Converting a covariance matrix from local currencies to a common currency

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Abstract

This short note demonstrates how a covariance matrix estimated using log-returns of multiple assets in their respective base currencies can be converted directly into a covariance matrix in a single common currency using basic matrix multiplication. This approach eliminates the need to convert returns in a common currency, simplifying the estimation process. In addition to describing the conversion process, this note also addresses the conversion of covariances between two currencies. By applying the proposed methodology, asset managers can efficiently analyze the covariance between assets denominated in diverse currencies, saving time and resources. It is thus a valuable tool for asset managers seeking to optimize portfolio allocation across different currencies.

JEL: C13 - Estimation: General; C58 - Financial Econometrics; G11 - Portfolio Choice; Investment Decisions

KEYWORDS: Covariance matrix, multicurrency, portfolio selection, currency risk, risk estimation

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I Introduction

The covariance matrix has been recognized as a fundamental tool for understanding the joint behaviour of multiple assets since the introduction of the portfolio selection model by Markowitz (1952). It is a key input in many quantitative models used by investors, analysts, and researchers, and is an essential component in modern financial theory. Typical applications are in value at risk computation and in Monte Carlo simulation to ensure that the simulated returns are consistent with the observed correlations between the assets, see for example (Meucci 2005). This paper proposes a straightforward approach for converting a covariance matrix estimated using log-returns of multiple assets in their respective base currencies into a covariance matrix in a single common currency. Unlike the commonly used approach, this method does not require historical data series in the desired currency, simplifying the estimation process.

Investing in assets denominated in different currencies can lead to distorted covariance estimates if the covariance matrix is calculated without converting the currencies. This is because changes in exchange rates can influence the returns on the investments, leading to a mix of both the underlying asset returns and the exchange rate fluctuations. This can result in inaccurate risk estimates and poor investment decisions. The new formula proposed in this paper allows to convert covariances into a common currency, such as USD or EUR, without to have to recompute the historical returns in the new currency. Having a covariance matrix in the same currency is essential for accurate risk estimation and effective risk management, particularly for banks, asset management firms, hedge funds and multinational corporations that conduct business across different countries and currencies need to measure their risk exposures in different currencies.

II Setup

Let $S_i(t)$ be the price at time t of an asset $i,\ i=1,\cdots,L$ in its base currency and suppose that the main base currency is the USD so all the currency rates are expressed versus the USD. Therefore, if we let $X_{k/\$}(t)$ be the units of currency $k,\ k=1,\cdots,K$ needed for buying 1 USD, then $S_i(t)X_i(t)$ is the time-t value of asset i in USD. To convert the price of asset i from its local currency to currency k and obtain the price $S_{i/k}(t)$ we apply the conversion formula

$$S_{i/k}(t) = S_i(t)\frac{X_{i/\$}(t)}{X_{k/\$}(t)}. \tag{1}$$

Let us assume that the asset prices in their own base currencies, at time $T = t + \Delta$, are given by

$$S_i(T) = S_i(t)\exp\left(\mu_i\Delta + \sigma_i\varepsilon_i\sqrt{\Delta}\right), \tag{2}$$

where ε_i is a random variable with zero mean and unit variance and μ_i and σ_i are constant parameters. In particular, μ_i is the annualized expected log-return

$$\mathbb{E}_t\left(\ln\frac{S_i(T)}{S_i(t)}\right) = \mu_i\Delta,$$

and σ_i the annualized volatility (square root of the variance) of the asset log-return

$$\mathbb{V}_t\left(\ln\frac{S_i(T)}{S_i(t)}\right) = \sigma_i^2\Delta.$$

Here we have assumed constant parameters The extension to time-varying but still deterministic parameters is straightforward: we simply replace $\mu_i\Delta$ and $\sigma_i^2\Delta$ by their time average.

The FX rate at time T is

$$X_{k/\$}(T) = X_{k/\$}(t)\exp\left(\mu_{k/\$}\Delta + \sigma_{k/\$}\varepsilon_{k/\$}\sqrt{\Delta}\right), \tag{3}$$

where $\varepsilon_{k/\$}$ is a random variable with zero mean and unit variance and therefore $\mu_{k/\$}$ is the expected log-return of the FX rate $k/\$$, and $\sigma_{k/\$}$ is the volatility, i.e.

$$\mathbb{E}_t\left(\ln\frac{X_{k/\$}(T)}{X_{k/\$}(t)}\right) = \mu_{k/\$}\Delta, \quad \mathbb{V}_t\left(\ln\frac{X_{k/\$}(T)}{X_{k/\$}(t)}\right) = \sigma_{k/\$}^2\Delta.$$

Combining Eqs. (1), (2) and (3), we get

$$S_{i/k}(T) = S_{i/k}(t)\exp\left(\left(\mu_i + \mu_{i/\$} - \mu_{k/\$}\right)\Delta + \left(\sigma_i\varepsilon_i + \sigma_{i/\$}\varepsilon_{i/\$} - \sigma_{k/\$}\varepsilon_{k/\$}\right)\sqrt{\Delta}\right), \tag{4}$$

and the same equation holds for asset j. These formulae with indexes i and j represent the basis to convert the covariance matrix from base currencies to the same currency, as discussed in the next section.

III Converting from local currencies to the same currency

Let $\sigma_{i/k,j/k}$ be the covariance between the log-returns of assets i and j in currency k, i.e.

$$\sigma_{i/k,j/k} = \frac{\mathbb{C}ov\left(\ln\left(\frac{S_{i/k}(T)}{S_{i/k}(t)}\right), \ln\left(\frac{S_{j/k}(T)}{S_{j/k}(t)}\right)\right)}{\Delta}, \tag{5}$$

and using Eq. (4), we have

$$\sigma_{i/k,j/k} = \mathbb{C}ov\left(\sigma_i\varepsilon_i + \sigma_{i/\$}\varepsilon_{i/\$} - \sigma_{k/\$}\varepsilon_{k/\$}, \sigma_j\varepsilon_j + \sigma_{j/\$}\varepsilon_{j/\$} - \sigma_{k/\$}\varepsilon_{k/\$}\right).$$

Given that the covariance is a bilinear operator, $\sigma_{i/k,j/k}$ can be computed according to the following formula

$$\sigma_{i/k,j/k} = \sigma_{i,j} + \sigma_{i,j/\$} - \sigma_{i,k/\$} + \sigma_{i/\$,j} + \sigma_{i/\$,j/\$} - \sigma_{i/\$,k/\$} - \sigma_{k/\$,j} - \sigma_{k/\$,j/\$} + \sigma^2_{k/\$}, \tag{6}$$

where the different terms appearing in equation Eq. (6) are as follows

$$\sigma_{i,j}\Delta = \mathbb{C}ov\left(\ln\left(\frac{S_i(T)}{S_i(t)}\right), \ln\left(\frac{S_j(T)}{S_j(t)}\right)\right)$$
$$\sigma_{i,j/\$}\Delta = \mathbb{C}ov\left(\ln\left(\frac{S_i(T)}{S_i(t)}\right), \ln\left(\frac{S_{j/\$}(T)}{S_{j/\$}(t)}\right)\right)$$

$$\sigma_{i,k/\$}\Delta = \mathbb{C}ov\left(\ln\left(\frac{S_i(T)}{S_i(t)}\right), \ln\left(\frac{X_{k/\$}(T)}{X_{k/\$}(t)}\right)\right)$$
$$\sigma_{i/\$,k/\$}\Delta = \mathbb{C}ov\left(\ln\left(\frac{S_{i/\$}(T)}{S_{i/\$}(t)}\right), \ln\left(\frac{X_{k/\$}(T)}{X_{k/\$}(t)}\right)\right)$$
$$\sigma_{i/\$,j/\$}\Delta = \mathbb{C}ov\left(\ln\left(\frac{S_{i/\$}(T)}{S_{i/\$}(t)}\right), \ln\left(\frac{S_{j/\$}(T)}{S_{j/\$}(t)}\right)\right)$$
$$\sigma^2_{k/\$}\Delta = \mathbb{C}ov\left(\ln\left(\frac{X_{k/\$}(T)}{X_{k/\$}(t)}\right), \ln\left(\frac{X_{k/\$}(T)}{X_{k/\$}(t)}\right)\right).$$

III.1 Covariance conversion via matrix multiplication

The computation of (6) admits a simple representation as matrix multiplication. Given the L assets denominated in K different currencies, consider the $(L+K)\times(L+K)$ covariance matrix:

$$\Sigma = \begin{bmatrix} \Sigma_{L,L} & \Sigma_{L,X} \\ \Sigma_{X,L} & \Sigma_{X,X} \end{bmatrix} \tag{7}$$

where $\Sigma_{L,L}$ is the $L\times L$ covariance matrix of all the assets in local currency, $\Sigma_{X,X}$ is the $K\times K$ covariance matrix of all the FX rates versus USD, $\Sigma_{L,X}$ is the $L\times K$ covariance matrix between the L assets and the K currencies and $\Sigma_{X,L} = \Sigma'_{L,X}$. The last row and column in $\Sigma_{X,X}$ refer to the variance and covariance of the USDUSD FX rate and therefore the last row and last column turn have all elements equal to 0. Suppose we want to calculate the $L\times L$ covariance matrix Σ_k in currency k. In order to do this, we define a matrix $\mathbf{A}$ of dimension $(L+K)\times L$ having the $L\times L$ identity matrix on the top of a $K\times L$ matrix $\mathbf{M}$ as follows

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_L \\ \mathbf{M} \end{bmatrix},$$

where $\mathbf{M}$ is an array with element $m_{k,i} = 1$ if asset i is denominated in currency k and 0 elsewhere. Let us also define a matrix $\mathbf{B}_k$ having the same dimension as $\mathbf{A}$ and with a row of 1's in the position of the specific currency k and zero elsewhere. It is then easy to verify that Σ_k can be obtained from

Σ via:

$$\Sigma_k = (\mathbf{A} - \mathbf{B}_k)' \Sigma (\mathbf{A} - \mathbf{B}_k). \tag{8}$$

Eq. (8) provides an immediate and efficient way to convert a covariance matrix estimated using base currencies to the same currency k without the need of converting time series from one currency to another.

III.2 Converting from one currency to another currency

If we aim to convert the covariance matrix from a common currency, e.g. currency k, to another, e.g. USD, Eq. (6) modifies into

$$\sigma_{i/\$,j/\$} = \sigma_{i/k,j/k} - \sigma_{i/k,k/\$} - \sigma_{j/k,k/\$} + \sigma^2_{k/\$}. \tag{9}$$

In this formula, we have a term, i.e. $\sigma_{i/k,k/\$}\Delta = \mathbb{C}ov\left(\ln\left(\frac{S_{i/k}(T)}{S_{i/k}(t)}\right), \ln\left(\frac{X_{k/\$}(T)}{X_{k/\$}(t)}\right)\right)$, that is not included in the matrix Σ_k. So we define the new matrix $\tilde{\Sigma}_k$ of size $(L+1)\times(L+1)$

$$\tilde{\Sigma}_k = \mathbf{D}'\Sigma\mathbf{D},$$

where the matrix $\mathbf{D}$ is built stacking the matrix $\mathbf{A} - \mathbf{B}_k$ and the column vector $\mathbf{e}_k$, having 1 in position k and 0 elsewhere, as follows

$$\mathbf{D} = \begin{bmatrix} \mathbf{A} - \mathbf{B}_k & -\mathbf{e}_k \end{bmatrix}.$$

Finally, the matrix in currency \$ is obtained as

$$\Sigma_\$ = \mathbf{E}'\tilde{\Sigma}_k\mathbf{E},$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{I}_L \\ \mathbf{1}_L' \end{bmatrix},$$

where $\mathbf{1}_L$ is a $L \times 1$ column vector with all elements equal to 1.

IV A numerical example

To demonstrate the conversion of a covariance matrix from a multi-currency environment to a common currency, this section provides an illustrative example featuring three assets denominated in different currencies: USD, EUR, and British Pound (GBP). Specifically, the analysis considers the monthly time series of three assets, shares in Apple (AAPL) in USD, Volkswagen (VOW) in EUR, and Unilever (ULVR) in GBP, from 29 January 2010 to 30 September 2022, as well as the time series for the EUR/USD and GBP/USD exchange rates during the same period. The resulting covariance matrix Σ is presented in Table I, with, for instance, the covariance between the log-returns of AAPL/USD and VOW/EUR equal to 2.065.

To convert this covariance matrix to a common currency (GBP), we define the matrices $\mathbf{A}$ and $\mathbf{B}_{£}$ in Tables II and III, respectively. The resulting covariance matrix $\Sigma_{£}$, which expresses the log-returns in £, is presented in Table IV.

To convert $\Sigma_{£}$ to $\Sigma_{\$}$, we define the matrix $\mathbf{D}$ (as presented in Table V) and compute $\tilde{\Sigma}_{£}$ (as shown in Table VI). The matrix $\mathbf{E}$ (presented in Table VII) is then used to compute $\Sigma_{\$}$, which appears in Table VIII. To provide a comprehensive analysis, we also present $\Sigma_{€}$ in Table IX.

V Conclusion

This note showcases a straightforward approach to convert a covariance matrix estimated using log-returns in the base currencies of multiple assets to a covariance matrix in a single currency. The conversion process involves basic matrix multiplication, making it a valuable tool for asset managers who aim to bypass the need to transform historical returns into a common currency, without requiring historical data series in the desired currency.

References

Markowitz, H. (1952), ‘Portfolio selection’, *The Journal of Finance* **7**(1), 77.

Meucci, A. (2005), *Risk and asset allocation*, New York: Springer.

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VII Statements and Declarations

The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

Table I: Covariance matrix in local currencies

	AAPL \$	VOW €	ULVR £	€/\$	£/\$	\$/\$
AAPL \$	6.041	2.065	0.505	0.248	0.479	0
VOW €	2.065	9.084	0.077	0.344	0.789	0
ULVR £	0.505	0.077	2.140	0.077	-0.183	0
€/\$	0.248	0.344	0.077	0.621	0.398	0
£/\$	0.479	0.789	-0.183	0.398	0.592	0
\$/\$	0	0	0	0	0	0

Table II: Matrix **A**

	AAPL \$	VOW €	ULVR £
AAPL \$	1	0	0
VOW €	0	1	0
ULVR £	0	0	1
€/\$	0	1	0
£/\$	0	0	1
\$/\$	1	0	0

Table III: Matrix $\mathbf{B}_{£}$

	AAPL	VOW	ULVR
AAPL	0	0	0
VOW	0	0	0
ULVR	0	0	0
€/\$	0	0	0
£/\$	1	1	1
\$/\$	0	0	0

Table IV: Covariance matrix $\Sigma_{£}$

	AAPL £	VOW £	ULVR £	£/$
AAPL £	5.674	1.238	0.688	0.113
VOW £	1.238	8.610	0.337	-0.596
ULVR £	0.688	0.337	2.140	0.183
£/$	0.113	-0.596	0.183	0.592

Table V: Matrix D

AAPL	1	0	0	0
VOW	0	1	0	0
ULVR	0	0	1	0
€/$	0	1	0	0
£/$	-1	-1	0	-1
$/$	1	0	0	0

Table VI: Covariance matrix $\tilde{\Sigma}_{£}$

	AAPL £	VOW £	ULVR £	£/$
AAPL £	5.674	1.238	0.688	0.479
VOW £	1.238	8.610	0.337	0.789
ULVR £	0.688	0.337	2.140	-0.183
£/$	0.479	0.789	-0.183	0.592

Table VII: Matrix $\mathbf{E}$

AAPL	1	0	0
VOW	0	1	0
ULVR	0	0	1
£/$	1	1	1

Table VIII: Covariance matrix $\Sigma_{\$}$

	AAPL $	VOW $	ULVR $
AAPL $	6.491	1.347	1.576
VOW $	1.347	8.011	0.516
ULVR $	1.576	0.516	3.099

Table IX: Covariance matrix $\Sigma_{€}$

	AAPL €	VOW €	ULVR €
AAPL €	6.167	1.721	0.883
VOW €	1.721	9.084	0.522
ULVR €	0.883	0.522	2.037