$,

$$\begin{aligned}\Lambda^\pi(X, Y) &\geq \frac{\mathbb{E}\left[X\mathbb{1}_{\{Y>q^+_\alpha(Y)\}}\right]}{\mathbb{P}(Y > q^+_\alpha(Y))}\\ &= \frac{\mathbb{E}\left[X\mathbb{1}_{\{\gamma Y>q^+_\alpha(\gamma Y)\}}\right]}{\mathbb{P}(\gamma Y > q^+_\alpha(\gamma Y))} = \mathbb{E}_{\mathbb{Q}_{\gamma Y}}[X],\end{aligned}$$

where the first equality holds by positive homogeneity of the quantile. Thus

$$\Lambda^\pi(X, Y) = \int_0^1 \Lambda^\pi(X, Y)\,d\gamma \geq \int_0^1 \mathbb{E}_{\mathbb{Q}_{\gamma Y}}[X]\,d\gamma = \Lambda^\pi_{AS}(X, Y).$$

(b) From Proposition 3.8 it follows that for any $X, Y \in L^\infty$

$$\widetilde{\Lambda}^\pi(X, Y) \geq \pi_\alpha(X) = \mathbb{E}_{\mathbb{Q}_X}[X] \geq \mathbb{E}_{\mathbb{Q}_Y}[X] = \Lambda^\pi_\nabla(X, Y)$$

where the second inequality is due to the fact that $\mathbb{Q}_X$ is the maximizer for X. By similar arguments it follows that

$$\begin{aligned}\widetilde{\Lambda}^\pi(X, Y) &= \int_0^1 \widetilde{\Lambda}^\pi(X, Y)\,d\gamma \geq \int_0^1 \mathbb{E}_{\mathbb{Q}_X}[X]\,d\gamma\\ &\geq \int_0^1 \mathbb{E}_{\mathbb{Q}_{\gamma Y}}[X]\,d\gamma = \Lambda^\pi_{AS}(X, Y)\end{aligned}$$

holds for any $X, Y \in L^\infty$. □

To conclude with the properties satisfied by the different CARs, it is also easy to see that none of the proposed methods satisfies full allocation, even when the allocation maps are CARs ($\Lambda(X, X) = \rho(X)$), as they are not linear in the first variable. However, one can always modify such CARs in order to get the desired property, as discussed in Section 2.2.

4. Robust versions

So far, no ambiguity on the choice of the probability measure $\mathbb{P}$ or on the choice of the Young function Φ has been considered. Following the approach of Bellini et al. (2018), who introduced robust Orlicz premia and robust HG risk measures, in this section we provide extensions and robust versions of the approaches presented in the paper, to deal with ambiguity with respect to the probabilistic model $\mathbb{P}$ as well as to the choice of the Young function Φ.

4.1. Ambiguity over $\mathbb{P}$

Ambiguity over the probabilistic model has been largely considered in decision theory, when facing the problem of maximizing the expected utility. This idea is commonly expressed by considering a set of probability measures, instead of assuming a single one. For a detailed treatment of the argument see, among others, Cerreia-Vioglio et al. (2011), Gilboa and Schmeidler (1989), and Maccheroni et al. (2006).

Quite recently, Bellini et al. (2018) have introduced robust versions of Orlicz risk premia for $\alpha = 0$. In particular, they consider the multiple priors, the variational preferences and the homothetic preferences approaches (see Cerreia-Vioglio et al., 2011; Chateauneuf and Faro, 2010; Gilboa and Schmeidler, 1989; Maccheroni et al., 2006 for details on robust versions of expected utility) and show that these three different approaches can be formulated in a unified way.

Since our aim is to generalize the capital allocations for Orlicz premia and HG risk measures introduced in Section 3, by taking into account ambiguity over $\mathbb{P}$, we need to consider robust Orlicz premia and robust HG risk measures for a general $\alpha \in [0,1)$ so to be able to introduce "robust" Orlicz quantiles. We focus here on the "variational preferences" approach in the general case where $\alpha \in [0,1)$ (referring to Bellini et al., 2018 for $\alpha = 0$) and where $\mathcal{Q}$ is the set of all probability measures absolutely continuous with respect to $\mathbb{P}$ over which there is ambiguity.

4.1.1. Robust Orlicz premia and robust HG risk measures for $\alpha \in [0,1)$

In this section, we generalize and study robust Orlicz risk premia and robust HG risk measures (introduced in Bellini et al., 2018 for $\alpha = 0$) to the general case of $\alpha \in [0,1)$.

Definition 4.1. Let a Young function Φ be given and let $\alpha \in [0,1)$ be fixed. The *robust Orlicz risk premium* of $X \in L^\infty_+$ is defined as

$$H_{c,\alpha}(X) := \inf\left\{k>0 \,\middle|\, \sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X}{k}\right)\right]-c(\mathbb{Q})\right\}\le 1-\alpha\right\} \tag{4.1}$$

where $c\colon \mathcal{Q}\to[0,+\infty]$ is convex and lower semi-continuous, satisfying
$\inf_{\mathbb{Q}\in\mathcal{Q}} c(\mathbb{Q})=0$; while the *robust Haezendonck-Goovaerts risk measure* of $X \in L^\infty$ is defined as

$$\pi_{c,\alpha}(X) := \inf_{x\in\mathbb{R}}\left\{x + H_{c,\alpha}\left((X-x)^+\right)\right\}. \tag{4.2}$$

For $\alpha = 0$, $H_c := H_{c,0}$ and $\pi_c := \pi_{c,0}$ correspond, respectively, to the robust Orlic premia and to the robust HG risk measure studied by Bellini et al. (2018). When $c\colon \mathcal{Q}\to[0,+\infty]$ is given by

$$c(\mathbb{Q}) = \begin{cases} 0, & \text{if } \mathbb{Q}\in\mathcal{S}, \\ +\infty, & \text{if } \mathbb{Q}\notin\mathcal{S}, \end{cases}$$

for $\mathcal{S}\subseteq\mathcal{Q}$, $H_{c,\alpha}$ reduces to the particular case of multiple priors, given by

$$H_{\mathcal{S},\alpha}(X) := \inf\left\{k>0 \,\middle|\, \sup_{\mathbb{Q}\in\mathcal{S}}\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X}{k}\right)\right]\le 1-\alpha\right\}, \tag{4.3}$$

while

$$\pi_{\mathcal{S},\alpha}(X) := \inf_{x\in\mathbb{R}}\left\{x + H_{\mathcal{S},\alpha}\left((X-x)^+\right)\right\}. \tag{4.4}$$

We now slightly generalize the results of Bellini et al. (2018) concerning the properties of robust Orlicz premia and the coherence of robust HG risk measures to the general case of $\alpha \in [0,1)$. The proof is omitted since it follows the same scheme of (Bellini et al., 2018, Lm 5, Thm 3) and of (Bellini and Rosazza Gianin, 2008, Prop. 12).

Proposition 4.1. *(a) For any $X \in L^\infty_+$ with $X \neq 0$, $H_{c,\alpha}(X)$ is the unique solution of*

$$\sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X}{H_{c,\alpha}(X)}\right)\right]-c(\mathbb{Q})\right\} = 1-\alpha. \tag{4.5}$$

Moreover, $H_{c,\alpha}$ is monotone, subadditive, positively homogeneous and satisfies $H_{c,\alpha}(b) = \frac{b}{\Phi^{-1}(1-\alpha)}$, for any $b \geq 0$.
(b) For any $\alpha \in [0,1)$, $\pi_{c,\alpha}$ is a coherent risk measure.

Our aim is now to generalize the approaches based on Orlicz quantiles of Section 3 to provide a robust version of CARs. In order to do so, we first need to go back to the definition of Orlicz quantiles and notice that it depends on the particular choice of the probability measure $\mathbb{P}$. Therefore, a robust version of Orlicz quantiles is needed for our purpose. We firstly have to establish whether the infimum of (4.2) is attained or not. In such a case, then, we will focus on the minimizers.

4.1.2. Existence of the minimum

To simplify the notation, for $X \in L^\infty$ and $x \in \mathbb{R}$ we set

$$\pi_{c,\alpha}(X,x) := x + H_{c,\alpha}\left((X-x)^+\right), \tag{4.6}$$

so that $\pi_{c,\alpha}(X) = \inf_{x\in\mathbb{R}} \pi_{c,\alpha}(X,x)$. We also set $\pi_c(X,x) := \pi_{c,0}(X,x)$.

We now summarize those properties of $\pi_{c,\alpha}(X,x)$ which will be useful in the following. The proof follows easily from the properties of $H_{c,\alpha}$.

Proposition 4.2. *Let $X \in L^\infty$, $\alpha \in [0,1)$ and $\pi_{c,\alpha}(X,x)$ be given by (4.6).*

(a) $\pi_{c,\alpha}(X,x)$ is convex in $x \in \mathbb{R}$.
(b) $\pi_{c,\alpha}(X+b,x) = \pi_{c,\alpha}(X,x-b)+b$, for any $x,b \in \mathbb{R}$.
(c) $\pi_{c,\alpha}(\lambda X,x) = \lambda\pi_{c,\alpha}\left(X,\frac{x}{\lambda}\right)$, for any $\lambda > 0$.

Similarly to the non-robust case, also in the robust case the infimum in (4.2) is always attained for any $\alpha \neq 0$.

Proposition 4.3. *If $\alpha \neq 0$ then the infimum in the definition of $\pi_{c,\alpha}$, given by (4.2), is always attained.*

Proof. The proof follows the scheme of (Bellini and Rosazza Gianin, 2008, Prop. 16).

Take $X \in L^\infty$, $\alpha \in (0,1)$ and $\pi_{c,\alpha}(X,x)$ as in (4.6). Since $\pi_{c,\alpha}(X,x)$ is convex in x (see Proposition 4.2) and $\pi_{c,\alpha}(X,x) = x$, for $x \geq \operatorname{ess\,sup}(X)$, it is enough to show that $\pi_{c,\alpha}(X,x)$ is decreasing on some interval, to prove the thesis. Take then $x < \operatorname{ess\,inf}(X)$; we are going to show that there exists a $b_0 \in \mathbb{R}$ such that $\pi_{c,\alpha}(X,x-b) - \pi_{c,\alpha}(X,x) > 0$ for any $b > b_0$.

First, we notice that, for $x < \operatorname{ess\,inf}(X)$ and $b > 0$, we have

$$\begin{aligned}&\pi_{c,\alpha}(X, x-b) - \pi_{c,\alpha}(X,x)\\&= H_{c,\alpha}(X-x+b) - H_{c,\alpha}(X-x) - b.\end{aligned}$$

It remains to compare $H_{c,\alpha}(X-x+b)$ and $H_{c,\alpha}(X-x)+b$.

On the one hand, for $b>0$

$$\begin{aligned}f(b) &:= \sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X-x+b}{H_{c,\alpha}(X-x)+b}\right)\right] - c(\mathbb{Q})\right\}\\&\geq \sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{\operatorname{ess\,inf}(X)-x+b}{H_{c,\alpha}(\operatorname{ess\,sup}(X)-x)+b}\right)\right] - c(\mathbb{Q})\right\}\\&= \Phi\left(\frac{\operatorname{ess\,inf}(X)-x+b}{\frac{\operatorname{ess\,sup}(X)-x}{\Phi^{-1}(1-\alpha)}+b}\right) \xrightarrow[b\to+\infty]{} \Phi(1)=1,\end{aligned}$$

since $\inf_{\mathbb{Q}\in\mathcal{Q}} c(\mathbb{Q}) = 0$, both $H_{c,\alpha}$ and Φ are monotone and Φ is continuous. On the other hand, it follows similarly that

$$\begin{aligned}f(b) &\leq \sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{\operatorname{ess\,sup}(X)-x+b}{H_{c,\alpha}(\operatorname{ess\,inf}(X)-x)+b}\right)\right] - c(\mathbb{Q})\right\}\\&= \Phi\left(\frac{\operatorname{ess\,sup}(X)-x+b}{\frac{\operatorname{ess\,inf}(X)-x}{\Phi^{-1}(1-\alpha)}+b}\right) \xrightarrow[b\to+\infty]{} \Phi(1)=1.\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{b\to+\infty} f(b) = 1 > 1-\alpha\\&\qquad= \sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X-x+b}{H_{c,\alpha}(X-x+b)}\right)\right] - c(\mathbb{Q})\right\}\end{aligned}$$

holds because $\alpha\in(0,1)$. Hence, since $\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X}{h}\right)\right]$ is decreasing in $h>0$ for any $\mathbb{Q}\in\mathcal{Q}$, it follows that there exists a $b_0\in\mathbb{R}$ such that $H_{c,\alpha}(X-x+b) > H_{c,\alpha}(X-x)+b$ for any $b>b_0$. The thesis then follows. □

The result above shows that the infimum of $\pi_{c,\alpha}$ is always attained for $\alpha\neq 0$, similarly to the non-robust case. We now consider the case $\alpha=0$, starting from the following.

Proposition 4.4. *If $\alpha=0$ then, for any $X\in L^\infty$, $\pi_c(X,x)$ is increasing in $x\in\mathbb{R}$.*

Proof. Let $X\in L^\infty$ be fixed. For any $x\geq \operatorname{ess\,sup}(X)$ it holds that $\pi_c(X,x)=x$. For any $x<\operatorname{ess\,sup}(X)$ and for any $b>0$

$$\begin{aligned}&\pi_c(X,x-b) - \pi_c(X,x)\\&\leq H_c\left((X-x)^+ + b\right) - H_c\left((X-x)^+\right) - b\\&\leq H_c\left((X-x)^+\right) + b - H_c\left((X-x)^+\right) - b = 0\end{aligned}$$

because of $H_c(b)=b$ and of subadditivity of the positive part and of H_c. □

It follows from the proposition above that, for $\alpha=0$, either the infimum in (4.2) is not attained or it is attained at any point of $(-\infty, x_0]$ for some $x_0\leq \operatorname{ess\,sup}(X)$. The following result investigates the existence of the minimum for $\alpha=0$ when $\Phi(x)=x$, in terms of conditions on the penalty function c.

Proposition 4.5. *Let $\Phi(x)=x$ and let $X\in L^\infty$ be non-constant.*

If $\inf_{\mathbb{Q}\in\mathcal{Q}} \frac{c(\mathbb{Q})}{1+c(\mathbb{Q})} > 0$, then the infimum in $\pi_c(X)$ is not attained.

Proof. If $\Phi(x)=x$, then $H_c(X)=\sup_{\mathbb{Q}\in\mathcal{Q}}\frac{\mathbb{E}_{\mathbb{Q}}[X]}{1+c(\mathbb{Q})}$ for any $X\in L^\infty_+$. By Proposition 4.4, $\pi_c(X,x)$ is increasing in $x\in\mathbb{R}$.

For any $X\in L^\infty$, it is therefore enough to consider $x\leq \operatorname{ess\,inf}(X)$. Then for any $b>0$ we have

$$\begin{aligned}&\pi_c(X,x-b)-\pi_c(X,x)\\&= H_c(X-x+b) - H_c(X-x) - b\\&= \sup_{\mathbb{Q}\in\mathcal{Q}}\frac{\mathbb{E}_{\mathbb{Q}}[X-x+b]}{1+c(\mathbb{Q})} - \sup_{\mathbb{Q}\in\mathcal{Q}}\frac{\mathbb{E}_{\mathbb{Q}}[X-x]}{1+c(\mathbb{Q})} - b\\&\leq \sup_{\mathbb{Q}\in\mathcal{Q}}\frac{b}{1+c(\mathbb{Q})} - b\\&= -b\inf_{\mathbb{Q}\in\mathcal{Q}}\frac{c(\mathbb{Q})}{1+c(\mathbb{Q})} < 0,\end{aligned}$$

where the last inequality holds by hypothesis. So, $\pi_c(X,x)$ is strictly increasing in $(-\infty, \operatorname{ess\,inf}(X))$. The thesis then follows. □

Whenever $\inf_{\mathbb{Q}\in\mathcal{Q}}\frac{c(\mathbb{Q})}{1+c(\mathbb{Q})}=0$, it can be easily checked that the infimum in $\pi_c(X)$ may be attained or not.

4.1.3. Robust Orlicz quantiles

Before introducing the notion of robust Orlicz quantiles we present an illustrative and motivating example.

Example 4.1. Given a probability space $(\Omega,\mathcal{F},\mathbb{P})$, with $\Omega=\{\omega_1,\omega_2,\omega_3\}$ and $\mathbb{P}(\omega_i)>0$ for any $i=1,2,3$, we consider the random variable

$$X=\begin{cases}-4, & \text{on } \omega_1,\\ 4, & \text{on } \omega_2,\\ 8, & \text{on } \omega_3;\end{cases}$$

and the set $\mathcal{S}=\{\mathbb{Q}_1,\mathbb{Q}_2\}$ of probability measures with $\mathbb{Q}_1(\omega_1)=\mathbb{Q}_1(\omega_2)=\frac{1}{4}$, $\mathbb{Q}_1(\omega_3)=\frac{1}{2}$ and $\mathbb{Q}_2(\omega_1)=\frac{1}{8}$, $\mathbb{Q}_2(\omega_2)=\frac{1}{2}$, $\mathbb{Q}_2(\omega_3)=\frac{3}{8}$.

For $\Phi(x)=x$, it follows that (see Bellini et al., 2018)

$$\pi_{\mathcal{S},\alpha}(X)=\inf_{x\in\mathbb{R}}\left\{x+\sup_{\mathbb{Q}\in\mathcal{S}}\frac{\mathbb{E}_{\mathbb{Q}}[(X-x)^+]}{1-\alpha}\right\}.$$

For $\alpha=\frac{1}{4}$, it can be easily checked that $\pi_{\mathcal{S},\alpha}(X)=\frac{20}{3}$ and the infimum in (4.4) is attained at any point of the interval $[0,4]$. For $\alpha=\frac{1}{2}$, instead, $\pi_{\mathcal{S},\alpha}(X)=8$ and the infimum in (4.4) is attained at any point of the interval $[4,8]$. Therefore, the infimum is not unique in such cases.

Something similar can be found also for the strictly convex $\Phi(x)=x^2$.

The example above justifies the extension of the definition of Orlicz quantiles (depending on the probability $\mathbb{P}$ given a priori) to the present setting dealing with ambiguity on the choice of $\mathbb{P}$. Differently from the non-robust case, where for $\Phi(x)=x$ the minimizers x^*_α reduce to classical quantiles with respect to $\mathbb{P}$, in the present setting (corresponding to ambiguity and to multiple priors – penalized by c or not) the minimizers take into account all the multiple priors, hence they can be interpreted as "robust quantiles". Referring to the previous example, indeed, while $q_{\mathbb{Q}_1,\alpha}=[-4,4]$ and $q_{\mathbb{Q}_2,\alpha}=\{4\}$ for $\alpha=\frac{1}{4}$, $q_{\mathbb{Q}_1,\alpha}=[4,8]$ and $q_{\mathbb{Q}_2,\alpha}=\{4\}$ for $\alpha=\frac{1}{2}$ (where $q_{\mathbb{Q},\alpha}$ denotes the set of α-quantiles with respect to $\mathbb{Q}$), in the robust case $q_{\mathcal{S},\alpha}=[0,4]$ for $\alpha=\frac{1}{4}$ and $q_{\mathcal{S},\alpha}=[4,8]$ for $\alpha=\frac{1}{2}$ (where $q_{\mathcal{S},\alpha}$ denotes the set of minimizers in the multiple prior case, later called robust α-quantiles).

Motivated by the previous example and by Proposition 4.3, guaranteeing that the infimum in $\pi_{c,\alpha}$ is always attained for $\alpha \neq 0$, it looks then natural to follow the same scheme of the non-robust case and call any

$$x^*_{c,\alpha}(X) \in \arg\min_{x\in\mathbb{R}} \pi_{c,\alpha}(X,x)$$

a *robust Orlicz quantile* at level α of X. In analogy with the non-robust case, we use the notations:

$$x^{*,-}_{c,\alpha}(X) := \inf \arg\min_{x\in\mathbb{R}} \pi_{c,\alpha}(X,x),$$
$$x^{*,+}_{c,\alpha}(X) := \sup \arg\min_{x\in\mathbb{R}} \pi_{c,\alpha}(X,x).$$

Proposition 4.6. *For any $\alpha \in (0,1)$ and $X \in L^\infty$, the set of robust Orlicz quantiles is a closed interval satisfying: cash-additivity, positive homogeneity and riskless. Moreover, robust Orlicz quantiles are bounded from above, i.e. $x^{*,+}_{c,\alpha}(X) \le \operatorname{ess\,sup}(X)$ for any $X \in L^\infty$.*

Proof. The proof follows from Propositions 4.2 and 4.3, similarly to the non-robust case. □

Since robust Orlicz quantiles satisfy most of the properties of the non-robust ones, we extend now the definitions of CARs given in Section 3 to the robust case. The following definitions have a similar motivation and interpretation of the non-robust case where we focus on allocating the risk capital to the sub-portfolios when their losses exceed their liability threshold and the aggregate loss also exceeds its liability threshold. Differently from the non-robust case, here we use robust Orlicz quantiles, that is, we assume that each sub-portfolio faces ambiguity over the probabilistic model $\mathbb{P}$ in the variational preferences approach.

Definition 4.2. Given the robust Orlicz risk premium $H_{c,\alpha}$ and the robust HG risk measure $\pi_{c,\alpha}$, we define $\Lambda^H_c : L^\infty_+ \times L^\infty \to \mathbb{R}^+$ as

$$\Lambda^H_c(X,Y) := H_{c,\alpha}\left(X \mathbb{1}_{\{Y \ge x^*_{c,\alpha}(Y)\}}\right)$$

and the map $\Lambda^\pi_c : L^\infty \times L^\infty \to \mathbb{R}$ as

$$\Lambda^\pi_c(X,Y) := x^*_{c,\alpha}(X) + \Lambda^H\left(\left(X - x^*_{c,\alpha}(X)\right)^+, Y\right)$$
$$= x^*_{c,\alpha}(X) + H_{c,\alpha}\left(\left(X - x^*_{c,\alpha}(X)\right)^+ \mathbb{1}_{\{Y \ge x^*_{c,\alpha}(Y)\}}\right) \quad (4.7)$$

where $x^*_{c,\alpha}(X)$ is a robust Orlicz quantile at level α of X.

As already remarked for Definition 3.1, if all the sub-portfolios X are positive, then it is reasonable to assume that the whole portfolio Y is positive a fortiori, so Λ^H_c could be defined only on $L^\infty_+ \times L^\infty_+$. Nevertheless, we have decided to define $\Lambda^H(X,Y)$ for any $Y \in L^\infty$ in view of (4.7) where Y is not necessarily positive. The result below, however, holds as well also for the restriction of Λ^H_c to $L^\infty_+ \times L^\infty_+$. As previously, in the following we fix $x^*_{c,\alpha}(X)$ to be the upper robust Orlicz quantile at level α of X.

As shown in the following result, Λ^H_c and Λ^π_c have similar properties to the non-robust case. The proof is omitted because similar to the non-robust case.

Proposition 4.7. *The map Λ^H_c is an audacious CAR for $H_{c,\alpha}$ satisfying: no-undercut, monotonicity, 1-positive homogeneity and 2-translation-invariance, while Λ^π_c is a CAR for $\pi_{c,\alpha}$ satisfying no-undercut, riskless, 1-cash-additivity, 1-positive homogeneity, 2-translation-invariance and cash-additivity.*

Similarly to the non-robust case, it is possible to extend also $\bar{\Lambda}^\pi$ of Definition 3.3 and $\widetilde{\Lambda}^\pi$ of Definition 3.4 to the robust case.

4.2. The case of multiple Φ

In this section, we consider the situation where the decision-maker is uncertain about the Young function to be used, while we assume there is only one $\mathbb{P}$. As before, we follow the scheme of Bellini et al. (2018) and take a worst-case approach for the multiplicity of possible Young functions.

We begin by clarifying which set $\mathcal{P}$ of Young functions is suitable for the purpose. The choice below guarantees that $\sup \mathcal{P}$ still remains a Young function.

Definition 4.3. A non-empty set $\mathcal{P}$ of Young functions, equipped with the pointwise order (i.e. $\Psi \ge \Phi :\Longleftrightarrow \Psi(x) \ge \Phi(x),\ \forall x > 0$), is said to be proper if $(\sup \mathcal{P})(x) = \sup_{\Phi\in\mathcal{P}} \Phi(x) < +\infty$ for all $x > 0$.

Here below, we generalize to $\alpha \in [0,1)$ the definitions introduced by Bellini et al. (2018) for $\alpha = 0$.

Definition 4.4. Let $\mathcal{P}$ be a proper set of Young functions and let $\alpha \in [0,1)$ be fixed. The Φ-*robust Orlicz risk premium* of $X \in L^\infty_+$ is defined as

$$H_{\mathcal{P},\alpha}(X) := \inf\left\{k > 0 \,\middle|\, \sup_{\Phi\in\mathcal{P}} \mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] \le 1-\alpha\right\}, \quad (4.8)$$

while the Φ-*robust Haezendonck-Goovaerts risk measure* of $X \in L^\infty$ as

$$\pi_{\mathcal{P},\alpha}(X) := \inf_{x\in\mathbb{R}} \left\{x + H_{\mathcal{P},\alpha}\left((X-x)^+\right)\right\}. \quad (4.9)$$

It then follows that $H_{\mathcal{P}}(0) = 0$ and that, for $\alpha = 0$, $H_{\mathcal{P}} := H_{\mathcal{P},0}$ reduces to that of Bellini et al. (2018).

The key result is contained in the next proposition, where we will use – and so in the following – the notation $H^\Phi_\alpha(X) = H_\alpha(X)$ of Definition 2.1 to emphasize the dependence on Φ.

Proposition 4.8. *Let $\mathcal{P}$ be a proper set of Young functions. Then, for any $X \in L^\infty_+$, we have*

$$H_{\mathcal{P},\alpha}(X) = H^{\sup \mathcal{P}}_\alpha(X).$$

Proof. Take $X \in L^\infty_+$. By Proposition 25(b) in Bellini et al. (2018), $H_{\mathcal{P}}(X) = \sup_{\Phi\in\mathcal{P}} H^\Phi(X)$ holds. Since the same argument remains valid for $\alpha \neq 0$, we assume such a result here too. Therefore, we only need to prove that $\sup_{\Phi\in\mathcal{P}} H^\Phi_\alpha(X) = H^{\sup\mathcal{P}}_\alpha(X)$. To this end, since $\mathcal{P}$ is proper, it is enough to prove that H^Φ_α is monotone increasing in Φ. Take any $\Psi, \Phi \in \mathcal{P}$, with $\Psi \ge \Phi$, then

$$\mathbb{E}\left[\Phi\left(\frac{X}{H^\Psi_\alpha(X)}\right)\right] \le \mathbb{E}\left[\Psi\left(\frac{X}{H^\Psi_\alpha(X)}\right)\right]$$
$$= 1-\alpha = \mathbb{E}\left[\Phi\left(\frac{X}{H^\Phi_\alpha(X)}\right)\right]$$

that implies $H^\Psi_\alpha(X) \ge H^\Phi_\alpha(X)$, since $\mathbb{E}\left[\Phi\left(\frac{X}{h}\right)\right]$ is decreasing in $h > 0$. □

The properties of $H_{\mathcal{P},\alpha}$ and of $\pi_{\mathcal{P},\alpha}$ then follow from Proposition 4.8.

Proposition 4.9. *Let $\mathcal{P}$ be a proper set of Young functions and $\alpha \in [0,1)$. Then $H_{\mathcal{P},\alpha}$ is monotone, subadditive and positively homogeneous and $\pi_{\mathcal{P},\alpha}$ is a law invariant coherent risk measure.*

Proof. By Proposition 4.8, we have $H_{\mathcal{P},\alpha} = H_\alpha^{\sup \mathcal{P}}$, where $\sup \mathcal{P}$ is a Young function, since $\mathcal{P}$ is proper. The thesis then follows by Propositions 2 and 12 of Bellini and Rosazza Gianin (2008), since they hold for any Young function. □

As in the case of ambiguity about the probabilistic model, to pursue our purpose we need to establish if the infimum of $\pi_{\mathcal{P},\alpha}$ is attained or not. However, in this case, it is clear that the infimum is always attained for $\alpha \neq 0$, since $\pi_{\mathcal{P},\alpha}$ is simply π_α with $\sup \mathcal{P}$ as Young function.

We define, indeed, a Φ-*robust Orlicz quantile* at level α of X as any

$$x^*_{\mathcal{P},\alpha}(X) \in \arg\min_{x \in \mathbb{R}} \pi_{\mathcal{P},\alpha}(X, x).$$

It is clear that Φ-robust Orlicz quantiles satisfy the same properties of non-robust ones, as in Proposition 3.1. By means of Φ-robust Orlicz quantiles, we therefore introduce the following definitions of Φ-robust CARs with similar motivations and interpretations of the non-robust Λ^H and Λ^π. In the present case, however, $H_{\mathcal{P},\alpha}$ takes into account ambiguity over Φ.

Definition 4.5. Given the Φ-robust Orlicz risk premium $H_{\mathcal{P},\alpha}$ and the Φ-robust HG risk measure $\pi_{\mathcal{P},\alpha}$, we define $\Lambda^H_{\mathcal{P}}: L^\infty_+ \times L^\infty \to \mathbb{R}^+$ as

$$\Lambda^H_{\mathcal{P}}(X,Y) := H_{\mathcal{P},\alpha}\left(X \mathbb{1}_{\left\{Y \geq x^*_{\mathcal{P},\alpha}(Y)\right\}}\right)$$

and $\Lambda^\pi_{\mathcal{P}}: L^\infty \times L^\infty \to \mathbb{R}$ as

$$\begin{aligned}\Lambda^\pi_{\mathcal{P}}(X,Y) & \\ := x^*_{\mathcal{P},\alpha}(X) + \Lambda^H\left(\left(X - x^*_{\mathcal{P},\alpha}(X)\right)^+, Y\right) & \quad (4.10)\\ = x^*_{\mathcal{P},\alpha}(X) + H_{\mathcal{P},\alpha}\left(\left(X - x^*_{\mathcal{P},\alpha}(X)\right)^+ \mathbb{1}_{\{Y \geq x^*_{\mathcal{P},\alpha}(Y)\}}\right) &\end{aligned}$$

where $x^*_{\mathcal{P},\alpha}(X)$ a Φ-robust Orlicz quantile at level α of X.

Thus, according to the above definition, the liability threshold of each sub-portfolio and of the aggregated loss Y is given by their corresponding Φ-robust Orlicz quantiles. As previously, we fix $x^*_{\mathcal{P},\alpha}(X)$ to be the upper Φ-robust Orlicz quantile at level α of X.

The main properties of the Φ-robust CARs above are provided in the following result that can be proved similarly to the non-robust case.

Proposition 4.10. *$\Lambda^H_{\mathcal{P}}$ is an audacious CAR for $H_{\mathcal{P},\alpha}$ which satisfies no-undercut, monotonicity, 1-law invariance, 1-positive homogeneity and 2-translation-invariance, while $\Lambda^\pi_{\mathcal{P}}$ is a CAR for $\pi_{\mathcal{P},\alpha}$ satisfying no-undercut, riskless, 1-cash-additivity, 1-law invariance, 1-positive homogeneity, 2-translation-invariance and cash-additivity.*

As previously, it is possible to extend also $\bar{\Lambda}^\pi$ of Definition 3.3 and $\widetilde{\Lambda}^\pi$ of Definition 3.4 to the Φ-robust case. Furthermore, similarly to the non-robust case, it is also possible to define robust versions of the linking-based approach. We refer the interested reader to Canna et al. (2020b) for a full treatment.

Declaration of competing interest

The authors declare not to have competing interest.

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