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Flexible-bandwidth Needlets

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We investigate here a generalized construction of spherical wavelets/needlets which admits extra-flexibility in the harmonic space, i.e., it allows the corresponding support in multipole (frequency) space to vary in more general forms than in the standard constructions. We study the analytic properties of this system and we investigate its behaviour when applied to isotropic random fields: more precisely, we establish asymptotic localization and uncorrelation properties (in the high-frequency sense) under broader assumptions than typically considered in the literature.

Keywords: Spherical wavelets; needlets; spherical random fields; high-frequency asymptotics.

1. Introduction and Statement of the Results

The statistical analysis of spherical random fields has become a rather important research topic in the last 15 years. In particular, strong motivations have come from a variety of fields, most notably Cosmology and Astrophysics, Geophysics, Climate Sciences: at the same time, it has become clear that the analysis of spherical data can lead to a number of deep mathematical issues, which have independent interest (see [19, 24, 31, 43] and the references therein). Among these issues, a very important role has been played by the investigation of spherical wavelet systems, and the analysis of their properties when applied to spherical random fields.

Among spherical wavelets, one of the most successful proposals is certainly the needlet system, which was introduced by [32, 33] and then applied to random fields and cosmological data immediately after by [3, 28, 36]; applications to spherical density estimation were considered in [4] (see also [5]), whereas extensions to more general harmonic kernels were discussed by [17]. Needlets on one hand represent a tight-frame system and hence satisfy classical requirements of approximation theory; on the other hand under some regularity conditions needlet coefficients have been shown to enjoy asymptotic uncorrelation properties (in the high-resolution sense) which makes their application to random fields extremely powerful. Extensions of the needlet construction to more general manifolds were given for instance by [23, 18, 12, 22, 11]. Statistical applications are currently too many to be recalled in any reasonable completeness: we refer for instance to [20, 21] or more recently [7, 10, 13, 14, 25, 42, 44, 16, 26]. Applications in Cosmology and Astrophysics are discussed for instance in [9, 30, 35, 39, 45, 46] and the references therein.

Our purpose in this paper is to generalize the needlet construction, allowing for a much more flexible form of the kernel function in the harmonic space; we then proceed to investigate the properties of these generalized needlet transforms when applied to isotropic spherical random fields. In particular, we establish explicit bounds on the decay of the correlation function for needlet coefficients under much broader conditions than given in the existing literature; these results make possible asymptotic statistical inference in the high-frequency sense for a much greater family of random models. In order to make these statements more precise, however, we need first to review some notation and background results.

1.1. Some Background Results and Notation

Let us recall first some standard background material on harmonic analysis on the sphere; we refer for instance to [2, 29] for more discussion and details. We write as usual $L^2\left(\mathbb{S}^d\right)$ to denote the space of square-integrable functions on the sphere (with respect to Lebesgue measure), where $\mathbb{S}^d$ is the $d-$dimensional sphere embedded in $\mathbb{R}^{d+1}$; $\omega_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}$ denotes the d-dimensional spherical surface measure, with $\Gamma(\cdot)$ the usual Gamma function. The following decomposition holds:

$$L^2\left(\mathbb{S}^d\right) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_{\ell;d} \ ,$$

where $\mathcal{H}_{\ell;d}$ is the restriction to $\mathbb{S}^d$ of the space of harmonic and homogeneous polynomials of degree ℓ on $\mathbb{R}^{d+1}$. The spaces $\mathcal{H}_{\ell;d}$ have dimension

$$N_{\ell;d} = \frac{\ell+\eta_d}{\eta_d}\binom{\ell+2\eta_d-1}{\ell} = \frac{2\ell^{d-1}}{(d-1)!}\left(1+o_\ell\left(1\right)\right), \quad \eta_d = \frac{(d-1)}{2} \ ;$$

the elements $\left\{f_\ell \in \mathcal{H}_{\ell;d}\right\}$ are the eigenfunctions of the Laplace-Beltrami operator

$$\Delta_{\mathbb{S}^d} f_\ell = -\ell(\ell+d-1) f_\ell \ , \ \ell = 0,1,2,...$$

On $\mathcal{H}_{\ell;d}$, we can choose a (real or complex-valued) orthonormal basis, which we write as $\left\{Y_{\ell,m} : m = 1,\ldots,N_{\ell;d}\right\}$, omitting the dependence on the dimension d. More explicitly, this entails that every function $f \in L^2\left(\mathbb{S}^d\right)$ admits the L^2 expansion

$$f\left(x\right) = \sum_{\ell\geq 0}\sum_{m=1}^{N_{\ell;d}} a_{\ell,m} Y_{\ell,m}\left(x\right), \tag{1.1}$$

where, for $\ell \geq 0$ and $m = 1,\ldots,N_{\ell;d}$,

$$a_{\ell,m} = \int_{\mathbb{S}^d} \overline{Y}_{\ell,m}\left(x\right) f\left(x\right) dx \in \mathbb{C} \ ,$$

are the so-called spherical harmonic coefficients, whereas dx denotes the surface measure. For any choice of an orthonormal basis, the following *addition formula* holds

$$\begin{aligned} Z_{\ell;d}\left(x_1,x_2\right) &= \sum_{m=1}^{N_{\ell,d}} \overline{Y}_{\ell,m}\left(x_1\right) Y_{\ell,m}\left(x_2\right) \\ &= \frac{\ell+\eta_d}{\eta_d \omega_d} G_\ell^{(\eta_d)}\left(\langle x_1,x_2\rangle\right), \quad \text{for } x_1,x_2 \in \mathbb{S}^d, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product over $\mathbb{R}^{d+1}$, $G_\ell^{(\eta_d)}$ is the Gegenbauer polynomial of degree ℓ and parameter η_d (see [2], Chapter 2) defined by

$$G_\ell^{(\eta_d)}(x) = \frac{(-1)^\ell}{2^\ell \ell!} \frac{\Gamma(\eta_d+\frac{1}{2})\Gamma(\ell+2\eta_d)}{\Gamma(2\eta_d)\Gamma(\eta_d+\ell+\frac{1}{2})} (1-x^2)^{-\eta_d+1/2} \frac{d^\ell}{dx^\ell}[(1-x^2)^{\ell+\eta_d-1/2}].$$

With some abuse of notation, we shall write both $Z_{\ell;d}(x,y)$ or $Z_{\ell;d}(\langle x,y\rangle)$, depending on the context. For instance, for $d=2$ we have

$$Z_{\ell;2}(x_1,x_2)=\frac{2\ell+1}{4\pi}P_\ell(\langle x_1,x_2\rangle)\ ,$$

where $P_0(t)\equiv 1$ and

$$P_\ell(\cdot):[-1,1]\to\mathbb{R}\ ,\ P_\ell(t):=\frac{d^\ell}{dt^\ell}(t^2-1)^\ell\ ,\ \ell=1,2,...,$$

is the usual Legendre polynomial.

The following *reproducing kernel property* holds

$$\int_{\mathbb{S}^d}Z_{\ell;d}(\langle x,y\rangle)P_{\ell';d}(\langle y,z\rangle)\,dy=Z_{\ell;d}(\langle x,z\rangle)\,\delta_{\ell'}^{\ell},\ \text{for all } \ell,\ell'\in\mathbb{N}\ ,$$

where $\delta_\cdot^\cdot$ is the Kronecker delta. Clearly for any $f\in L^2\left(\mathbb{S}^d\right)$ its projection over the space $\mathcal{H}_{\ell;d}$ is given by

$$f_\ell(x)=Z_{\ell;d}[f](x)=\int_{\mathbb{S}^d}Z_{\ell;d}(\langle x,y\rangle)f(y)\,dy=\sum_{m=1}^{N_{\ell,d}}a_{\ell,m}Y_{\ell,m}(x)\,.$$

The standard needlet kernel, as introduced by [32], can then be defined as follows; for any $j=1,2,...$

$$\Psi_j(x,y)=\sum_{\ell\geq 0}b\left(\frac{\ell}{B^j}\right)Z_{\ell,d}(\langle x,y\rangle)\,,$$

where $B>1$ is a fixed (bandwidth) parameter, whereas $b(\cdot):\mathbb{R}\to\mathbb{R}$ is a weight function which satisfies three properties: a) it is compactly supported in $[\frac{1}{B},B]$; b) it is C^∞; c) it satisfies the *Partition of Unity* property, namely, $\sum_{j\geq 0}b^2(\frac{\ell}{B^j})=1$, for all $\ell\in\mathbb{N}$.

Under these conditions, in [32] the following nearly-exponential localization property is established; for all $x,y\in\mathbb{S}^2$ and for all integers M, there exists a constant c_M (depending on $b(\cdot)$, but not on x,y or j) such that

$$\left|\Psi_j(x,y)\right|\leq\frac{c_M B^{dj}}{\left[1+B^j d_{\mathbb{S}^2}(x,y)\right]^M}\ ,\tag{1.2}$$

where $d_{\mathbb{S}^2}(x,y):=\arccos(\langle x,y\rangle)$ is the standard geodesic distance on the sphere. This key localization property can then be exploited to derive a number of extremely important features of the needlet system; indeed the needlet projectors are simply defined by

$$\psi_{j,k}(x)=\sqrt{\lambda_{j,k}}\sum_{\ell\geq 0}b\left(\frac{\ell}{B^j}\right)Z_{\ell,d}\left(\langle x,\xi_{j,k}\rangle\right)\,,\tag{1.3}$$

where $\left\{\xi_{j,k}:j\geq 0,k=1,\dots,K_j\right\}$ and $\left\{\lambda_{j,k}:j\in\mathbb{N},k=1,\dots,K_j\right\}$ are cubature points and weights respectively, see also [32]. The corresponding needlet coefficients are defined as

$$\beta_{j,k}=\langle f,\psi_{j,k}\rangle_{L^2\left(\mathbb{S}^d\right)},\quad j\geq 0,k=1,\dots,K_j,\tag{1.4}$$

where $f(\cdot)$ denotes any (random or deterministic) function in $L^2\left(\mathbb{S}^d\right)$.

As mentioned above, a key ingredient for the interest that needlet transforms have drawn when applied to the analysis of spherical random fields are their asymptotic uncorrelation properties. We can recall them briefly as follows.

Recall that a random field f on the unit sphere $\mathbb{S}^d$ is isotropic if its distribution law is invariant with respect to the action of the group of rotations $SO(d+1)$. Assume we have a zero-mean, finite variance, isotropic random field on $\mathbb{S}^d$; then the spectral representation (1.1) holds in the $L^2\left(\Omega \times \mathbb{S}^d\right)$ sense, where the family of zero-mean random coefficients $\left\{a_{\ell,m}\right\}_{\ell\in\mathbb{N},m=1,\dots,N_{\ell;d}}$ satisfies

$$\mathbb{E}\left[a_{\ell,m}\overline{a}_{\ell',m'}\right] = \delta_\ell^{\ell'}\delta_m^{m'}C_\ell\,,\,\ell,\ell'\in\mathbb{N},m,m'=1,\dots,N_{\ell;d}.$$

The sequence $\{C_\ell\}_{\ell\in\mathbb{N}}$ is labelled the angular power spectrum of the random field. In [3] and many subsequent papers (starting from [5, 27]), it is assumed that the angular power spectrum obeys some regularity condition such as

$$C_\ell = g(\ell)\ell^{-\alpha}, \alpha > 2 \tag{1.5}$$

for some positive $g \in C^\infty$ such that its r-derivative $g^{(r)}(u) = \frac{d^r g(u)}{du^r} = O(u^{-r})$, $r\in\mathbb{N}$, , as $u\to\infty$ (for $\ell = 0$ we allow $C_\ell = C_0$ to take any finite, nonnegative value). For instance, $g(\cdot)$ could be any slowly-varying function, in the sense of [6]. Now write

$$\beta_j(x) := \int_{\mathbb{S}^d} f(y)\Psi_j\left(x,y\right)dy\;; \tag{1.6}$$

up to a normalization, (1.6) can be simply interpreted as a continuous version of (1.4): note indeed that $\beta_{j,k} = \beta_j(\xi_{j,k})\sqrt{\lambda_{j,k}}$. Assuming that $\{f(\cdot)\}$ is an isotropic spherical random field whose angular power spectrum satisfies (1.5), it was shown in [3] that for all positive integers N there exists $c_N > 0$ such that

$$\left|\mathrm{Corr}(\beta_j(x),\beta_j(y))\right| \le \frac{c_N}{(1+B^j d_{\mathbb{S}^2}(x,y))^N} \text{ for all } j\in\mathbb{N}. \tag{1.7}$$

In words, (1.7) is stating that for any two fixed points on the sphere, the correlation between the standard needlet transforms of order j at these two points is going to zero nearly-exponentially (i.e., faster than any polynomials) as j diverges. This uncorrelation property is equivalent to high-frequency independence in the Gaussian case, and hence it makes possible the implementation of a number of statistical procedures whose properties can be rigorously established, in the high-frequency regime.

Notation

From now on we will define $\theta_{xy} = \theta := d_{\mathbb{S}^d}(x,y)$, for $x,y\in\mathbb{S}^d$ and $\theta\in[0,\pi]$. With some abuse of notation we will write equivalently $\Psi_j(x,y)$ and $\Psi_j(\cos\theta_{xy}) = \Psi_j(\cos\theta)$. As usual, given $f(x)$ a C^n function, we will use $f^{(n)}(x)$ to denote its $n-$derivative, whereas c, c_k will be used to denote positive constants, whose values can change from line to line.

1.2. Main Results

As mentioned above, our plan in this paper is to introduce a further degree of flexibility in the needlet construction, by allowing the scale width in the multipole space to cover a much broader spectrum of possibilities than in the existing literature. More precisely, as illustrated in the previous Section in the standard needlet construction the $j-$order transform is supported in the harmonic space over the interval (B^{j-1}, B^{j+1}).

There are several reasons, we believe, why it is of interest to consider needlet-like transforms with more general support in the harmonic space. For instance, practitioners may be interested in multipoles ranging over more general domains than $\ell \in (B^{j-1}, B^{j+1})$ for physical reasons related to their model of interests; otherwise, experimental settings may put specific constraints on the multipoles on which needlet transforms can be computed. Indeed, the largest possible multipole for which data are available is determined by experimental conditions; implementing needlets on a fixed bandwidth range with extrema of the form $\ell \in (B^{j-1}, B^{j+1})$ may entail either overcoming the limit on which multipoles are actually available, or else dropping a subset of frequencies which have been actually observed. On the other hand, it is also sometimes the case that the range of values (B^{j-1}, B^{j+1}) is considered to grow too rapidly for large values of j, and data analysts/applied scientists therefore prefer to reduce it to achieve better frequency-domain resolution in their analysis. Moreover, the range of multipoles of interest may be dictated by physical considerations: for instance, in Cosmological experiments a crucial issue is the determination of peaks in the angular power spectrum, and hence experimentalists may wish to calibrate the needlet domain in harmonic space to make sure features of interest are covered. All these situations have actually taken place, for instance, in the analysis of Cosmic Microwave Background data, and it has been common to implement needlets on varying multipole windows, with no theoretical background to justify these choices, see e.g. [39] and the references therein.

Our plan is then to consider needlet projectors of the following form:

$$\Psi_j(x,y) = \sum_{\ell \geq 0} b_j(\ell) Z_{\ell,d}(\langle x,y \rangle) \ , \tag{1.8}$$

where $\left\{b_j(\cdot)\right\}_{j\in\mathbb{N}}$ is a sequence of weight functions which generalize the sequence $\left\{b\left(\frac{\cdot}{B^j}\right)\right\}_{j\in\mathbb{N}}$ characterizing standard needlets. To make our statements more precise, we will need some more tools and notation; in particular, we need to introduce a *scale sequence* $\left\{S_j\right\}_{j\in\mathbb{N}}$ starting with $S_0 = 1$, that is, an increasing positive-valued sequence such that the support of $b_j(\cdot)$ is included in $\Lambda_j = [S_{j-1}, S_{j+1}]$ for all $j \in \mathbb{N}$. We are therefore implicitly maintaining the semi-orthogonality properties of standard needlets, that is, the support of $b_j(\cdot)$ and $b_{j'}(\cdot)$ are disjoint whenever $\left|j-j'\right| \geq J$, with $J \geq 2$. For notational simplicity, we shall always assume in the sequel that the sequence $S_j - S_{j-1}$ is increasing, i.e.,

$$S_j - S_{j-1} \leq S_{j+1} - S_j \text{ , for all } j \in \mathbb{N};$$

this will allow us to avoid some less elegant statement of results in terms of the largest between $(S_{j+1} - S_j)$ and $(S_j - S_{j-1})$ - the substance of the approach is clearly unaltered. The other key ingredient in the construction is a sequence of kernel functions $\left\{b_j(\cdot)\right\}_{j\in\mathbb{N}}$ on multipole space, depending on the sequence $\left\{S_j\right\}_{j\in\mathbb{N}}$, for which we require the following conditions:

Assumption 1.1. *The sequence of functions $\left\{b_j(\cdot)\right\}$ is such that*

1. *for all $n, j \in \mathbb{N}$*

$$|b_j^{(n)}(u)| \leq K(n) \frac{1}{(S_j - S_{j-1})^n} \ ,$$

where the constant $K(n)$ does not depend on j;

2. *b_j has a compact support in $\Lambda_j = \left[S_{j-1}, S_{j+1}\right]$, with*

$$b_j\left(S_{j-1}\right) = b_j\left(S_{j+1}\right) = 0\;,\; b_j\left(S_j\right) = S_0 = 1\;.$$

Note that for $j = 0$, we set $S_{-1} = 0$ and $\Lambda_0 = [0,1]$;

3. *the* partition of unity property *holds, that is,*

$$\sum_{j\geq 0} b_j^2\left(u\right) = 1, \quad \textit{for all } u \geq 1\;.$$

In the case of standard needlets, the sequence $\left\{b_j\left(\cdot\right)\right\}_{j\in\mathbb{N}}$ can be obtained by scaling a function $b(\cdot)$, which is compactly supported in $\left[B^{-1}, B\right]$ for some $B > 1$ and with bounded derivatives of any order. In particular, in the standard construction we have

$$b_j(u) := b\left(\frac{u}{B^j}\right)\;,\; S_j := B^j\;,$$

and hence

$$|b_j^{(n)}(u)| = \frac{1}{B^{nj}}\left|b^{(n)}\left(\frac{u}{B^j}\right)\right| \leq \frac{1}{\left(B^j - B^{j-1}\right)^n}\sup_u\left|b^{(n)}\left(u\right)\right|\;.$$

The following localization property is the first main result of this paper:

Theorem 1.2 (Localization property). *As $j \to \infty$, for all $\theta \in [0,\pi]$ and $M \in \mathbb{N}$, with $M > d$, there exists a constant $c_M > 0$ (i.e., $\theta-$uniform and independent from j) such that*

$$\left|\Psi_j\left(\cos\theta\right)\right| \leq c_M(S_{j+1}^d - S_{j-1}^d)\max\left\{\frac{1}{(1+S_{j-1}\theta)^{2M}}, \frac{1}{\left(1+\left(S_j - S_{j-1}\right)\theta\right)^{2M}}\right\}\;. \tag{1.9}$$

It is important to note that in the standard case (i.e., for $\left\{S_j := B^j\right\}_{j\in\mathbb{N}}$, some $B > 1$) the bound (1.9) can be written as

$$\begin{aligned}\left|\Psi_j\left(\cos\theta\right)\right| &\leq c_M(B^{(j+1)d} - B^{(j-1)d})\max\left\{\frac{1}{(1+B^{(j-1)}\theta)^{2M}}, \frac{1}{(1+\left(B^j - B^{j-1}\right)\theta)^{2M}}\right\} \\ &\leq c_M^{\prime}(B - \frac{1}{B})B^{jd}\max\left\{\frac{B^{2M}}{(1+B^j\theta)^{2M}}, \frac{(B/(B-1))^{2M}}{(1+B^j\theta)^{2M}}\right\} \\ &= c_M^{\prime\prime}\frac{B^{jd}}{(1+B^j\theta)^{2M}}\;,\end{aligned}$$

so that Theorem 1.2 yields the estimate (1.2) which was established in the pioneering papers [32, 33].

The system of *flexible-bandwidth needlets* $\left\{\psi_{j,k}\left(\cdot\right)\right\}_{j,k}$ (or flexible needlets for short) can now be defined, analogously to (1.3), as $\psi_{j,k}\left(\cdot\right) : \mathbb{S}^d \to \mathbb{R}$ such that

$$\psi_{j,k}\left(\cdot\right) = \sqrt{\lambda_{j,k}}\sum_{\ell\geq 0} b_j\left(\ell\right)Z_{\ell,d}\left(\left\langle\cdot, \xi_{j,k}\right\rangle\right)\;,\; j \geq 0,\; k = 1, \ldots, K_j,$$

$\left\{\xi_{j,k},\lambda_{j,k}\right\}_{j\in\mathbb{N},k=1,...,K_j}$ representing as before sets of cubature points and weights such that

$$\int_{\mathbb{S}^2} Y_{\ell,m}(x)\overline{Y}_{\ell',m'}(x)dx = \sum_{k=1}^{K_j} Y_{\ell,m}(\xi_{j,k})\overline{Y}_{\ell',m'}(\xi_{j,k})\lambda_{j,k} \text{ , for all } \ell,\ell' \leq S_{j+1} \, .$$

Starting from the seminal contribution [33], needlets have been very widely used to characterize decomposition spaces (e.g., Besov and Triebel-Lizorkin). We do not address these issues in this paper, but we focus instead on high-frequency uncorrelation of needlet coefficients evaluated on random fields; more precisely, we investigate the corrrelation of the field (1.6), evaluated by means of (1.8). Assumption (1.5) requires a form of scale invariance of the angular power spectrum at very large multipoles/very small scales. In applications, it is often the case that power spectra may exhibit more complex behaviour, for instance with sinusoidal oscillations as those which characterize the angular power spectrum of Cosmic Microwave Background radiation (see [38]). In the present paper, we hence extend and generalize the previous uncorrelation results (1.7) considering a much broader class of angular power spectra for random fields in $\mathbb{S}^d$; more precisely, we consider power spectra taking the form

Assumption 1.3. *The angular power spectrum satisfies $C_\ell = \ell^{-\alpha}g(\ell)$, where $\ell \geq 1$ and $\alpha > 2$, and the function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is such that*

$$g_1 \leq g(u) \leq g_2, \textit{ for some } g_2 \geq g_1 > 0$$

and for some $\beta \in [0,1)$ and $r \in \mathbb{N}$

$$g^{(r)}(u) = O_{u\to\infty}(u^{-(1-\beta)r}),\ g^{(r)}(u) = \frac{d^r g(u)}{du^r} \, .$$

For instance, Assumption 1.3 covers angular power of the form

$$C_\ell = \sum_{p=1}^{P} c_p \left\{d_p + \sin(\ell^{\beta_p}/M_p)\right\}\ell^{-\alpha},\ d_p > 1 \text{ , } c_p, M_p > 0 \text{ , } 0 < \beta_p < 1 \text{ for } p = 1, ..., P,$$

thus exhibiting much richer oscillations than allowed in (1.5).

We now investigate uncorrelation properties in this broader framework, and we establish our second main result.

Theorem 1.4 (Uncorrelation property). *Under Assumptions 1.1 and 1.3, there exists positive constants c_N (depending on α, d and $g(\cdot)$), such that, as $j \to \infty$ we have*

$$|\mathrm{Corr}(\beta_j(x),\beta_j(y))| \leq c_N \times \max\left\{\frac{1}{(S_{j-1}^{(1-\beta)}\theta)^{2N}}, \frac{1}{((S_j - S_{j-1})\theta)^{2N}}\right\} \, . \tag{1.10}$$

As we discuss in the subsection below, this result generalizes uncorrelation properties in the literature even in the standard needlet case $S_j = B^j$ for $j = 1, 2, ...,$ and hence we believe it can have considerable importance for applications. More precisely, the uncorrelation properties of standard needlets have been used in the Cosmological literature to develop estimators of power asymmetries ([37]), local estimators of nonGaussianity ([41]), estimators of geometric functionals to search for anisotropies ([39]), point-source detection ([9]) and many other related issues. Our present results justify the applicability of these techniques under much broader assumptions than so far considered.

1.3. Discussion

Some remarks are in order:

- Given the localization result established in Theorem 1.2, and the details of the construction of the needlet kernel, it can be easily verified that flexible needlets form a *tight frame* and they allow for exact reconstruction formulae. More formally, for all $f \in L^2\left(\mathbb{S}^d\right)$ it is standard to show that the corresponding needlet coefficients satisfy

$$\begin{aligned}
\sum_{j\in\mathbb{N}}\sum_{k=1}^{K_j}\beta_{j,k}^2 &= \sum_{j\in\mathbb{N}}\sum_{k=1}^{K_j}\lambda_{j,k}\left[\sum_{\ell\in\Lambda_j} b_j(\ell)a_{\ell,m}Y_{\ell,m}(\xi_{j,k})\right]^2\\
&= \sum_{j\in\mathbb{N}}\sum_{\ell_1,\ell_2\in\Lambda_j}\sum_{m_1=1}^{N_{\ell_1;d}}\sum_{m_2=1}^{N_{\ell_2;d}} b_j(\ell_1)b_j(\ell_2)a_{\ell_1,m_1}\overline{a}_{\ell_2,m_2}\\
&\quad\times\sum_{k=1}^{K_j}\lambda_{j,k}Y_{\ell_1,m_1}(\xi_{j,k})\overline{Y}_{\ell_2,m_2}(\xi_{j,k})\\
&= \sum_{j\in\mathbb{N}}\sum_{\ell_1,\ell_2\in\Lambda_j}\sum_{m_1=1}^{N_{\ell_1;d}}\sum_{m_2=1}^{N_{\ell_2;d}} b_j(\ell_1)b_j(\ell_2)a_{\ell_1,m_1}\overline{a}_{\ell_2,m_2}\delta_{\ell_1}^{\ell_2}\delta_{m_1}^{m_2}\\
&= \sum_{\ell\in\mathbb{N}}\sum_{m=1}^{N_{\ell;d}}\sum_{j\in\mathbb{N}} b_j^2(\ell)\left|a_{\ell,m}\right|^2 = \sum_{\ell\in\mathbb{N}}\sum_{m=1}^{N_{\ell;d}}\left|a_{\ell,m}\right|^2 = \|f\|^2_{L^2\left(\mathbb{S}^d\right)}\ ,
\end{aligned}$$

  where the last equality is due to Parseval's identity. Likewise, it can be shown that the following reconstruction formula holds:

$$f(\cdot) = \sum_{j\in\mathbb{N}}\sum_{k=1}^{K_j}\beta_{j,k}\psi_{j,k}(\cdot) \text{ in } L^2\left(\mathbb{S}^d\right)\ .$$

  The details here are identical to those in the seminal papers by [32, 33], and are therefore omitted for brevity's sake.
- In the standard needlet case and under (1.5), (1.10) leads to the following bound:

$$c_N \times \max\left\{\frac{1}{(B^{(j-1)(1-\beta)}\theta)^{2N}}, \frac{1}{((B^j - B^{j-1})\theta)^{2N}}\right\} \leq \frac{c_N}{(B^{(j-1)(1-\beta)}\theta)^{2N}}\ .$$

  As mentioned above, this result generalizes to all $\beta \in [0,1)$ the uncorrelation bound for needlet coefficients which was given for $\beta = 0$ by [3] and then exploited in a number of subsequent papers to construct statistical procedures with an asymptotic justification, in the high-frequency sense.
- In the general case, asymptotic uncorrelation can continue to hold for $\beta > 0$, but the upper bound is less and less efficient as β grows; indeed assuming that $S_{j-1}^{1-\beta} < (S_j - S_{j-1})$, for all $\theta \in [0,\pi]$

we get

$$|\mathrm{Corr}(\beta_j(x),\beta_j(y))| \leq \max\left\{1, \frac{c_N}{(S_{j-1}^{1-\beta}\theta)^{2N}}\right\} .$$

It should be noted that in the discretized case the construction of cubature points is such that their minimum distance decays as

$$d_j := \min_{k,k'} d_{\mathbb{S}^2}(\xi_{jk},\xi_{jk'}) \simeq S_{j-1}^{-1} \text{ , as } j \to \infty .$$

For $\beta = 0$, it then follows that needlet coefficients have correlations decaying to zero (as $j \to \infty$) when evaluated on any pair of locations whose distance decays more slowly than d_j. This is no longer the case for less regular power spectra: indeed for $\beta > 0$ to ensure asymptotic uncorrelation we must consider pair of coefficients whose distance is fixed or decays to zero more slowly than $d_j^{1-\beta}$.

1.4. Some Simple Applications

The uncorrelation properties of spherical needlets have allowed for an enormous amount of applications in statistical inference in the last few years, among which we mention subsampling techniques in [5], Whittle estimation of the model parameters in [15], point source detection in [10], testing for isotropy in [39], and many others. For brevity's sake we do not develop these applications in the broader framework considered in this paper; we just include a simple examples on goodness of fit testing.

As it is often the case in the analysis (for instance) of CMB data, we assume a Gaussian isotropic random field $\{f(.)\}$ is observed on a region $D \subset \mathbb{S}^2$, and out of the observations in this region we need to check goodness of fit for some given model for the angular power spectrum, $\{C_\ell = C_\ell(\theta)\}_{\ell\in\mathbb{N}}$. For any $j \in \mathbb{N}$, let Ξ_j denote the grid of cubature points $\left\{\xi_{j,k}\right\}_{k=1,\dots,K_j}$. Consider the following testing procedure:

a) take a needlet construction such that $S_{j-1}^{1-\beta} < (S_j - S_{j-1})$, for $j = 1, 2, \dots$, so that we impose a lower bound on the width of $(S_j - S_{j-1})$ (i.e., $(S_j - S_{j-1})/S_{j-1}$ can shrink to zero, but $\{(S_j - S_{j-1})/S_{j-1}^{1-\beta}\}$ cannot); compute the needlet coefficients $\left\{\beta_{j,k}\right\}_{k=1,\dots,K_j}$

b) choose a subset D_j of these coefficients such that, for $k \in D_j$, $\xi_{j,k} \in \Xi_j \cap D$, and for all $k, k' \in D_j$ one has $d_{\mathbb{S}^2}(\xi_{j,k},\xi_{j,k'}) > \delta/S_{j-1}^{1-\beta-\varepsilon}$, for $\delta, \varepsilon > 0$, and at the same time $\mathrm{card}\left\{D_j\right\} \to \infty$ as $j \to \infty$ (the elements of D_j can be viewed as a subsampling of the cubature points in the grid Ξ_j with some constraints on their distance). Note that for all $M > 0$, there exist c_M such that

$$|\mathrm{Corr}(\beta_{j,k},\beta_{j,k'})| \leq \frac{c_M}{(S_{j-1}^{1-\beta} d_{\mathbb{S}^2}(\xi_{j,k},\xi_{j,k'}))^M} \leq c_M \delta^{-M} S_{j-1}^{-M\varepsilon} \text{ for all } j \in \mathbb{N},$$

hence in particular $\mathrm{Corr}(\beta_{j,k},\beta_{j,k'}) \to 0$ as $j \to \infty$ for all $k, k' \in D_j$, with $k \neq k'$.

c) now compute

$$I_j = \frac{1}{\sqrt{2\mathrm{card}\left\{D_j\right\}}} \sum_{k\in D_j} \left\{ \frac{\beta_{j,k}^2}{\mathbb{E}\left\{\beta_{j,k}^2\right\}} - 1 \right\} \text{ , where } \mathbb{E}\left\{\beta_{j,k}^2\right\} = \sum_{\ell\in\Lambda_j} b_j^2(\ell) \frac{2\ell+1}{4\pi} C_\ell \, .$$

It is immediate to see that $\mathbb{E}\left\{I_j\right\} = 0$ and

$$\begin{aligned}
\operatorname{Var}\left\{I_j\right\} &= \frac{1}{2\mathrm{card}\left\{D_j\right\}} \sum_{k \in D_j} \operatorname{Var}\left\{\frac{\beta_{j,k}^2}{\mathbb{E}\left\{\beta_{j,k}^2\right\}}\right\} \\
&\quad + \frac{1}{2\mathrm{card}\left\{D_j\right\}} \sum_{k,k' \in D_j, k \neq k'} \operatorname{Cov}\left\{\frac{\beta_{j,k}^2}{\mathbb{E}\left\{\beta_{j,k}^2\right\}}, \frac{\beta_{j,k'}^2}{\mathbb{E}\left\{\beta_{j,k'}^2\right\}}\right\} \\
&= 1 + A_j
\end{aligned}$$

where, using the Diagram (Wick's) Formula (see [34], p. 202)

$$\begin{aligned}
A_j &= \frac{1}{2\mathrm{card}\left\{D_j\right\}} \sum_{k,k' \in D_j, k \neq k'} \operatorname{Cov}\left\{\frac{\beta_{j,k}^2}{\mathbb{E}\left\{\beta_{j,k}^2\right\}}, \frac{\beta_{j,k'}^2}{\mathbb{E}\left\{\beta_{j,k'}^2\right\}}\right\} \\
&= \frac{1}{\mathrm{card}\left\{D_j\right\}} \sum_{k,k' \in D_j, k \neq k'} \operatorname{Corr}^2\left\{\frac{\beta_{j,k}}{\sqrt{\mathbb{E}\left\{\beta_{j,k}^2\right\}}}, \frac{\beta_{j,k'}}{\sqrt{\mathbb{E}\left\{\beta_{j,k'}^2\right\}}}\right\} \\
&\leq \mathrm{card}\left\{D_j\right\} \times c_M^2 \delta^{-2M} S_{j-1}^{-2M\varepsilon} \to 0 \text{ , as } j \to \infty \text{ ,}
\end{aligned}$$

by recalling $\mathrm{card}\left\{D_j\right\} = O(S_j)$ and choosing M such that $S_j = o(S_{j-1}^{2M\varepsilon})$. We have thus shown that $\lim_{j\to\infty} \operatorname{Var}\left(I_j\right) = 1$.

d) finally, recalling that for a zero mean random variable X the fourth cumulant is defined by $\mathrm{Cum}_4(X) = \mathbb{E}[X^4] - 3\mathbb{E}[X^2]^2$, it is now a standard computation to show that

$$\begin{aligned}
\mathrm{Cum}_4\left\{I_j\right\} &= \frac{1}{\left\{2\mathrm{card}\left\{D_j\right\}\right\}^2} \mathrm{Cum}_4\left\{\sum_{k \in D_j}\left\{\frac{\beta_{j,k}^2}{\mathbb{E}\left\{\beta_{j,k}^2\right\}} - 1\right\}\right\} \\
&= \frac{1}{\left\{2\mathrm{card}\left\{D_j\right\}\right\}^2}
\end{aligned}$$

$$\times O\left\{\sum_{k_1,k_2,k_3,k_4 \in D_j} \operatorname{Corr}^2\left\{\frac{\beta_{j,k_1}^2}{\mathbb{E}\left\{\beta_{j,k_1}^2\right\}}, \frac{\beta_{j,k_2}^2}{\mathbb{E}\left\{\beta_{j,k_3}^2\right\}}\right\} \ldots \operatorname{Corr}^2\left\{\frac{\beta_{j,k_4}^2}{\mathbb{E}\left\{\beta_{j,k_4}^2\right\}}, \frac{\beta_{j,k_2}^2}{\mathbb{E}\left\{\beta_{j,k_1}^2\right\}}\right\}\right\}$$

$$= O\left(\frac{1}{\mathrm{card}\left\{D_j\right\}}\right) .$$

It is then an immediate application of the Malliavin-Stein method (see [34] and the references therein) to prove that a (quantitative) Central Limit Theorem holds for the sequence $\left\{I_j\right\}_{j\in\mathbb{N}}$, thus making well-principled goodness of fit tests available.

In a similar manner, under these broader circumstances extensions can be implemented for needlet based-procedures in a number of areas of theoretical and applied interest: we mention for instance high-frequency maximum likelihood estimates (as investigated by [15] in the standard needlet case), polyspectra estimation (see e.g., [8]), isotropy testing (see [39]), power spectrum estimation (see [5, 38]), point source detection (see [9, 10]) and many others. For brevity's sake, we do not discuss the implementation details here. It should be noted that the uncorrelation properties that we established for the needlet coefficients do not require by any means the Gaussianity assumption: they hold for general isotropic random fields. However, we do need Gaussianity to apply the Diagram formula and the fourth cumulant results in points c) and d) above; hence the non Gaussian framework is considerably more challenging.

1.5. Plan of the Paper

The properties of flexible-bandwidth needlets in terms of localization in real space are discussed in Section 2, while uncorrelation properties are investigated in Section 3; an explicit construction for the sequence of kernels $\left\{b_j(\cdot)\right\}_{j\in\mathbb{N}}$ is given in the Appendix (Section 4).

2. Localization Properties

In this section we will establish a localization property which generalizes analogous results for standard needlets in [32], Mexican needlets in [14, 17] and scale-directional wavelets in [30]. Throughout this section S_j, Λ_j, b_j are defined as in Section 1.

Let us first recall some useful notation. Consider a real-valued sequence $\{r_\ell : \ell \geq 0\}$ and let the *discrete difference operators* Δ^+, Δ^- be defined by

$$\Delta^+ r_\ell := r_{\ell+1} - r_\ell \; , \; \Delta^- r_\ell := r_\ell - r_{\ell-1}.$$

These operators can be viewed as discrete versions of derivation on sequences (see also [27, Definition 2.1]), and can be used to define

$$\Upsilon_d\left(\ell\right) := \upsilon_{1;d}\left(\ell\right)\Delta^-\Delta^+ + \upsilon_{0;d}\left(\ell\right)\Delta^+, \; d\geq 2\,,$$

where

$$\begin{aligned}
\upsilon_{1,d}\left(\ell\right) &:= \frac{\ell}{2\left(\ell+\eta_d\right)} = \frac{\ell}{2\ell+d-1} = \frac{1}{2} - \frac{d-1}{4\ell+2d-2}\;, \\
\upsilon_{0,d}\left(\ell\right) &:= \frac{2\eta_d}{2\left(\ell+\eta_d\right)} = \frac{d-1}{2\ell+d-1} \leq \frac{d-1}{2\ell}\;.
\end{aligned}$$

Our main result is the following.

Proposition 2.1 (Localization). *Let $\Psi_j(\cdot)$ be defined as*

$$\Psi_j\left(\cos\theta\right) := \sum_{\ell\in\Lambda_j} b_j\left(\ell\right) Z_{\ell;d}\left(\cos\theta\right) \;,\; j\in\mathbb{N},$$

where for all $M>0$ there exists a positive constant $c_M>0$ such that

$$\left|\left(\Delta^-\right)^M\left(\Delta^+\right)^M b_j\left(\ell\right)\right| \leq c_M \frac{1}{\left(S_j - S_{j-1}\right)^{2M}}. \tag{2.1}$$

Then, it holds that

$$\left|(\cos\theta-1)^M\Psi_j(\cos\theta)\right|\leq c_M\left(S_{j+1}^d-S_{j-1}^d\right)\max\left\{\frac{1}{S_{j-1}^{2M}},\frac{1}{\left(S_j-S_{j-1}\right)^{2M}}\right\}$$

and hence, because $\theta^2=O(|\cos\theta-1|)$ *for* $\theta\in(0,\pi)$

$$\left|\Psi_j(\cos\theta)\right|\leq c_M(S_{j+1}^d-S_{j-1}^d)\max\left\{\frac{1}{S_{j-1}^{2M}\theta^{2M}},\frac{1}{\left(S_j-S_{j-1}\right)^{2M}\theta^{2M}}\right\}.$$

***Remark* 2.2.** The literature on localization properties for needlet-like constructions is now very rich. To the best of our knowledge, the most general result is given in Theorem 3.1 by [23], in the framework of Dirichlet spaces with a doubling measure and local scale-invariant Poincaré inequality, and hence much broader circumstances than we consider here. However, in [23] needlets are implemented by "dilating" a single function $f(.)$ at various scales, i.e. by studying the kernel $f(\delta\sqrt{L})(x,y)$, where L is a differential operator, δ is a scaling parameter (which can be taken as $\delta=2^{-j}$ in the standard needlet case) and the function f is the same for every value of δ. Our situation here is different because of the varying bandwidth; in the notation of [23], our kernel should take a form such as $f_\delta(\delta\sqrt{L})$, that is f_δ itself has to vary with the parameter δ (to ensure the partition of unity and reconstruction properties). Most probably, our result could also be obtained as a consequence of Theorem 3.1 of [23], by a construction of $f_\delta(.)$ similar to the one we implemented and by establishing proper bounds (which will depend on δ) on the derivatives of $f_\delta(.)$; it can indeed be noted that in the case of the sphere and for $S_j=B^j$, Theorem 3.1 in [23] and our result do yield the same bounds. Such a new construction could even allow a generalization of flexible needlets to more general Dirichlet spaces; we leave this as a topic for future research.

The proof of the previous results requires the following two lemmas, which are generalizations to $\mathbb{S}^d$ of [27, Lemma 4.1], where $\mathbb{S}^2$ was considered.

Lemma 2.3. *Let*

$$q(\cos\theta):=\sum_{\ell\geq0}r_\ell\frac{(\ell+\eta_d)}{\eta_d\omega_d}G_\ell^{(\eta_d)}(\cos\theta)=\sum_{\ell\geq0}r_\ell Z_{\ell;d}(\cos\theta),$$

where $\{r_\ell:\ell\geq0\}$ *is a real-valued sequence. Then, for any* $N\in\mathbb{N}$*,*

$$(\cos\theta-1)^N q(\cos\theta)=\sum_{\ell\in\mathbb{N}}r_{\ell;d}^{[N]}Z_{\ell;d}(\cos\theta),\tag{2.2}$$

where $r_{\ell;d}^{[N]}:=\Upsilon_d^N(\ell)\,r_\ell$*.*

Proof of Lemma 2.3. Recall first the identity, valid for $x\in[-1,1]$, $\ell\in\mathbb{N}_0$

$$\begin{aligned}&(x-1)\left[2(\ell+\eta_d)G_\ell^{(\eta_d)}(x)\right]\\&\quad=(\ell+1)\,G_{\ell+1}^{(\eta_d)}(x)-2\,(\ell+2\eta_d)\,G_\ell^{(\eta_d)}(x)+(\ell+2\eta_d-1)\,G_{\ell-1}^{(\eta_d)}(x),\end{aligned}$$

see [1, Equation 22.7.3]. With the convention $G_{-1}^{(\eta_d)}(x) = 0$ for any $x \in [-1,1]$, $r_{-1} = 0$, and writing $Z_{\ell;d}(\cos\theta) = 2(\ell+\eta_d)G_{\ell}^{(\eta_d)}(\cos\theta)$, we have

$$\begin{aligned}
&\sum_{\ell\geq 0} r_\ell \left[(x-1)\, Z_{\ell;d}(x)\right] \\
&\quad = \sum_{\ell\geq 0} r_\ell \left[\frac{\ell+1}{2\left((\ell+1)+\eta_d\right)} Z_{\ell+1;d}(x) - Z_{\ell;d}(x) + \frac{\ell+2\eta_d-1}{2\left((\ell-1)+\eta_d\right)} Z_{\ell-1;d}(x)\right] \\
&\quad = \sum_{\ell\geq 1} r_{\ell-1} \frac{\ell}{2(\ell+\eta_d)} Z_{\ell;d}(x) - \sum_{\ell\geq 0} r_\ell Z_{\ell;d}(x) + \sum_{\ell\geq -1} r_{\ell+1} \frac{\ell+2\eta_d}{2(\ell+\eta_d)} Z_{\ell;d}(x) \\
&\quad = \sum_{\ell\geq 0} \left[\frac{\ell}{2(\ell+\eta_d)} r_{\ell-1} - \frac{2(\ell+\eta_d)}{2(\ell+\eta_d)} r_\ell + \frac{\ell+2\eta_d}{2(\ell+\eta_d)} r_{\ell+1}\right] Z_{\ell;d}(x) \\
&\quad = \sum_{\ell\geq 0} \left[\frac{\ell}{2(\ell+\eta_d)} (r_{\ell-1} - 2r_\ell + r_{\ell+1}) + \frac{2\eta_d}{2(\ell+\eta_d)} (r_{\ell+1} - r_\ell)\right] Z_{\ell;d}(x) \\
&\quad = \sum_{\ell\geq 0} r_\ell^{[1]} Z_{\ell;d}(x)\,.
\end{aligned}$$

Now, fixing $x = \cos\theta$ and dividing by $2\eta_d\omega_d$, we obtain that

$$(\cos\theta - 1)\, q(\cos\theta) = \sum_{\ell\geq 0} r_\ell^{[1]} Z_{\ell;d}(\cos\theta)\,.$$

Iterating, we obtain (2.2).

□

***Remark* 2.4.** Lemma 2.3 exploits the natural fact that if a function $q(u)$ can be expanded into Gegenbauer polynomials with coefficients $\{r_\ell : \ell \geq 0\}$, then also $(u-1)^N q(u)$ can also be expanded with coefficients which can explicitly computed by properly applying iteratively the difference operators to the sequence $\{r_\ell : \ell \geq 0\}$. In some sense, this can be viewed as an extension to the spherical domain of the classical duality relationships between Fourier transforms and derivatives.

Let us prove now that $b_j(\ell)$ satisfies (2.1).

Lemma 2.5. *For any* $N \in \mathbb{N}$

$$|\Upsilon_d^N(\ell)\, b_j(\ell)| \leq \frac{1}{2^N}(2N)!|\max_u\{b_j^{(2N)}(u)\}| + \sum_{i=0}^{2N-1} \frac{C(i)}{\ell^{2N-i}} |\max_u\{b_j^{(i)}(u)\}|.$$

Proof. Let us consider first $N = 1$. Then we have

$$\begin{aligned}
\Upsilon_d(\ell)\, b_j(\ell) &= \left(\upsilon_{1;d}(\ell)\, \Delta^- \Delta^+ + \upsilon_{0;d}(\ell)\, \Delta^+\right) b_j(\ell) \\
&= \upsilon_{1;d}(\ell)\, \Delta^- \left(b_j(\ell+1) - b_j(\ell)\right) + \upsilon_{0;d}(\ell) \left(b_j(\ell+1) - b_j(\ell)\right) \\
&= \frac{\ell}{2(\ell+\eta_d)} \left(b_j(\ell+1) - b_j(\ell) - (b_j(\ell) - b_j(\ell-1))\right) + \frac{2\eta_d}{2(\ell+\eta_d)} \left(b_j(\ell+1) - b_j(\ell)\right).
\end{aligned}$$

The Mean Value Theorem implies that there exists $u_1 \in (\ell, \ell+1)$ and $u_2 \in (\ell-1, \ell)$ such that

$$\Upsilon_d\,(\ell)\, b_j(\ell) = \frac{\ell}{2(\ell+\eta_d)} \left(b_j^{(1)}(u_1) - b_j^{(1)}(u_2) \right) + \frac{2\eta_d}{2(\ell+\eta_d)} b_j^{(1)}(u_1). \tag{2.3}$$

Applying once more the Mean Value Theorem we have that there exists $a_1 \in (u_2, u_1)$ such that

$$\Upsilon_d\,(\ell)\, b_j(\ell) = \frac{\ell}{2(\ell+\eta_d)} \left(b_j^{(2)}(a_1)(u_1-u_2) \right) + \frac{2\eta_d}{2(\ell+\eta_d)} b_j^{(1)}(u_1).$$

Hence

$$|\Upsilon_d\,(\ell)\, b_j(\ell)| \leq \frac{2\ell}{2(\ell+\eta_d)} \max \left| b_j^{(2)}(u) \right| + \frac{2\eta_d}{2(\ell+\eta_d)} \max \left| b_j^{(1)}(u) \right|.$$

Our assumptions on $b_j(\ell)$ and its derivatives allow to complete the proof for $N=1$. The general case follows applying Υ_d^{N-1} on (2.3) and using induction, for $N \in \mathbb{N}$.

***Remark* 2.6.** Observe that

$$\frac{2\ell}{2(\ell+\eta_d)} \max \left| b_j^{(2)}(u) \right| \leq \frac{2\ell}{2(\ell+\eta_d)} \frac{1}{(S_j - S_{j-1})^2} \leq \frac{c_d}{(S_j - S_{j-1})^2}$$

$$\frac{2\eta_d}{2(\ell+\eta_d)} \max \left| b_j^{(1)}(u) \right| \leq \frac{2\eta_d}{2(\ell+\eta_d)} \frac{1}{(S_j - S_{j-1})} \leq \frac{c'_d}{\ell(S_j - S_{j-1})},$$

where $c_d, c'_d > 0$ depend only on d. Then

$$|\Upsilon_d\,(\ell)\, b_j(\ell)| \leq c \max \left\{ \frac{1}{S_{j-1}^2}, \frac{1}{(S_j - S_{j-1})^2} \right\}.$$

More generally,

$$|\Upsilon_d^N\,(\ell)\, b_j(\ell)| \leq c(2N) \max \left\{ \frac{1}{S_{j-1}^{2N}}, \frac{1}{(S_j - S_{j-1})^{2N}} \right\}. \tag{2.4}$$

□

***Remark* 2.7.** It is immediate to see that, as $j \to \infty$,

$$\sum_{\ell \in \Lambda_j} \ell^{d-1} = \frac{1}{d}(S_{j+1}^d - S_{j-1}^d) + O(S_{j+1}^{d-1}) = O(S_{j+1}^d - S_{j-1}^d)\,.$$

***Proof of Proposition 2.1*.** For any $j \in \mathbb{N}_0$, it suffices to note that applying Lemma 2.3 yields

$$\left| (\cos\theta - 1)^N \, \Psi_j\,(\cos\theta) \right| = \left| \sum_{\ell \geq 0} b_j(\ell)^{[N]} Z_{\ell,d}(\cos\theta) \right|.$$

Lemma 2.5, (2.4) and the conditions on $b_j(\cdot)$ imply that for all $M > 0$

$$\begin{aligned}
&\left|(\cos\theta-1)^M\,\Psi_j\,(\cos\theta)\right| \\
&\leq c_M \max\left\{\frac{1}{S_{j-1}^{2M}},\frac{1}{\left(S_j-S_{j-1}\right)^{2M}}\right\}\sum_{\ell\in\Lambda_j}\frac{\ell+\eta_d}{\eta_d\omega_d}\left|G_\ell^{(\eta_d)}(\cos\theta)\right|.
\end{aligned}$$

In view of Remark 2.7, because $\theta^2=O(|\cos\theta-1|)$, we have

$$\left|\Psi_j\,(\cos\theta)\right|\leq c_M'\max\left\{\frac{1}{(S_{j-1})^{2M}},\frac{1}{(S_j-S_{j-1})^{2M}}\right\}\frac{(S_{j+1}^d-S_{j-1}^d)}{\theta^{2M}}$$

as claimed. □

3. Uncorrelation Properties

Our last step consists in showing that kernels of the type

$$\Phi_j\,(\cos\theta)=\sum_{\ell\in\Lambda_j}b_j^2\,(\ell)\,C_\ell Z_{\ell;d}\,(\cos\theta)\,,$$

satisfy a localization property under the conditions on the power spectrum C_ℓ specified in Assumption 1.3. This result will allow us to show that needlet coefficients are asymptotically uncorrelated for $j\to\infty$.

Recall first that, for all $d=1,2,\ldots$

$$\begin{aligned}
\left|Z_{\ell;d}\,(\cos\theta)\right| &\leq \frac{2\ell+d-1}{(d-1)}\binom{\ell+d-2}{\ell} \\
&\leq c_d\times\ell^{d-1},
\end{aligned}$$

where the constant c_d depends only on d. Now note that

$$\begin{aligned}
\frac{d^N}{du^N}(b_j(u)^2u^{-\alpha}g(u)) &= \sum_{k=0}^N\binom{N}{k}\frac{d^k}{du^k}b_j(u)^2\frac{d^{N-k}}{du^{N-k}}(u^{-\alpha}g(u)) \\
&= \sum_{k=0}^N\binom{N}{k}\frac{d^k}{du^k}(a_{j+1}(u)-a_j(u))\sum_{i=0}^{N-k}\binom{N-k}{i}\frac{d^i}{du^i}u^{-\alpha}\frac{d^{N-k-i}}{du^{N-k-i}}g(u) \\
&= \sum_{k=0}^N\binom{N}{k}\frac{d^k}{du^k}(a_{j+1}(u)-a_j(u))\sum_{i=0}^{N-k}\binom{N-k}{i}[-\alpha]_iu^{-\alpha-i}\frac{d^{N-k-i}}{du^{N-k-i}}g(u),
\end{aligned}$$

where

$$[-\alpha]_i:=-\alpha(-\alpha-1)\ldots(-\alpha-i+1)\,.$$

It follows that, for all ℓ such that $S_{j-1}\leq\ell\leq S_{j+1}$, we have

$$\left|\frac{d^N}{du^N}(b_j(u)^2u^{-\alpha}g(u))\right|_{u=\ell} \leq c_{N,\alpha}\sum_{k=0}^{N}\frac{1}{(S_j-S_{j-1})^k}\sum_{i=0}^{N-k}\ell^{-\alpha-i}\ell^{-(N-k-i)(1-\beta)}$$
$$\leq c_{N,\alpha}\ell^{-\alpha}\ell^{-N(1-\beta)}\sum_{k=0}^{N}\frac{\ell^{k(1-\beta)}}{(S_j-S_{j-1})^k}$$
$$=\sum_{k=0}^{N}\frac{c_{N,\alpha}\ell^{-\alpha}}{(S_j-S_{j-1})^k\ell^{(N-k)(1-\beta)}}\ .$$

Note that for $(S_j - S_{j-1}) \geq S_{j-1}^{(1-\beta)}$ the denominator is bounded below by $S_{j-1}^{N(1-\beta)}$, whereas for $(S_j - S_{j-1}) < S_{j-1}^{(1-\beta)}$ we have the smaller bound $(S_j - S_{j-1})^N < S_{j-1}^{-N(1-\beta)}$. The bottom line is hence

$$\left|\frac{d^N}{du^N}(b_j(u)^2u^{-\alpha}g(u))\right|_{u=\ell} \leq c_{N,\alpha}\times\ell^{-\alpha}\times\max\left\{\frac{1}{S_{j-1}^{N(1-\beta)}},\frac{1}{(S_j-S_{j-1})^N}\right\}\ ,$$

where $c_{N,\alpha} > 0$.

Now consider the correlation function

$$\Phi(\cos\theta)=\sum_{\ell\in\Lambda_j}b_j(\ell)^2\ell^{-\alpha}g(\ell)Z_{\ell,d}(\cos\theta)\ ;$$

we have the bound

$$|\cos\theta-1|^N\Phi(\cos\theta)=\sum_{\ell\in(S_{j-1},S_{j+1})}\left\{\Upsilon_d^N(\ell)\,b_j(\ell)^2\ell^{-\alpha}g(\ell)\right\}Z_{\ell,d}(\cos\theta)$$
$$\leq C\times\max\left\{\frac{1}{S_{j-1}^{2N(1-\beta)}},\frac{1}{(S_j-S_{j-1})^{2N}}\right\}\sum_{\ell\in(S_{j-1},S_{j+1})}\ell^{-\alpha}Z_{\ell,d}(\cos\theta)$$
$$\leq C\times\max\left\{\frac{1}{S_{j-1}^{2N(1-\beta)}},\frac{1}{(S_j-S_{j-1})^{2N}}\right\}$$
$$\times\min\left\{(S_{j+1}-S_{j-1})S_{j-1}^{d-\alpha-1},S_{j-1}^{d-\alpha}\right\}\ ,$$

where $C > 0$. It is easy to check that the denominator (i.e., the variance of the field $\beta_j(\cdot)$) is given by

$$\sum_{\ell\in\Lambda_j}b_j(\ell)^2\ell^{-\alpha}g(\ell)\frac{\ell+\eta_d}{\eta_d\omega_d}G_\ell^{(\eta_d)}(1).$$

Because $b_j^{(1)} \leq K/(S_j - S_{j-1})$ and $b_j(\ell) = 1$ for some $\ell \in \Lambda_j$, by a simple first-order Taylor expansion it is readily seen that there exist S'_{j-1}, S'_{j+1} which satisfy the following conditions:

$$S_{j-1} < S'_{j-1} < S'_{j+1} < S_{j+1} \ ,$$
$$(S'_{j+1} - S'_{j-1}) > c_1(S_{j+1} - S_{j-1}) \text{ , some } c_1 > 0 \ ,$$
$$b_j(\ell) > c_2 > 0 \text{ for all } \ell \in (S'_{j-1}, S'_{j+1}) \ ,$$

where the constants c_1, c_2 are uniform (they do not depend on j). Hence we have the lower bound

$$\sum_{\ell \in \Lambda_j} b_j(\ell)^2 \ell^{-\alpha} g(\ell) \frac{\ell + \eta_d}{\eta_d \omega_d} G_\ell^{(\eta_d)}(1) \geq c_2^2 \sum_{\ell \in (S'_{j-1}, S'_{j+1})} \ell^{-\alpha} g(\ell) \frac{\ell + \eta_d}{\eta_d \omega_d} G_\ell^{(\eta_d)}$$
$$\geq C \times \min \left\{ (S_{j+1} - S_{j-1}) S_{j-1}^{d-\alpha-1}, S_{j-1}^{d-\alpha} \right\} \ ,$$

where $C > 0$. Then, we have

$$|\mathrm{Corr}(\beta_j(x), \beta_j(y))|$$
$$\leq C \times \max \left\{ \frac{1}{S_{j-1}^{2N(1-\beta)}}, \frac{1}{(S_j - S_{j-1})^{2N}} \right\} \frac{1}{|\cos\theta - 1|^N}$$
$$\leq C' \times \max \left\{ \frac{1}{(S_{j-1}^{(1-\beta)}\theta)^{2N}}, \frac{1}{((S_j - S_{j-1})\theta)^{2N}} \right\},$$

with $C, C' > 0$, as claimed.

4. Appendix : an Explicit Construction for $\{b_j(\cdot)\}_{j \in \mathbb{N}}$

In this Appendix, we will provide an explicit construction of $\{b_j(\cdot)\}_{j \in \mathbb{N}}$. Most of the steps are a generalization under the more general circumstances considered in this paper of the procedure which was suggested in [3] for the standard needlet case.

Let us define a non-decreasing sequence of functions $a_j : \mathbb{R}^+ \to [0,1]$ such that

$$a_j \in C^\infty \left(\mathbb{R}^+\right), \ a_j(u) = 1 \text{ for } |u| \leq S_{j-1} \text{ for } j \geq 1,$$

(so that $a_j(0) = 1$, for every $j \in \mathbb{N}$), and

$$0 < a_j(u) \leq 1 \text{ for } u \in \left[S_{j-1}, S_j\right] \ .$$

The support of $a_j(\cdot)$ is contained in $[0, S_j]$. We introduce now a sequence of window functions $\{b_j : j \in \mathbb{N}\}$ given by

$$b_j(u) := \sqrt{a_{j+1}(u) - a_j(u)}. \tag{4.1}$$

Observe that

$$b_j(u) = \begin{cases} \sqrt{1 - a_j(u)} & S_{j-1} < u \leq S_j \\ \sqrt{a_{j+1}(u)} & S_j < u < S_{j+1} \\ 0 & \text{otherwise} \end{cases} . \tag{4.2}$$

Lemma 4.1. *For any $j \in \mathbb{N}$, it holds that $b_j \in C^\infty$.*

Proof. For any $j \in \mathbb{N}$, it follows from Equation (4.2) that $b_j(u) \in C^\infty$ in $(0, S_{j-1}) \cup (S_{j-1}, S_{j+1}) \cup (S_{j+1}, \infty)$. Indeed note that in S_j the function $b_j(u)$ is continuous (since $a_{j+1}(S_j) = 1$ and $a_j(S_j) = 0$). To establish the smoothness of $b_j(\cdot)$ we need to study the behaviour of a_j (and, consequently, b_j) in S_{j-1} and S_{j+1}. In order to do so we prove that left and right derivatives coincide in these two points. Let us start by proving that $b_j(\cdot)$ is C^∞ in S_{j+1}.

The Taylor series of a_{j+1} centered at S_{j+1} can be written as

$$a_{j+1}(u) = a_{j+1}(S_{j+1}) + \cdots + \frac{a_{j+1}^{(n)}(S_{j+1})}{n!}(u - S_{j+1})^n + o((u - S_{j+1})^n) \text{ as } u \to S_{j+1}$$

for all n. Since $a_j(u) \in C^\infty$ and $a_{j+1}^{(k)}(S_{j+1}^+) = 0$ we get that $a_{j+1}^{(k)}(S_{j+1}) = 0$ for all $k = 0, \ldots, n$ and then

$$a_{j+1}(u) = o((u - S_{j+1})^n)$$

for all n, as $u \to S_{j+1}$.

Moreover, we note that $b_j(u) = \sqrt{a_{j+1}(u)}$ for all $u < S_{j+1}$ and then since $a_j(S_{j+1}^-) = 0$, we get that

$$\frac{b_j(u) - b_j(S_{j+1})}{u - S_{j+1}} = \frac{\sqrt{a_{j+1}(u)} - 0}{u - S_{j+1}} = \frac{o(u - S_{j+1})}{u - S_{j+1}} = o(1)$$

and then $b_j \in C^1$ in S_{j+1}.

A similar argument can be implemented for $u = S_{j-1}$. Indeed, we note that $a_j(S_{j-1}) = 1$ and since $a_j(u)$ is C^∞ and it is zero on S_{j-1}^-, we have that $a_j^{(k)}(S_{j-1}) = 0$ for all $k = 1, \ldots, n$. Then a Taylor series expansion leads to

$$a_j(u) = 1 + o((u - S_{j-1})^n)$$

for all n. Moreover, since a_j is continuous and it is equal to 1 in S_{j-1}^- we have that $a_j(S_{j-1}^+) = 1$ and also $a_{j+1}(S_{j-1}) = 1$. Hence in a neighborhood of S_{j-1} we have that $b_j(u) = \sqrt{1 - a_j(u)}$ so that the quotient derivative of $b_j(\cdot)$ from the right is

$$\frac{\sqrt{1 - a_j(u)} - 0}{u - S_{j-1}} = \frac{o(u - S_{j-1})}{u - S_{j-1}} = o(1).$$

Then $b_j \in C^1$ in S_{j-1} which implies $b_j \in C^1$ in $[0, \infty)$; iterating the procedure proves that $b_j \in C^\infty$. □

We propose here a numerical recipe for $b_j(\cdot)$, which is largely analogous to the proposal developed in [3] for the standard needlet construction. First introduce the function $\phi \in C_c^\infty$, given by

$$\phi(t) = \begin{cases} \exp\left(-\frac{1}{1-t^2}\right) & \text{for } t \in [-1, 1] \\ 0 & \text{otherwise .} \end{cases}$$

Consider now

$$\Phi(u) = \begin{cases} 0 & u \leq -1 \\ \frac{\int_{-1}^{u} \phi(t) dt}{c_{\Phi}} & u \in (-1, 1) \\ 1 & u \geq 1 \end{cases},$$

where

$$c_{\Phi} = \int_{-1}^{1} \phi(t)\, dt = \int_{-1}^{1} \exp\left(-\frac{1}{1-t^2}\right) dt \simeq 0.444.$$

Also, for any $j \in \mathbb{N}$, define

$$a_j(u) = \begin{cases} 1 & \text{for } u \in \left[0, S_{j-1}\right] \\ \Phi\left(\frac{S_j + S_{j-1} - 2u}{S_j - S_{j-1}}\right) & \text{for } u \in \left(S_{j-1}, S_j\right] \\ 0 & \text{for } u \in \left[S_j, \infty\right) \end{cases}. \tag{4.3}$$

Note that in $[S_{j-1}, S_j]$

$$a_j(u) = \Phi(\tau_j(u))$$

where $\tau_j : [S_{j-1}, S_{j+1}] \to [-1, 1]$ is a linear transformation defined by

$$\tau_j(u) = m_j u + q_j$$

with

$$m_j = -\frac{2}{S_j - S_{j-1}}; \quad q_j = \frac{S_j + S_{j-1}}{S_j - S_{j-1}}.$$

***Remark* 4.2.** It follows that, for any $r \in \mathbb{N}$,

$$\begin{aligned} a_j^{(r)}(u) &= \frac{d^r}{du^r} a_j(u) = \tau_j^{(r)}(u) \Phi^{(r)}\left(\tau_j(u)\right) \\ &= \frac{(-2)^r}{\left(S_j - S_{j-1}\right)^r} \frac{\phi^{(r-1)}\left(\tau_j(u)\right)}{c_{\Phi}}. \end{aligned} \tag{4.4}$$

Finally, according to (4.1), we can define a sequence of window functions $\left\{b_j : j \in \mathbb{N}\right\}$, where $b_j : \mathbb{R}^+ \to [0, 1]$ is such that

$$b_j(u) := \sqrt{a_{j+1}(u) - a_j(u)}. \tag{4.5}$$

Proposition 4.3. *For any $a_j(\cdot)$ defined as in 4.3 and $n \geq 1$*

$$|a_j^{(n)}(u)| \leq k(n-1) 2^n \frac{1}{(S_j - S_{j-1})^n}$$

where $k(n-1)$ does not depend on j.

Proof. Let us rewrite (4.4) as

$$a_j^{(r)}(u) = \frac{(-2)^r}{(S_j - S_{j-1})^r} \frac{\phi^{(r-1)}(\tau_j(u))}{c_\Phi}. \tag{4.6}$$

In order to study the behavior of $\phi^{(r-1)}(\tau_j(u))$, let us start focusing on the $C^\infty(\mathbb{R})$ function $s : \mathbb{R} \to \mathbb{R}$ given by

$$s(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

We can explicitly compute its derivatives for any $n \in \mathbb{N}$ as

$$s^{(n)}(t) = \begin{cases} \frac{Q_n(t)}{t^{2n}} s(t) & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases},$$

where Q_n is a polynomial of degree $n-1$ defined recursively by the following formula

$$\begin{aligned} Q_1(t) &= 1 \\ Q_{n+1}(t) &= t^2 Q_n'(t) - (2nt - 1) Q_n(t). \end{aligned}$$

Since

$$\phi(\tau_j(u)) = \begin{cases} \exp\left(-\frac{1}{1-\tau_j(u)^2}\right) & \text{if } \tau_j(u) \in [-1,1] \\ 0 & \text{otherwise} \end{cases},$$

we can rewrite

$$\phi(\tau_j(u)) = s(g(\tau_j(u))) \quad \text{with } g(y) = 1 - y^2.$$

Applying the chain rule for high order derivatives for composite functions, the so-called *Faà di Bruno's formula* (see paragraph 4.3, [40]) yields for $\tau_j(u) \in [-1,1]$,

$$\begin{aligned} \frac{d^n}{du^n}\phi(\tau_j(u)) &= n! \sum_{\nu=1}^{n} \frac{s^{(\nu)}(g(\tau_j(u)))}{\nu!} \sum_{h_1+\cdots+h_\nu=n} \frac{(1-\tau_j(u)^2)^{(h_1)}}{h_1!} \cdots \frac{(1-\tau_j(u)^2)^{(h_\nu)}}{h_\nu!} \\ &= n! \sum_{\nu=1}^{n} \frac{Q_n(g(\tau_j(u)))}{g(\tau_j(u))^{2\nu}} \frac{s(g(\tau_j(u)))}{\nu!} \sum_{h_1+\cdots+h_\nu=n} \frac{(1-\tau_j(u)^2)^{(h_1)}}{h_1!} \cdots \frac{(1-\tau_j(u)^2)^{(h_\nu)}}{h_\nu!} \\ &= n! \sum_{\nu=1}^{n} \frac{Q_n(1-\tau_j(u)^2)}{(1-\tau_j(u)^2)^{2\nu}} \frac{e^{-\frac{1}{1-\tau_j(u)^2}}}{\nu!} \sum_{h_1+\cdots+h_\nu=n} \frac{(1-\tau_j(u)^2)^{(h_1)}}{h_1!} \cdots \frac{(1-\tau_j(u)^2)^{(h_\nu)}}{h_\nu!} \end{aligned}$$

where $h_i \geq 1$.

Before we proceed further, we need to recall a couple of immediate facts. First note that if Q_n is a polynomial of degree n, then since $|\tau_j(u)| \leq 1$

$$\left| Q_n(1 - \tau_j(u)^2) \right| \leq C(n).$$

Also, it holds that

$$\sum_{h_1+\cdots+h_\nu=n} \frac{(1-\tau_j(u)^2)^{(h_1)}}{h_1!} \cdots \frac{(1-\tau_j(u)^2)^{(h_\nu)}}{h_\nu!} \leq \binom{n+\nu-1}{n} (2)^\nu (\tau_j(u))^\nu .$$

Indeed inside the sum we have the first and second derivatives of $(1-\tau_j(u)^2)$ and hence we are summing terms of the form $2^\alpha(2\tau_j(u))^\beta$ with $\alpha+\beta=\nu$. The binomial coefficient counts all the possible combinations such that $h_1+\cdots+h_\nu=n$.

Thus we have that

$$\phi^{(n)}(\tau_j(u)) \leq n! C(n) \exp\left(-\frac{1}{1-\tau_j(u)^2}\right) \sum_{\nu=1}^{n} \frac{\tau_j(u)^\nu}{(1-\tau_j(u)^2)^{2\nu}} \frac{2^\nu}{\nu!} \binom{n+\nu-1}{n}.$$

Now, considering that

$$\frac{\tau_j(u)^\nu}{(1-\tau_j(u)^2)^{2\nu}} \leq \frac{1}{(1-\tau_j(u)^2)^{2n}},$$

$$\sum_{\nu=1}^{n} \frac{(2)^\nu}{\nu!} \binom{n+\nu-1}{n} = \frac{2^n n \binom{2n}{n}}{n+1};$$

it follows that

$$\phi^{(n)}(\tau_j(u)) \leq n! \frac{2^n n \binom{2n}{n}}{n+1} C(n) \exp\left(-\frac{1}{1-\tau_j(u)^2}\right) \frac{1}{(1-\tau_j(u)^2)^{2n}}.$$

Finally, observe that

$$\left| \exp\left(-\frac{1}{1-\tau_j(u)^2}\right) \frac{1}{(1-\tau_j(u)^2)^{2n}} \right| \leq \max\left\{ \exp\left(-\frac{1}{1-\tau_j(u)^2}\right) \frac{1}{(1-\tau_j(u)^2)^{2n}} \right\} = \frac{k(n)}{e^{2n}}$$

for $\tau_j(u) \in [-1,1]$, leading to

$$\left| \phi^{(n)}(\tau_j(u)) \right| \leq k(n)$$

where $k(n)$ does not depend on j. Substituting in (4.6) the proof of the proposition is completed. □

The next result is similar.

Lemma 4.4. *For any $b_j(\cdot)$ defined as in 4.5 and $n=1,2,....$, we have that*

$$|b_j^{(n)}(u)| \leq K(n) \frac{1}{(S_j-S_{j-1})^n},$$

where $K(n)$ does not depend on j.

Proof. We study $b_j(u) = \sqrt{a_{j+1}(u) - a_j(u)}$ in the interval $u \in [S_{j-1}, S_{j+1}]$. Recalling (4.2), we focus first on $[S_{j-1}, S_j]$. Again, *Faà di Bruno's formula* (see paragraph 4.3 [40]) implies

$$b_j^{(n)}(u) = n! \sum_{\nu=1}^{n} \frac{(\sqrt{1-a_j(u)})^{(\nu)}}{\nu!} \sum_{h_1+\cdots+h_\nu=n} \frac{a_j^{(h_1)}(u)}{h_1!} \cdots \frac{a_j^{(h_\nu)}(u)}{h_\nu!}$$

From Proposition 4.3 it follows that

$$|b_j^{(n)}(u)| \leq$$

$$n! \sum_{\nu=1}^{n} \left| \frac{(\sqrt{1-a_j(u)})^{(\nu)}}{\nu!} \right| \sum_{h_1+\cdots+h_\nu=n} \frac{C(h_1)}{h_1!} \left(\frac{2}{S_j - S_{j-1}} \right)^{h_1} \left(\frac{1}{1-\tau_j(u)^2} \right)^{2h_1} \cdots \times$$

$$\times \frac{C(h_\nu)}{h_\nu!} \left(\frac{2}{S_j - S_{j-1}} \right)^{h_\nu} \left(\frac{2}{1-\tau_j(u)^2} \right)^{2h_\nu} \exp\left(-\frac{2}{1-\tau_j(u)^2} \right)^{\nu}$$

$$\leq n! C(n) \left(\frac{2}{S_j - S_{j-1}} \right)^n \sum_{\nu=1}^{n} \binom{n+\nu-1}{n} \left| \frac{(\sqrt{1-a_j(u)})^{(\nu)}}{\nu!} \right| \exp\left(-\frac{1}{1-\tau_j(u)^2} \nu \right) \left(\frac{1}{1-\tau_j(u)^2} \right)^{2n}$$

$$= n! C(n) \left(\frac{2}{S_j - S_{j-1}} \right)^n \sum_{\nu=1}^{n} \binom{n+\nu-1}{n} \left| \frac{1}{\nu!} \frac{1}{(1-a_j(u))^{\nu-1/2}} \right| \exp\left(-\frac{1}{1-\tau_j(u)^2} \nu \right) \left(\frac{1}{1-\tau_j(u)^2} \right)^{2n}.$$

Now we have that

$$\left| \frac{\exp\left(-\frac{1}{1-\tau_j(u)^2}\right)}{1 - \exp\left(-\frac{1}{1-\tau_j(u)^2}\right)} \right| \leq \frac{1}{e-1}, \quad \left| \frac{\exp\left(-\frac{1}{1-\tau_j(u)^2}\right)}{(1-\tau_j(u)^2)^{2n}} \right| \leq \frac{k(n)}{e^{2n}}.$$

Proceeding similarly in $[S_j, S_{j+1}]$, the thesis follows. $\square$

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