



Cartan structure equations and Levi-Civita connection in noncommutative geometry

Paolo Aschieri ^{a,b,*}

^a Dipartimento di Scienze e Innovazione Tecnologica, Università del Piemonte Orientale, Viale T. Michel - 15121, Alessandria, Italy

^b INFN, Sezione di Torino, via P. Giuria 1, 10125 Torino, Italy



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ABSTRACT

We study the differential and Riemannian geometry of algebras A endowed with an action of a triangular Hopf algebra H and noncommutativity compatible with the associated braiding. The modules of one forms and of braided derivations are modules in a compact closed category of H -equivariant A -bimodules, whose internal morphisms correspond to tensor fields. Vector fields and forms approaches to curvature and torsion are proven to be equivalent by extending the Cartan calculus to left (right) A -module (not necessarily A -bimodule) connections. The Cartan structure equations and the Bianchi identities are derived.

Existence and uniqueness of the Levi-Civita connection for *arbitrary* pseudo-Riemannian metrics is proven via a Koszul formula. The general theory includes Drinfeld twists of commutative geometries and also cotriangular Hopf algebras. It is illustrated with the example of the tensor square of Sweedler Hopf algebra which becomes a noncommutative Einstein manifold via a non-central metric.

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* Correspondence to: Dipartimento di Scienze e Innovazione Tecnologica, Università del Piemonte Orientale, Viale T. Michel - 15121, Alessandria, Italy.
E-mail address: paolo.aschieri@uniupo.it.

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1. Introduction

Noncommutative Riemannian geometry is an active and interdisciplinary research field. One line of study follows Connes' approach, focusing on spectral geometry and the analysis of the Laplacian (see [25] for a recent review). Another line adopts a more algebraic perspective, beginning with a differential calculus on a noncommutative algebra, introducing a notion of Riemannian metric, and addressing the existence and uniqueness of the Levi-Civita connection. Interest in the field is also driven by the possibility that gravity on noncommutative spacetimes may capture aspects of a quantum theory of gravity.

Given a noncommutative algebra A and an associated differential calculus $(d, \Omega(A))$, there are different approaches to the notion of metric tensor and to that of Levi-Civita connection. On one hand one can consider metrics g that are compatible with the noncommutative structure, i.e. that are central elements of the bimodule $\Omega(A) \otimes_A \Omega(A)$ ($ag = ga$ for all $a \in A$), or of the dual bimodule. On the other hand one can relax this condition and study arbitrary metrics, which is useful when considering g as a dynamical field. Similar considerations hold for the study of connections. In particular the notion of Levi-Civita connection relies on the possibility of imposing the metric compatibility condition $\nabla g = 0$. This requires lifting the connection from the module $\Omega(A)$ of one forms to the tensor product module $\Omega(A) \otimes_A \Omega(A)$. However, lifting connections on tensor products modules is a nontrivial problem. In the literature it is typically overcome by constraining the connection to be compatible with the noncommutative structure in the sense of being a bimodule connection [18]. For approaches to Riemannian geometry using central metrics and bimodule Levi-Civita connections see [11], [12], [14], [15], [33], [29], see also the monograph [13].

For a selected class of noncommutative algebras on the other hand it is possible to relax this centrality constraint on the metric and consider g just as a (properly defined symmetric) element in $\Omega(A) \otimes_A \Omega(A)$. As said, this is relevant when g is seen as a dynamical field, so that for example ag is also a metric for any invertible $a \in A$. In case of \mathbb{R}^n with Moyal-Weyl noncommutativity the Levi-Civita connection of an arbitrary symmetric metric was constructed in [3] using a noncommutative Koszul formula (see also [6, §3.4, §8.5]). A similar result holds for the noncommutative torus [31]. These results and those in [2] rely on the existence of (undeformed) derivations of the noncommutative algebra A generating the A -module of vector fields (dual to that of one forms). A generalization via local charts with Moyal-Weyl noncommutativity is in [4] in the deformation quantization context.

In these examples the noncommutative algebra A is endowed with a representation of a triangular Hopf algebra H and the noncommutativity is compatible with the braiding τ given by the triangular structure. Triangularity of H implies that τ is a representation of the permutation group. We shall call these algebras braided commutative H -module algebras, they are commutative algebras A in the category of H -module algebras with H triangular. Besides noncommutative tori and Connes-Landi spheres [17], belong to this class the algebras studied in [16], the noncommutative algebras obtained from Drinfeld twists of commutative ones and cotriangular Hopf algebras (see [19] for their classification in the finite dimensional case).

In this paper we develop a general theory of Riemannian geometry for such algebras A . We study arbitrary pseudo-Riemannian metrics g (not only central) and prove existence and uniqueness of the Levi-Civita connection (which need not be a bimodule connection). This is achieved by providing a Koszul formula, see (6.4), that leads to an explicit expression of the Levi-Civita connection, see (6.9).

Noncommutative differential geometry is studied primarily in the language of forms, the complementary vector fields approach being more problematic since the linear space of derivations of a noncommutative algebra A is not generally an A -bimodule. For A a braided commutative H -module algebra, vector fields are braided derivations and a braided derivation based differential calculus and geometry can be constructed. The general theory of algebras, Lie algebras and differential operators in symmetric monoidal categories was outlined in the late '80s in [28, §13.5]. The differential and Cartan calculus was pioneered in [22], for a braided derivations approach see [33]. When considering connections one can contract them with vector fields to obtain covariant derivatives along vector fields. These were used in [5] in the context of twist deformation of commutative algebras, and led to define curvature and torsion as left A -linear maps on tensor products of vector fields, a quantum analogue of the standard general relativity definitions. These constructions did not require lifting

covariant derivatives from vector fields to tensor fields. In [7] such a lifting, within the setting of braided commutative H -module algebras, was provided for connections that are not invariant under the H -action, these are, in general, not bimodule connections. Concerning covariant derivatives, the lifting is understood when the vector field is a derivation, or when the connection is a bimodule connection, or when it is H -equivariant so that the braiding acts trivially.

We study here the covariant derivative and its lifting to tensor products in full generality. This framework allows us to: *i*) Extend the Cartan calculus to include covariant derivatives; *ii*) Establish the equivalence between the formulations of curvature and torsion in terms of forms and in terms of vector fields; *iii*) Derive the Cartan structure equations and the Bianchi identities; and *iv*) Obtain a general Koszul formula determining the Levi-Civita connection via the metric compatibility condition $\nabla_u g = 0$ for any vector field (braided derivation) u .

Our analysis is based on considering a new perspective on operators on H -equivariant A -bimodules: such operators can be expressed as compositions of operators acting from the left and from the right. This is not merely a matter of notation but reflects covariance requirements with respect to the (generally) non-cocommutative Hopf algebra H . In particular, the covariant derivative ∇_u is a composition of a left connection ∇ (acting from the right) and an inner derivative (acting from the left). Only when the vector field u is a derivation of the algebra can ∇_u be seen as an operator acting from the right. Similarly, its extension d_{∇_u} from modules over A to modules over the exterior algebra $\Omega^*(A)$ is with operators u and d acting from the left, while ∇ acts from the right. It is precisely with this composition that we extend the Cartan calculus to include covariant derivatives, proving the braided Cartan calculus relation $[d_{\nabla_u}, i_v] = i_{[u, v]}$. This establishes result *i*) and is the key identity underpinning *ii*), the equivalence between the vector field and the exterior form formulations of curvature and torsion. Thus, for example, the curvature tensor defined in [5] as the braided commutator of covariant derivatives along vector fields corresponds to the standard definition given by squaring the connection ∇ (cf. Theorem 4.6 and Theorem 4.7 for the analogous result on torsion). A similar correspondence for the torsion tensor was established in [15] within the specific framework of tame differential calculi (where the A -bimodule of 1-forms is generated by central 1-forms, and vector fields are derivations). The relation between (left) connections on vector fields and the dual (right) connections on one forms then leads to result *iii*): the Cartan structure equations for curvature and torsion and the associated Bianchi identities. We also lift the covariant derivative along vector fields to act on tensor fields using the results of [7], where as mentioned, connections were lifted to tensor product modules without assuming the bimodule connection property.

Finally, upon introducing a pseudo-Riemannian metric, this noncommutative differential geometry is used to provide *iv*), a Koszul formula for metric compatible torsion free connections, yielding both the existence and uniqueness of the Levi-Civita connection, as well as an explicit and global constructive method. This result is very general, as it assumes neither H -equivariant metrics (as in [33]) nor a preferred set of derivations of the noncommutative algebra. When H -equivariant metrics are considered we obtain bimodule Levi-Civita connections and recover the results of [33]. It also significantly extends the result of [4] in the context of formal Drinfeld twist deformation quantization, where the Koszul formula was provided either locally, subordinated to open charts with Moyal–Weyl noncommutativity, or globally, in implicit form as a formal power series in \hbar starting with the commutative Koszul formula and determined recursively in \hbar .

In the present work, pseudo-Riemannian metrics are just braided symmetric non-degenerate contravariant tensors. This is the natural context where to formulate a noncommutative gravity theory where the metric is the dynamical field. We here present in vacuum Einstein equations leading to noncommutative Einstein manifolds. This general theory is illustrated with the example of the noncommutative torus (and the Moyal–Weyl plane), which is sketched in order to make contact with previous literature, and the example of the tensor square of Sweedler Hopf algebra. This is a cotriangular finite dimensional Hopf algebra with abelian two dimensional Lie algebra of braided derivations. The Levi-Civita connection, curvature, Ricci tensor and scalar curvature are explicitly presented for any metric. We solve the noncommutative Einstein manifold condition for a non-central non-equivariant metric.

The algebraic structure underlying this study is that of the categories of H -modules and of H -equivariant A -bimodules (compatible H -modules A -bimodules). Following [7] and the sharpened results in [9], [10] we recall the different structures of modules and module maps we need in noncommutative Riemannian geometry. This clarifies the constructions and the different general properties needed in the progress of the paper. For example, left (right) connections are linear maps but are not morphism in the category ${}^H\mathcal{M}$ of H -modules, in categorical terms they are internal morphisms. Left connections have different H -action from right connections; they are different internal morphisms in ${}^H\mathcal{M}$. The covariant derivative associated with a left connection, in turn, can be viewed both as a left A -linear map and an internal morphism in the category ${}^H_A\mathcal{M}_A$ of H -equivariant A -bimodules (cf. Remark 4.3). Torsion and curvature exhibits further structural properties: they are internal morphisms (left A -linear maps transforming under the H -adjoint action) in the subcategory of symmetric H -equivariant A -bimodules ${}^H_A\mathcal{M}_A^{\text{sym}}$. In particular, we study internal morphisms associated with tensor products of finitely generated and projective modules and their duals. This allows, as in classical differential geometry, to understand noncommutative tensors fields in all their different forms, as elements of A -bimodules (sections), or as various left (right) A -module maps. The underlying category in this richest case is the compact closed category of symmetric H -equivariant A -bimodules finitely generated and projective as A -modules (it is a ribbon category with trivial twist isomorphisms since H is triangular).

This underlying categorical context is presented in Section 2 with some details, in particular on the biclosed and rigid structures, so that the paper is self-contained. The section closes with the examples where A is a cotriangular Hopf algebra and where it is the Drinfeld twist deformation of the algebra of smooth functions on a manifold M . In Section 3 the braided derivations formulation of the differential and Cartan calculus is revisited. We provide the example of the calculus on cotriangular Hopf algebras and show its equivalence with the (categorical) exterior algebra approach of [21]. In Section 4

we review known properties of right connections and left connections, since both are relevant for understanding curvature and torsion. We then introduce the covariant derivative, establish the Cartan formula $[d_{\nabla_u}, i_v] = i_{[u,v]}$ and show that the curvature tensor, defined as the square of the connection, can be equivalently defined via the commutator of covariant derivatives along vector fields. Similarly, two different definitions of torsion are shown to be equivalent. In Section 5 the relation between connections on modules and dual modules is recalled and that between curvatures studied. This leads to relate the vector field formulation of curvature and torsion to the one involving the dual connection on forms. These are the Cartan structure equations for curvature and torsion since choosing a basis (a trivialization of the frame bundle) we obtain those structure equations. The associated Bianchi identities are also obtained. In Section 6 existence and uniqueness of the Levi-Civita connection is proven for any braided symmetric metric. The Ricci tensor and the scalar curvature are canonically defined, this leads to in vacuum Einstein field equations giving noncommutative Einstein manifolds. In Section 7 the Riemannian geometry of the tensor square of Sweedler Hopf algebra is presented.

2. Hopf algebras, braidings and representations

We work in the category of \mathbb{k} -modules, with \mathbb{k} a fixed field of characteristic zero or the ring of formal power series in a variable \hbar over such field; much of what follows holds for a commutative unital ring. The tensor product over \mathbb{k} is denoted \otimes . Algebras over \mathbb{k} are assumed associative and unital. Hopf algebras are assumed with invertible antipode.

In Section 2.1 we study right (left) \mathbb{k} -linear maps and A -linear maps between H -modules and between H -equivariant A -bimodules, hence introducing biclosed (left and right closed) monoidal categories and, when the Hopf algebra H is triangular, symmetric biclosed monoidal categories. In Section 2.2 we continue the study of right (left) A -linear maps considering, for an arbitrary Hopf algebra H , the case of a finitely generated and projective A -module, this is the same as a rigid module. In Section 2.3 we study these rigid H -equivariant A -bimodules when H is triangular, the corresponding category is an example of a compact closed category and hence of a ribbon category. Most of these results are covered (albeit sometimes implicitly) either in the literature on quantum groups (see e.g. [23,27]) or in the related one on tensor categories (see e.g. [20]). We here set the notation (following [7,9,10]) and present the main results that will be used for the later sections, spelling out in particular the properties of \mathbb{k} -linear and of right (left) A -linear maps transforming under different left H -adjoint actions. A last section is devoted to the example of the category of bicovariant bimodules of a cotriangular Hopf algebra and to that of noncommutative vector bundles obtained via Drinfeld twist deformation.

2.1. Closed monoidal categories

We start recalling basic Hopf algebra notions, the category of H -modules for an arbitrary Hopf algebra H and also for a triangular Hopf algebra H . In this simple context we introduce \mathbb{k} -linear maps that are not invariant under the H -action, they come with two different H -actions structuring them as \mathbb{k} -linear maps acting from the right or from the left. Their categorical interpretation, as internal morphisms, is also discussed. They structure the monoidal category of H -modules ${}^H\mathcal{M}$ as a biclosed monoidal category. If H is a triangular Hopf algebra then $({}^H\mathcal{M}, \otimes, \mathbb{k}\text{-hom}, \text{hom}_{\mathbb{k}})$ is furthermore a symmetric biclosed monoidal category and there is a tensor product $\otimes_{\mathcal{R}}$ of internal morphisms.

Given an H -module algebra A we then study H -equivariant A -bimodules and the associated right A -linear maps and left A -linear maps (internal morphisms). This is the biclosed monoidal category $({}^H_A\mathcal{M}_A, \otimes_A, {}_A\text{hom}, \text{hom}_A)$. If H is triangular and the product in A is compatible with the braiding, restricting to modules where the A -bimodule structure is compatible with the braiding (symmetric A -bimodules) we have the biclosed monoidal subcategory $({}^H_A\mathcal{M}_A^{\text{sym}}, \otimes_A, {}_A\text{hom}, \text{hom}_A)$, which is symmetric.

This chain of results holds as well when we consider a graded algebra instead of A , and graded modules.

2.1.1. Modules over a Hopf algebra

Let H be a Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$ over \mathbb{k} . We denote by ${}^H\mathcal{M}$ the category of left H -modules, where objects in ${}^H\mathcal{M}$ are \mathbb{k} -modules V with a left H -action $\triangleright : H \otimes V \rightarrow V$, while morphisms in ${}^H\mathcal{M}$ are \mathbb{k} -module maps $f : V \rightarrow W$ that are H -equivariant, i.e.,

$$h \triangleright f(v) = f(h \triangleright v), \tag{2.1}$$

for all $h \in H$ and $v \in V$; we write $f \in \text{Hom}_{{}^H\mathcal{M}}(V, W)$. In this paper H -modules will be always left H -modules and will be simply called H -modules.

Since H is a bialgebra ${}^H\mathcal{M}$ is a (strict) monoidal category. Given two H -modules V and W their tensor product $V \otimes W$ is an H -module with H -action

$$\triangleright : H \otimes V \otimes W \longrightarrow V \otimes W, \quad h \otimes v \otimes w \longmapsto h \triangleright (v \otimes w) := (h_{(1)} \triangleright v) \otimes (h_{(2)} \triangleright w), \tag{2.2}$$

where we have used the Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ (with summation understood) for the coproduct of H . The tensor product of two morphisms in ${}^H\mathcal{M}$, $f : V \rightarrow V', g : W \rightarrow W'$ is the morphism in ${}^H\mathcal{M}$ defined by $f \otimes g : V \otimes W \rightarrow V' \otimes W', v \otimes w \mapsto f(v) \otimes g(w)$. The tensor product functor \otimes is associative. The unit object in ${}^H\mathcal{M}$ is \mathbb{k} with left H -action given by the counit of H , $\triangleright : H \otimes \mathbb{k} \rightarrow \mathbb{k}, h \otimes \lambda \mapsto \varepsilon(h)\lambda$.

Since the bialgebra H is a Hopf algebra, ${}^H\mathcal{M}$ is a closed monoidal category. For any V, W in ${}^H\mathcal{M}$, we denote by $\text{hom}_{\mathbb{k}}(V, W)$ in ${}^H\mathcal{M}$ the \mathbb{k} -module $\text{Hom}_{\mathbb{k}}(V, W)$ of \mathbb{k} -linear maps $L : V \rightarrow W$ equipped with the adjoint H -action

$$\triangleright : H \otimes \text{hom}_{\mathbb{k}}(V, W) \longrightarrow \text{hom}_{\mathbb{k}}(V, W), \quad h \otimes L \longmapsto h \triangleright L := h_{(1)} \triangleright \circ L \circ S(h_{(2)}) \triangleright, \tag{2.3}$$

i.e., $(h \triangleright L)(v) = h_{(1)} \triangleright (L(S(h_{(2)}) \triangleright v))$. Given morphisms $f^{op} : V \rightarrow V', g : W \rightarrow W'$ (where $f^{op} : V \rightarrow V'$ is just $f : V' \rightarrow V$ thought as morphisms in the opposite category $({}^H\mathcal{M})^{op}$) we have the morphism

$$\text{hom}_{\mathbb{k}}(f^{op}, g) : \text{hom}_{\mathbb{k}}(V, W) \longrightarrow \text{hom}_{\mathbb{k}}(V', W'), \quad L \longmapsto g \circ L \circ f. \tag{2.4}$$

This way we have defined the so-called internal-hom functor $\text{hom}_{\mathbb{k}} : ({}^H\mathcal{M})^{op} \times {}^H\mathcal{M} \longrightarrow {}^H\mathcal{M}$. This is compatible with the tensor product functor \otimes , indeed since any H -equivariant map $f : V \otimes W \rightarrow Z$ can be considered as an H -equivariant map $\zeta(f) : V \rightarrow \text{hom}(W, Z)$ via $\zeta(f)(v) := f(v, -)$, we have that the functor $- \otimes W$ is left adjoint to $\text{hom}_{\mathbb{k}}(W, -)$, thus $({}^H\mathcal{M}, \otimes, \text{hom}_{\mathbb{k}})$ is a closed monoidal category.

Let V, W be modules in ${}^H\mathcal{M}$, we can define another H -adjoint action on \mathbb{k} -linear maps $V \rightarrow W$. We denote by ${}_{\mathbb{k}}\text{hom}(V, W)$ the \mathbb{k} -module $\text{Hom}_{\mathbb{k}}(V, W)$ with H -action \triangleright^{cop} defined by

$$\begin{aligned} \triangleright^{cop} : H \otimes {}_{\mathbb{k}}\text{hom}(V, W) &\longrightarrow {}_{\mathbb{k}}\text{hom}(V, W), \quad h \otimes \tilde{L} \longmapsto h \triangleright^{cop} \tilde{L} := h_{(2)} \triangleright \circ \tilde{L} \circ S^{-1}(h_{(1)}) \triangleright, \\ (h \triangleright^{cop} \tilde{L})(v) &= h_{(2)} \triangleright (\tilde{L}(S^{-1}(h_{(1)}) \triangleright v)). \end{aligned} \tag{2.5}$$

This gives the monoidal structure $({}^H\mathcal{M}, \otimes, {}_{\mathbb{k}}\text{hom})$, with the functor $V \otimes -$ that is left adjoint to ${}_{\mathbb{k}}\text{hom}(V, -)$ (via $\zeta(f) : W \rightarrow {}_{\mathbb{k}}\text{hom}(V, Z)$, $\zeta(f)(w) := f(-, w)$, for any $f \in \text{Hom}_{{}^H\mathcal{M}}(V \otimes W, Z)$).

While \mathbb{k} -linear maps $L \in \text{hom}_{\mathbb{k}}(V, W)$ naturally act from the left, indeed the \triangleright adjoint action satisfies, for all $h \in H$, $v \in V$, $h \triangleright (L(v)) = (h_{(1)} \triangleright L)(h_{(2)} \triangleright v)$, \mathbb{k} -linear maps $\tilde{L} \in {}_{\mathbb{k}}\text{hom}(V, W)$ naturally act from the right, indeed the \triangleright^{cop} adjoint action satisfies,

$$h \triangleright (\tilde{L}(v)) = (h_{(2)} \triangleright^{cop} \tilde{L})(h_{(1)} \triangleright v), \tag{2.6}$$

that, evaluating \tilde{L} on v from the right, reads $h \triangleright ((v)(\tilde{L})) = (h_{(1)} \triangleright v)(h_{(2)} \triangleright^{cop} \tilde{L})$.

In summary, associated with the Hopf algebra H , we have the biclosed monoidal category

$$({}^H\mathcal{M}, \otimes, \text{hom}_{\mathbb{k}}, {}_{\mathbb{k}}\text{hom}).$$

The submodules $\text{hom}_{\mathbb{k}}(V, W)^H \subset \text{hom}_{\mathbb{k}}(V, W)$ and ${}_{\mathbb{k}}\text{hom}(V, W)^H \subset {}_{\mathbb{k}}\text{hom}(V, W)$ of H -invariant elements, i.e., $h \triangleright L = \varepsilon(h)L$ and $h \triangleright^{cop} \tilde{L} = \varepsilon(h)\tilde{L}$, coincide with that of H -equivariant maps $V \rightarrow W$ (compare for example (2.6) with (2.1)) and are hence identified with $\text{Hom}_{{}^H\mathcal{M}}(V, W)$.

2.1.2. Modules over a triangular Hopf algebra

Let now H be a triangular Hopf algebra with universal \mathcal{R} -matrix $\mathcal{R} \in H \otimes H$. We recall that it satisfies

$$\begin{aligned} \Delta^{cop}(h) &= \mathcal{R}\Delta(h)\mathcal{R}^{-1} \text{ for all } h \in H, \\ (\Delta \otimes \text{id})\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \end{aligned}$$

and the triangularity condition $\mathcal{R}_{21} = \mathcal{R}^{-1}$. Because of the triangular structure the monoidal category ${}^H\mathcal{M}$ is symmetric: the braiding $\tau := \tau(\mathcal{R})$ is the natural isomorphism $\tau : \otimes \Rightarrow \otimes^{op}$ with components defined by,

$$\tau_{V,W} : V \otimes W \longrightarrow W \otimes V, \quad v \otimes w \longmapsto (\bar{R}^\alpha \triangleright w) \otimes (\bar{R}_\alpha \triangleright v) \tag{2.7}$$

where we used the notation $\mathcal{R} = R^\alpha \otimes R_\alpha$, $\mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha$. The category ${}^H\mathcal{M}$ is symmetric because for all V, W , $\tau_{W,V} \circ \tau_{V,W} = \text{id}_{V \otimes W}$, hence τ provides a representation of the permutation group. With slight abuse of notation we shall frequently omit the indices in the isomorphisms $\tau_{V,W}$ and simply write τ .

The functors $- \otimes W$ and $W \otimes -$ in this triangular case are naturally isomorphic via the braiding; correspondingly, the two internal-hom functors $\text{hom}_{\mathbb{k}}$ and ${}_{\mathbb{k}}\text{hom}$ are naturally isomorphic via the family of isomorphisms, for all $V, W \in {}^H\mathcal{M}$,

$$\mathcal{D}_{V,W} : \text{hom}_{\mathbb{k}}(V, W) \longrightarrow {}_{\mathbb{k}}\text{hom}(V, W), \quad L \longmapsto \mathcal{D}_{V,W}(L) := (\bar{R}^\alpha \triangleright L) \circ \bar{R}_\alpha \triangleright \tag{2.8}$$

with inverse given by (cf. [7, §3.2, Rmk. 3.11] and recall that a triangular structure $\mathcal{R} \in H \otimes H$ is in particular a twist or 2-cocycle, cf. [27, Ex. 2.3.6])

$$\mathcal{D}_{V,W}^{-1} : {}_{\mathbb{k}}\text{hom}(V, W) \longrightarrow \text{hom}_{\mathbb{k}}(V, W), \quad \tilde{L} \longmapsto \mathcal{D}_{V,W}^{-1}(\tilde{L}) = (R^\alpha \triangleright^{cop} \tilde{L}) \circ R_\alpha \triangleright. \tag{2.9}$$

Here we just prove H -equivariance: for all $h \in H$, $h \triangleright \mathcal{D}_{V,W}^{-1}(\tilde{L}) = \mathcal{D}_{V,W}^{-1}(h \triangleright^{cop} \tilde{L})$. This equality is equivalent to $h \triangleright \mathcal{D}_{V,W}^{-1}(\tilde{L}) \circ h_{(2)} \triangleright = \mathcal{D}_{V,W}^{-1}(h_{(1)} \triangleright^{cop} \tilde{L}) \circ h_{(2)} \triangleright$ (use \mathbb{k} -linearity and $h_{(1)} \otimes h_{(2)} S(h_{(3)}) = h$). Recalling (2.3) the left hand side equals $h \triangleright$

$\circ \mathcal{D}_{V,W}^{-1}(\tilde{L})$. Recalling (2.5), that \triangleright^{cop} and \triangleright are actions and using quasi-cocommutativity in the form $\mathcal{R}\Delta(h) = \Delta^{cop}(h)\mathcal{R}$, the right hand side too equals this expression: $\mathcal{D}_{V,W}^{-1}(h_{(1)} \triangleright^{cop} \tilde{L}) \circ h_{(2)} \triangleright = (R^\alpha h_{(1)} \triangleright^{cop} \tilde{L}) \circ R_\alpha h_{(2)} \triangleright = (h_{(2)} R^\alpha \triangleright^{cop} \tilde{L}) \circ h_{(1)} R_\alpha \triangleright = h_{(2)} \triangleright^{cop} (R^\alpha \triangleright^{cop} \tilde{L}) \circ h_{(1)} R_\alpha \triangleright = h \triangleright \circ \mathcal{D}_{V,W}^{-1}(\tilde{L})$.

In summary, when the Hopf algebra H is triangular the quadruple

$$({}^H\mathcal{M}, \otimes, \text{hom}_{\mathbb{k}}, {}_{\mathbb{k}}\text{hom})$$

is a symmetric biclosed monoidal category.

In a braided closed monoidal category we can evaluate, compose and consider tensor products not just of morphisms but also of internal morphisms. For quasitriangular Hopf algebras and hence for triangular Hopf algebras, internal morphisms evaluation and composition are the usual ones of \mathbb{k} -linear maps on \mathbb{k} -modules. The composition of internal morphisms is easily seen to be an internal morphism; we give a proof for internal morphisms carrying the \triangleright^{cop} adjoint action. Let $\tilde{L} \in {}_{\mathbb{k}}\text{hom}(W, Z)$ and $\tilde{L}' \in {}_{\mathbb{k}}\text{hom}(V, W)$, for all $v \in V$, iterating expression (2.6) we have,

$$\begin{aligned} h \triangleright (\tilde{L} \circ \tilde{L}'(v)) &= h \triangleright (\tilde{L}(\tilde{L}'(v))) = (h_{(2)} \triangleright^{cop} \tilde{L})(h_{(1)} \triangleright (\tilde{L}'(v))) \\ &= (h_{(3)} \triangleright^{cop} \tilde{L}) \circ (h_{(2)} \triangleright^{cop} \tilde{L}') (h_{(1)} \triangleright v) \\ &= (h_{(2)} \triangleright^{cop} (\tilde{L} \circ \tilde{L}'))(h_{(1)} \triangleright v) \end{aligned} \tag{2.10}$$

where in the last equality we used the definition (2.5) of \triangleright^{cop} . This shows $\tilde{L} \circ \tilde{L}' \in {}_{\mathbb{k}}\text{hom}(V, Z)$.

The tensor product of morphisms in ${}^H\mathcal{M}$ is also the usual one, for all $f \in \text{Hom}_{{}^H\mathcal{M}}(V, W)$, $f' \in \text{Hom}_{{}^H\mathcal{M}}(V', W')$, $(f \otimes f')(v \otimes v') := f(v) \otimes f'(v')$, for any $v \in V, v' \in V'$. On the other hand the tensor product of internal morphisms differs from that of the category of \mathbb{k} -modules (cf. [27, Coroll. 9.3.16]). Given \mathbb{k} -linear maps $L \in \text{hom}_{\mathbb{k}}(V, W)$, $L' \in \text{hom}_{\mathbb{k}}(V', W')$ the tensor product $L \otimes_{\mathcal{R}} L'$ is the \mathbb{k} -linear map

$$L \otimes_{\mathcal{R}} L' := (L \circ \bar{R}^\alpha \triangleright) \otimes (\bar{R}_\alpha \triangleright L') \in \text{hom}_{\mathbb{k}}(V \otimes V', W \otimes W'), \tag{2.11}$$

i.e., $(L \otimes_{\mathcal{R}} L')(v \otimes v') = L(\bar{R}^\alpha \triangleright v) \otimes (\bar{R}_\alpha \triangleright L')(v')$, for any $v \in V, v' \in V'$. It is this associative tensor product that is compatible with the H -module structure: $h \triangleright (L \otimes_{\mathcal{R}} L') = h_{(1)} \triangleright L \otimes_{\mathcal{R}} h_{(2)} \triangleright L'$. From the definition it follows that

$$L \otimes_{\mathcal{R}} L' = (L \otimes \text{id}) \circ (\bar{R}^\alpha \triangleright \otimes \bar{R}_\alpha \triangleright L') = (L \otimes_{\mathcal{R}} \text{id}) \circ (\text{id} \otimes_{\mathcal{R}} L'). \tag{2.12}$$

While $L \otimes_{\mathcal{R}} \text{id} = L \otimes \text{id}$, we have $\text{id} \otimes_{\mathcal{R}} L' = \bar{R}^\alpha \triangleright \otimes \bar{R}_\alpha \triangleright L' = \tau \circ (L' \otimes \text{id}) \circ \tau^{-1}$.

Similarly, it can be proven that given \mathbb{k} -linear maps $\tilde{L} \in {}_{\mathbb{k}}\text{hom}(V, W)$, $\tilde{L}' \in {}_{\mathbb{k}}\text{hom}(V', W')$ we have the corresponding tensor product $\tilde{L} \tilde{\otimes}_{\mathcal{R}} \tilde{L}'$

$$\tilde{L} \tilde{\otimes}_{\mathcal{R}} \tilde{L}' := (\bar{R}^\alpha \triangleright^{cop} \tilde{L}) \otimes (\tilde{L}' \circ \bar{R}_\alpha \triangleright) \in {}_{\mathbb{k}}\text{hom}(V \otimes V', W \otimes W'), \tag{2.13}$$

i.e., for all $v \in V, v' \in V'$, $(\tilde{L} \tilde{\otimes}_{\mathcal{R}} \tilde{L}')(v \otimes v') = (\bar{R}^\alpha \triangleright^{cop} \tilde{L})(v) \otimes \tilde{L}'(\bar{R}_\alpha \triangleright v')$. This tensor product is associative and compatible with the H -adjoint action \triangleright^{cop} , that is, we have $h \triangleright^{cop} (\tilde{L} \tilde{\otimes}_{\mathcal{R}} \tilde{L}') = (h_{(1)} \triangleright^{cop} \tilde{L}) \tilde{\otimes}_{\mathcal{R}} (h_{(2)} \triangleright^{cop} \tilde{L}')$.

2.1.3. Algebras and bimodules

Let H be a Hopf algebra. A left H -module algebra A is an algebra with a compatible H -module structure,

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \quad h \triangleright 1_A = \epsilon(h)1_A$$

for all $h \in H$ and $a, b \in A$. We denote by ${}^H_A\mathcal{M}_A$ the category of H -equivariant A -bimodules. An object V in ${}^H_A\mathcal{M}_A$ is an A -bimodule with a compatible H -module structure, i.e., $H \otimes V \rightarrow V, h \otimes v \mapsto h \triangleright v; h \triangleright av = (h_{(1)} \triangleright a)(h_{(2)} \triangleright v)$ and similarly for the right A -module structure. Morphisms in ${}^H_A\mathcal{M}_A$ are H -equivariant maps that are also A -bimodule morphisms.

The category ${}^H_A\mathcal{M}_A$ becomes a monoidal category with the balanced tensor product \otimes_A (where by definition $V \otimes_A W$, with V, W in ${}^H_A\mathcal{M}_A$, is the quotient of $V \otimes W$ in ${}^H_A\mathcal{M}_A$ with the obvious left and right A -actions inherited from those of V and W respectively).

If V, W are modules in ${}^H_A\mathcal{M}_A$ also $\text{hom}_{\mathbb{k}}(V, W)$ and ${}_{\mathbb{k}}\text{hom}(V, W)$ are modules in ${}^H_A\mathcal{M}_A$ with H -action as in (2.3) and (2.5) respectively. The A -bimodule structure of $\text{hom}_{\mathbb{k}}(V, W)$ is given via the left A -module structure of V and W , that of ${}_{\mathbb{k}}\text{hom}(V, W)$ via the right A -module structure of V and W : For all $a \in A, v \in V, L \in \text{hom}_{\mathbb{k}}(V, W), \tilde{L} \in {}_{\mathbb{k}}\text{hom}(V, W)$,

$$(aL)(v) = a(L(v)), \quad (La)(v) = L(av), \tag{2.14}$$

$$(\tilde{L}a)(v) = \tilde{L}(v)a, \quad (a\tilde{L})(v) = \tilde{L}(va). \tag{2.15}$$

Let $\text{hom}_A(V, W) \subset \text{hom}_{\mathbb{k}}(V, W)$ be the submodule in ${}^H_A\mathcal{M}_A$ of right A -linear maps: for all $a \in A, L(va) = L(v)a$, and let ${}_A\text{hom}(V, W) \subset {}_{\mathbb{k}}\text{hom}(V, W)$ be the submodule in ${}^H_A\mathcal{M}_A$ of left A -linear maps: for all $a \in A, \tilde{L}(av) = a\tilde{L}(v)$.

Associated with $\text{hom}_A(V, W) \subset \text{hom}_{\mathbb{k}}(V, W)$ and ${}_A\text{hom}(V, W) \subset {}_{\mathbb{k}}\text{hom}(V, W)$ we have the functors $\text{hom}_A : ({}^H_A\mathcal{M}_A)^{op} \times {}^H_A\mathcal{M}_A \rightarrow {}^H_A\mathcal{M}_A$ and ${}_A\text{hom} : ({}^H_A\mathcal{M}_A)^{op} \times {}^H_A\mathcal{M}_A \rightarrow {}^H_A\mathcal{M}_A$, with the action on morphisms (f^{op}, g) in $({}^H_A\mathcal{M}_A)^{op} \times {}^H_A\mathcal{M}_A$ as in (2.4). The functor $- \otimes_A W$ is left adjoint to $\text{hom}_A(W, -)$ [9], and similarly, $V \otimes_A -$ is left adjoint to ${}_A\text{hom}(V, -)$, thus

$$({}^H_A\mathcal{M}_A, \otimes_A, \text{hom}_A, {}_A\text{hom})$$

is a biclosed monoidal category.

When H has a triangular structure \mathcal{R} we consider A to be braided commutative (also called symmetric or quasi-commutative) if, for all $a, b \in A$,

$$ab = (\bar{R}^\alpha \triangleright b)(\bar{R}_\alpha \triangleright a). \tag{2.16}$$

Similarly, V in ${}^H_A\mathcal{M}_A$ is symmetric if

$$av = (\bar{R}^\alpha \triangleright v)(\bar{R}_\alpha \triangleright a). \tag{2.17}$$

We denote by ${}^H_A\mathcal{M}_A^{\text{sym}}$ the full subcategory of symmetric modules in ${}^H_A\mathcal{M}_A$. Let V, W be modules in ${}^H_A\mathcal{M}_A^{\text{sym}}$, then $V \otimes_A W$, $\text{hom}_A(V, W)$ and ${}_A\text{hom}(V, W)$ are also in ${}^H_A\mathcal{M}_A^{\text{sym}}$; for example it is easy to see that for all $a \in A$, $L \in \text{hom}_A(V, W)$, $\tilde{L} \in {}_A\text{hom}(V, W)$, $La = (\bar{R}^\alpha \triangleright a)(\bar{R}_\alpha \triangleright L)$ and $\tilde{L}a = (\bar{R}^\alpha \triangleright a)(\bar{R}_\alpha \triangleright \tilde{L})$. Extending to the biclosed case the results of [9] we have that

$$({}^H_A\mathcal{M}_A^{\text{sym}}, \otimes_A, \text{hom}_A, {}_A\text{hom})$$

is a full closed monoidal subcategory of $({}^H_A\mathcal{M}_A, \otimes_A, \text{hom}_A, {}_A\text{hom})$ which is symmetric.

The braiding is induced from that in ${}^H_A\mathcal{M}_A$ and the isomorphisms in (2.8) restrict to isomorphisms

$$\mathcal{D}_{V,W} : \text{hom}_A(V, W) \rightarrow {}_A\text{hom}(V, W) \tag{2.18}$$

in ${}^H_A\mathcal{M}_A^{\text{sym}}$, thus proving that hom_A and ${}_A\text{hom}$ are naturally isomorphic functors (cf. [7, §5.6], the A -module actions (2.15) there are denoted \cdot^{op} , so that e.g. $a \cdot^{op} \tilde{L}(v) = \tilde{L}(va)$).

Since $({}^H_A\mathcal{M}_A^{\text{sym}}, \otimes_A, \text{hom}_A)$ is a symmetric closed monoidal category, besides the usual evaluation and composition, we have the tensor product of morphisms, denoted \otimes_A . For all $f \in \text{Hom}_{{}^H_A\mathcal{M}_A^{\text{sym}}}(V, W)$, $f' \in \text{Hom}_{{}^H_A\mathcal{M}_A^{\text{sym}}}(V', W')$, $(f \otimes_A f')(v \otimes_A v') := f(v) \otimes_A f'(v')$, for any $v \in V, v' \in V'$. We also have the tensor product of internal morphisms that with slight abuse of notation we still denote $\otimes_{\mathcal{R}}$. Indeed, similarly to the braiding, this can be seen as induced from the tensor product of internal morphisms in $({}^H_A\mathcal{M}_A, \otimes, \text{hom}_{\mathbb{k}})$. Let $L \in \text{hom}_A(V, W) \subset \text{hom}_{\mathbb{k}}(V, W)$ and $L' \in \text{hom}_A(V', W') \subset \text{hom}_{\mathbb{k}}(V', W')$, then $L \otimes_{\mathcal{R}} L' \in \text{hom}_{\mathbb{k}}(V \otimes V', W \otimes W')$, as defined in (2.11) is trivially right A -linear and induces a well-defined right A -linear map in $\text{hom}_A(V \otimes_A V', W \otimes_A W')$ that we still denote $L \otimes_{\mathcal{R}} L'$ (cf. [7, Thm. 5.16]). Associativity is straightforward. Moreover, for each quadruple V, W, V', W' of modules in ${}^H_A\mathcal{M}_A^{\text{sym}}$ the map

$$\otimes_{\mathcal{R}V,W,V',W'} : \text{hom}_A(V, W) \otimes_A \text{hom}_A(V', W') \rightarrow \text{hom}_A(V \otimes_A V', W \otimes_A W'), \quad L \otimes_A L' \mapsto L \otimes_{\mathcal{R}} L'$$

is a morphism in the category (cf. [27, Prop. 9.3.13], [9, §5.6]).

Similarly, in $({}^H_A\mathcal{M}_A^{\text{sym}}, \otimes_A, {}_A\text{hom})$ the tensor product of internal morphisms is denoted $\tilde{\otimes}_{\mathcal{R}}$ and can be seen as induced from (2.13). Here too for each quadruple V, W, V', W' of modules in ${}^H_A\mathcal{M}_A^{\text{sym}}$ we have that

$$\tilde{\otimes}_{\mathcal{R}V,W,V',W'} : {}_A\text{hom}(V, W) \otimes_A {}_A\text{hom}(V', W') \rightarrow {}_A\text{hom}(V \otimes_A V', W \otimes_A W'), \quad \tilde{L} \otimes_A \tilde{L}' \mapsto \tilde{L} \tilde{\otimes}_{\mathcal{R}} \tilde{L}'$$

is a morphism in ${}^H_A\mathcal{M}_A^{\text{sym}}$.

2.1.4. Graded algebras and bimodules

The results of the previous subsection can be extended to the case of \mathbb{Z} -graded modules $V = \bigoplus_{n \in \mathbb{Z}} V^n$. We consider H to be \mathbb{Z} -graded and nontrivial only in degree zero. Let Ω^\bullet be a graded algebra and an H -module algebra, it is graded braided commutative if

$$\theta \wedge \theta' = (-1)^{|\theta||\theta'|} (\bar{R}^\alpha \triangleright^{cop} \theta') \wedge (\bar{R}_\alpha \triangleright^{cop} \theta),$$

where \wedge denotes the product in Ω^\bullet and θ, θ' are arbitrary elements in Ω^\bullet of homogeneous degree $|\theta|$ and $|\theta'|$. Correspondingly, ${}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}}$ denotes the category of \mathbb{Z} -graded modules that are relative H -modules Ω^\bullet -bimodules (with grade compatible Ω^\bullet -module actions) and that are graded symmetric: $V = \bigoplus_{n \in \mathbb{Z}} V^n$ is in ${}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}}$ if

$$\theta v = (-1)^{|\theta||v|} (\bar{R}^\alpha \triangleright v)(\bar{R}_\alpha \triangleright^{cop} \theta),$$

where $|v|$ is the degree of the homogeneous element $v \in V$.

The category ${}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}}$ is monoidal with tensor product \otimes_{Ω^\bullet} . It is also closed, indeed first observe that for each V, W in ${}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}}$, we have that ${}_{\mathbb{k}}\text{hom}(V, W)$ and $\text{hom}_{\mathbb{k}}(V, W)$ are naturally graded H -modules. Then define

$$\Omega^\bullet\text{hom}(V, W)$$

to be the graded H -submodule of ${}_{\mathbb{k}}\text{hom}(V, W)$ spanned by *graded* left Ω^\bullet -linear maps; these are maps $\tilde{L} \in {}_{\mathbb{k}}\text{hom}(V, W)$ of homogenous degree $|\tilde{L}|$ (i.e. $\tilde{L} : V^n \rightarrow W^{n+|\tilde{L}|}$, $n \in \mathbb{Z}$) such that

$$\tilde{L}(\theta v) = (-1)^{|\tilde{L}||\theta|}\theta\tilde{L}(v). \tag{2.19}$$

Similarly, $\text{hom}_{\Omega^\bullet}(V, W)$ is the module of right Ω^\bullet -linear maps, $L(v\theta) = L(v)\theta$. The modules ${}_{\Omega^\bullet}\text{hom}(V, W)$ and $\text{hom}_{\Omega^\bullet}(V, W)$ are in ${}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}}$; their Ω^\bullet -bimodule structure reads, for all $L \in \text{hom}_{\Omega^\bullet}(V, W)$, and for all $\tilde{L} \in {}_{\Omega^\bullet}\text{hom}(V, W)$, $\theta \in \Omega^\bullet$, $v \in V$, respectively of homogenous degree $|\tilde{L}|$, $|\theta|$ and $|v|$,

$$\begin{aligned} (\theta L)(v) &= \theta(L(v)), & (L\theta)(v) &= L(\theta v), \\ (\tilde{L}\theta)(v) &= (-1)^{|\theta||v|}\tilde{L}(v)\theta, & (\theta\tilde{L})(v) &= (-1)^{|\theta|(|\tilde{L}|+|v|)}\tilde{L}(v\theta). \end{aligned} \tag{2.20}$$

This defines the functors $\text{hom}_{\Omega^\bullet} : ({}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}})^{\text{op}} \times {}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}} \rightarrow {}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}}$ and ${}_{\Omega^\bullet}\text{hom} : ({}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}})^{\text{op}} \times {}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}} \rightarrow {}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}}$, where their action on morphisms (f^{op}, g) in $({}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}})^{\text{op}} \times {}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}}$ is as in (2.4).

Similarly to the ungraded case we have the symmetric biclosed monoidal category

$$({}^H_{\Omega^\bullet}\mathcal{M}_{\Omega^\bullet}^{\text{sym}}, \otimes_{\Omega^\bullet}, \text{hom}_{\Omega^\bullet}, {}_{\Omega^\bullet}\text{hom})$$

with tensor product of internal morphisms respectively denoted $\otimes_{\mathcal{R}}$ and $\tilde{\otimes}_{\mathcal{R}}$.

2.2. Finitely generated projective modules and their duals

In this section H is a Hopf algebra (no triangularity structure is assumed). We study finitely generated and projective left (right) A -modules in ${}^H_A\mathcal{M}_A$ and their duals. It is known that finitely generated and projective \mathbb{k} -modules are rigid modules. Similarly, finitely generated and projective right (left) A -modules in ${}^H_A\mathcal{M}_A$ are rigid modules in ${}^H_A\mathcal{M}_A$. We continue the study of internal morphisms proving key canonical isomorphisms for internal morphisms in ${}^H_A\mathcal{M}_A$ arising from tensor products with a rigid module (cf. 2 in Theorem 2.5).

Among the various equivalent definitions of finitely generated projective module (see e.g. the monograph [26]) we use the convenient characterization in terms of a pair of dual bases

Lemma 2.1. (Dual Basis Lemma). *Let A be an algebra. A left A -module Γ is finitely generated and projective if and only if there exists a family of elements $\{s_i \in \Gamma : i = 1, \dots, n\}$ and left A -linear maps $\{s^i \in {}^*\Gamma := {}_A\text{Hom}(\Gamma, A) : i = 1, \dots, n\}$ with $n \in \mathbb{N}$, such that for any $s \in \Gamma$ we have (sum over repeated indices understood)*

$$s = {}^*s^i(s) s_i. \tag{2.21}$$

A right A -module Σ is finitely generated and projective if and only if there exists a family of elements $\{\sigma^i \in \Sigma : i = 1, \dots, n\}$ and right A -linear maps $\{\sigma_i^* \in \Sigma^* := \text{Hom}_A(\Sigma, A) : i = 1, \dots, n\}$ with $n \in \mathbb{N}$, such that for any $\sigma \in \Sigma$ we have

$$\sigma = \sigma^i \sigma_i^*(\sigma). \tag{2.22}$$

The set $\{s_i, {}^*s^i : i = 1, \dots, n\}$ is loosely referred to a “pair of dual bases” for the left A -module Γ , even though $\{s_i\}$ is just a generating set of Γ and not necessarily a basis. Similarly $\{\sigma_i^*, \sigma^i : i = 1, \dots, n\}$ is a pair of dual bases for the right A -module Σ .

The dual ${}^*\Gamma := {}_A\text{Hom}(\Gamma, A)$ of a finitely generated and projective left A -module Γ is a finitely generated and projective right A -module, with right A -action as in (2.15). Moreover, the dual $({}^*\Gamma)^* := \text{Hom}_A({}^*\Gamma, A)$ of the dual is a left A -module canonically identified with the original module Γ . Similarly, we have the left A -module $\Sigma^* := \text{Hom}_A(\Sigma, A)$ dual to the right A -module Σ and the canonical identification ${}^*(\Sigma^*) := {}_A\text{Hom}(\Sigma^*, A) \simeq \Sigma$. We state these properties for left A -modules (the proof can be easily derived from e.g. [26, §2B]).

Proposition 2.2. *Let Γ be a finitely generated and projective left A -module. Denote by $\{s_i, {}^*s^i : i = 1, \dots, n\}$ a pair of dual bases. For any $s \in \Gamma$, let $\iota(s) \in ({}^*\Gamma)^* := \text{Hom}_A({}^*\Gamma, A)$ be defined by $\iota(s)({}^*s) := {}^*s(s)$, for all ${}^*s \in {}^*\Gamma$. We have*

1. $\{{}^*s^i, \iota(s_i) : i = 1, \dots, n\}$ is a pair of dual bases for ${}^*\Gamma$,
2. ${}^*\Gamma$ is a finitely generated and projective right A -module,
3. The canonical map $\iota : \Gamma \rightarrow ({}^*\Gamma)^*$, $s \mapsto \iota(s)$ is an isomorphism of left A -modules.

If we consider modules in ${}^H_A\mathcal{M}_A$, Proposition 2.2 holds in ${}^H_A\mathcal{M}_A$. Let Γ be a module in ${}^H_A\mathcal{M}_A$ that is finitely generated and projective as left A -module, then

- 1'. $*\Gamma := {}_A\text{hom}(\Gamma, A)$ is in ${}^H_A\mathcal{M}_A$ and is finitely generated and projective as right A -module,
- 2'. $(*\Gamma)^* := \text{hom}_A(*\Gamma, A)$ is in ${}^H_A\mathcal{M}_A$ and is finitely generated and projective as left A -module,
- 3'. The canonical map $\iota : \Gamma \rightarrow (*\Gamma)^*$, $s \mapsto \iota(s)$ is an isomorphism in ${}^H_A\mathcal{M}_A$.

We further recall (cf. e.g. [7, Prop. 6.17]),

Proposition 2.3. *Let Γ be a finitely generated and projective left A -module, so that $*\Gamma$ is a finitely generated and projective right A -module. Let W be an A -bimodule. Then there exist right A -module and left A -module isomorphisms (evaluation maps)*

$$\begin{aligned} \flat : *\Gamma \otimes_A W &\rightarrow {}_A\text{Hom}(\Gamma, W) & \iota : W \otimes_A \Gamma &\rightarrow \text{Hom}_A(*\Gamma, W) \\ *s \otimes_A w &\mapsto (*s \otimes_A w)^\flat(s) := *s(s)w & w \otimes_A s &\mapsto \iota(w \otimes_A s)(*s) := w*s(s). \end{aligned} \tag{2.23}$$

The inverses are $\tilde{L} \mapsto \tilde{L}^{\flat^{-1}} = *s^i \otimes_A \tilde{L}(s_i)$ and $L \mapsto \iota^{-1}(L) = L(*s^i) \otimes_A s_i$ (sum on i understood). If in addition Γ and W are modules in ${}^H_A\mathcal{M}_A$ then

$$\flat : *\Gamma \otimes_A W \rightarrow {}_A\text{hom}(\Gamma, W), \quad \iota : W \otimes_A \Gamma \rightarrow \text{hom}_A(*\Gamma, W)$$

are module isomorphisms in ${}^H_A\mathcal{M}_A$.

A pairing between modules Γ, Σ in ${}^H_A\mathcal{M}_A$ is a morphism $\Gamma \otimes_A \Sigma \rightarrow A$. We denote by

$$\langle \cdot, \cdot \rangle : \Gamma \otimes_A *\Gamma \rightarrow A, \quad s \otimes_A *s \mapsto \langle s, *s \rangle := *s(s) = \iota(s)(*s) \tag{2.24}$$

the pairing due to the evaluation of $*\Gamma = {}_A\text{hom}(\Gamma, A)$ on Γ . It is well-defined on the balanced tensor product \otimes_A because of the left A -module structure of $*\Gamma$. It is easily seen to be left and right A -linear, indeed the notation $\langle \cdot, \cdot \rangle$ conveniently takes into account the A -bimodule structures of Γ and $*\Gamma$, as well as that $*\Gamma$ are left A -linear maps while $\Gamma \simeq (*\Gamma)^*$ are right A -linear maps: For all $a \in A, s \in \Gamma, *s \in *\Gamma, \langle as, *s \rangle = a\langle s, *s \rangle, \langle sa, *s \rangle = \langle s, a*s \rangle, \langle s, *sa \rangle = \langle s, *s \rangle a$. Furthermore, H -equivariance

$$h \triangleright \langle s, *s \rangle = \langle h_{(1)} \triangleright s, h_{(2)} \triangleright^{cop} *s \rangle \tag{2.25}$$

is due to the H -module structure of $*\Gamma$, cf. (2.6).

We extend the pairing $\langle \cdot, \cdot \rangle : \Gamma \otimes_A *\Gamma \rightarrow A$ to the morphisms in ${}^H_A\mathcal{M}_A$

$$\langle \cdot, \cdot \rangle : \Gamma \otimes_A *\Gamma \otimes_A W \rightarrow W, \quad s \otimes_A *s \otimes_A w \mapsto \langle s, *s \otimes_A w \rangle := \langle s, *s \rangle w \tag{2.26}$$

$$\langle \cdot, \cdot \rangle : W \otimes_A \Gamma \otimes_A *\Gamma \rightarrow W, \quad w \otimes_A s \otimes_A *s \mapsto \langle w \otimes_A s, *s \rangle := w \langle s, *s \rangle \tag{2.27}$$

so that the internal morphisms $(*s \otimes_A w)^\flat$ and $\iota(w \otimes_A s)$ of Proposition 2.3 are respectively simply denoted as $\langle \cdot, *s \otimes_A w \rangle$ and $\langle w \otimes_A s, \cdot \rangle$.

Definition 2.4. Given modules $\Gamma \in {}^H_A\mathcal{M}_A, \Sigma \in {}^H_A\mathcal{M}_A$ we say that Γ has right dual Σ , or equivalently that Σ has left dual Γ , if we have maps $\text{ev} : \Gamma \otimes_A \Sigma \rightarrow A$ (evaluation map) and $\text{coev} : A \rightarrow \Sigma \otimes_A \Gamma$ (coevaluation map) in ${}^H_A\mathcal{M}_A$ such that the compositions

$$\begin{aligned} \Gamma &\simeq \Gamma \otimes_A A \xrightarrow{\text{id}_\Gamma \otimes_A \text{coev}} \Gamma \otimes_A \Sigma \otimes_A \Gamma \xrightarrow{\text{ev} \otimes_A \text{id}_\Gamma} \Gamma, \\ \Sigma &\simeq A \otimes_A \Sigma \xrightarrow{\text{coev} \otimes_A \text{id}_\Sigma} \Sigma \otimes_A \Gamma \otimes_A \Sigma \xrightarrow{\text{id}_\Sigma \otimes_A \text{ev}} \Sigma, \end{aligned} \tag{2.28}$$

are respectively the identity maps id_Γ and id_Σ . If Γ has a right dual we say that it is right rigid. If Σ has a left dual we similarly say that it is left rigid.

We will also denote the evaluation and coevaluation maps of the right rigid module Γ by $\text{ev}_\Gamma : \Gamma \otimes_A \Sigma \rightarrow A$ and $\text{coev}_\Gamma : A \rightarrow \Sigma \otimes_A \Gamma$, and frequently use the notations $\langle \cdot, \cdot \rangle : \Gamma \otimes_A \Sigma \rightarrow A$ and $\langle \cdot, \cdot \rangle_\Gamma$ for the evaluation map.

Left (right) duals are unique up to isomorphisms, in this sense we can simply speak of *the* left (right) dual of a module, and we say that the module is rigid. This implies that if there exists a coevaluation map, it is uniquely determined by the pairing $\text{ev} : \Gamma \otimes_A \Sigma \rightarrow A$. A pairing $\langle \cdot, \cdot \rangle : \Gamma \otimes_A \Sigma \rightarrow A$ is exact if there exists a map $\text{coev} : A \rightarrow \Sigma \otimes_A \Gamma$ fulfilling the conditions of Definition 2.4.

The pairing in (2.24) is exact if and only if Γ in ${}^H_A\mathcal{M}_A$ is finitely generated and projective as left A -module.

Theorem 2.5. Let Γ in ${}^H_A\mathcal{M}_A$ and $*\Gamma := {}_A\text{hom}(\Gamma, A)$.

1. Γ is finitely generated and projective as left A -module in ${}^H_A\mathcal{M}_A$ if and only if Γ is right rigid.
2. If Γ is right rigid, for all $V, W \in {}^H_A\mathcal{M}_A$,

$${}_A\text{hom}(\Gamma \otimes_A V, W) \simeq {}_A\text{hom}(V, *\Gamma \otimes_A W), \quad \text{hom}_A(V \otimes_A *\Gamma, W) \simeq \text{hom}_A(V, W \otimes_A \Gamma) \tag{2.29}$$

are isomorphisms in ${}^H_A\mathcal{M}_A$.

3. Let Υ be also right rigid in ${}^H_A\mathcal{M}_A$ and $*\Upsilon := {}_A\text{hom}(\Upsilon, A)$, then so is $\Gamma \otimes_A \Upsilon$, and we further have the isomorphism in ${}^H_A\mathcal{M}_A$, $*\Upsilon \otimes_A *\Gamma \simeq *(\Gamma \otimes_A \Upsilon)$.

Proof. 1. Let $W = \Gamma$ in Proposition 2.3. Since $\text{id}_\Gamma \in \text{Hom}_{{}^H_A\mathcal{M}_A}(\Gamma, \Gamma) \subset {}_A\text{hom}(\Gamma, \Gamma)$ is central, i.e., $a \text{id}_\Gamma = \text{id}_\Gamma a$ (recall the A -bimodule structure (2.15) of ${}_A\text{hom}(\Gamma, \Gamma)$) so is its image $\text{id}_\Gamma^{b^{-1}} = *s^i \otimes_A s_i \in *\Gamma \otimes_A \Gamma$. Thus

$$\text{coev} : A \rightarrow *\Gamma \otimes_A \Gamma, \quad a \mapsto a *s^i \otimes_A s_i = *s^i \otimes_A s_i a \tag{2.30}$$

is a well defined H -equivariant A -bimodule map. The coherence conditions (2.28) follow using (2.30) and recalling that $\{s_i, *s^i\}$ and $\{*s^i, \iota(s_i)\}$ are a pair of dual bases for Γ and $*\Gamma$, respectively (cf. Proposition 2.2). Vice versa, if Γ is rigid then (2.30) determines a dual basis.

2. For all $\tilde{L} \in {}_A\text{hom}(\Gamma \otimes_A V, W)$, define

$$\tilde{L}^\sharp := (\text{id}_{*\Gamma} \otimes_A \tilde{L}) \circ (\text{coev} \otimes_A \text{id}_V) : V \rightarrow *\Gamma \otimes_A W, \quad v \mapsto \tilde{L}^\sharp(v) = *s^i \otimes_A \tilde{L}(s_i \otimes_A v). \tag{2.31}$$

The map \tilde{L}^\sharp is left A -linear because so is \tilde{L} and because coev is A -bilinear. Hence we have a well-defined map

$$\sharp : {}_A\text{hom}(\Gamma \otimes_A V, W) \rightarrow {}_A\text{hom}(V, *\Gamma \otimes_A W)$$

which is H -equivariant since so is coev . Its inverse is $\flat : {}_A\text{hom}(V, *\Gamma \otimes_A W) \rightarrow {}_A\text{hom}(\Gamma \otimes_A V, W)$, $s \otimes_A v \mapsto \tilde{P}^b(s \otimes_A v) = (s, \tilde{P}(v))$, (recall equation (2.26)). Indeed we have $\tilde{L}^\sharp \flat(s \otimes_A v) = \langle s, \tilde{L}^\sharp(v) \rangle = \langle s, *s^i \otimes_A \tilde{L}(s_i \otimes_A v) \rangle = \langle s, *s^i \rangle \tilde{L}(s_i \otimes_A v) = \tilde{L}(\langle s, *s^i \rangle s_i \otimes_A v) = \tilde{L}(s \otimes_A v)$ where in the third equality we used left A -linearity of \tilde{L} . We similarly have $\tilde{P}^b \sharp(s_i \otimes_A v) = *s^i \otimes_A \tilde{P}^b(s_i \otimes_A v) = *s^i \otimes_A (s_i, \tilde{P}(v)) = \tilde{P}(v)$. The second isomorphism in (2.29) is similarly proven.

3. Let $\{u_j, *u^j : j = 1, \dots, m\}$ be a dual basis of Υ , then a dual basis of $\Gamma \otimes_A \Upsilon$ is given by the elements $t_{ij} = s_i \otimes_A u_j \in \Gamma \otimes_A \Upsilon$, and the elements $*t^{ij} \in *(\Gamma \otimes_A \Upsilon)$ defined by, for all $s \in \Gamma, u \in \Upsilon$, $*t^{ij}(s \otimes_A u) := \langle s \otimes_A u, *u^j \rangle, *s^i$.

In order to prove the isomorphism $*\Upsilon \otimes_A *\Gamma \simeq *(\Gamma \otimes_A \Upsilon)$, define $\varphi_{\Gamma, \Upsilon} : *\Upsilon \otimes_A *\Gamma \rightarrow *(\Gamma \otimes_A \Upsilon)$, $*u \otimes_A *s \mapsto \varphi_{\Gamma, \Upsilon}(*u \otimes_A *s) = *t^{ij} \langle t_{ij}, *u \rangle, *s$. The map $\varphi_{\Gamma, \Upsilon}$ is easily seen to be in ${}^H_A\mathcal{M}_A$; its inverse is $\varphi_{\Gamma, \Upsilon}^{-1} : *(\Gamma \otimes_A \Upsilon) \rightarrow *\Upsilon \otimes_A *\Gamma$, $*t \mapsto \varphi_{\Gamma, \Upsilon}^{-1}(*t) = *u^j \otimes_A *s^i \langle s_i \otimes_A u_j, *t \rangle$. \square

Remark 2.6. Due to the isomorphism $*\Upsilon \otimes_A *\Gamma \simeq *(\Gamma \otimes_A \Upsilon)$, in the following the right dual of $\Gamma \otimes_A \Upsilon$ will be considered to be $*\Upsilon \otimes_A *\Gamma$, with evaluation and coevaluation maps

$$\langle \cdot, \cdot \rangle : \Gamma \otimes_A \Upsilon \otimes_A *\Upsilon \otimes_A *\Gamma \rightarrow A, \quad s \otimes_A u \otimes_A *u \otimes_A *s \mapsto \langle s \otimes_A u, *u \otimes_A *s \rangle = \langle s \otimes_A u, *s \rangle,$$

$$\text{coev} : A \rightarrow *\Upsilon \otimes_A *\Gamma \otimes_A \Gamma \otimes_A \Upsilon, \quad a \mapsto a *u^j \otimes_A *s^i \otimes_A s_i \otimes_A u_j,$$

where $\{s_i, *s^i : i = 1, \dots, n\}$ and $\{u_j, *u^j : j = 1, \dots, m\}$ are dual basis of Γ and Υ , respectively.

Remark 2.7. Definition 2.4 holds in a generic monoidal category. For example also when the tensor product is topological. In this case rigid modules are projective and topologically finitely generated (i.e., there exists a finite number of elements that span a dense subset of the module). This is the case of Example 2.12.

The right dual (or adjoint) of a morphism $f : \Gamma \rightarrow \Upsilon$ of right rigid modules in ${}^H_A\mathcal{M}_A$ is the morphism $*f : *\Upsilon \rightarrow *\Gamma$ defined by $\langle s, *f(*u) \rangle = \langle f(s), *u \rangle$, for all $s \in \Gamma, *u \in *\Upsilon$. Explicitly, $*f(*u) = *s^i \langle f(s_i), *u \rangle$ (using a pair of dual bases for Γ). Similarly, the left dual of a morphism $g : \Sigma \rightarrow \Lambda$ of left rigid modules in ${}^H_A\mathcal{M}_A$ is the morphism $g^* : \Lambda^* \rightarrow \Sigma^*$ defined by $\langle g^*(\lambda^*), \sigma \rangle = \langle \lambda^*, g(\sigma) \rangle$ for all $\lambda^* \in \Lambda^*, \sigma \in \Sigma$. Duals of internal morphisms will be studied in Section 5.1.

2.3. Compact closed categories

If the Hopf algebra H has a triangular structure \mathcal{R} and A is braided commutative (cf. (2.16)) we can consider finitely generated (left or right) A -modules in the symmetric category ${}^H_A\mathcal{M}_A^{\text{sym}}$ of symmetric modules; we recall that by definition these satisfy $av = (\bar{R}^\alpha \triangleright v)(\bar{R}_\alpha \triangleright a)$ for all $a \in A, v \in V$, cf. (2.17).

Let Γ in ${}^H_A\mathcal{M}_A^{\text{sym}}$ be finitely generated and projective as a left A -module, hence it is right rigid in ${}^H_A\mathcal{M}_A^{\text{sym}}$, with right dual ${}^*\Gamma = {}_A\text{hom}(\Gamma, A)$ which is finitely generated and projective as a right A -module in ${}^H_A\mathcal{M}_A^{\text{sym}}$. All the results in section 2.2 concerning modules in ${}^H_A\mathcal{M}_A^{\text{sym}}$ hold true in the full subcategory ${}^H_A\mathcal{M}_A^{\text{sym}}$ of symmetric modules.

In a symmetric category a right rigid module Γ is also left rigid, and vice versa a left rigid module is also right rigid. We give an explicit proof for our category of interest ${}^H_A\mathcal{M}_A^{\text{sym}}$, where we recall that the braiding τ is defined in (2.7).

Proposition 2.8. *Let $\Gamma \in {}^H_A\mathcal{M}_A^{\text{sym}}$ be right rigid, with right dual ${}^*\Gamma$, evaluation $\langle \cdot, \cdot \rangle : \Gamma \otimes_A {}^*\Gamma \rightarrow A$, and coevaluation $\text{coev} : A \rightarrow {}^*\Gamma \otimes_A \Gamma$. Then Γ is left rigid with left dual ${}^*\Gamma$ and evaluation and coevaluation maps*

$$\begin{aligned} \langle \cdot, \cdot \rangle' &:= \langle \cdot, \cdot \rangle \circ \tau_{*,\Gamma,\Gamma} : {}^*\Gamma \otimes_A \Gamma \rightarrow A, \\ \text{coev}' &:= \tau_{*,\Gamma,\Gamma} \circ \text{coev} : A \rightarrow \Gamma \otimes_A {}^*\Gamma. \end{aligned} \tag{2.32}$$

Proof. We have to show that the compositions

$$\begin{aligned} \Gamma &\simeq A \otimes_A \Gamma \xrightarrow{\text{coev}' \otimes_A \text{id}_\Gamma} \Gamma \otimes_A {}^*\Gamma \otimes_A \Gamma \xrightarrow{\text{id}_\Gamma \otimes_A \langle \cdot, \cdot \rangle'} \Gamma, \\ {}^*\Gamma &\simeq {}^*\Gamma \otimes_A A \xrightarrow{\text{id}_{* \Gamma} \otimes_A \text{coev}'} {}^*\Gamma \otimes_A \Gamma \otimes_A {}^*\Gamma \xrightarrow{\langle \cdot, \cdot \rangle' \otimes_A \text{id}_{* \Gamma}} {}^*\Gamma, \end{aligned} \tag{2.33}$$

equal id_Γ and $\text{id}_{* \Gamma}$, respectively. Using a pair of dual bases we have $\text{coev}(1_A) = {}^*s^i \otimes_A s_i \in {}^*\Gamma \otimes_A \Gamma$, $\text{coev}'(1_A) = \bar{R}^\alpha \triangleright s_i \otimes_A \bar{R}_\alpha \triangleright {}^*s^i \in \Gamma \otimes_A {}^*\Gamma$ so that these conditions read, for all $s \in \Gamma$, $*s \in {}^*\Gamma$,

$$\bar{R}^\alpha \triangleright s_i \langle \bar{R}^\beta \triangleright s, \bar{R}_\beta \bar{R}_\alpha \triangleright {}^*s^i \rangle = s, \quad \langle \bar{R}^\beta \bar{R}^\alpha \triangleright s_i, \bar{R}_\beta \triangleright *s \rangle \bar{R}_\alpha \triangleright {}^*s^i = *s. \tag{2.34}$$

We prove the first equation

$$\begin{aligned} \bar{R}^\alpha \triangleright s_i \langle \bar{R}^\beta \triangleright s, \bar{R}_\beta \bar{R}_\alpha \triangleright {}^*s^i \rangle &= \bar{R}^\gamma \triangleright \langle \bar{R}^\beta \triangleright s, \bar{R}_\beta \bar{R}_\alpha \triangleright {}^*s^i \rangle \bar{R}_\gamma \bar{R}^\alpha \triangleright s_i \\ &= \langle \bar{R}^\gamma \bar{R}^\beta \triangleright s, \bar{R}^\delta \bar{R}_\beta \bar{R}_\alpha \triangleright {}^*s^i \rangle \bar{R}_\delta \bar{R}_\gamma \bar{R}^\alpha \triangleright s_i \\ &= \langle \bar{R}^\gamma \bar{R}^\beta \triangleright s, \bar{R}^\delta \bar{R}_\alpha \bar{R}_\gamma \triangleright {}^*s^i \rangle \bar{R}_\delta \bar{R}^\alpha \bar{R}_\beta \triangleright s_i \\ &= \langle \bar{R}^\beta \triangleright s, \bar{R}_{\beta(1)} \triangleright {}^*s^i \rangle \bar{R}_{\beta(2)} \triangleright s_i \\ &= \langle s, {}^*s^i \rangle s_i \\ &= s \end{aligned} \tag{2.35}$$

where in the second line we used $(\Delta \otimes \text{id})\mathcal{R}^{-1} = \mathcal{R}_{23}^{-1}\mathcal{R}_{13}^{-1}$, in the third line the Yang–Baxter equation $\mathcal{R}_{12}^{-1}\mathcal{R}_{13}^{-1}\mathcal{R}_{23}^{-1} = \mathcal{R}_{23}^{-1}\mathcal{R}_{13}^{-1}\mathcal{R}_{12}^{-1}$, in the fourth that $\bar{R}^\delta \bar{R}_\alpha \bar{R}_\gamma \triangleright {}^*s^i = 1 \otimes 1$ due to triangularity of \mathcal{R} and then $(\text{id} \otimes \Delta)\mathcal{R}^{-1} = \mathcal{R}_{12}^{-1}\mathcal{R}_{13}^{-1}$. In the fifth we used that coev is H -equivariant so that for all $h \in H$, $h \triangleright ({}^*s^i \otimes s_i) = h \triangleright (\text{coev}(1_A)) = \text{coev}(h \triangleright 1_A) = \varepsilon(h) {}^*s^i \otimes s_i$ and in the last line the hypothesis that ${}^*\Gamma$ is right dual to Γ . The proof of the second equation in (2.34) is similarly obtained. \square

Vice versa, if Γ is left rigid with left dual ${}^*\Gamma$, evaluation and coevaluation maps $\langle \cdot, \cdot \rangle'$ and coev' , then ${}^*\Gamma$ is also right dual to Γ with evaluation and coevaluation maps $\langle \cdot, \cdot \rangle$ and coev implicitly defined by (2.32), i.e., $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle' \circ \tau_{\Gamma, * \Gamma}$, $\text{coev} := \tau_{\Gamma, * \Gamma} \circ \text{coev}'$; (just read (2.35) from the fifth line to the first and then use (2.34)). We therefore speak of finitely generated and projective A -modules in ${}^H_A\mathcal{M}_A^{\text{sym}}$, there is no need to specify if they are left or right A -modules.

We denote by ${}^H_A\mathcal{M}_A^{\text{sym,fp}}$ the full subcategory in ${}^H_A\mathcal{M}_A^{\text{sym}}$ of finitely generated and projective modules. Every module has a left and a right dual and therefore the category ${}^H_A\mathcal{M}_A^{\text{sym,fp}}$ is a rigid category. Rigid symmetric monoidal categories are called compact closed categories. We have shown that the symmetric monoidal category ${}^H_A\mathcal{M}_A^{\text{sym,fp}}$ is a compact closed category.

Remark 2.9. The exact pairing $\langle \cdot, \cdot \rangle' : {}^*\Gamma \otimes_A \Gamma \rightarrow A$ induces the isomorphisms in ${}^H_A\mathcal{M}_A^{\text{sym}}$ (cf. Proposition 2.3)

$$\begin{aligned} \langle \cdot, \cdot \rangle' : \Gamma \otimes_A W &\longrightarrow {}_A\text{hom}({}^*\Gamma, W), \quad s \otimes_A w \mapsto \langle \cdot, s \otimes_A w \rangle' := \langle \cdot, s \rangle' \otimes_A w, \\ \langle \cdot, \cdot \rangle' : W \otimes_A \Gamma &\longrightarrow \text{hom}_A(\Gamma^*, W), \quad w \otimes_A s \mapsto \langle w \otimes_A s, \cdot \rangle' := w \otimes_A \langle s, \cdot \rangle'. \end{aligned} \tag{2.36}$$

For later use we notice that the dual of the braiding isomorphism equals the braiding on the dual modules: In the notations of Remark 2.6 (and recalling the definition of right dual morphism given after Remark 2.7),

$${}^*(\tau_{\Gamma, \Upsilon}) = \tau_{* \Gamma, * \Upsilon}, \tag{2.37}$$

that is, for all $s \in \Gamma, u \in \Upsilon, *s \in *\Gamma, *u \in *\Upsilon, \langle s \otimes_A u, \tau_{*\Gamma, *\Upsilon}(*s \otimes_A *u) \rangle = \langle \tau_{\Gamma, \Upsilon}(s \otimes_A u), *s \otimes_A *u \rangle$. This is equivalent to $\langle \tau_{\Gamma, \Upsilon}^{-1}(u \otimes_A s), \tau_{*\Gamma, *\Upsilon}(*s \otimes_A *u) \rangle = \langle u \otimes_A s, *s \otimes_A *u \rangle$ and is easily proven,

$$\begin{aligned} \langle \tau_{\Gamma, \Upsilon}^{-1}(u \otimes_A s), \tau_{*\Gamma, *\Upsilon}(*s \otimes_A *u) \rangle &= \langle R_\alpha \triangleright s, \langle R^\alpha \triangleright u, \bar{R}^\beta \triangleright *u \rangle \bar{R}_\beta \triangleright *s \rangle \\ &= \bar{R}^\gamma \triangleright \langle R^\alpha \triangleright u, \bar{R}^\beta \triangleright *u \rangle \langle \bar{R}_\gamma R_\alpha \triangleright s, \bar{R}_\beta \triangleright *s \rangle \\ &= \langle \bar{R}^\gamma R^\alpha \triangleright u, \bar{R}^\delta \bar{R}^\beta \triangleright *u \rangle \langle \bar{R}_\delta \bar{R}_\gamma R_\alpha \triangleright s, \bar{R}_\beta \triangleright *s \rangle \\ &= \langle u, \bar{R}^\delta \bar{R}^\beta \triangleright *u \rangle \langle \bar{R}_\delta \triangleright s, \bar{R}_\beta \triangleright *s \rangle \\ &= \langle u, \bar{R}^\beta \triangleright *u \rangle \bar{R}_\beta \triangleright \langle s, *s \rangle \\ &= \langle u \otimes_A s, *s \otimes_A *u \rangle . \end{aligned}$$

2.4. Examples

Let H be a triangular Hopf algebra and A a braided commutative H module algebra. We study examples of compact closed categories ${}^H_A \mathcal{M}_A^{\text{sym}, \text{fp}}$ of symmetric relative H -equivariant A -bimodules finitely generated and projective as A -modules. The first example arises when $A = K$ is a cotriangular Hopf algebra and $H = U^{op} \otimes U$ is the triangular Hopf algebra obtained from the triangular Hopf algebra U dual to K . Another example is that of equivariant vector bundles on a manifold and a further one (adapting the treatment in [9, §6] to the compact closed category context) is obtained via noncommutative Drinfeld twist deformation of equivariant vector bundles.

Example 2.10. Cotriangular Hopf Algebra. Let $A = K$ be a finite dimensional Hopf algebra over a field \mathbb{k} and let U be the dual Hopf algebra. Right (left) K -coactions correspond to left (right) U -actions on modules (using Sweedler like notation, given a right K -coaction $V \rightarrow V \otimes K, v \mapsto v_0 \otimes v_1$ we have the left U -action $\triangleright : U \otimes V \rightarrow V, \xi \triangleright v = v_0 \xi(v_1)$, while given a left K -coaction $V \rightarrow K \otimes V, v \mapsto v_{-1} \otimes v_0$ we have the right U -action $\triangleleft : V \otimes U \rightarrow V, v \triangleleft \xi = \xi(v_{-1}) v_0$). Vice versa, since K is finite dimensional over the field $\mathbb{k}, \text{Hom}_{\mathbb{k}}(U, V) \simeq V \otimes K$ (cf. (2.23)); this implies that given a left U -action, the map $V \rightarrow \text{Hom}_{\mathbb{k}}(U, V), v \mapsto \Delta_v; \Delta_v(\xi) = \xi \triangleright v$ defines a right K -coaction $\Delta_R : V \rightarrow V \otimes K$; similarly, right U -actions define left K -coactions. Moreover, right (left) K -comodule algebras are equivalently left (right) U -module algebras. In particular, since K is a K -bicomodule algebra via the coproduct, then it is a U -bimodule algebra. Recall that a K -bicovariant bimodule [34, Def. 2.3] is a K -bimodule with compatible and commuting left and right K -coactions. The duality between K -coactions and U -actions implies that this category is equivalent to that of relative U -bimodules K -bimodules (these are relative left U -modules K -bimodules and relative right U -modules K -bimodules with commuting left and right U -actions). Now U -bimodules (U -bimodule algebras) are equivalently left $U^{op} \otimes U$ -modules (left $U^{op} \otimes U$ -module algebras), where U^{op} is the Hopf algebra with opposite product and $U^{op} \otimes U$ is the tensor product Hopf algebra; similarly, relative U -bimodules K -bimodules are equivalently relative left $U^{op} \otimes U$ -modules K -bimodules (the $U^{op} \otimes U$ -action on a K -bicovariant bimodule V reads $(\zeta \otimes \xi) \triangleright v = \zeta(v_{-1}) v_0 \xi(v_1)$ and on K itself $(\zeta \otimes \xi) \triangleright k = \zeta(k_1) k_2 \xi(k_3)$). Hence the monoidal category of K -bicovariant bimodules is equivalent to the monoidal category ${}^{U^{op} \otimes U}_K \mathcal{M}_K$ of relative $U^{op} \otimes U$ -modules K -bimodules. These are free K -modules since K -bicovariant bimodules are free K -modules (cf. [34, Thm. 2.1]).

Let now K be cotriangular, this is the case if and only if its dual U is triangular. Let \mathcal{R} be the triangular structure of U , i.e., the cotriangular structure of K ; in particular we have the quasi-commutativity property, for all $k, k' \in K$,

$$k'k = \mathcal{R}(k_{(1)} \otimes k'_{(1)})k_{(2)}k'_{(2)}\mathcal{R}^{-1}(k_{(3)} \otimes k'_{(3)}) . \tag{2.38}$$

Furthermore U^{op} is triangular with \mathcal{R} -matrix \mathcal{R}^{-1} and $U^{op} \otimes U$ is also triangular with \mathcal{R} -matrix $\mathcal{R} = (\text{id} \otimes \text{flip} \otimes \text{id})(\mathcal{R}^{-1} \otimes \mathcal{R})$. We show that the quasi-commutativity property (2.38) of K is just the braided commutativity property of K with respect to the \mathcal{R} -matrix of $U^{op} \otimes U$, for all $k, k' \in K$,

$$\begin{aligned} k'k &= \bar{R}^\alpha \triangleright k \bar{R}_\alpha \triangleright k' = k_{(2)}k'_{(2)} \bar{R}^\alpha(k_{(1)} \otimes k_{(3)}) \bar{R}_\alpha(k'_{(1)} \otimes k'_{(3)}) \\ &= k_{(2)}k'_{(2)} \mathcal{R}^\alpha(k_{(1)}) \bar{\mathcal{R}}^\beta(k_{(3)}) \mathcal{R}_\alpha(k'_{(1)}) \bar{\mathcal{R}}_\beta(k'_{(3)}) \\ &= \mathcal{R}(k_{(1)} \otimes k'_{(1)})k_{(2)}k'_{(2)}\mathcal{R}^{-1}(k_{(3)} \otimes k'_{(3)}) , \end{aligned} \tag{2.39}$$

where we used the notation $\mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha = (\text{id} \otimes \text{flip} \otimes \text{id})(\mathcal{R} \otimes \mathcal{R}^{-1}) = \mathcal{R}^\alpha \otimes \bar{\mathcal{R}}^\beta \otimes \mathcal{R}_\alpha \otimes \bar{\mathcal{R}}_\beta$.

In conclusion, if K is a finite dimensional cotriangular Hopf algebra over a field then K is a braided commutative algebra with respect to the triangular Hopf algebra $U^{op} \otimes U$ and, recalling also the previous section, the category ${}^{U^{op} \otimes U}_K \mathcal{M}_K^{\text{sym}, \text{f}}$ of finitely generated symmetric relative $U^{op} \otimes U$ -modules K -modules is a rigid symmetric category (compact closed category) of free K -modules.

Similarly, we can consider K a cotriangular quantum group, for example of the A, B, C, D series given via multiparametric \mathcal{R} matrices, with dually paired triangular topological Hopf algebra U over $\mathbb{C}[[\hbar]]$ [30]. As before, K is braided commutative

and the category $U^{op} \otimes_U \mathcal{M}_K^{sym,fp}$ of finitely generated and projective symmetric $U^{op} \otimes U$ -equivariant K -modules is a compact closed category. ■

Compact closed categories are easily obtained via Drinfeld twists of compact closed categories. Recall that a Drinfeld twist is an invertible element $\mathcal{F} \in H \otimes H$ satisfying the cocycle and normalization properties:

$$(\mathcal{F} \otimes id)(\Delta \otimes id)\mathcal{F} = (id \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F}, \quad (\varepsilon \otimes id)\mathcal{F} = (id \otimes \varepsilon)\mathcal{F} = 1_H. \tag{2.40}$$

Given a twist we deform the triangular Hopf algebra H with universal \mathcal{R} -matrix \mathcal{R} in the triangular Hopf algebra $H^{\mathcal{F}}$ that as algebra is the same as H , while it has coproduct $\Delta^{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}$ and universal \mathcal{R} -matrix $\mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$. We deform the braided commutative algebra A in the braided commutative algebra $A^{\mathcal{F}}$ that equals A as \mathbb{k} -module, while it has multiplication, for all $a, b \in A$, $a \cdot^{\mathcal{F}} b := (\bar{f}^\alpha \triangleright a)(\bar{f}_\alpha \triangleright b)$, where $\mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha$. The normalization conditions imply that the unit of A is also the unit of $A^{\mathcal{F}}$. We further associate to any module Γ in ${}^H_A \mathcal{M}_A^{sym,fp}$ a new module $\Gamma^{\mathcal{F}}$ in ${}^{H^{\mathcal{F}}}_{A^{\mathcal{F}}} \mathcal{M}_{A^{\mathcal{F}}}^{sym}$ that has the same \mathbb{k} -module structure. The $H^{\mathcal{F}}$ -module structure is given by the H -action (indeed as algebras $H^{\mathcal{F}}$ equals H) and the left and right $A^{\mathcal{F}}$ -actions are given by, for all $a \in A^{\mathcal{F}}$, $s \in \Gamma$, $a \cdot^{\mathcal{F}} s := (\bar{f}^\alpha \triangleright a)(\bar{f}_\alpha \triangleright s)$ and $s \cdot^{\mathcal{F}} a = (\bar{f}^\alpha \triangleright s)(\bar{f}_\alpha \triangleright a)$. Due to these assignments the compact closed category ${}^H_A \mathcal{M}_A^{sym,fp}$ is equivalent to the compact closed category ${}^{H^{\mathcal{F}}}_{A^{\mathcal{F}}} \mathcal{M}_{A^{\mathcal{F}}}^{sym,fp}$. (They are equivalent as symmetric monoidal categories and hence, since any monoidal functor preserves rigidity of objects, as compact closed categories.)

Example 2.11. *Ug-equivariant $C^\infty(M)$ -bimodules from G-equivariant vector bundles on M.* Let G be a Lie group and M a G -manifold with right G -action $r : M \times G \rightarrow M$, $(m, g) \mapsto mg$. We assume manifolds to be finite dimensional and second countable. In this case every finite rank vector bundle has a finite open covering of the base trivializing the bundle.

Recall that a G -equivariant vector bundle $E \rightarrow M$ is a vector bundle $E \rightarrow M$ where E and M are right G -manifolds, the G -actions on E and M are compatible with the projection $E \rightarrow M$ and G acts via linear isomorphisms on the fibers of E .

The right G -action on M pulls back to a left G -action $G \times C^\infty(M) \rightarrow C^\infty(M)$ on the algebra $C^\infty(M)$ of smooth complex valued functions on M . Using the G -action on E we also have a left G -action on the $C^\infty(M)$ -module of sections $\Gamma(E)$ (for all $m \in M, s \in \Gamma(E)$, $(g \triangleright s)(m) = s(mg)g^{-1}$). With this action $\Gamma(E)$ is a relative G -module $C^\infty(M)$ -module (or G -equivariant $C^\infty(M)$ -module), i.e., for all $a \in C^\infty(M), s \in \Gamma(E), g \in G, g \triangleright as = (g \triangleright a)(g \triangleright s)$. Since $E \rightarrow M$ is trivialized by a finite open covering of M then $\Gamma(E)$ is finitely generated and projective as a $C^\infty(M)$ -module. This construction leads to a functor $\Gamma : G\text{-VecBun}_M \rightarrow {}^G_{C^\infty(M)} \mathcal{M}_{C^\infty(M)}^{sym,fp}$ from the category of G -equivariant vector bundles over M to that of G -equivariant $C^\infty(M)$ -modules that are finitely generated and projective over $C^\infty(M)$. (If G is the trivial group the functor is an equivalence proving Serre-Swan theorem in the smooth context.) Both categories are rigid symmetric monoidal, hence are compact closed categories and the functor is symmetric monoidal.

Associated with the G -action on M we have a (smooth) action of the Lie algebra $\mathfrak{g} = \text{Lie } G$ on $A = C^\infty(M)$ via derivations. It canonically lifts to an action of the universal enveloping algebra $U\mathfrak{g}$ on A that therefore becomes a $U\mathfrak{g}$ -module algebra. Similarly, $\Gamma(E)$ is a relative $U\mathfrak{g}$ -module A -bimodule. (Technically, $C^\infty(M)$ and the finitely generated projective module $\Gamma(E)$ are nuclear Fréchet spaces with respect to the usual C^∞ -topology. Furthermore, $C^\infty(M)$ is a unital Fréchet algebra with the usual pointwise product $\mu := \text{diag}_M^* : C^\infty(M) \hat{\otimes} C^\infty(M) \rightarrow C^\infty(M)$ given by pull-back of the diagonal map $\text{diag}_M : M \rightarrow M \times M$. Here $C^\infty(M) \hat{\otimes} C^\infty(M) \simeq C^\infty(M \times M)$ denotes the completed tensor product. Therefore $C^\infty(M)$ is an algebra object in the category of $U\mathfrak{g}$ -Fréchet spaces.)

It follows that we have a symmetric monoidal functor $ULie \circ \Gamma$ from the category $G\text{-VecBun}_M$ to the category of $U\mathfrak{g}$ -equivariant A -bimodules. The functor is valued in the sub-category ${}^{U\mathfrak{g}}_A \mathcal{M}_{A\text{geom}}^{sym,fp}$, where objects are the modules of sections $\Gamma(E)$ of G -equivariant vector bundles $E \rightarrow M$ and morphisms are $U\mathfrak{g}$ -equivariant A -bimodule morphisms. This is a compact closed category. ■

Example 2.12. *Formal braided equivariant vector bundles.* Associated with the ring $\mathbb{C}[[\hbar]]$ of formal power series in \hbar with coefficients in \mathbb{C} we have the formal power series extension of the \mathbb{C} -modules $U\mathfrak{g}$, A and $\Gamma(E)$, denoted as usual $U\mathfrak{g}[[\hbar]]$, $A[[\hbar]]$ and $\Gamma(E)[[\hbar]]$. The tensor product in the definition of the $\mathbb{C}[[\hbar]]$ -Hopf algebra $U\mathfrak{g}[[\hbar]]$ is the completed one in the \hbar -adic topology (cf. e.g. [23, §XVI]), so that $U\mathfrak{g}[[\hbar]] \hat{\otimes} U\mathfrak{g}[[\hbar]] \simeq (U\mathfrak{g} \otimes U\mathfrak{g})[[\hbar]]$ (and the tensor product in the definition of the $\mathbb{C}[[\hbar]]$ -algebra $A[[\hbar]]$ is the completed one in the Fréchet and \hbar -adic topologies, so that $A[[\hbar]] \hat{\otimes} A[[\hbar]] \simeq C^\infty(M \times M)[[\hbar]]$). The $A[[\hbar]]$ -bimodule $\Gamma(E)[[\hbar]]$ is topologically finitely generated and projective and we denote by ${}^{U\mathfrak{g}[[\hbar]]}_{A[[\hbar]]} \mathcal{M}_{A[[\hbar]]\text{geom}}^{sym,fp}$ the associated compact closed category. It has been obtained via the change of base ring $\mathbb{C} \rightarrow \mathbb{C}[[\hbar]]$ that induces the braided monoidal functor $\mathbb{C}[[\hbar]] \hat{\otimes} : {}^U_{A\text{geom}} \mathcal{M}_A^{sym,fp} \rightarrow {}^{U\mathfrak{g}[[\hbar]]}_{A[[\hbar]]} \mathcal{M}_{A[[\hbar]]\text{geom}}^{sym,fp}$ that to $\Gamma(E)$ associates the extension $\mathbb{C}[[\hbar]] \hat{\otimes} \Gamma(E) \simeq \Gamma(E)[[\hbar]]$ and extends morphisms by $\mathbb{C}[[\hbar]]$ -linearity.

By considering a twist \mathcal{F} of $U\mathfrak{g}[[\hbar]]$ we obtain the rigid braided commutative monoidal category ${}^{U\mathfrak{g}[[\hbar]]^{\mathcal{F}}}_{A[[\hbar]]^{\mathcal{F}}} \mathcal{M}_{A[[\hbar]]^{\mathcal{F}}\text{geom}}^{sym,fp}$. This provides a deformation quantization of G -equivariant bundles on $A = C^\infty(M)$. ■

A further example, as shown in the next section, is provided by the symmetric monoidal category of the modules of covariant and contravariant tensor fields canonically associated with the differential calculus on a braided commutative H -module algebra A .

3. Differential and Cartan calculus

We give a self contained thorough exposition of the differential and Cartan calculus on a braided commutative H -module algebra A (cf. (2.16)), where H is a triangular Hopf algebra. For later use, details on the duality between covariant and contravariant vector fields are provided. The presentation is constructive and when A is a cotriangular Hopf algebra we obtain a bicovariant differential calculus à la Woronowicz [34]. We then show its equivalence with that in [21]. The notations and conventions adopted stem from Section 2 and will be used throughout the paper. They differ from [22], where the construction of a Kähler differential and Cartan calculus of an algebra in a symmetric rigid category was outlined, and from [33] where, as here, a braided derivations approach is pursued. Apart from the example and the notions of Lie algebra and bimodule of braided derivations (found in the beginning) the expert reader can skip this section and go back to relevant formulae and theorems when they will be referred to in later sections.

3.1. Braided derivations and differential calculus

Let $\text{Der}_{\mathcal{R}}(A)$ be the \mathbb{k} -module of braided derivations. These are \mathbb{k} -linear maps $u \in \text{hom}_{\mathbb{k}}(A, A)$ that satisfy the braided Leibniz rule, for all $a, b \in A$,

$$u(ab) = u(a)b + \bar{R}^\alpha \triangleright a (\bar{R}_\alpha \triangleright u)(b) . \tag{3.1}$$

$\text{Der}_{\mathcal{R}}(A)$ is an H -submodule of $\text{hom}_{\mathbb{k}}(A, A)$. Indeed, for all $h \in H, u \in \text{Der}_{\mathcal{R}}(A)$ and $a, b \in A$

$$\begin{aligned} (h \triangleright u)(ab) &= h_{(1)} \triangleright (u(S(h_{(3)}) \triangleright a S(h_{(2)}) \triangleright b)) \\ &= h_{(1)} \triangleright (u(S(h_{(3)}) \triangleright a) S(h_{(2)}) \triangleright b) + h_{(1)} \triangleright ((\bar{R}^\alpha S(h_{(3)}) \triangleright a) (\bar{R}_\alpha \triangleright u)(S(h_{(2)}) \triangleright b)) \\ &= h_{(1)} \triangleright (u(S(h_{(2)}) \triangleright a)) b + \bar{R}^\alpha h_{(2)} S(h_{(3)}) \triangleright a (\bar{R}_\alpha h_{(1)} \triangleright u)(b) \\ &= (h \triangleright u)(a) b + \bar{R}^\alpha \triangleright a (\bar{R}_\alpha \triangleright (h \triangleright u))(b) \end{aligned}$$

where in the third line we used (2.3) in the form $h \triangleright L(v) = (h_{(1)} \triangleright L)(h_{(2)} \triangleright v)$ (with L given by $\bar{R}_\alpha \triangleright u$, h given by $h_{(1)}$ and v given by $S(h_{(2)}) \triangleright b$) and then that $\Delta(h)\mathcal{R}^{-1} = \mathcal{R}^{-1}\Delta^{\text{cop}}(h)$.

As for derivations on a commutative algebra, it is not difficult to see (cf. [33, Lemma 3.1]) that the braided commutator

$$[\ , \] : \text{Der}_{\mathcal{R}}(A) \otimes \text{Der}_{\mathcal{R}}(A) \rightarrow \text{Der}_{\mathcal{R}}(A) , \quad u \otimes v \mapsto [u, v] := uv - \bar{R}^\alpha \triangleright v \bar{R}_\alpha \triangleright u ,$$

(where composition of operators is understood) closes in $\text{Der}_{\mathcal{R}}(A)$, is an H -equivariant map (for all $h \in H, u, v, \in \text{Der}_{\mathcal{R}}(A)$, $h \triangleright [u, v] = [h_{(1)} \triangleright u, h_{(2)} \triangleright v]$) and structures the H -module $\text{Der}_{\mathcal{R}}(A)$ as an (H, \mathcal{R}) -Lie algebra, i.e., we have the braided antisymmetry property and the Jacobi identity, for all $u, v, z \in \text{Der}_{\mathcal{R}}(A)$,

$$[u, v] = -[\bar{R}^\alpha \triangleright v, \bar{R}_\alpha \triangleright u] , \quad [u, [v, z]] = [[u, v], z] + [\bar{R}^\alpha \triangleright v, [\bar{R}_\alpha \triangleright u, z]] . \tag{3.2}$$

Braided derivations are furthermore a module in ${}^H_A \mathcal{M}_A^{\text{sym}}$ by defining, for all $a, b \in A$,

$$(au)(b) = a u(b) , \quad ua = (\bar{R}^\alpha \triangleright a) \bar{R}_\alpha \triangleright u ; \tag{3.3}$$

au is again a braided derivation because of the braided commutative property (2.16) of A , for all $a, b, c \in A, u \in \text{Der}_{\mathcal{R}}(A)$,

$$\begin{aligned} (au)(bc) &= a u(bc) \\ &= a u(b)c + a \bar{R}^\alpha \triangleright b (\bar{R}_\alpha \triangleright u)(c) \\ &= (au)(b)c + (\bar{R}^\beta \bar{R}^\alpha \triangleright b) (\bar{R}_\beta \triangleright a) (\bar{R}_\alpha \triangleright u)(c) \\ &= (au)(b)c + (\bar{R}^\alpha \triangleright b) (\bar{R}_\alpha \triangleright (au))(c) , \end{aligned}$$

where in the last passage we used the quasitriangularity property $\mathcal{R}_{12}^{-1} \mathcal{R}_{13}^{-1} = (\text{id} \otimes \Delta) \mathcal{R}^{-1}$. One similarly proves the compatibility $[u, av] = u(a)v + \bar{R}^\alpha \triangleright a [\bar{R}_\alpha \triangleright u, v]$.

From now on we set

$$\mathfrak{X}(A) := \text{Der}_{\mathcal{R}}(A)$$

and refer to it as the bimodule of (braided) vector fields.

We denote by $\underline{\Omega}(A) := {}^* \mathfrak{X}(A) = {}_A \text{hom}(\mathfrak{X}(A), A) \subset {}_{\mathbb{k}} \text{hom}(\mathfrak{X}(A), A)$ the right dual module of left A -linear maps $\mathfrak{X}(A) \rightarrow A$ with H -action \triangleright^{cop} defined in (2.5) and A -bimodule structure defined in (2.15). It is a module in ${}^H_A \mathcal{M}_A^{\text{sym}}$ and as in (2.24) we denote the evaluation of an element in $\underline{\Omega}(A)$ on a vector field via the bracket

$$\langle \cdot, \cdot \rangle : \mathfrak{X}(A) \otimes_A \underline{\Omega}(A) \rightarrow A, \quad u \otimes_A \omega \mapsto \langle u, \omega \rangle \tag{3.4}$$

that is H -equivariant (cf. (2.25)).

We define the map $d : A \rightarrow \underline{\Omega}(A)$ by

$$\langle u, da \rangle = u(a), \tag{3.5}$$

for all $u \in \mathfrak{X}(A)$. This definition is well-defined since both $\langle \cdot, da \rangle : \mathfrak{X}(A) \rightarrow A$ and $\hat{a} : \mathfrak{X}(A) \rightarrow A, u \mapsto u(a)$ are left A -linear maps. The map d is H -equivariant, indeed, for all $h \in H$ the identities

$$h \triangleright \langle u, da \rangle = \langle h_{(1)} \triangleright u, h_{(2)} \triangleright^{cop} da \rangle, \quad h \triangleright \langle u(a) \rangle = \langle h_{(1)} \triangleright u \rangle \langle h_{(2)} \triangleright a \rangle$$

imply $h \triangleright^{cop} \langle da \rangle = \langle d(h \triangleright a) \rangle$. Next we prove the undeformed Leibniz rule $d(ab) = (da)b + adb$:

$$\begin{aligned} \langle u, d(ab) \rangle &= u(ab) = u(a)b + \bar{R}^\alpha \triangleright a (\bar{R}_\alpha \triangleright u)(b) = \langle u, (da)b \rangle + (\bar{R}^\alpha \triangleright a) \langle \bar{R}_\alpha \triangleright u, db \rangle \\ &= \langle u, (da)b \rangle + \langle u, adb \rangle, \end{aligned}$$

where we used (3.3) and that the pairing $\langle \cdot, \cdot \rangle$ is well-defined on the balanced tensor product $\mathfrak{X}(A) \otimes_A \underline{\Omega}(A)$.

The module of one-forms $\Omega(A)$ is the submodule of $\underline{\Omega}(A)$ in ${}^H_A \mathcal{M}_A^{\text{sym}}$ defined by

$$\Omega(A) := AdA = \{ \omega \in \underline{\Omega}(A) ; \omega = a^i da_i \} \tag{3.6}$$

for all $a^i, a_i \in A$, with finite sum over the index i understood (the right A -action closes in $\Omega(A)$ due to the Leibniz rule). A first order differential calculus on an algebra A is an A -bimodule $\Omega(A)$ and a map $d : A \rightarrow \Omega(A)$ that satisfies the Leibniz rule and such that every element of $\Omega(A)$ is of the form $a^i da_i$. We thus see that $(\Omega(A), d)$ as defined in (3.5) and (3.6) is a first order differential calculus on A . It is an H -equivariant differential calculus since A and $\Omega(A)$ are in ${}^H_A \mathcal{M}_A^{\text{sym}}$ and $d : A \rightarrow \Omega(A)$ is H -equivariant.

The contraction operator is the morphism in ${}^H_A \mathcal{M}_A^{\text{sym}}$ defined by

$$i : \mathfrak{X}(A) \rightarrow \Omega(A)^* := \text{hom}_A(\Omega(A), A), \quad u \mapsto i_u, \quad i_u((da^i)a_i) := \langle u, (da^i)a_i \rangle = u(a^i)a_i. \tag{3.7}$$

Proposition 3.1. *The contraction $i : \mathfrak{X}(A) \rightarrow \Omega(A)^* := \text{hom}_A(\Omega(A), A)$ is an isomorphism in ${}^H_A \mathcal{M}_A^{\text{sym}}$.*

Proof. The \mathbb{k} -linear map $v : \text{hom}_A(\Omega(A), A) \rightarrow \text{hom}_{\mathbb{k}}(A, A), \psi \mapsto v_\psi$, given by, for all $a \in A, v_\psi(a) := \psi(da)$ is H -equivariant, for all $h \in H, a \in A$,

$$h \triangleright (v_\psi(a)) = h \triangleright (\psi(da)) = \langle h_{(1)} \triangleright \psi, h_{(2)} \triangleright^{cop} da \rangle = \langle h_{(1)} \triangleright \psi, d(h_{(2)} \triangleright a) \rangle = v_{h_{(1)} \triangleright \psi}(h_{(2)} \triangleright a).$$

We show $v(\text{hom}_A(\Omega(A), A)) \subseteq \mathfrak{X}(A)$, indeed, for all $\psi \in \text{hom}_A(\Omega(A), A), v_\psi$ is a braided derivation: for all $a, b \in A$,

$$\begin{aligned} v_\psi(ab) &= \psi((da)b + adb) = \psi(da)b + \psi((\bar{R}^\alpha \triangleright^{cop} db)\bar{R}_\alpha \triangleright a) \\ &= v_\psi(a)b + \psi(\bar{R}^\alpha \triangleright^{cop} db)\bar{R}_\alpha \triangleright a \\ &= v_\psi(a)b + (\bar{R}^\beta \bar{R}_\alpha \triangleright a)\bar{R}_\beta \triangleright (\psi(\bar{R}^\alpha \triangleright^{cop} db)) \\ &= v_\psi(a)b + (\bar{R}^\beta \bar{R}^\gamma \bar{R}_\alpha \triangleright a)(\bar{R}_\beta \triangleright \psi)(\bar{R}_\gamma \bar{R}^\alpha \triangleright^{cop} db) \\ &= v_\psi(a)b + (\bar{R}^\beta \triangleright a)(\bar{R}_\beta \triangleright \psi)(db) \\ &= v_\psi(a)b + (\bar{R}^\beta \triangleright a)v_{\bar{R}_\beta \triangleright \psi}(b) \\ &= v_\psi(a)b + (\bar{R}^\beta \triangleright a)(\bar{R}_\beta \triangleright v_\psi)(b), \end{aligned} \tag{3.8}$$

where in the fourth line we used $(\text{id} \otimes \Delta)\mathcal{R}^{-1} = \mathcal{R}_{12}^{-1}\mathcal{R}_{13}^{-1}$ and in the fifth triangularity of \mathcal{R} .

The induced map $v : \text{hom}_A(\Omega(A), A) \rightarrow \mathfrak{X}(A)$ is the inverse of the contraction operator: The equality $v \circ i = \text{id}_{\mathfrak{X}(A)}$ is easily proven, for all $u \in \mathfrak{X}(A), a \in A, v_{i_u}a = i_u(da) = u(a)$. We are left to prove $i \circ v = \text{id}_{\Omega(A)^*}$; for all $\psi \in \Omega(A)^*, (da^i)a_i \in \Omega(A), i_{v_\psi}((da^i)a_i) = v_\psi(a^i)a_i = \psi(da^i)a_i = \psi((da^i)a_i)$. \square

Corollary 3.2. *The pairing $\langle \cdot, \cdot \rangle : \mathfrak{X}(A) \otimes_A \Omega(A) \rightarrow A$ is non-degenerate: i) $\langle u, \omega \rangle = 0$ for all $\omega \in \Omega(A)$ implies $u = 0$; ii) $\langle u, \omega \rangle = 0$ for all $u \in \mathfrak{X}(A)$ implies $\omega = 0$.*

Proof. i) for all $a \in A$, $0 = \langle u, da \rangle = u(a)$ implies $u = 0$. ii) Follows from $\mathfrak{X}(A) \simeq \Omega(A)^*$. \square

Corollary 3.3. *If $\Omega(A)$ is finitely generated and projective over A then $\underline{\Omega}(A) = \Omega(A)$.*

Proof. If $\Omega(A)$ is in ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$ so is $\mathfrak{X}(A) \simeq \Omega(A)^*$ and the canonical isomorphism $*(\Omega(A)^*) \simeq \Omega(A)$ (cf. paragraph before Proposition 2.2) implies $\underline{\Omega}(A) := *\mathfrak{X}(A) \simeq \Omega(A)$; henceforth, since $\Omega(A) \subseteq \underline{\Omega}(A)$, $\underline{\Omega}(A) = \Omega(A)$. \square

Associated with the modules $\Omega(A)$ and $\mathfrak{X}(A)$ in ${}^H_A \mathcal{M}_A^{\text{sym}}$ we have the modules $\mathcal{T}^{p,0} = \Omega(A)^{\otimes p}$ and $\mathcal{T}^{0,q} = \mathfrak{X}(A)^{\otimes q}$, $p, q \in \mathbb{N}$, with $\mathcal{T}^{0,0} = A$, and the graded H -module algebras of contravariant tensor fields $\mathcal{T}^{\bullet,0} = \bigoplus_{p \in \mathbb{N}} \mathcal{T}^{p,0}$ and of covariant tensor fields $\mathcal{T}^{0,\bullet} = \bigoplus_{q \in \mathbb{N}} \mathcal{T}^{0,q}$. We also have the graded H -module algebra $\mathcal{T}^{\bullet,\bullet} = \bigoplus_{p,q \in \mathbb{N}} \mathcal{T}^{p,q}$ with product that on elements of homogeneous degree is defined by

$$\otimes_A : \mathcal{T}^{p,q} \otimes \mathcal{T}^{p',q'} \rightarrow \mathcal{T}^{p+p',q+q'}, \quad \theta \otimes_A \nu \otimes \theta' \otimes_A \nu' \mapsto \theta \otimes_A \bar{R}^\alpha \triangleright^{cop} \theta' \otimes_A \bar{R}_\alpha \triangleright \nu \otimes_A \nu', \tag{3.9}$$

where $\theta \in \mathcal{T}^{p,0}$, $\nu \in \mathcal{T}^{0,q}$, $\theta' \in \mathcal{T}^{p',0}$, $\nu' \in \mathcal{T}^{0,q'}$.

We define the module $\Omega^r(A)$ of r -forms as the submodule of $\mathcal{T}^{r,0}$ of completely braided antisymmetric tensors. This is obtained by writing every permutation \wp of r elements as composition of elementary nearest neighbour transpositions $\wp = \tau_{k_1} \circ \tau_{k_2} \dots \circ \tau_{k_n-1}$ (where $\tau_k : (1, 2, \dots, k, k+1, \dots, n) \rightarrow (1, 2, \dots, k+1, k, \dots, n)$, $k = 1, 2, \dots, r-1$) and by associating to every permutation \wp the morphism in ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$ given by

$$\Pi_\wp = \tau_{k_1} \circ \tau_{k_2} \dots \circ \tau_{k_r} : \mathcal{T}^{r,0} \rightarrow \mathcal{T}^{r,0},$$

where $\tau_k = \text{id} \otimes_A \dots \otimes_A \text{id} \otimes_A \tau \otimes_A \text{id} \otimes_A \dots \otimes_A \text{id}$ is a product of $r-1$ factors with the braiding $\tau = \tau(\mathcal{R})$ occurring in the k^{th} factor. Since the universal \mathcal{R} matrix is triangular we have $\tau_k^2 = \text{id}$ besides the braid relations $\tau_{k+1} \circ \tau_k \circ \tau_{k+1} = \tau_k \circ \tau_{k+1} \circ \tau_k$ and $\tau_k \circ \tau_\ell = \tau_\ell \circ \tau_k$ for $|k - \ell| \geq 2$. Thus the map $\wp \rightarrow \Pi_\wp$ is a well-defined representation of the permutation group.

The module $\Omega^r(A)$ of r -forms is then the image of the projector

$$A_r := \frac{1}{r!} \sum_{\wp} \text{sign}(\wp) \Pi_\wp : \mathcal{T}^{r,0} \rightarrow \mathcal{T}^{r,0}, \tag{3.10}$$

where the sum is over all the $r!$ permutations and $\text{sign}(\wp)$ is 1 or -1 depending on the even or odd number of elementary transpositions occurring in \wp .

The module of exterior forms is $\Omega^*(A) = \bigoplus_{r \in \mathbb{N}} \Omega^r(A)$ with $\Omega^0(A) = A$, $\Omega^1(A) = \Omega(A)$. From $\Pi_\wp \circ \tau_k = \Pi_{\wp \circ \tau_k}$ it follows that $A_r \circ \tau_k = -A_r$ and hence $A_r \circ \Pi_\wp = \text{sign}(\wp) A_r$ and $A_r = A_r \circ (A_\ell \otimes_A \text{id}^{\otimes_A(r-\ell)}) = A_r \circ (\text{id}^{\otimes_A \ell} \otimes_A A_{r-\ell})$ for all $\ell = 0, 1, \dots, r$. As in the classical case (see e.g. [1, §6.1]) these properties imply that the wedge product $\wedge : \Omega^*(A) \otimes \Omega^*(A) \rightarrow \Omega^*(A)$ defined on exterior forms of homogenous degree as

$$\wedge : \Omega^r(A) \otimes \Omega^{r'}(A) \longrightarrow \Omega^{r+r'}(A), \quad \theta \otimes \theta' \mapsto \theta \wedge \theta' := \frac{(r+r')!}{r!r'} A_{r+r'}(\theta \otimes_A \theta')$$

is associative and graded braided commutative: $\theta \wedge \theta' = (-1)^{r'r'} \bar{R}^\alpha \triangleright^{cop} \theta' \wedge \bar{R}_\alpha \triangleright^{cop} \theta$. The module of exterior forms with the wedge product becomes the graded braided commutative H -module algebra $(\Omega^*(A), \wedge)$. It is generated by $\Omega(A)^0 = A$ and $\Omega^1(A) = \Omega(A)$ since so is the tensor algebra $\mathcal{T}^{\bullet,\bullet}$, explicitly, let $\theta_1, \theta_2, \dots, \theta_r \in \Omega(A)$, then $\theta_1 \wedge \theta_2 \dots \wedge \theta_r = r! A_r(\theta_1 \otimes \theta_2 \dots \otimes \theta_r)$.

For later use we also observe that the wedge product of exterior forms $\theta \in \Omega^r(A)$ and $\theta' \in \Omega^{r'}(A)$ also reads

$$\theta \wedge \theta' = A_{r,r'}(\theta \otimes_A \theta'). \tag{3.11}$$

Here $A_{r,r'} := \sum \text{sign}(\mathcal{A}_{r,r'}) \Pi_{\mathcal{A}_{r,r'}}$ with the sum running over all (r, r') -shuffles $\mathcal{A}_{r,r'}$, i.e., permutations $(1, \dots, r, r+1, \dots, r+r') \rightarrow (\wp(1), \dots, \wp(r), \wp(r+1), \dots, \wp(r+r'))$ with $\wp(1) < \wp(2) < \dots < \wp(r)$ and $\wp(r+1) < \wp(r+2) < \dots < \wp(r+r')$. (This follows from the decomposition of any permutation of $r+r'$ elements as $\mathcal{A}_{r,r'} \circ \wp_r \circ \wp_{r'}$ where \wp_r permutes the first r elements, $\wp_{r'}$ the last r').

The first order differential calculus $(\Omega(A), d)$ defined in (3.5) and (3.6) uniquely extends to the graded algebra of exterior forms $\Omega^*(A)$ by defining, for all $a, a_1, a_2, \dots, a_r \in A$, $d(da) = 0$ and $d(da_1 \wedge da_2 \wedge \dots \wedge da_r) = da \wedge da_1 \wedge da_2 \wedge \dots \wedge da_r$. One has to prove however that the definition is well-defined, i.e., if $\sum_{i_1, \dots, i_r} a_{i_1, \dots, i_r} da_{i_1} \wedge da_{i_2} \wedge \dots \wedge da_{i_r} = 0$ then $\sum_{i_1, \dots, i_r} da_{i_1, \dots, i_r} \wedge da_{i_1} \wedge da_{i_2} \wedge \dots \wedge da_{i_r} = 0$ (finite sum understood).

Theorem 3.4. *Let H be a Hopf algebra with triangular structure \mathcal{R} and A a braided commutative H -module algebra. Let $\Omega(A)$ be the module in ${}^H_A \mathcal{M}_A^{\text{sym}}$ of one-forms as defined in (3.6) and $(\Omega^*(A), \wedge)$ the graded braided commutative algebra of exterior forms. There exists one and only one H -equivariant map of degree one $d : \Omega^*(A) \rightarrow \Omega^{*+1}(A)$ such that:*

1. d restricted to the degree zero subalgebra A is the original derivative $d : A \rightarrow \Omega(A)$.

2. It satisfies the graded Leibniz rule

$$d(\theta \wedge \theta') = d\theta \wedge \theta' + (-1)^k \theta \wedge d\theta' \tag{3.12}$$

for any $\theta \in \Omega(A)^k$ and $\theta' \in \Omega^*(A)$.

3. It is nilpotent, $d \circ d = 0$.

Proof. We follow the extended bimodule method of [34, Thm 4.1]. Let AX be the free left A -module with one generator X . The direct sum $\tilde{\Omega}(A) := AX \oplus \Omega(A)$ is a left A -module. Any element in $\tilde{\Omega}(A)$ has a unique decomposition as $bX + \theta$ with $b \in A, \theta \in \Omega(A)$. The left A -module $\tilde{\Omega}(A)$ becomes an A -bimodule by defining, for any $bX + \omega \in \tilde{\Omega}(A)$,

$$(bX + \omega)a = baX + bda + \omega a .$$

In particular we have $da = Xa - aX$. The bimodule $\tilde{\Omega}(A)$ is canonically a bimodule in ${}^H_A \mathcal{M}_A^{\text{sym}}$ by defining, for all $h \in H$, $h \triangleright (bX + \omega) = (h \triangleright b)X + h \triangleright^{\text{cop}} \omega$, so that the generator X is H -equivariant, for all $h \in H$, $h \triangleright X = \epsilon(h)X$.

Let $\tilde{\mathcal{T}}^{r,0} = \tilde{\Omega}(A)^{\otimes_{A^r}}$ and $\tilde{\Omega}^*(A)$ be the exterior algebra associated with A and $\tilde{\Omega}(A)$; it is defined via the projector $\tilde{A}_r : \tilde{\mathcal{T}}^{r,0} \rightarrow \tilde{\mathcal{T}}^{r,0}$ based on the braiding $\tau : \tilde{\Omega}(A) \otimes_A \tilde{\Omega}(A) \rightarrow \tilde{\Omega}(A) \otimes_A \tilde{\Omega}(A)$. We show that $\Omega^*(A)$ is a graded subalgebra of $\tilde{\Omega}^*(A)$. The degree zero subalgebras of $\Omega^*(A)$ and of $\tilde{\Omega}^*(A)$ equal A . The module $\Omega(A)$ is a submodule in ${}^H_A \mathcal{M}_A^{\text{sym}}$ of $\tilde{\Omega}(A)$, in particular the braiding of $\tilde{\Omega}(A) \otimes_A \tilde{\Omega}(A)$ restricts to the braiding of $\Omega(A) \otimes_A \Omega(A)$. This implies that $\tilde{A}_r|_{\Omega^n(A)} = A_r$ and therefore that the wedge product of $\tilde{\Omega}^*(A)$ restricts to the wedge product of $\Omega^*(A)$, i.e., $\tilde{\wedge}|_{\Omega^*(A) \otimes_A \Omega^*(A)} = \wedge$.

For any $\theta \in \Omega^*(A)$ of homogeneous degree k we define

$$d\theta := X \tilde{\wedge} \theta - (-1)^k \theta \tilde{\wedge} X .$$

On A we recover the derivative d of the first order differential calculus. The graded Leibniz rule (3.12) is easily verified. The property $d \circ d = 0$ follows from $d(d\theta) = 0$ for any $\theta \in \Omega^k(A)$ and is due to the H -equivariance of X , $X \tilde{\wedge} X = X \otimes X - \tilde{R}_\alpha \triangleright X \otimes \tilde{R}^\alpha \triangleright X = 0$. From the graded Leibniz rule and $d \circ d = 0$ we obtain, for all $a_1, a_2, \dots, a_r \in A$,

$$d(a_1 da_2 \wedge da_3 \wedge \dots \wedge da_r) = da_1 \wedge da_2 \wedge da_3 \wedge \dots \wedge da_r .$$

This shows that $d : \Omega^*(A) \rightarrow \Omega^{\bullet+1} \subset \tilde{\Omega}^{\bullet+1}$; it also shows uniqueness of $d : \Omega^*(A) \rightarrow \Omega^{\bullet+1}$. \square

The triple $(\Omega^*(A), \wedge, d)$ with $\Omega^*(A) = \bigoplus_{r \in \mathbb{N}} \Omega^r(A)$, $\Omega^0(A) = A$, is an H -equivariant braided commutative differential graded algebra because $(\Omega^*(A), \wedge)$ is a graded braided commutative H -module algebra and $d : \Omega^*(A) \rightarrow \Omega^{\bullet+1}(A)$ is an H -equivariant map of degree one that satisfies $d \circ d = 0$ and the graded Leibniz rule (3.12). It is an H -equivariant differential calculus on A because it is generated in degree zero and one.

3.2. Cartan calculus

In this section and in the rest of the paper we assume $\Omega(A)$ to be finitely generated and projective, hence $\Omega(A) = \underline{\Omega}(A)$ is in ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$ with left dual $\mathfrak{X}(A)$ (cf. Lemma 3.1 and Corollary 3.3) and the contraction operator becomes just the canonical isomorphism $\mathfrak{X}(A) \simeq (*\mathfrak{X}(A))^*$ of Proposition 2.2. Besides the evaluation map $\langle \cdot, \cdot \rangle : \mathfrak{X}(A) \otimes_A \Omega(A) \rightarrow A$ we have the coevaluation map

$$\text{coev} : A \rightarrow \Omega(A) \otimes_A \mathfrak{X}(A) , \quad a \mapsto a \omega^i \otimes_A e_i$$

where $\{e_i \in \mathfrak{X}(A), \omega^i \in \Omega(A) : i = 1, \dots, n\}$ is a pair of dual bases for $\mathfrak{X}(A)$.

We study the contraction operator on tensors and the inner derivative on forms, the Lie derivative and the Cartan calculus. While the existence of these operators and of the Cartan calculus is independent from assuming $\Omega(A)$ in ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$, if this is the case, all the relevant modules will be in ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$.

Proposition 3.5. *The module $\mathcal{T}^{0,r} = \mathfrak{X}(A)^{\otimes_{A^r}}$ in ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$ is left dual to $\mathcal{T}^{r,0} = \Omega(A)^{\otimes_{A^r}}$, with evaluation and coevaluation maps*

$$\langle \cdot, \cdot \rangle : \mathcal{T}^{0,r} \otimes_A \mathcal{T}^{r,0} \rightarrow A , \quad \langle v_r \otimes_A \dots \otimes_A v_2 \otimes_A v_1, \omega_1 \otimes_A \omega_2 \otimes_A \dots \otimes_A \omega_r \rangle = \langle v_r, \dots \langle v_2, \langle v_1, \omega_1 \rangle \omega_2 \rangle \dots \omega_r \rangle$$

$$\text{coev} : A \rightarrow \mathcal{T}^{r,0} \otimes_A \mathcal{T}^{0,r} , \quad \text{coev}(a) = a \omega^{i_1} \otimes_A \omega^{i_2} \dots \otimes_A \omega^{i_r} \otimes_A e_{i_r} \otimes_A \dots \otimes_A e_{i_2} \otimes_A e_{i_1} .$$

Proof. For $r = 1$ this is rigidity of $\Omega(A)$. The proposition follows from $\Omega(A) = *\mathfrak{X}(A)$ and, by iteration, from Remark 2.6. \square

Recalling the algebra structure (3.9) of $\mathcal{T}^{\bullet,\bullet}$, we have that $\mathcal{T}^{p,q} = \mathcal{T}^{r,0} \otimes_A \mathcal{T}^{p-r,q}$ (if $p \geq r$) and we immediately extend the pairing $\langle \cdot, \cdot \rangle$ to the evaluation of $\mathcal{T}^{0,r}$ on $\mathcal{T}^{p,q}$. This is the morphism in ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$ defined to be trivial if $r > p$ and otherwise given by

$$\langle \cdot, \cdot \rangle : \mathcal{T}^{0,r} \otimes_A \mathcal{T}^{p,q} \rightarrow \mathcal{T}^{p-r,q}, \quad \langle \nu, \theta \otimes_A \eta \rangle := \langle \nu, \theta \rangle \eta \tag{3.13}$$

for all $\nu \in \mathcal{T}^{0,r}$, $\theta \in \mathcal{T}^{r,0}$, $\eta \in \mathcal{T}^{p-r,q}$. In particular, for $r = 1$ we obtain the extension of the contraction operator to $i : \mathfrak{X}(A) \rightarrow \text{hom}_A(\mathcal{T}^{p,q}, \mathcal{T}^{p-1,q})$. Hence the evaluation (3.13) is just the iteration of the contraction operator r -times: $\langle \nu_r \otimes_A \dots \otimes_A \nu_1, \eta \rangle = i_{\nu_r} \circ \dots \circ i_{\nu_2} \circ i_{\nu_1}(\eta)$.

The exterior algebra construction applied to $\mathfrak{X}(A)$ rather than Ω gives the graded braided antisymmetric H -module algebra $\mathfrak{X}^\bullet(A) := \bigoplus_{r \in \mathbb{N}} \mathfrak{X}^r(A)$ of polyvector fields. The duality of Proposition 3.5 then implies the duality between the graded modules $\Omega^\bullet(A)$ and $\mathfrak{X}^\bullet(A)$.

Proposition 3.6. *The module $\mathfrak{X}(A)^r \subset \mathfrak{X}(A)^{\otimes_A r} = \mathcal{T}^{0,r}$ in ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$ is left dual, for any $r \in \mathbb{N}$, to $\Omega^r(A) \subset \Omega(A)^{\otimes_A r} = \mathcal{T}^{r,0}$. The evaluation map $\langle \cdot, \cdot \rangle_{\mathfrak{X}(A)^r} : \mathfrak{X}(A)^r \otimes_A \Omega^r(A) \rightarrow A$ is the restriction to $\mathfrak{X}(A)^r \otimes_A \Omega^r(A)$ of the evaluation map in Proposition 3.5, the coevaluation map is*

$$\text{coev}_{\mathfrak{X}(A)^r} : A \rightarrow \Omega^r(A) \otimes_A \mathfrak{X}(A)^r, \quad \text{coev}_{\mathfrak{X}(A)^r}(a) = \frac{1}{(r!)^2} a \omega^{i_1} \wedge \omega^{i_2} \dots \wedge \omega^{i_r} \otimes_A e_{i_r} \wedge \dots \wedge e_{i_2} \wedge e_{i_1}.$$

Proof. We begin by observing that

$$\begin{aligned} \omega^{s_1} \wedge \omega^{s_2} \dots \wedge \omega^{s_r} \otimes_A e_{s_r} \otimes_A \dots \otimes_A e_{s_2} \otimes_A e_{s_1} &= \\ = \omega^{i_1} \otimes_A \omega^{i_2} \dots \otimes_A \omega^{i_r} \otimes_A (e_{i_r} \otimes_A \dots \otimes_A e_{i_2} \otimes_A e_{i_1}, \omega^{s_1} \wedge \omega^{s_2} \dots \otimes_A \omega^{s_r}) e_{s_r} \otimes_A \dots \otimes_A e_{s_2} \otimes_A e_{s_1} & \\ = \omega^{i_1} \otimes_A \omega^{i_2} \dots \otimes_A \omega^{i_r} \otimes_A (e_{i_r} \wedge \dots \wedge e_{i_2} \wedge e_{i_1}, \omega^{s_1} \otimes_A \omega^{s_2} \dots \otimes_A \omega^{s_r}) e_{s_r} \otimes_A \dots \otimes_A e_{s_2} \otimes_A e_{s_1} & \\ = \omega^{s_1} \otimes_A \omega^{s_2} \dots \otimes_A \omega^{s_r} \otimes_A e_{s_r} \wedge \dots \wedge e_{s_2} \wedge e_{s_1} & \end{aligned} \tag{3.14}$$

where we first rewrote $\omega^{s_1} \wedge \omega^{s_2} \dots \wedge \omega^{s_r}$ using that $(\text{id}_{\mathcal{T}^{r,0}} \otimes_A \langle \cdot, \cdot \rangle) \circ (\text{coev} \otimes_A \text{id}_{\mathcal{T}^{r,0}}) = \text{id}_{\mathcal{T}^{r,0}}$ (cf. (2.28)); then we used that the adjoint of the projector $A_r : \mathcal{T}^{r,0} \rightarrow \mathcal{T}^{r,0}$ is the projector $A_r : \mathcal{T}^{0,r} \rightarrow \mathcal{T}^{0,r}$, cf. (2.37), and finally that $(\langle \cdot, \cdot \rangle_{\mathfrak{X}(A)^r} \otimes_A \text{id}_{\mathcal{T}^{0,r}}) \circ (\text{id}_{\mathcal{T}^{0,r}} \otimes_A \text{coev}) = \text{id}_{\mathcal{T}^{0,r}}$.

We now prove that $\langle \cdot, \cdot \rangle_{\mathfrak{X}(A)^r}$ and $\text{coev}_{\mathfrak{X}(A)^r}$ are evaluation and coevaluation maps. We first prove the equality $(\langle \cdot, \cdot \rangle_{\mathfrak{X}(A)^r} \otimes_A \text{id}_{\mathfrak{X}(A)^r}) \circ (\text{id}_{\mathfrak{X}(A)^r} \otimes_A \text{coev}_{\mathfrak{X}(A)^r}) = \text{id}_{\mathfrak{X}(A)^r}$, indeed, for all $u_r \wedge \dots \wedge u_2 \wedge u_1 \in \mathfrak{X}(A)^r$,

$$\begin{aligned} \frac{1}{(r!)^2} \langle u_r \wedge \dots \wedge u_2 \wedge u_1, \omega^{s_1} \wedge \omega^{s_2} \dots \wedge \omega^{s_r} \rangle_{\mathfrak{X}(A)^r} e_{s_r} \wedge \dots \wedge e_{s_2} \wedge e_{s_1} &= \\ = \frac{1}{r!} \langle u_r \wedge \dots \wedge u_2 \wedge u_1, A_r(\omega^{s_1} \otimes_A \omega^{s_2} \dots \otimes_A \omega^{s_r}) \rangle_{\mathfrak{X}(A)^r} e_{s_r} \wedge \dots \wedge e_{s_2} \wedge e_{s_1} & \\ = \frac{1}{r!} \langle A_r(u_r \wedge \dots \wedge u_2 \wedge u_1), \omega^{s_1} \otimes_A \omega^{s_2} \dots \otimes_A \omega^{s_r} \rangle_{\mathfrak{X}(A)^r} e_{s_r} \wedge \dots \wedge e_{s_2} \wedge e_{s_1} & \\ = \frac{1}{r!} \langle u_r \wedge \dots \wedge u_2 \wedge u_1, \omega^{s_1} \wedge \omega^{s_2} \dots \wedge \omega^{s_r} \rangle_{\mathfrak{X}(A)^r} e_{s_r} \otimes_A \dots \otimes_A e_{s_2} \otimes_A e_{s_1} & \\ = \langle A_r(u_r \wedge \dots \wedge u_2 \wedge u_1), \omega^{s_1} \otimes_A \omega^{s_2} \dots \otimes_A \omega^{s_r} \rangle_{\mathfrak{X}(A)^r} e_{s_r} \otimes_A \dots \otimes_A e_{s_2} \otimes_A e_{s_1} & \\ = \langle u_r \wedge \dots \wedge u_2 \wedge u_1, \omega^{s_1} \otimes_A \omega^{s_2} \dots \otimes_A \omega^{s_r} \rangle_{\mathfrak{X}(A)^r} e_{s_r} \otimes_A \dots \otimes_A e_{s_2} \otimes_A e_{s_1} & \\ = u_r \wedge \dots \wedge u_2 \wedge u_1, & \end{aligned}$$

where in the third line we used that the adjoint of the projector $A_n : \mathcal{T}^{n,0} \rightarrow \mathcal{T}^{n,0}$ is the projector $A_n : \mathcal{T}^{0,n} \rightarrow \mathcal{T}^{0,n}$, cf. (2.37), in the fourth, equation (3.14), and in the fifth we considered again the adjoint of the projector $A_n : \mathcal{T}^{n,0} \rightarrow \mathcal{T}^{n,0}$.

Similarly one proves the equality $(\text{id}_{\Omega^r(A)} \otimes_A \langle \cdot, \cdot \rangle_{\mathfrak{X}(A)^r}) \circ (\text{coev}_{\mathfrak{X}(A)^r} \otimes_A \text{id}_{\Omega^r(A)}) = \text{id}_{\Omega^r(A)}$. \square

As in the commutative case the contraction operator restricted to forms becomes the inner derivative.

Proposition 3.7. *The contraction operator $i : \mathfrak{X}(A) \otimes_A \Omega^{\otimes_A r} \rightarrow \Omega^{\otimes_A r-1}$ restricts to*

$$i : \mathfrak{X}(A) \otimes_A \Omega^r \rightarrow \Omega^{r-1}, \tag{3.15}$$

giving, for all $u \in \mathfrak{X}(A)$, the (graded) braided inner derivative $i_u : \Omega^\bullet(A) \rightarrow \Omega^{\bullet-1}(A)$,

$$i_u(\theta \wedge \theta') = i_u(\theta) \wedge \theta' + (-1)^{|\theta|} (\bar{R}^\alpha \triangleright^{\text{cop}} \theta) \wedge i_{\bar{R}_{\alpha \triangleright u}} \theta', \tag{3.16}$$

where $|\theta| \in \mathbb{N}$ is the degree of the homogeneous form θ , while $\theta' \in \Omega^\bullet(A)$. Furthermore, for all $u, v \in \mathfrak{X}(A)$, on $\Omega^\bullet(A)$ we have

$$i_u \circ i_v + i_{\bar{R}^{\alpha \triangleright v}} \circ i_{\bar{R}^{\alpha \triangleright u}} = 0. \tag{3.17}$$

Proof. Use that $\Omega(A)$ is a braided commutative A -bimodule, K -equivariance of i , the property $(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$ and triangularity of the universal \mathcal{R} -matrix to prove the first of the following equalities, for all $u \in \mathfrak{X}(A)$, $\omega \in \Omega(A)$, $\eta \in \mathcal{T}^{0,\bullet}$ and integers $k \geq 2$,

$$i_u \circ \tau_1(\omega \otimes_A \eta) = \bar{R}^{\alpha \triangleright \text{cop}} \omega \otimes_A i_{\bar{R}^{\alpha \triangleright u}} \eta ; \quad i_u \circ \tau_k = \tau_{k-1} \circ i_u \tag{3.18}$$

where τ_1 is the braiding in the first two factors of the tensor product, while τ_k is that on the k^{th} and $k^{\text{th}} + 1$ factor in $\mathcal{T}^{0,r}$, $r \geq k + 1$. The second equality is the commutativity of the contraction with the braiding, which holds whenever the braiding does not involve the first factor of the tensor product.

We now prove (3.15) by induction on the degree of exterior forms. Equation (3.15) is trivially true for $r = 0, 1$. We assume it holds for r and show it holds for $r + 1$. Let $\omega \in \Omega(A)$, $\theta \in \Omega^r(A)$; from (3.11) we have $\omega \wedge \theta = A_{1,r}(\omega \otimes_A \theta)$ where $A_{1,r}$ represents the alternating sum of all $(1, r)$ -shuffles,

$$\begin{aligned} A_{1,r} &= \text{id}^{\otimes A^r} - \tau_1 + \tau_2 \circ \tau_1 - \tau_3 \circ \tau_2 \circ \tau_1 + \dots (-1)^{r-1} \tau_r \circ \tau_{r-1} \dots \circ \tau_1 \\ &= \text{id}^{\otimes A^r} - (\text{id} \otimes_A A_{1,r-1}) \circ \tau_1. \end{aligned}$$

From the second equality in (3.18), $i_u \circ A_{1,r} = i_u - i_u \circ (\text{id} \otimes_A A_{1,r-1}) \circ \tau_1 = i_u - A_{1,r-1} \circ i_u \circ \tau_1$, hence, for all $u \in \mathfrak{X}(A)$,

$$\begin{aligned} i_u(\omega \wedge \theta) &= i_u \circ A_{1,r}(\omega \otimes_A \theta) \\ &= i_u(\omega)\theta - A_{1,r-1}(\bar{R}^{\alpha \triangleright \text{cop}} \omega \otimes_A i_{\bar{R}^{\alpha \triangleright u}} \theta) \\ &= i_u(\omega)\theta - (\bar{R}^{\alpha \triangleright \text{cop}} \omega) \wedge i_{\bar{R}^{\alpha \triangleright u}} \theta \end{aligned} \tag{3.19}$$

where in the second line we used the first equality in (3.18). The inductive hypothesis implies that this expression is in $\Omega^\bullet(A)$.

The braided Leibniz rule (3.16) is also easily proven by induction. It holds for $|\theta| = 0, 1$ (cf. (3.19)). We assume that it holds for $|\theta| \leq r$ and show that it holds also for forms of homogenous degree $r + 1$. This follows from, for all $\omega \in \Omega(A)$, $\theta' \in \Omega^\bullet(A)$, $u \in \mathfrak{X}(A)$,

$$\begin{aligned} i_u(\omega \wedge \theta \wedge \theta') &= i_u(\omega)\theta \wedge \theta' - (\bar{R}^{\alpha \triangleright \text{cop}} \omega) \wedge i_{\bar{R}^{\alpha \triangleright u}}(\theta \wedge \theta') \\ &= i_u(\omega)\theta \wedge \theta' - (\bar{R}^{\alpha \triangleright \text{cop}} \omega) \wedge i_{\bar{R}^{\alpha \triangleright u}}(\theta) \wedge \theta' + (-1)^{k+1}(\bar{R}^{\alpha \triangleright \text{cop}} \omega) \wedge (\bar{R}^{\beta \triangleright \text{cop}} \theta) \wedge i_{\bar{R}^{\beta \triangleright u}} \theta' \\ &= i_u(\omega \wedge \theta) \wedge \theta' + (-1)^{k+1}(\bar{R}^{\alpha \triangleright \text{cop}}(\omega \wedge \theta)) \wedge i_{\bar{R}^{\alpha \triangleright u}} \theta' \end{aligned}$$

where in the first equality we used (3.19), in the second the inductive hypothesis, in the third again (3.19) and where we have rewritten the last addend using $(\Delta \otimes \text{id})\mathcal{R}^{-1} = \mathcal{R}_{23}^{-1}\mathcal{R}_{13}^{-1}$.

Finally, equality (3.17) trivially holds on forms of homogenous degree 0 and 1, and for higher forms $\theta \in \Omega^r(A)$, $r \geq 2$ it follows from

$$(i_u \circ i_v + i_{\bar{R}^{\alpha \triangleright v}} \circ i_{\bar{R}^{\alpha \triangleright u}})\theta = \langle (\text{id} + \tau)(u \otimes_A v), \theta \rangle = \langle u \otimes_A v, (\text{id} + \tau_1)\theta \rangle = 0$$

where in the second passage we used (2.37) and in the last that $\tau_1 \circ A_r = \frac{1}{r!} \sum_{\wp} \text{sign}(\wp) \tau_1 \circ \Pi_{\wp} = \frac{1}{r!} \sum_{\wp} \text{sign}(\wp) \Pi_{\tau_1 \circ \wp} = -A_r$. \square

We next study the action of the (H, \mathcal{R}) -Lie algebra of vector fields $\mathfrak{X}(A)$ on tensor fields, i.e., the Lie derivative. On A and $\mathfrak{X}(A)$ we define

$$\mathcal{L} : \mathfrak{X}(A) \otimes A \rightarrow A, \quad \mathcal{L}_u(a) := u(a); \quad \mathcal{L} : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \rightarrow \mathfrak{X}(A), \quad \mathcal{L}_u(v) := [u, v].$$

Since for any $h \in H$, $h \triangleright (\mathcal{L}_u(a)) = \mathcal{L}_{h(1) \triangleright u}(h(2) \triangleright a)$ and $h \triangleright (\mathcal{L}_u(v)) = \mathcal{L}_{h(1) \triangleright u}(h(2) \triangleright v)$, \mathcal{L} is H -equivariant. It is not difficult to check its compatibility with the A -bimodule structure (3.3) of $\mathfrak{X}(A)$, $\mathcal{L}_u(av) = \mathcal{L}_u(a)v + \bar{R}^{\alpha \triangleright a} \mathcal{L}_{\bar{R}^{\alpha \triangleright u}}(v)$, $\mathcal{L}_u(va) = \mathcal{L}_u(v)a + \bar{R}^{\alpha \triangleright v} \mathcal{L}_{\bar{R}^{\alpha \triangleright u}}(a)$. We extend \mathcal{L} to an H -equivariant action of $\mathfrak{X}(A)$ on the tensor product $\mathfrak{X}(A) \otimes \mathfrak{X}(A)$ by defining, $\mathcal{L}_u(v \otimes z) = \mathcal{L}_u(v) \otimes z + \bar{R}^{\alpha \triangleright v} \otimes \mathcal{L}_{\bar{R}^{\alpha \triangleright u}} z$ that is, the action of $u \in \mathfrak{X}(A)$ is given by $\mathcal{L}_u \otimes_{\mathcal{R}} \text{id}_{\mathfrak{X}(A)} + \text{id}_{\mathfrak{X}(A)} \otimes_{\mathcal{R}} \mathcal{L}_u$; this is a well defined internal morphism in the symmetric category ${}^H\mathcal{M}$ in accordance with (2.11), in particular it commutes with the braiding τ . The compatibility of \mathcal{L} with the A -bimodule structure of $\mathfrak{X}(A)$ implies that also the following action on the balanced tensor product over A is well defined (that is, independent from the representatives): $\mathcal{L}_u(v \otimes_A z) =$

$\mathcal{L}_u(v) \otimes_A z + \bar{R}^\alpha \triangleright v \otimes_A \mathcal{L}_{\bar{R}^\alpha \triangleright u} z$. It is extended to the whole tensor algebra $\mathcal{T}^{0,\bullet}$ by iterating this braided derivation rule, i.e., for all $v, v' \in \mathcal{T}^{0,\bullet}$,

$$\mathcal{L}_u(v \otimes_A v') = \mathcal{L}_u(v) \otimes_A v' + \bar{R}^\alpha \triangleright v \otimes_A \mathcal{L}_{\bar{R}^\alpha \triangleright u}(v').$$

The Lie derivative on contravariant tensor fields is canonically defined by duality, for all $v \in \mathcal{T}^{0,r}$ and $\theta \in \mathcal{T}^{r,0}$,

$$\mathcal{L}_u \langle v, \theta \rangle = \langle \mathcal{L}_u v, \theta \rangle + \langle \bar{R}^\alpha \triangleright v, \mathcal{L}_{\bar{R}^\alpha \triangleright u} \theta \rangle \tag{3.20}$$

i.e., $\langle v, \mathcal{L}_u \theta \rangle := \mathcal{L}_{\bar{R}^\alpha \triangleright u} \langle \bar{R}^\alpha \triangleright v, \theta \rangle - \langle \mathcal{L}_{\bar{R}^\alpha \triangleright u} \bar{R}^\alpha \triangleright v, \theta \rangle$. It follows that vector fields act on the tensor algebra $\mathcal{T}^{\bullet,\bullet}$ as braided derivations. On tensor fields $\mathcal{T}^{\bullet,\bullet}$ we have, for all $u, v \in \mathfrak{X}(A)$,

$$\mathcal{L}_u \circ \mathcal{L}_v - \mathcal{L}_{\bar{R}^\alpha \triangleright v} \circ \mathcal{L}_{\bar{R}^\alpha \triangleright u} = \mathcal{L}_{[u,v]}. \tag{3.21}$$

On A this is the definition of the braided commutator $[u, v]$; on $\mathfrak{X}(A)$ it is just the Jacobi identity in (3.2), then (3.20) implies that it holds on $\Omega(A)$. Since both sides of (3.21) are braided derivations this equality extends to all $\mathcal{T}^{\bullet,\bullet}$. Equation (3.21) shows that the Lie derivative $\mathcal{L} : \mathfrak{X}(A) \otimes \mathcal{T}^{\bullet,\bullet} \rightarrow \mathcal{T}^{\bullet,\bullet}$ is an action of the (H, R) -Lie algebra of derivations $\mathfrak{X}(A)$ on $\mathcal{T}^{\bullet,\bullet}$.

The commutativity with the braiding, $\mathcal{L}_u \circ \tau = \tau \circ \mathcal{L}_u$, and the wedge product expression (3.11) imply that vector fields also act as braided derivations on the exterior algebras $\Omega^\bullet(A)$ and $\mathfrak{X}^\bullet(A) := \bigoplus_{n \in \mathbb{N}} \mathfrak{X}^n(A)$ (that are both H -submodules of $\mathcal{T}^{\bullet,\bullet}$). From (3.20) it is immediate to compute, for all $u, v \in \mathfrak{X}(A)$, $\omega \in \Omega(A)$,

$$(\mathcal{L}_u \circ i_v - i_{\bar{R}^\alpha \triangleright v} \circ \mathcal{L}_{\bar{R}^\alpha \triangleright u})\omega = i_{[u,v]}\omega; \tag{3.22}$$

since both left hand side and right hand side are braided derivations on $\Omega^\bullet(A)$, this relation extends to arbitrary exterior forms.

The Lie derivative commutes with the exterior derivative on A , for all $a \in A$, $v \in \mathfrak{X}(A)$, $\mathcal{L}_v da = d\mathcal{L}_v a$, indeed, for all $u \in \mathfrak{X}(A)$,

$$\begin{aligned} \langle u, \mathcal{L}_v da \rangle &= \mathcal{L}_{\bar{R}^\alpha \triangleright v} \langle \bar{R}^\alpha \triangleright u, da \rangle - \langle [\bar{R}^\alpha \triangleright v, \bar{R}^\alpha \triangleright u], da \rangle = \mathcal{L}_{\bar{R}^\alpha \triangleright v} \mathcal{L}_{\bar{R}^\alpha \triangleright u} a - \mathcal{L}_{[\bar{R}^\alpha \triangleright v, \bar{R}^\alpha \triangleright u]} a \\ &= \mathcal{L}_u \mathcal{L}_v a = \langle u, d\mathcal{L}_v a \rangle. \end{aligned}$$

Using induction on the form degree we have that $d\mathcal{L}_v \theta = \mathcal{L}_v d\theta$ for any $\theta \in \Omega^\bullet(A)$.

Similarly, for all $v \in \mathfrak{X}(A)$,

$$\mathcal{L}_v = i_v \circ d + d \circ i_v$$

trivially holds on A and by induction on the form degree it holds on $\Omega^\bullet(A)$ since both the right hand side and the left hand side are braided derivations of $\Omega^\bullet(A)$.

The equations $\mathcal{L}_z \circ d = d \circ \mathcal{L}_z$, $\mathcal{L}_v = i_v \circ d + d \circ i_v$ and (3.17), (3.21) (restricted to exterior forms), (3.22), $d^2 = 0$, constitute the Cartan calculus of the exterior, Lie and inner derivatives [22,33]. Notice that the derivation of these equations holds true also if $\Omega(A)$ is not finitely generated and projective over A (indeed we never used coevaluation maps, just nondegeneracy of the pairing, cf. Corollary 3.2). We summarize these equations in the following theorem,

Theorem 3.8 (Braided Cartan calculus). *Let A be a braided commutative left H -module algebra and consider the associated braided differential algebra $(\Omega^\bullet(A), \wedge, d)$. The exterior derivative, the Lie derivative and inner derivative along vector fields $u, v \in \mathfrak{X}(A)$ are graded braided derivations of $\Omega^\bullet(A)$ (respectively of degree 1, 0, -1) that satisfy*

$$\begin{aligned} [\mathcal{L}_u, \mathcal{L}_v] &= \mathcal{L}_{[u,v]}, \quad [i_u, i_v] = 0, \\ [\mathcal{L}_u, i_v] &= i_{[u,v]}, \quad [i_u, d] = \mathcal{L}_u, \\ [\mathcal{L}_u, d] &= 0, \quad [d, d] = 0, \end{aligned}$$

where $[L, L'] = L \circ L' - (-1)^{|L||L'|} \bar{R}^\alpha \triangleright L' \circ \bar{R}^\alpha \triangleright L$ is the graded braided commutator of \mathbb{k} -linear maps L, L' on $\Omega^\bullet(A)$ of degree $|L|$ and $|L'|$.

3.3. Examples

Example 3.9. *Braided derivations of a cotriangular Hopf algebra K define a bicovariant differential calculus à la Woronowicz. Let $A = K$ be a finite dimensional cotriangular Hopf algebra over a field \mathbb{k} . Let U be the dual triangular Hopf algebra with \mathcal{R} -matrix $\mathcal{R} = \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha$, inverse $\mathcal{R}^{-1} = \bar{\mathcal{R}}^\alpha \otimes \bar{\mathcal{R}}_\alpha$ and antipode S . Recall from Example 2.10 that K is a $U^{op} \otimes U$ -module algebra and is braided commutative with $\mathcal{R} = R^\alpha \otimes R_\alpha = (\text{id} \otimes \text{flip} \otimes \text{id})(\mathcal{R}^{-1} \otimes \mathcal{R})$. The braided derivations*

$$\text{Der}_{\mathcal{R}}(K) = \{u \in \text{hom}_{\mathbb{k}}(K, K) \mid u(k\ell) = u(k)\ell + \bar{R}^\alpha \triangleright k(\bar{R}_\alpha \triangleright u)(\ell), \text{ for all } k, \ell \in K\} \tag{3.23}$$

are then a relative $U^{op} \otimes U$ -module K -bimodule. We set $\mathfrak{X}(K) := \text{Der}_{\mathcal{R}}(K)$ and call it the module of (braided) vector fields. As in Example 2.10, since K is finite dimensional over the field \mathbb{k} , $\mathfrak{X}(K)$ is equivalently a K -bicovariant bimodule, it is therefore free over K . In particular, the adjoint left U^{op} -action $\triangleright : U^{op} \otimes \mathfrak{X}(K) \rightarrow \mathfrak{X}(K)$, $\zeta \otimes u \mapsto \zeta \triangleright u$, where $(\zeta \triangleright u)(k) = \zeta_{(1) \triangleright} (u(S^{-1}(\zeta_{(2)} \triangleright k))$ for all $k \in K$, with $\zeta \triangleright k = \zeta(k_{(1)})k_{(2)}$ and S^{-1} the antipode in U^{op} , is dual to the left K -coaction

$$\Delta_L : \mathfrak{X}(K) \rightarrow K \otimes \mathfrak{X}(K), \quad u \mapsto \Delta_L(u) = u_{-1} \otimes u_0 \tag{3.24}$$

defined by $u_{-1} \otimes u_0(k) := u(k_{(2)})_{(1)} S^{-1}(k_{(1)}) \otimes u(k_{(2)})_{(2)}$ for all $k \in K$. (In order to prove that $\zeta \triangleright u = \zeta(u_{-1})u_0$ for all $\zeta \in U^{op}$, evaluate both members on $k \in K$.)

Let ${}_{\text{inv}}\mathfrak{X}(K) \subset \mathfrak{X}(K)$ be the \mathbb{k} -submodule of left-invariant vector fields, i.e., of vector fields that under the adjoint left U^{op} -action transform in the trivial representation: $\zeta \triangleright u = \varepsilon(\zeta)u$, for all $\zeta \in U^{op}$; equivalently, of vector fields invariant under the coaction Δ_L : $\Delta_L(u) = 1_K \otimes u$. The K -bicovariant bimodule structure of $\mathfrak{X}(K)$ implies

$$\mathfrak{X}(K) = K \otimes {}_{\text{inv}}\mathfrak{X}(K). \tag{3.25}$$

The dual K -module of one-forms is the K -bicovariant bimodule $\Omega(K) := {}^*\mathfrak{X}(K)$. The differential $d : K \rightarrow \Omega(K)$ is defined as in (3.5); left K -linearity of the pairing $\langle \cdot, \cdot \rangle : \mathfrak{X}(K) \otimes_K \Omega(K) \rightarrow K$ implies that d is determined by the left-invariant vector fields ${}_{\text{inv}}\mathfrak{X}(K)$. Let $\{u_j\}$, $j = 1, 2, \dots, n$, be a basis of ${}_{\text{inv}}\mathfrak{X}(K)$ and $\{\omega^j\}$ the dual basis of left-invariant one-forms, $\langle u_j, \omega^j \rangle = \delta_j^i$; from $\omega^j \langle u_j, \cdot \rangle = \text{id}_{\Omega(K)}$ (sum over j understood) we have

$$dk = \omega^j u_j(k). \tag{3.26}$$

We study the module of left-invariant vector fields ${}_{\text{inv}}\mathfrak{X}(K)$ and prove that $d : K \rightarrow \Omega(K)$ defines a bicovariant differential calculus à la Woronowicz (in particular this implies the surjectivity property $\Omega(K) = KdK$, cf. (3.6)).

Due to left-invariance, the $U^{op} \otimes U$ -action on ${}_{\text{inv}}\mathfrak{X}(K)$ reduces to the U -action, so that these vector fields satisfy the braided derivation property, for all $k, \ell \in K$, $u \in {}_{\text{inv}}\mathfrak{X}(K)$,

$$u(k\ell) = u(k)\ell + \bar{R}^\alpha \triangleright k(\bar{R}_\alpha \triangleright u)(\ell) = u(k)\ell + \overline{\mathcal{R}}^\alpha \triangleright k(\overline{\mathcal{R}}_\alpha \triangleright u)(\ell). \tag{3.27}$$

This shows that the U -module ${}_{\text{inv}}\mathfrak{X}(K)$ is a (U, \mathcal{R}) -Lie algebra: that of left-invariant vector fields. Let \mathfrak{g} be the image of ${}_{\text{inv}}\mathfrak{X}(K)$ under the \mathbb{k} -linear map ${}_{\text{inv}}\mathfrak{X}(K) \rightarrow U$, $u \mapsto \chi_u := \varepsilon \circ u$. The identity $u(k) = k_1 \chi_u(k_2)$ for all $k \in K$ provides the inverse map, hence the isomorphism ${}_{\text{inv}}\mathfrak{X}(K) \simeq \mathfrak{g}$. This identity follows applying $\text{id} \otimes \varepsilon$ to $\Delta(u(k)) = k_{(1)} \otimes u(k_{(2)})$. This latter identity in turn is equivalent to the left-invariance condition $\Delta_L(u) = 1_K \otimes u$. Indeed, for all $k \in K$, $u_{-1} \otimes u_0(k) = 1_K \otimes u(k)$ is equivalent to $u_{-1}k_{(1)} \otimes u_0(k_{(2)}) = k_{(1)} \otimes u(k_{(2)})$ which, recalling the definition of Δ_L , reads $\Delta(u(k)) = k_{(1)} \otimes u(k_{(2)})$.

Furthermore, ${}_{\text{inv}}\mathfrak{X}(K) \simeq \mathfrak{g}$ is a U -module isomorphism, that is, for all $\xi \in U$, $u \in {}_{\text{inv}}\mathfrak{X}(K)$, we have the U -equivariance $\chi_{\xi \triangleright u} = \xi \triangleright \chi_u$, where $\triangleright : U \otimes U \rightarrow U$, $\xi \otimes \zeta \mapsto \xi \triangleright \zeta = \xi_1 \zeta S(\xi_2)$ is the adjoint U -action. Indeed, for all $k \in K$ (and using the standard pairing notation $\langle \cdot, \cdot \rangle : U \otimes K \rightarrow \mathbb{k}$),

$$\begin{aligned} \chi_{\xi \triangleright u}(k) &= \varepsilon(\langle \xi \triangleright u, k \rangle) \\ &= \varepsilon(\langle \xi_{(1)} \triangleright (u(S(\xi_{(2)} \triangleright k))) \rangle) \\ &= \varepsilon(u(k_{(1)})_{(1)} \langle \xi_{(1)}, u(k_{(1)})_{(2)} \rangle \langle S(\xi_{(2)}), k_{(2)} \rangle) \\ &= \langle \xi_{(1)}, u(k_{(1)}) \rangle \langle S(\xi_{(2)}), k_{(2)} \rangle \end{aligned}$$

equals

$$\begin{aligned} (\xi \triangleright \chi_u)(k) &= (\xi_{(1)} \chi_u S(\xi_{(2)}))(k) \\ &= \langle \xi_{(1)}, k_{(1)} \rangle \varepsilon(u(k_{(2)})) \langle S(\xi_{(2)}), k_{(3)} \rangle \\ &= \langle \xi_{(1)}, u(k_{(1)})_{(1)} \rangle \varepsilon(u(k_{(1)})_{(2)}) \langle S(\xi_{(2)}), k_{(2)} \rangle \\ &= \langle \xi_{(1)}, u(k_{(1)}) \rangle \langle S(\xi_{(2)}), k_{(2)} \rangle \end{aligned}$$

where in the third line we used that $u \in {}_{\text{inv}}\mathfrak{X}(K)$ is left-invariant, i.e., $\Delta(u(k)) = k_{(1)} \otimes u(k_{(2)})$.

U -equivariance of the isomorphism ${}_{\text{inv}}\mathfrak{X}(K) \simeq \mathfrak{g}$ and the braided derivation property (3.27) imply the braided derivation property $\chi_u(k\ell) = \chi_u(k)\varepsilon(\ell) + \overline{\mathcal{R}}^\alpha(k) \langle \overline{\mathcal{R}}_\alpha \triangleright \chi_u, \ell \rangle$ for all $k, \ell \in K$, hence \mathfrak{g} is the U -module and (U, \mathcal{R}) -Lie algebra

$$\mathfrak{g} = \{\chi \in U; \Delta_U \chi = \chi \otimes 1_U + \overline{\mathcal{R}}^\alpha \otimes \overline{\mathcal{R}}_\alpha \triangleright \chi\}. \tag{3.28}$$

From $(\varepsilon_U \otimes \text{id})\Delta_U \chi = \chi$ and (3.28) we see that $\mathfrak{g} \subset \ker \varepsilon_U$. In terms of the unit and counit of K we have that the elements of the U -module \mathfrak{g} satisfy, for all $\chi \in \mathfrak{g}$, $\chi(1_K) = 0$, $\Delta \chi - \chi \otimes \varepsilon_K \in U \otimes \mathfrak{g}$. Equivalently, $S(\mathfrak{g})$ is a right U -module under the adjoint action $\chi' \triangleleft \xi := S(\xi_{(1)})\chi' \xi_{(2)}$ and its elements satisfy: for all $\chi' \in S(\mathfrak{g})$, $\chi'(1_K) = 0$, $\Delta \chi' - \varepsilon_K \otimes \chi' \in S(\mathfrak{g}) \otimes U$. This

proves that $S(\mathfrak{g})$ is a quantum Lie algebra associated with a bicovariant differential calculus à la Woronowicz on the Hopf algebra K [24, §14.2.3, Coroll. 10]. The differential calculus is the one defined by $\mathfrak{g} \simeq \text{inv}\mathfrak{X}(K)$ as in (3.26). This follows from [34, Thm. 5.2] where it is proven that $dk = (\chi'_j \triangleright k)\omega^j$ with $\{\chi'_j\}$ a basis of $S(\mathfrak{g})$. Indeed choosing $\chi'_j = -S(\chi_{u_j})$ we have $dk = (\chi'_j \triangleright k)\omega^j = \omega^j u_j(k)$ as in (3.26); for the last equality see for example [8, eq. (2.17)]. ■

Remark 3.10. In Section 7.2.1 the example of the Sweedler Hopf algebra is explicitly presented.

Remark 3.11. The bicovariant calculus on the cotriangular Hopf algebra (K, \mathcal{R}) that we have constructed and which is determined by \mathfrak{g} is equivalent to the one defined in [21, §4.3]. Indeed we have,

$$\begin{aligned} \mathfrak{g} &= \{ \chi \in U; \chi(k\ell) = \chi(k)\ell + (\overline{\mathcal{R}}^\alpha \triangleright k)(\overline{\mathcal{R}}_\alpha \triangleright \chi)(\ell), \text{ for all } k, \ell \in K \} \\ &= \{ \chi \in U; \chi(k\ell) = \chi(k)\ell + \mathcal{R}^{-1}(k \otimes \ell_{(1)}S(\ell_{(3)}))\chi(\ell_{(2)}), \text{ for all } k, \ell \in K \} \end{aligned} \tag{3.29}$$

$$= \{ \chi \in U; \chi(1_K) = 0 \text{ and } \chi(\ker\varepsilon \cdot \ker\varepsilon) = 0 \} \tag{3.30}$$

where $\ker\varepsilon \cdot \ker\varepsilon$ is the left ideal in K considered in [21, Prop. 4.8], with, for all $\ell, k \in K$, $\ell \cdot k := \ell_{(1)}k_{(2)}\mathcal{R}(k_{(1)}S(k_{(3)}) \otimes S(\ell_{(2)}) = k_{(2)}\ell_{(2)}\mathcal{R}(k_{(1)} \otimes \ell_{(1)}S(\ell_{(3)}))$ (see also Proposition 4.7 for this last equality, wherein the notation used is $\mathbf{r}_2(\ell) = \mathcal{R}_\alpha(\ell)\mathcal{R}^\alpha$). The equality (3.29) follows from duality, for all $\zeta \in U$,

$$\langle \zeta \triangleright \chi \rangle(\ell) = \langle \zeta_{(1)}\chi S(\zeta_{(2)}) \rangle, \ell = \langle \zeta_{(1)} \otimes S(\zeta_{(2)}) \otimes \chi \rangle, \ell_{(1)} \otimes \ell_{(3)} \otimes \ell_{(2)} = \langle \zeta \otimes \chi, \ell_{(1)}S(\ell_{(3)}) \otimes \ell_{(2)} \rangle.$$

We prove the equality in (3.30) by proving the inclusions \subseteq and \supseteq . If $\chi \in \mathfrak{g}$ then we have already seen that $\chi(1) = 0$; we have $\chi(\ker\varepsilon \cdot \ker\varepsilon) = 0$ because $\ell \mapsto \ell_{(1)}S(\ell_{(3)}) \otimes \ell_{(2)} =: \ell_{-1} \otimes \ell_0$ is the adjoint left coaction of K on K , so that:

$$\begin{aligned} \chi(\ell \cdot k) &= \chi(k_{(2)}\ell_0)\mathcal{R}(k_{(1)} \otimes \ell_{-1}) \\ &= \chi(k_{(2)})\varepsilon(\ell_0)\mathcal{R}(k_{(1)} \otimes \ell_{-1}) + \mathcal{R}^{-1}(k_{(2)} \otimes \ell_{-1})\chi(\ell_0)\mathcal{R}(k_{(1)} \otimes \ell_{-2}) \\ &= \chi(k_{(2)})\mathcal{R}(k_{(1)} \otimes \varepsilon(\ell)1) + \chi(\ell_0)\mathcal{R}\mathcal{R}^{-1}(k \otimes \ell_{-1}) \\ &= \chi(k)\varepsilon(\ell) + \chi(\ell)\varepsilon(k), \end{aligned}$$

where we used (3.29) in the second line. This shows the inclusion \subseteq in line (3.30). For the other inclusion, if $\chi(1) = 0$ and $\chi(\ker\varepsilon \cdot \ker\varepsilon) = 0$ then from $\chi((\ell - \varepsilon(\ell)1) \cdot (k - \varepsilon(k)1)) = 0$ we have $\mathcal{R}(k_{(1)} \otimes \ell_{-1})\chi(k_{(2)}\ell_0) = \chi(\ell \cdot k) = \chi(k)\varepsilon(\ell) + \varepsilon(k)\chi(\ell)$ and equivalently

$$k_{(1)} \otimes \ell_{-2}\mathcal{R}(k_{(2)} \otimes \ell_{-1})\chi(k_{(3)}\ell_0) = k_{(1)} \otimes \ell_{-1}(\chi(k_{(2)})\varepsilon(\ell_0) + \varepsilon(k_{(2)})\chi(\ell_0)).$$

Applying \mathcal{R}^{-1} to this expression we obtain

$$\begin{aligned} \chi(k\ell) &= \mathcal{R}^{-1}(k_{(1)} \otimes \ell_{-1})\chi(k_{(2)})\varepsilon(\ell_0) + \mathcal{R}^{-1}(k \otimes \ell_{(1)}S(\ell_{(3)}))\chi(\ell_{(2)}) \\ &= \mathcal{R}^{-1}(k_{(1)} \otimes \varepsilon(\ell)1)\chi(k_{(2)}) + \overline{\mathcal{R}}^\alpha(k)\overline{\mathcal{R}}_\alpha(\ell_{(1)}S(\ell_{(3)}))\chi(\ell_{(2)}) \\ &= \chi(k)\varepsilon(\ell) + \overline{\mathcal{R}}^\alpha(k)(\overline{\mathcal{R}}_{\alpha(1)} \otimes S(\overline{\mathcal{R}}_{\alpha(2)}) \otimes \chi, \ell_{(1)} \otimes \ell_{(3)} \otimes \ell_{(2)}) \\ &= \chi(k)\varepsilon(\ell) + \overline{\mathcal{R}}^\alpha(k)(\overline{\mathcal{R}}_\alpha \triangleright \chi)(\ell), \end{aligned}$$

that shows $\chi \in \mathfrak{g}$.

Remark 3.12. A slight variation of the construction in Example 3.9 allows to consider $A = K$ an infinite dimensional Hopf algebra over a field \mathbb{k} with cotriangular structure $\mathcal{R} : K \otimes K \rightarrow \mathbb{k}$. This is obtained by rewriting the U^{op} and U -actions in terms of K -coactions, cf. Example 2.10. In this infinite dimensional case we consider the rational morphisms $\text{HOM}_{\mathbb{k}}(K, K)$, see e.g. [32], i.e., those \mathbb{k} -linear maps $L : K \rightarrow K$ such that the left and right coadjoint actions $\Delta_L : \text{HOM}_{\mathbb{k}}(K, K) \rightarrow K \otimes \text{HOM}_{\mathbb{k}}(K, K)$ and $\Delta_R : \text{HOM}_{\mathbb{k}}(K, K) \rightarrow \text{HOM}_{\mathbb{k}}(K, K) \otimes K$ are well defined by, for all $k \in K$, $L_{-1} \otimes L_0(k) := L(k_{(2)})_{(1)}S^{-1}(k_{(1)}) \otimes L(k_{(2)})_{(2)}$ and $L_0(k) \otimes L_1 := L(k_{(1)})_{(1)} \otimes L(k_{(1)})_{(2)}S(k_{(2)})$, cf. (3.24). Then, setting $U := K^\circ$ (the Hopf dual of K) and recalling the definition of U -adjoint action \triangleright in (2.3) and the associated U^{op} one just before (3.24), we have, for all $\zeta \otimes \xi \in U^{op} \otimes U$, $(\zeta \otimes \xi) \triangleright L = \zeta(L_{-1})L_0\xi(L_1)$. Recall that K is a right $K^{cop} \otimes K$ -comodule where $K^{cop} \otimes K$ has cotriangular structure $\mathcal{R} = (\mathcal{R}^{-1} \otimes \mathcal{R}) \circ (\text{id} \otimes \text{flip} \otimes \text{id})$. Braided derivations $u \in \text{Der}_{\mathcal{R}}(K)$ are then defined as in (3.23) but with $u \in \text{HOM}_{\mathbb{k}}(K, K)$ and where, for all $k \in K$, $\overline{R}^\alpha \triangleright k \overline{R}_\alpha \triangleright u = \mathcal{R}^{-1} \triangleright (k \otimes u)$ stands now for $k_{(2)}u_0\mathcal{R}^{-1}(k_{(1)} \otimes k_{(3)} \otimes u_{-1} \otimes u_1)$, (cf. (2.39)). They span a K -bicovariant bimodule so that they are a free left K -module as in (3.25). The right K -comodule of left invariant vector fields $\text{inv}\mathfrak{X}(K)$ is isomorphic to the right K -comodule \mathfrak{g} as defined in (3.28) where $\overline{\mathcal{R}}^\alpha \otimes \overline{\mathcal{R}}_\alpha \triangleright \chi$ is the element in $U \otimes U$ defined by $\mathcal{R}^{-1}(- \otimes \chi_1) \otimes \chi_0$. (That, for all $k \in K$, $\mathcal{R}^{-1}(- \otimes k) \in U$ follows from cotriangularity of \mathcal{R} .) We then proceed as after equation (3.28) and, if U separates the elements of K , we conclude that $S(\mathfrak{g})$ defines a quantum Lie algebra. The bicovariant differential calculus is as in (3.26).

Example 3.13. Let G be a Lie group and M a G -manifold as in Example 2.11. Recall the Drinfeld twist deformation of vector bundles on M of Example 2.11. Drinfeld twist deformations of the differential and Cartan calculus on M have been considered in [5] and in more generality in [33]. They exemplify the constructions presented in this section. ■

4. Right connections, left connections, curvature and torsion

We review connections on modules $\Gamma \in {}^H_A\mathcal{M}_A^{\text{sym}}$ considered as right A -modules or as left A -modules and study their extensions to ${}^H_{\Omega^\bullet(A)}\mathcal{M}_{\Omega^\bullet(A)}^{\text{sym}}$, where as usual H is a triangular Hopf algebra, A a braided commutative H -module algebra and $(\Omega^\bullet(A), \wedge, d)$ the associated differential calculus constructed in Section 3. This first explanatory part can be found e.g. in [13, §4], the braided commutativity aspects being in [7]. Covariant derivatives along vector fields are then introduced, their braided bracket with inner derivatives is an inner derivative generalizing the braided Cartan identity $[\mathcal{L}_u, i_v] = i_{[u, v]}$ to $[d_{\nabla_u}, i_v] = i_{[u, v]}$. The latter implies that the curvature tensor, defined as the square of the connection, can be equivalently defined via the commutator of covariant derivatives along vector fields. This extends to the braided commutative geometry setting the usual two equivalent definitions of curvature. Similarly for the torsion tensor.

Right (left) connections on modules in ${}^H_A\mathcal{M}_A^{\text{sym}}$ are more general connections than the bimodule connections considered in [18]. As shown in the last subsection, using the braided commutativity property of A and of Γ they can be summed to give connections on tensor product modules.

4.1. Connections and Cartan formula

Let H be a triangular Hopf algebra, A be a braided commutative H -module algebra and $(\Omega^\bullet(A), \wedge, d)$ the associated H -equivariant differential calculus constructed in Section 3.

Definition 4.1. A right connection on a module Γ in ${}^H_A\mathcal{M}_A^{\text{sym}}$ is a \mathbb{k} -linear map

$$\nabla : \Gamma \rightarrow \Gamma \otimes_A \Omega(A) \tag{4.1}$$

in $\text{hom}_{\mathbb{k}}(\Gamma, \Gamma \otimes_A \Omega(A))$, which satisfies the Leibniz rule, for all $s \in \Gamma, a \in A$,

$$\nabla(sa) = \nabla(s)a + s \otimes_A da . \tag{4.2}$$

A left connection on Γ is a \mathbb{k} -linear map

$$\nabla : \Gamma \rightarrow \Omega(A) \otimes_A \Gamma \tag{4.3}$$

in ${}_{\mathbb{k}}\text{hom}(\Gamma, \Omega(A) \otimes_A \Gamma)$, which satisfies the Leibniz rule,

$$\nabla(as) = da \otimes_A s + a \nabla(s) . \tag{4.4}$$

We denote by $\text{Con}_A(\Gamma)$ and ${}_A\text{Con}(\Gamma)$ the set of all right, respectively left connections. Notice that in the definition of $\text{Con}_A(\Gamma)$ (respectively ${}_A\text{Con}(\Gamma)$), no compatibility condition with the left (right) A -module structure of Γ is required.

Given any connection $\nabla \in \text{Con}_A(\Gamma)$ and any right A -linear map $L \in \text{hom}_A(\Gamma, \Gamma \otimes_A \Omega(A))$, the sum $\nabla + L$ is a connection in $\text{Con}_A(\Gamma)$. The action $\nabla \mapsto \nabla + L$ is free and transitive and hence $\text{Con}_A(\Gamma)$ is an affine space over the module $\text{hom}_A(\Gamma, \Gamma \otimes_A \Omega(A))$ in ${}^H_A\mathcal{M}_A^{\text{sym}}$.

Similarly, the difference $\nabla - \nabla'$ of two left connections is a left A -linear map, and ${}_A\text{Con}(\Gamma)$ is an affine space over the module ${}_A\text{hom}(\Gamma, \Gamma \otimes_A \Omega(A))$ in ${}^H_A\mathcal{M}_A^{\text{sym}}$. Notice that left connections, as left A -linear maps, act from the right, cf. (2.5) and (2.6). Their properties become more intuitive by evaluating them on the right of elements $s \in \Gamma$, hence writing $(s)^\nabla$ rather than $\nabla(s)$.

We can act with the H -adjoint action \triangleright defined in (2.3) on $\nabla \in \text{Con}_A(\Gamma) \subset \text{hom}_{\mathbb{k}}(\Gamma, \Gamma \otimes_A \Omega(A))$, for all $h \in H$,

$$h \triangleright \nabla := h_1 \triangleright \circ \nabla \circ S(h_2) \triangleright . \tag{4.5}$$

This \mathbb{k} -linear map $h \triangleright \nabla \in \text{hom}_{\mathbb{k}}(\Gamma, \Gamma \otimes_A \Omega(A))$ is easily seen to satisfy (cf. [7, §6.2]), for all $s \in \Gamma$ and $a \in A$,

$$(h \triangleright \nabla)(sa) = (h \triangleright \nabla)(s)a + s \otimes_A \varepsilon(h)da . \tag{4.6}$$

In particular we see that if $\varepsilon(h) = 0$ then $h \triangleright \nabla \in \text{hom}_A(\Gamma, \Gamma \otimes_A \Omega(A))$, while if $\varepsilon(h) = 1$ then $h \triangleright \nabla \in \text{Con}_A(\Gamma)$. Similarly for $\nabla \in {}_A\text{Con}(\Gamma)$. Using this action and the braided commutativity of the A -bimodule Γ , a right connection ∇ on Γ is shown to be also a braided left connection, cf. [7, Prop. 6.8], and similarly a left connection ∇ on Γ is also a braided right connection, for all $a \in A, s \in \Gamma$,

$$\begin{aligned} \nabla(as) &= (\bar{R}^\alpha \triangleright a)(\bar{R}_\alpha \triangleright \nabla)(s) + \bar{R}^\alpha \triangleright s \otimes_A \bar{R}_\alpha \triangleright da , \\ \nabla(sa) &= (\bar{R}^\alpha \triangleright^{cop} \nabla)(s)(\bar{R}_\alpha \triangleright a) + \bar{R}^\alpha \triangleright da \otimes_A \bar{R}_\alpha \triangleright s . \end{aligned} \tag{4.7}$$

If ∇ is H -equivariant we have $\nabla(as) = a \nabla(s) + \bar{R}^\alpha \triangleright s \otimes_A \bar{R}_\alpha \triangleright da$, and thus recover the notion of bimodule connection studied in [18].

The H -module algebra homomorphism (injection) $A \rightarrow \Omega^\bullet(A)$ allows to associate to modules in ${}^H_A \mathcal{M}_A^{\text{sym}}$ modules in ${}^H_{\Omega^\bullet(A)} \mathcal{M}_{\Omega^\bullet(A)}^{\text{sym}}$ via the change of base ring $\Gamma \rightarrow \Omega^\bullet(A) \otimes_A \Gamma$. Indeed, given Γ in ${}^H_A \mathcal{M}_A^{\text{sym}}$, the bimodule in ${}^H_{\Omega^\bullet(A)} \mathcal{M}_{\Omega^\bullet(A)}^{\text{sym}}$

$$\Omega^\bullet(A) \otimes_A \Gamma \tag{4.8}$$

is naturally a left $\Omega^\bullet(A)$ -module by defining, for all $\theta, \theta' \in \Omega^\bullet(A)$, $s \in \Gamma$, $\theta' \wedge (\theta \otimes_A s) := (\theta' \wedge \theta) \otimes_A s$. Since $\Omega^\bullet(A)$ is a graded braided commutative H -module algebra, $\Omega^\bullet(A) \otimes_A \Gamma = \bigoplus_{n \in \mathbb{N}} \Omega^n(A) \otimes_A \Gamma$ becomes a graded braided commutative $\Omega^\bullet(A)$ -bimodule by defining $(\theta \otimes_A s) \wedge \theta' := \theta \wedge \bar{R}^\alpha \triangleright^{\text{cop}} \theta' \otimes_A \bar{R}_\alpha \triangleright s$. Hence $\Omega^\bullet(A) \otimes_A \Gamma$ is a module in ${}^H_{\Omega^\bullet(A)} \mathcal{M}_{\Omega^\bullet(A)}^{\text{sym}}$. We notice that, for any Γ, Γ' in ${}^H_A \mathcal{M}_A^{\text{sym}}$, the canonical isomorphism in ${}^H_{\Omega^\bullet(A)} \mathcal{M}_{\Omega^\bullet(A)}^{\text{sym}}$,

$$(\Omega^\bullet(A) \otimes_A \Gamma) \otimes_{\Omega^\bullet(A)} (\Omega^\bullet(A) \otimes_A \Gamma') \simeq \Omega^\bullet(A) \otimes_A (\Gamma \otimes_A \Gamma'),$$

given by $(\theta \otimes_A s) \otimes_{\Omega^\bullet(A)} (\theta' \otimes_A s') = (\theta \otimes_A s) \otimes_{\Omega^\bullet(A)} \theta' \wedge (1_{\Omega^\bullet(A)} \otimes_A s') \mapsto \theta \wedge \bar{R}^\alpha \triangleright^{\text{cop}} \theta' \otimes_A (\bar{R}_\alpha \triangleright s \otimes_A s')$ implies that the association $\Gamma \rightarrow \Omega^\bullet(A) \otimes_A \Gamma$ defines a strict monoidal functor from ${}^H_A \mathcal{M}_A^{\text{sym}}$ to ${}^H_{\Omega^\bullet(A)} \mathcal{M}_{\Omega^\bullet(A)}^{\text{sym}}$, and hence from ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$ to ${}^H_{\Omega^\bullet(A)} \mathcal{M}_{\Omega^\bullet(A)}^{\text{sym,fp}}$.

More generally, given W in ${}^H_{\Omega^\bullet(A)} \mathcal{M}_{\Omega^\bullet(A)}^{\text{sym}}$ (and hence in ${}^H_A \mathcal{M}_A^{\text{sym}}$) and Γ in ${}^H_A \mathcal{M}_A^{\text{sym}}$, we have $W \otimes_A \Gamma$ in ${}^H_A \mathcal{M}_A^{\text{sym}}$ and we structure it as a module in ${}^H_{\Omega^\bullet(A)} \mathcal{M}_{\Omega^\bullet(A)}^{\text{sym}}$ with the obvious left $\Omega^\bullet(A)$ -module structure and with right $\Omega^\bullet(A)$ -module structure determined by requiring $W \otimes_A \Gamma$ to be graded symmetric. Notice that if $\Sigma \in {}^H_A \mathcal{M}_A^{\text{sym}}$, the associativity $(W \otimes_A \Gamma) \otimes_A \Sigma = W \otimes_A (\Gamma \otimes_A \Sigma)$ in ${}^H_A \mathcal{M}_A^{\text{sym}}$ lifts to ${}^H_{\Omega^\bullet(A)} \mathcal{M}_{\Omega^\bullet(A)}^{\text{sym}}$.

Analogously, we define the module $\Gamma \otimes_A \Omega^\bullet(A)$ in ${}^H_{\Omega^\bullet(A)} \mathcal{M}_{\Omega^\bullet(A)}^{\text{sym}}$. The $\Omega^\bullet(A)$ -bimodule actions read, for all $s \in \Gamma$, $\theta, \theta' \in \Omega^\bullet(A)$,

$$(s \otimes_A \theta) \wedge \theta' := s \otimes_A (\theta \wedge \theta'), \quad \theta' \wedge (s \otimes_A \theta) := (\bar{R}^\alpha \triangleright s) \otimes_A (\bar{R}_\alpha \triangleright^{\text{cop}} \theta') \wedge \theta. \tag{4.9}$$

Every left A -linear map $\tilde{L} \in {}_A \text{hom}(\Gamma, \Omega^n(A) \otimes_A \Gamma)$, uniquely extends to a graded left $\Omega^\bullet(A)$ -linear map in ${}_{\Omega^\bullet(A)} \text{hom}(\Omega^\bullet(A) \otimes_A \Gamma, \Omega^{\bullet+n}(A) \otimes_A \Gamma)$, that we still denote \tilde{L} and with degree $|\tilde{L}| = n$; it is given by $\tilde{L}(\theta \otimes_A s) = (-1)^{|\tilde{L}||\theta|} \theta \wedge \tilde{L}(s)$ for all $\theta \in \Omega^\bullet(A)$ of homogeneous degree $|\theta|$ and $s \in \Gamma$. This provides an isomorphism ${}_A \text{hom}(\Gamma, \Omega(A) \otimes_A \Gamma) \simeq {}_{\Omega^\bullet(A)} \text{hom}(\Omega^\bullet(A) \otimes_A \Gamma, \Omega^{\bullet+1}(A) \otimes_A \Gamma)$ of modules in ${}^H_A \mathcal{M}_A^{\text{sym}}$. Similarly, $\text{hom}_A(\Gamma, \Gamma \otimes_A \Omega(A)) \simeq \text{hom}_{\Omega^\bullet(A)}(\Gamma \otimes_{\Omega^\bullet(A)} \Omega^\bullet(A), \Gamma \otimes_A \Omega^{\bullet+1}(A))$ as modules in ${}^H_A \mathcal{M}_A^{\text{sym}}$.

Correspondingly, right and left connections on Γ uniquely extend as \mathbb{k} -linear maps of degree one on the modules $\Gamma \otimes_A \Omega^\bullet(A)$ and $\Omega^\bullet(A) \otimes_A \Gamma$.

Lemma 4.2. *The connections $\nabla \in \text{Con}_A(\Gamma)$ and $\nabla \in {}_A \text{Con}(\Gamma)$ extend to the graded maps $d_\nabla \in \text{hom}_{\mathbb{k}}(\Gamma \otimes_A \Omega^\bullet(A), \Gamma \otimes_A \Omega^{\bullet+1}(A))$ and $d_\nabla \in \mathbb{k} \text{hom}(\Omega^\bullet(A) \otimes_A \Gamma, \Omega^{\bullet+1}(A) \otimes_A \Gamma)$ well-defined by*

$$d_\nabla : \Gamma \otimes_A \Omega^\bullet(A) \longrightarrow \Gamma \otimes_A \Omega^{\bullet+1}(A), \tag{4.10}$$

$$s \otimes_A \theta \longmapsto d_\nabla(s \otimes_A \theta) := \nabla(s) \wedge \theta + s \otimes_A d\theta,$$

and, for all $k \in \mathbb{N}$,

$$d_\nabla : \Omega^k(A) \otimes_A \Gamma \longrightarrow \Omega^{k+1}(A) \otimes_A \Gamma, \tag{4.11}$$

$$\theta \otimes_A s \longmapsto d_\nabla(\theta \otimes_A s) := d\theta \otimes_A s + (-1)^k \theta \wedge \nabla(s).$$

More generally, for any $h \in H$, the maps $h \triangleright \nabla$ and $h \triangleright^{\text{cop}} \nabla$ extend to the well-defined internal morphisms in ${}^H \mathcal{M}$,

$$d_{h \triangleright \nabla}(s \otimes_A \theta) := (h \triangleright \nabla)(s) \wedge \theta + s \otimes_A \varepsilon(h) d\theta, \tag{4.12}$$

$$d_{h \triangleright^{\text{cop}} \nabla}(\theta \otimes_A s) := \varepsilon(h) d\theta \otimes_A s + (-1)^k \theta \wedge h \triangleright^{\text{cop}} \nabla(s).$$

The H -action on d_∇ and d_∇ reads, for all $h' \in H$, $h' \triangleright d_\nabla = d_{h' \triangleright \nabla}$, $h' \triangleright^{\text{cop}} d_\nabla = d_{h' \triangleright^{\text{cop}} \nabla}$, and we have the Leibniz rule, for all $\zeta \in \Gamma \otimes_A \Omega^k(A)$, $\vartheta \in \Omega^\bullet(A)$ and for all $\sigma \in \Omega^\bullet(A) \otimes_A \Gamma$ and $\theta \in \Omega^k(A)$,

$$d_\nabla(\zeta \wedge \vartheta) = d_\nabla \zeta \wedge \vartheta + (-1)^k \zeta \wedge d\vartheta, \quad d_\nabla(\theta \wedge \sigma) = d\theta \wedge \sigma + (-1)^k \theta \wedge d_\nabla \sigma. \tag{4.13}$$

Proof. We give the proof for left connections, the other case is similar. The definitions in (4.11) and (4.12) are well-defined because independent from the representative chosen for the balanced tensor product over A . We now show that $d_\nabla \in \mathbb{k} \text{hom}(\Omega^\bullet(A) \otimes_A \Gamma, \Omega^{\bullet+1}(A) \otimes_A \Gamma)$. Let $V, W \in {}^H_A \mathcal{M}_A$ and $\pi_{V,W} : V \otimes W \rightarrow V \otimes_A W$ be the projection to the balanced tensor product; consider the \mathbb{k} -linear map

$$\pi_{\Omega^{k+1}, \Gamma} \circ (d \otimes \text{id}_\Gamma) + (-1)^k (\wedge \otimes_A \text{id}_\Gamma) \circ \pi_{\Omega^k, \Omega(A) \otimes_A \Gamma} \circ (\text{id}_{\Omega^k(A)} \tilde{\otimes}_{\mathcal{R}} \nabla) : \Omega^k(A) \otimes \Gamma \longrightarrow \Omega^{k+1}(A) \otimes_A \Gamma \tag{4.14}$$

where $\wedge \otimes_A \text{id}_\Gamma$ is a tensor product of morphisms in ${}^H_A \mathcal{M}_A$ while $d \otimes \text{id}_\Gamma$ is a tensor product of morphisms in ${}^H \mathcal{M}$ and $\text{id}_{\Omega^k(A)} \tilde{\otimes}_{\mathcal{R}} \nabla$ is a tensor product of internal morphisms in ${}^H \mathcal{M}$. The maps in (4.14) are sums of compositions of \mathbb{k} -linear maps carrying the \triangleright^{cop} adjoint action (recall that a morphism in ${}^H \mathcal{M}$ or ${}^H_A \mathcal{M}_A$ can be seen as an internal morphism carrying trivial \triangleright^{cop} adjoint action, cf. the last paragraph of Section 2.1.1). Since the composition of internal morphisms is an internal morphism (cf. equation (2.10)), the map in (4.14) is therefore in $\mathbb{k}\text{hom}(\Omega^*(A) \otimes \Gamma, \Omega^{*+1}(A) \otimes_A \Gamma)$. By evaluating these internal morphisms on elements $\theta \otimes s \in \Omega^k \otimes \Gamma$ we see that they equal $d_\nabla \circ \pi_{\Omega^k(A), \Gamma}$, thus showing that $d_\nabla \in \mathbb{k}\text{hom}(\Omega^*(A) \otimes_A \Gamma, \Omega^{*+1}(A) \otimes_A \Gamma)$. Similarly we prove that for all $h \in H$, $d_{h \triangleright^{cop} \nabla} \in \mathbb{k}\text{hom}(\Omega^*(A) \otimes_A \Gamma, \Omega^{*+1}(A) \otimes_A \Gamma)$.

The H -action on the connection follows observing that the only map in (4.14) that carries a nontrivial H -action is ∇ . The Leibniz rule is easily proven, indeed because of \mathbb{k} -linearity it is enough to consider $\sigma = \eta \otimes_A s$ with $\eta \in \Omega^k(A)$. \square

Vice versa, internal morphism $d_\nabla \in \text{hom}_{\mathbb{k}}(\Gamma \otimes_A \Omega^*(A), \Gamma \otimes_A \Omega^{*+1}(A))$, $d_\nabla \in \mathbb{k}\text{hom}(\Omega^*(A) \otimes_A \Gamma, \Omega^{*+1}(A) \otimes_A \Gamma)$ that satisfy the Leibniz rule (4.13) are uniquely defined by connections $\nabla, \tilde{\nabla}$, respectively. In view of this bijection we also call d_∇ and $d_{\tilde{\nabla}}$ connections.

We canonically extend the inner derivative $i : \mathfrak{X}(A) \rightarrow \text{hom}_A(\Omega^*(A), \Omega^{*-1}(A))$ to $i : \mathfrak{X}(A) \rightarrow \text{hom}_A(\Omega^*(A) \otimes_A \Gamma, \Omega^{*-1}(A) \otimes_A \Gamma)$ by defining, for all $u \in \mathfrak{X}(A)$,

$$i_u : \Omega^*(A) \otimes_A \Gamma \rightarrow \Omega^{*-1}(A) \otimes_A \Gamma, \quad \theta \otimes_A s \mapsto i_u(\theta \otimes_A s) = i_u(\theta) \otimes_A s. \tag{4.15}$$

The covariant derivative along a vector field $u \in \mathfrak{X}(A)$ of a left connection is the \mathbb{k} -linear operator of zero degree defined by $d_{\nabla_u} : \Omega^*(A) \otimes_A \Gamma \rightarrow \Omega^*(A) \otimes_A \Gamma$,

$$d_{\nabla_u} := i_u \circ d_\nabla + d_\nabla \circ i_u, \tag{4.16}$$

in particular, on Γ we have $d_{\nabla_u} = i_u \circ \nabla$ that as usual we denote by ∇_u , hence $\nabla_u := i_u \circ \nabla$. Notice however that ∇_u is the composition of a \mathbb{k} -linear map ∇ acting from the right and a \mathbb{k} -linear map i_u acting from the left. From (4.11) we have that the covariant derivative satisfies, for all $u, v \in \mathfrak{X}(A)$, $a \in A$, $s, t \in \Gamma$,

$$\begin{aligned} \nabla_{u+v} s &= \nabla_u s + \nabla_v s, \\ \nabla_{au} s &= a \nabla_u s, \\ \nabla_u (s + t) &= \nabla_u (s) + \nabla_u (t), \end{aligned} \tag{4.17}$$

and

$$\nabla_u (as) = \mathcal{L}_u(a)s + (\tilde{R}^\alpha \triangleright a) \nabla_{\tilde{R}^\alpha \triangleright u} s. \tag{4.18}$$

Remark 4.3. We also term covariant derivative a map $\nabla^{cd} : \mathfrak{X}(A) \times \Gamma \rightarrow \Gamma$ that satisfies (4.17) and (4.18); equivalently, since (4.17) and (4.18) imply \mathbb{k} -bilinearity, we term covariant derivative a left A -linear map $\nabla^{cd} : \mathfrak{X}(A) \otimes \Gamma \rightarrow \Gamma$ that satisfies the Leibniz rule $\nabla^{cd}(u \otimes as) = \mathcal{L}_u(a)s + (\tilde{R}^\alpha \triangleright a) \nabla^{cd}(\tilde{R}^\alpha \triangleright u \otimes s)$. This latter more elegantly reads

$$\nabla^{cd}(u \otimes as) = \mathcal{L}_u(a)s + \nabla^{cd}(ua \otimes s). \tag{4.19}$$

Since $\nabla \in {}_A \text{Con}(\Gamma) \subset \mathbb{k}\text{hom}(\Gamma, \Omega(A) \otimes_A \Gamma)$, we also require $\nabla^{cd} \in \mathbb{k}\text{hom}(\mathfrak{X}(A) \otimes \Gamma, \Gamma)$. A covariant derivative is therefore a map $\nabla^{cd} \in {}_A \text{hom}(\mathfrak{X}(A) \otimes \Gamma, \Gamma)$ satisfying the Leibniz rule (4.19). Notice that ${}_A \text{hom}(\mathfrak{X}(A) \otimes \Gamma, \Gamma)$ is a module in ${}^H_A \mathcal{M}_A$, not in ${}^H_A \mathcal{M}_A^{\text{sym}}$, indeed $\mathfrak{X}(A) \otimes \Gamma$ is in ${}^H_A \mathcal{M}_A$ not in ${}^H_A \mathcal{M}_A^{\text{sym}}$.

Let $\pi : \mathfrak{X}(A) \otimes \Omega(A) \rightarrow \mathfrak{X}(A) \otimes_A \Omega(A)$ be the canonical projection to the balanced tensor product. To any connection $\nabla \in {}_A \text{Con}(\Gamma)$ we associate a covariant derivative via the map

$$\nabla \mapsto \nabla^b := (\langle \cdot, \cdot \rangle \circ \pi \otimes_A \text{id}_\Gamma) \circ (\text{id}_{\mathfrak{X}(A)} \otimes \nabla), \quad \nabla^b(u \otimes s) = i_u \circ \nabla(s).$$

If the module $\mathfrak{X}(A)$ is finitely generated and projective, this map is a bijection with inverse $\nabla^{cd} \mapsto (\nabla^{cd})^\sharp := (\text{id}_\Omega \otimes_A \nabla^{cd}) \circ (\text{coev} \circ \eta \otimes \text{id}_\Gamma)$, $(\nabla^{cd})^\sharp(s) = \omega^i \otimes_A \nabla^{cd}(e_i \otimes s)$; here as usual $\mathbb{k} \otimes \Gamma \simeq \Gamma$, $\eta : \mathbb{k} \rightarrow A$ is the unit $1_{\mathbb{k}} \mapsto \eta(1_{\mathbb{k}}) = 1_A$, $\{e_i, \omega^i : i = 1 \dots, n\}$ is a dual basis of $\mathfrak{X}(A)$ and $\text{coev}(1_A) = \omega^i \otimes_A e_i$.

It is easy to see that covariant derivatives form an affine space over the module ${}_A \text{hom}(\mathfrak{X}(A) \otimes_A \Gamma, \Gamma)$ in ${}^H_A \mathcal{M}_A^{\text{sym}}$. The bijection $b = \sharp^{-1}$ is compatible with the affine structures of the space of connections ${}_A \text{Con}(\Gamma)$ and the affine space of covariant derivatives; thus it lifts to an isomorphism of affine spaces over the isomorphic modules ${}_A \text{hom}(\Gamma, \Omega \otimes_A \Gamma)$ and ${}_A \text{hom}(\mathfrak{X}(A) \otimes_A \Gamma, \Gamma)$ in ${}^H_A \mathcal{M}_A^{\text{sym}}$ (cf. Theorem 2.5).

Lemma 4.4. *The covariant derivative d_{∇_u} along a vector field $u \in \mathfrak{X}(A)$ of a left connection satisfies the braided Leibniz rule, for all $\theta \in \Omega^\bullet(A)$, $\sigma \in \Omega^\bullet(A) \otimes_A \Gamma$*

$$d_{\nabla_u}(\theta \wedge \sigma) = \mathcal{L}_u(\theta) \wedge \sigma + (\bar{R}^\alpha \triangleright^{cop} \theta) \wedge d_{\nabla_{\bar{R}_\alpha \triangleright u}}(\sigma). \tag{4.20}$$

Proof. Because of \mathbb{k} -linearity it is enough to consider a form of homogeneous degree $\theta \in \Omega^k(A)$. We apply the definition and use the Leibniz rule for d_{∇} and the braided one for the inner derivative, $i_u(\theta \wedge \sigma) = i_u(\theta) \wedge \sigma + (-1)^k(\bar{R}^\alpha \triangleright^{cop} \theta) \wedge i_{\bar{R}_\alpha \triangleright u} \sigma$, that immediately follows from (3.16), thus obtaining

$$\begin{aligned} d_{\nabla_u}(\theta \wedge \sigma) &= (i_u \circ d_{\nabla} + d_{\nabla} \circ i_u)(\theta \wedge \sigma) \\ &= i_u(d\theta \wedge \sigma + (-1)^k \theta \wedge d_{\nabla} \sigma) + d_{\nabla}(i_u(\theta) \wedge \sigma + (-1)^k (\bar{R}^\alpha \triangleright^{cop} \theta) \wedge i_{\bar{R}_\alpha \triangleright u} \sigma) \\ &= i_u(d\theta) \wedge \sigma - (-1)^k (\bar{R}^\alpha \triangleright d\theta) \wedge i_{\bar{R}_\alpha \triangleright u} \sigma + (-1)^k i_u(\theta) \wedge d_{\nabla} \sigma + (\bar{R}^\alpha \triangleright^{cop} \theta) \wedge i_{\bar{R}_\alpha \triangleright u} d_{\nabla} \sigma \\ &\quad + d(i_u \theta) \wedge \sigma - (-1)^k i_u(\theta) \wedge d_{\nabla} \sigma + (-1)^k d(\bar{R}^\alpha \triangleright^{cop} \theta) \wedge i_{\bar{R}_\alpha \triangleright u} \sigma + (\bar{R}^\alpha \triangleright^{cop} \theta) \wedge d_{\nabla} i_{\bar{R}_\alpha \triangleright u} \sigma \\ &= \mathcal{L}_u(\theta) \wedge \sigma + (\bar{R}^\alpha \triangleright^{cop} \theta) \wedge d_{\nabla_{\bar{R}_\alpha \triangleright u}}(\sigma) \end{aligned}$$

where we used the identity $d(\bar{R}^\alpha \triangleright^{cop} \theta) \otimes \bar{R}_\alpha = \bar{R}^\alpha \triangleright (d\theta) \otimes \bar{R}_\alpha$ due to H -equivariance of the exterior derivative. \square

Theorem 4.5. *For all vector fields $u, v \in \mathfrak{X}(A)$ the covariant derivative $d_{\nabla_u} : \Omega^\bullet(A) \otimes_A \Gamma \rightarrow \Omega^\bullet(A) \otimes_A \Gamma$ and the inner derivative $i_v : \Omega^\bullet(A) \otimes_A \Gamma \rightarrow \Omega^{\bullet-1} \otimes_A \Gamma$, satisfy the braided Cartan relation*

$$d_{\nabla_u} \circ i_v - i_{\bar{R}^\alpha \triangleright v} \circ d_{\nabla_{\bar{R}_\alpha \triangleright u}} = i_{[u, v]}, \tag{4.21}$$

equivalently,

$$i_u \circ d_{\nabla_v} - d_{\nabla_{\bar{R}_\alpha \triangleright v}} \circ i_{\bar{R}_\alpha \triangleright u} = i_{[u, v]}. \tag{4.22}$$

Proof. Because of \mathbb{k} -linearity, in order to prove the first relation it is enough to evaluate it on $\theta \otimes_A s$, with $\theta \in \Omega^\bullet(A)$ and $s \in \Gamma$. We then compute

$$\begin{aligned} (d_{\nabla_u} \circ i_v - i_{\bar{R}^\alpha \triangleright v} \circ d_{\nabla_{\bar{R}_\alpha \triangleright u}})(\theta \otimes_A s) &= d_{\nabla_u}(i_v \theta \otimes_A s) - i_{\bar{R}^\alpha \triangleright v}(\mathcal{L}_{\bar{R}_\alpha \triangleright u} \theta \otimes_A s + (\bar{R}^\beta \triangleright^{cop} \theta) \otimes_A \nabla_{\bar{R}_\beta \bar{R}_\alpha \triangleright u} s) \\ &= \mathcal{L}_u \circ i_v(\theta) \otimes_A s + \bar{R}^\alpha \triangleright (i_v \theta) \otimes_A \nabla_{\bar{R}_\alpha \triangleright u} s \\ &\quad - i_{\bar{R}^\alpha \triangleright v} \circ \mathcal{L}_{\bar{R}_\alpha \triangleright u}(\theta) \otimes_A s - i_{\bar{R}^\alpha \triangleright v}(\bar{R}^\beta \triangleright^{cop} \theta) \otimes_A \nabla_{\bar{R}_\beta \bar{R}_\alpha \triangleright u} s \\ &= i_{[u, v]}(\theta \otimes_A s); \end{aligned}$$

where in the last passage the second and fourth term cancel because $\bar{R}^\alpha \triangleright (i_v \theta) \otimes \bar{R}_\alpha = \bar{R}^\alpha \triangleright (v, \theta) \otimes \bar{R}_\alpha = \langle \bar{R}^\alpha_{(1)} \triangleright v, \bar{R}^\alpha_{(2)} \triangleright^{cop} \theta \rangle \otimes \bar{R}_\alpha = \langle \bar{R}^\alpha \triangleright v, \bar{R}^\beta \triangleright^{cop} \theta \rangle \otimes \bar{R}_\beta \bar{R}_\alpha = i_{\bar{R}^\alpha \triangleright v}(\bar{R}^\beta \triangleright^{cop} \theta) \otimes \bar{R}_\beta \bar{R}_\alpha$.

The equivalent Cartan relation (4.22) is obtained by observing that both the left hand side and the right hand side in (4.21) are \mathbb{k} -linear expressions in u and in v , by considering the substitution $u \otimes v \rightarrow \bar{R}^\alpha \triangleright v \otimes \bar{R}_\alpha \triangleright u$, and by recalling the braided antisymmetry of the braided commutator (cf. the first identity in (3.2)). \square

The braided Cartan relation (4.21) equivalently reads $[d_{\nabla_u}, i_v] = i_{[u, v]}$ where the braided bracket, despite the connection d_{∇} is not H -equivariant, braids nontrivially only the vector fields u and v , as in the Cartan relation $[\mathcal{L}_u, i_v] = i_{[u, v]}$.

4.2. Curvature

The curvatures of the connections $\nabla \in \text{Con}_A(\Gamma)$ and $\nabla \in {}_A\text{Con}(\Gamma)$ are respectively defined by

$$d_{\nabla}^2 = d_{\nabla} \circ d_{\nabla}, \quad d_{\nabla}^2 = d_{\nabla} \circ d_{\nabla}. \tag{4.23}$$

These are respectively right and left $\Omega^\bullet(A)$ -linear maps,

$$d_{\nabla}^2 \in \text{hom}_{\Omega^\bullet(A)}(\Gamma \otimes_A \Omega^\bullet(A), \Gamma \otimes_A \Omega^{\bullet+2}), \quad d_{\nabla}^2 \in \Omega^\bullet(A) \text{hom}(\Omega^\bullet(A) \otimes_A \Gamma, \Omega^{\bullet+2} \otimes_A \Gamma).$$

Recalling that $\Omega(A) = {}_A \text{hom}(\mathfrak{X}(A), A)$ we have a morphism

$${}_A \text{hom}(\Gamma, \Omega^2(A) \otimes_A \Gamma) \longrightarrow {}_A \text{hom}(\mathfrak{X}^2(A) \otimes_A \Gamma, \Gamma) \tag{4.24}$$

(that becomes an isomorphism if the A -module $\mathfrak{X}(A)$ is finitely generated and projective in ${}^H_A \mathcal{M}_A^{\text{sym}}$, cf. Theorem 2.5 and Remark 2.6). In ${}_A \text{hom}(\mathfrak{X}^2(A) \otimes_A \Gamma, \Gamma)$ we have a second definition of curvature of a left connection $\nabla \in {}_A \text{Con}(\Gamma)$. As in [5] we define the curvature R_∇ to be the \mathbb{k} -linear map $R_\nabla : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \otimes \Gamma \rightarrow \Gamma, u \otimes v \otimes s \mapsto R_\nabla(u, v, s)$,

$$R_\nabla(u, v, s) := (\nabla_u \circ \nabla_v - \nabla_{\bar{R}^{\alpha \triangleright v}} \circ \nabla_{\bar{R}^{\alpha \triangleright u}} - \nabla_{[u, v]})(s). \tag{4.25}$$

We now show that it is a left A -linear map $\mathfrak{X}^2(A) \otimes_A \Gamma \rightarrow \Gamma$ and relate it to the curvature d_∇^2 . To this end we extend the evaluation $\langle \cdot, \cdot \rangle : \mathcal{T}^{0,r} \otimes_A \mathcal{T}^{p,q} \rightarrow \mathcal{T}^{p-r,q}$, of $\mathcal{T}^{0,r}$ on $\mathcal{T}^{p,q}$, defined in (3.13), to the evaluation of $\mathcal{T}^{0,r}$ on $\mathcal{T}^{p,q} \otimes_A \Gamma$,

$$\langle \cdot, \cdot \rangle : \mathcal{T}^{0,r} \otimes_A \mathcal{T}^{p,q} \otimes_A \Gamma \longrightarrow \mathcal{T}^{p-r,q} \otimes_A \Gamma, \quad v \otimes_A \tau \otimes_A s \mapsto \langle v, \tau \otimes_A s \rangle := \langle v, \tau \rangle s, \tag{4.26}$$

which is a morphism in ${}^H_A \mathcal{M}_A^{\text{sym}}$. Similarly, from the duality between $\mathfrak{X}^r(A)$ and $\Omega^r(A)$ (submodules of $\mathcal{T}^{0,r}$ and $\mathcal{T}^{r,0}$) we have the evaluation of $\mathfrak{X}^r(A)$ on $\Omega^r(A) \otimes_A \Gamma$, that we still denote $\langle \cdot, \cdot \rangle : \mathfrak{X}^r(A) \otimes_A \Omega^r(A) \otimes_A \Gamma \rightarrow \Gamma$.

Theorem 4.6. *Let $\mathfrak{X}(A)$ be finitely generated and projective as A -module in ${}^H_A \mathcal{M}_A^{\text{sym}}$ and let Γ in ${}^H_A \mathcal{M}_A^{\text{sym}}$. Consider a left connection $\nabla \in {}_A \text{Con}(\Gamma)$.*

i) *The curvature R_∇ defined in (4.25) satisfies, for all $u, v \in \mathfrak{X}(A)$ and $s \in \Gamma$,*

$$R_\nabla(u, v, s) = -i_u \circ i_v \circ d_\nabla^2(s). \tag{4.27}$$

ii) *The curvature $R_\nabla \in {}_A \text{hom}(\mathfrak{X}^2(A) \otimes_A \Gamma, \Gamma)$.*

Proof. Part i):

$$\begin{aligned} i_u \circ i_v \circ d_\nabla^2(s) &= i_u(i_v \circ d_\nabla(\nabla s)) = i_u(d_{\nabla v}(\nabla s) - d_\nabla(\nabla v s)) \\ &= i_{[u, v]}(\nabla s) + d_{\nabla \bar{R}^{\alpha \triangleright v}}(i_{\bar{R}^{\alpha \triangleright u}} \nabla s) - \nabla_u(\nabla v s) \\ &= -(\nabla_u \circ \nabla v s - \nabla_{\bar{R}^{\alpha \triangleright v}} \circ \nabla_{\bar{R}^{\alpha \triangleright u}} s - \nabla_{[u, v]} s) = -R_\nabla(u, v, s) \end{aligned}$$

where in the second equality we added and subtracted $d_\nabla \circ i_v$, in the third we used the Cartan relation (4.22) of Theorem 4.5.

Part ii): Another expression for the curvature is

$$\begin{aligned} R_\nabla(u, v, s) &= -i_u \circ i_v \circ d_\nabla^2(s) = -i_u \langle v, d_\nabla^2 s \rangle = -\langle u, \langle v, d_\nabla^2 s \rangle \rangle = -\langle u \otimes_A v, d_\nabla^2 s \rangle \\ &= -\frac{1}{2} \langle u \wedge v, d_\nabla^2 s \rangle, \end{aligned} \tag{4.28}$$

where in the last equality we used that $\mathfrak{X}(A) \otimes_A \mathfrak{X}(A)$ is the direct sum of braided antisymmetric plus braided symmetric vector fields, and that these latter have vanishing pairing with 2-forms. Since $d_\nabla^2 \in {}_A \text{hom}(\Gamma, \Omega^2(A) \otimes_A \Gamma)$ and $\langle \cdot, \cdot \rangle : \mathfrak{X}^2(A) \otimes_A \Omega^2(A) \otimes_A \Gamma \longrightarrow \Gamma$ we see that R_∇ is well-defined as a map $\mathfrak{X}^2(A) \otimes_A \Gamma \rightarrow \Gamma$. In other terms we can write

$$R_\nabla(u, v, s) = \frac{1}{2} R_\nabla(u \wedge v \otimes_A s).$$

Moreover, $\langle \cdot, \cdot \rangle$ is left A -linear because it is a morphism in ${}^H_A \mathcal{M}_A^{\text{sym}}$, hence also R_∇ is left A -linear: for every $a \in A, u, v \in \mathfrak{X}(A)$ and $s \in \Gamma, R_\nabla(au, v, s) = aR_\nabla(u, v, s)$. Finally, we have $R_\nabla \in {}_A \text{hom}(\mathfrak{X}^2(A) \otimes_A \Gamma, \Gamma)$ because R_∇ transforms according to the H -adjoint action $\triangleright^{\text{cop}}$, indeed, for all $h \in H$,

$$\begin{aligned} h \triangleright (R_\nabla(u, v, s)) &= -\frac{1}{2} \langle h_{(1)} \triangleright u \wedge h_{(2)} \triangleright v, (h_{(4)} \triangleright^{\text{cop}} d_\nabla^2)(h_{(3)} \triangleright s) \rangle \\ &= (h_{(4)} \triangleright^{\text{cop}} R_\nabla)(h_{(1)} \triangleright u, h_{(2)} \triangleright v, h_{(3)} \triangleright s), \end{aligned}$$

i.e., $h \triangleright (R_\nabla(u \wedge v \otimes_A s)) = (h_{(2)} \triangleright^{\text{cop}} R_\nabla)(h_{(1)} \triangleright (u \wedge v \otimes_A s))$. \square

An analogous theorem holds for right connections; it involves considering the A -bimodule of forms $\text{hom}_A(\mathfrak{X}(A), A)$. This is obtained via the isomorphism $\mathcal{D}_{\mathfrak{X}(A), A}^{-1} : \Omega(A) = {}_A \text{hom}(\mathfrak{X}(A), A) \rightarrow \text{hom}_A(\mathfrak{X}(A), A)$, cf. (2.18). Equivalently, one can use the evaluation map

$$\langle \cdot, \cdot \rangle' := \langle \cdot, \cdot \rangle \circ \tau_{\Omega(A), \mathfrak{X}(A)} : \Omega(A) \otimes_A \mathfrak{X}(A) \rightarrow A. \tag{4.29}$$

4.3. Torsion

In this subsection we set $\Gamma = \mathfrak{X}(A)$ with $\mathfrak{X}(A)$ finitely generated and projective as A -module in ${}^H_A \mathcal{M}_A^{\text{sym}}$. As usual $\Omega(A) = {}^* \mathfrak{X}(A) = {}_A \text{hom}(\mathfrak{X}(A), A)$ is the right dual module. Consider the canonical element $I := \text{coev}(1_A) \in \Omega(A) \otimes \mathfrak{X}(A)$. Notice that I is invariant under the H -action: for all $h \in H, h \triangleright I = \varepsilon(h)I$ (indeed, so is 1_A , and $\text{coev} : A \rightarrow \Omega(A) \otimes_A \mathfrak{X}(A)$ is a morphism in ${}^H_A \mathcal{M}_A^{\text{sym}}$).

Given a left connection ∇ , we define the associated torsion 2-form with values in vector fields to be the tensor field

$$d_{\nabla}(I) \in \Omega^2(A) \otimes_A \mathfrak{X}(A).$$

We also define the torsion T_{∇} as the \mathbb{k} -linear map $T_{\nabla} : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \rightarrow \mathfrak{X}(A), u \otimes v \mapsto T_{\nabla}(u, v)$,

$$T_{\nabla}(u, v) := \nabla_u v - \nabla_{\bar{R}\alpha \triangleright v} \bar{R}\alpha \triangleright u - [u, v]. \tag{4.30}$$

We show that it is a left A -linear map $\mathfrak{X}^2(A) \rightarrow \mathfrak{X}(A)$ and relate it to the torsion 2-form $d_{\nabla}(I)$.

Theorem 4.7. Consider a left connection $\nabla \in {}_A \text{Con}(\mathfrak{X}(A))$. For all $u, v \in \mathfrak{X}(A)$ we have

$$T_{\nabla}(u, v) = -i_u \circ i_v \circ d_{\nabla}(I), \tag{4.31}$$

hence $T_{\nabla} \in {}_A \text{hom}(\mathfrak{X}^2(A), \mathfrak{X}(A))$.

Proof. The first of the coherence conditions (2.28) between the evaluation and coevaluation maps $\langle \cdot, \cdot \rangle : \mathfrak{X}(A) \otimes_A \Omega(A) \rightarrow A$ and $\text{coev} : A \rightarrow \Omega(A) \otimes \mathfrak{X}(A)$ equivalently reads, for all $v \in \mathfrak{X}(A), i_v I = v$. We then recall the definition of the covariant derivative in equation (4.16), and compute, for all $u, v \in \mathfrak{X}(A)$,

$$\begin{aligned} i_u \circ i_v \circ d_{\nabla}(I) &= i_u \circ d_{\nabla_v}(I) - i_u \circ d_{\nabla} \circ i_v(I) \\ &= i_{[u, v]}(I) + d_{\nabla_{\bar{R}\alpha \triangleright v}} \circ i_{\bar{R}\alpha \triangleright u}(I) - i_u \circ d_{\nabla}(v) \\ &= -([u, v] + d_{\nabla_{\bar{R}\alpha \triangleright v}}(\bar{R}\alpha \triangleright u) - \nabla_u v) \\ &= -T_{\nabla}(u, v) \end{aligned} \tag{4.32}$$

where in the second line we used the Cartan relation (4.22) of Theorem 4.5. The equality

$$T_{\nabla}(u, v) = -i_u \circ i_v \circ d_{\nabla}(I) = -i_u \langle v, d_{\nabla} I \rangle = -\langle u \otimes_A v, d_{\nabla} I \rangle = -\frac{1}{2} \langle u \wedge v, d_{\nabla} I \rangle$$

shows that T_{∇} is a well-defined map $\mathfrak{X}^2(A) \rightarrow \mathfrak{X}(A)$; we can hence write $T_{\nabla}(u, v) = \frac{1}{2} T_{\nabla}(u \wedge v)$. Left A -linearity of T_{∇} immediately follows from left A -linearity of $\langle \cdot, \cdot \rangle$. In order to show that $T_{\nabla} \in {}_A \text{hom}(\mathfrak{X}^2(A), \mathfrak{X}(A))$ it remains to prove that T_{∇} transforms according to the H -adjoint action $\triangleright^{\text{cop}}$, indeed, recalling the H -coinvariance of the canonical element I we have, for all $h \in H$,

$$\begin{aligned} h \triangleright (T_{\nabla}(u, v)) &= -\frac{1}{2} \langle h_{(1)} \triangleright u \wedge h_{(2)} \triangleright v, (h_{(4)} \triangleright^{\text{cop}} d_{\nabla})(h_{(3)} \triangleright I) \rangle \\ &= -\frac{1}{2} \langle h_{(1)} \triangleright u \wedge h_{(2)} \triangleright v, (h_{(3)} \triangleright^{\text{cop}} d_{\nabla})(I) \rangle = (h_{(3)} \triangleright T_{\nabla})(h_{(1)} \triangleright u, h_{(2)} \triangleright v) \end{aligned}$$

i.e., $h \triangleright (T_{\nabla}(u \wedge v)) = (h_{(2)} \triangleright^{\text{cop}} T_{\nabla})(h_{(1)} \triangleright (u \wedge v))$. \square

Notice that recalling the second definition in (4.12) we also have $h \triangleright^{\text{cop}} T_{\nabla} = T_{h \triangleright^{\text{cop}} \nabla}$.

An analogous result holds for right connections. In this case we consider $\mathfrak{X}(A)$ right dual to $\Omega(A)$ with evaluation map as in (4.29) and coevaluation map $\text{coev}' := \tau_{\Omega(A), \mathfrak{X}(A)} \circ \text{coev} : A \rightarrow \mathfrak{X}(A) \otimes_A \Omega(A)$, cf. equation (2.32). We further consider the canonical element $I' = \text{coev}'(1_A) \in \mathfrak{X}(A) \otimes_A \Omega(A)$. The torsion of a right connection ∇ is then defined by $d_{\nabla}(I') \in \mathfrak{X}(A) \otimes_A \Omega^2(A)$.

4.4. Sum of connections

Let Γ and $\widehat{\Gamma}$ be two modules in ${}^H_A \mathcal{M}_A^{\text{sym}}$ with left (right) connections. The sum of these left (right) connections is the left (right) connection on the tensor product module $\Gamma \otimes_A \widehat{\Gamma}$.

Theorem 4.8 (Sum of connections [7]). Given connections $\nabla \in \text{Con}_A(\Gamma)$, $\widehat{\nabla} \in \text{Con}_A(\widehat{\Gamma})$ and $\nabla \in {}_A\text{Con}(\Gamma)$, $\widehat{\nabla} \in {}_A\text{Con}(\widehat{\Gamma})$ on the modules Γ and $\widehat{\Gamma}$ in ${}^H_A\mathcal{M}_A^{\text{sym}}$ their sums, respectively defined by

$$\begin{aligned} \nabla \oplus_{\mathcal{R}} \widehat{\nabla} : \Gamma \otimes_A \widehat{\Gamma} &\longrightarrow \Gamma \otimes_A \widehat{\Gamma} \otimes_A \Omega(A) \\ s \otimes_A \widehat{s} &\longmapsto \tau_{23} \circ (\nabla(s) \otimes_A \widehat{s}) + (\bar{R}^\alpha \triangleright s) \otimes_A (\bar{R}_\alpha \triangleright \widehat{\nabla})(\widehat{s}), \\ \nabla \tilde{\oplus}_{\mathcal{R}} \widehat{\nabla} : \Gamma \otimes_A \widehat{\Gamma} &\longrightarrow \Omega(A) \otimes_A \Gamma \otimes_A \widehat{\Gamma} \\ s \otimes_A \widehat{s} &\longmapsto (\bar{R}^\alpha \triangleright^{\text{cop}} \nabla)(s) \otimes_A (\bar{R}_\alpha \triangleright \widehat{s}) + \tau_{12} \circ (s \otimes_A \widehat{\nabla})(\widehat{s}), \end{aligned}$$

where $\tau_{23} : \Gamma \otimes_A \Omega(A) \otimes_A \widehat{\Gamma} \rightarrow \Gamma \otimes_A \widehat{\Gamma} \otimes_A \Omega(A)$ and $\tau_{12} : \Gamma \otimes_A \Omega(A) \otimes_A \widehat{\Gamma} \rightarrow \Omega(A) \otimes_A \Gamma \otimes_A \widehat{\Gamma}$ are the braiding isomorphisms, are well-defined connections $\nabla \oplus_{\mathcal{R}} \widehat{\nabla} \in {}_A\text{Con}(\Gamma \otimes_A \widehat{\Gamma})$ and $\nabla \tilde{\oplus}_{\mathcal{R}} \widehat{\nabla} \in {}_A\text{Con}(\Gamma \otimes_A \widehat{\Gamma})$. The sums $\oplus_{\mathcal{R}}$ and $\tilde{\oplus}_{\mathcal{R}}$ are associative and H -equivariant.

Corollary 4.9. Let $\bigoplus_{n \in \mathbb{N}} \Gamma^{\otimes n}$ be the tensor algebra generated by Γ in ${}^H_A\mathcal{M}_A^{\text{sym}}$. Connections ∇ and ∇ on Γ uniquely lift to connections on $\bigoplus_{n \in \mathbb{N}} \Gamma^{\otimes n}$ that we still denote ∇ and ∇ and that are given by the braided Leibniz rules of Theorem 4.8. In particular, connections on $\mathfrak{X}(A)$ lift to connections on $\mathcal{T}^{0,\bullet}$ and similarly, connections on $\Omega(A)$ lift to connections on $\mathcal{T}^{\bullet,0}$.

Associated with the braided Leibniz rule for connections on $\mathcal{T}^{0,\bullet}$ and on $\mathcal{T}^{\bullet,0}$ we have the braided Leibniz rule for covariant derivatives on $\mathcal{T}^{0,\bullet}$ and on $\mathcal{T}^{\bullet,0}$. The following corollary will be used in Section 6 in order to study connections compatible with a metric structure on A , these are connections on the tensor product $\Omega(A) \otimes_A \Omega(A)$.

Corollary 4.10. For any connection $\nabla \in \text{Con}(\mathfrak{X}(A))$ and vector field $u \in \mathfrak{X}(A)$ the covariant derivative $\nabla_u : \mathfrak{X}(A) \rightarrow \mathfrak{X}(A)$ lifts to the covariant derivative $\nabla_u : \mathcal{T}^{0,\bullet} \rightarrow \mathcal{T}^{0,\bullet}$ defined via the braided Leibniz rule

$$\begin{aligned} \nabla_u(v \otimes_A z) &= (\bar{R}^\alpha \triangleright^{\text{cop}} \nabla)_u v \otimes_A \bar{R}_\alpha \triangleright z + \bar{R}^\alpha \triangleright v \otimes_A \nabla_{\bar{R}_\alpha \triangleright u} z \\ &= \bar{R}^\alpha \triangleright (\nabla_{\bar{R}_\beta \triangleright u} \bar{R}_\gamma \triangleright v) \otimes_A \bar{R}_\alpha \bar{R}^\beta \bar{R}^\gamma \triangleright z + \bar{R}^\alpha \triangleright v \otimes_A \nabla_{\bar{R}_\alpha \triangleright u} z \end{aligned}$$

where $(\bar{R}^\alpha \triangleright^{\text{cop}} \nabla)_u := i_u \circ (\bar{R}^\alpha \triangleright^{\text{cop}} \nabla)$.

Proof. The first addend in the first equality is straightforward; the second addend follows by considering the identity, for all $\theta \in \Omega(A)$, $z' \in \mathcal{T}^{0,\bullet}$,

$$\begin{aligned} i_u \circ \tau_{12}(v \otimes_A \theta \otimes_A z') &= \langle u, \bar{R}^\beta \triangleright^{\text{cop}} \theta \rangle \bar{R}_\beta \triangleright v \otimes_A z' = \bar{R}^\gamma \bar{R}_\beta \triangleright v \bar{R}_\gamma \triangleright \langle u, \bar{R}^\beta \triangleright^{\text{cop}} \theta \rangle \otimes_A z' \\ &= \bar{R}^\alpha \triangleright v \otimes_A i_{\bar{R}_\alpha \triangleright u}(\theta \otimes_A z') \end{aligned}$$

that is due to quasitriangularity of the \mathcal{R} -matrix. The second equality follows from triangularity of the \mathcal{R} -matrix and the identities

$$\begin{aligned} R_\alpha \triangleright (\bar{R}_\beta \triangleright u, \nabla \bar{R}_\gamma \triangleright v) \otimes_A R^\alpha \bar{R}^\beta \bar{R}^\gamma \triangleright z &= \langle R_{\alpha(1)}, \bar{R}_\beta \triangleright u, R_{\alpha(2)} \triangleright (\nabla \bar{R}_\gamma \triangleright v) \rangle \otimes_A R^\alpha \bar{R}^\beta \bar{R}^\gamma \triangleright z \\ &= \langle u, R_\delta \triangleright (\nabla \bar{R}_\gamma \triangleright v) \rangle \otimes_A R^\delta \bar{R}^\gamma \triangleright z \end{aligned}$$

and $R_\delta \triangleright (\nabla \bar{R}_\gamma \triangleright v) \otimes_A R^\delta \bar{R}^\gamma \triangleright z = R_\delta \triangleright (\nabla S^{-1}(R_\gamma) \triangleright v) \otimes_A R^\delta R^\gamma \triangleright z = (R_\alpha \triangleright^{\text{cop}} \nabla)v \otimes_A R^\alpha \triangleright z$, both due to the quasitriangular structure. \square

5. Duality and Cartan structure equations for curvature and torsion

We study the standard duality notions of dual left or right A -linear (more generally graded $\Omega^\bullet(A)$ -linear) maps and dual connections in the context of modules in ${}^H_A\mathcal{M}_A^{\text{sym,fp}}$. This leads in Section 5.2 to the relation between the curvature on a module Γ and the associated curvature on the dual module ${}^*\Gamma$ and similarly for the torsion. These are the Cartan structure equations for curvature and torsion in braided noncommutative geometry in a global coordinate independent formalism. Globally defined curvature and torsion coefficients, with respect to a pair of dual bases for the finitely generated and projective module of vector fields, are introduced and the Bianchi identities proven.

5.1. Connections on dual modules

Let Γ be in ${}^H_A\mathcal{M}_A^{\text{sym,fp}}$, the dual module ${}^*\Gamma = {}_A\text{hom}(\Gamma, A)$ is in ${}^H_A\mathcal{M}_A^{\text{sym,fp}}$. The associated modules $\Omega^\bullet(A) \otimes_A \Gamma$ and ${}^*\Gamma \otimes_A \Omega^\bullet(A)$, defined in (4.8) and (4.9), are modules in ${}^H_{\Omega^\bullet(A)}\mathcal{M}_{\Omega^\bullet(A)}^{\text{sym,fp}}$, indeed ${}^*\Gamma \otimes_A \Omega^\bullet(A)$ is right dual to $\Omega^\bullet(A) \otimes_A \Gamma$ in

$H_{\Omega^*(A)} \mathcal{M}_{\Omega^*(A)}^{\text{sym}}$ with evaluation and coevaluation maps that are the $\Omega^*(A)$ -linear extensions of the ones of Γ ; explicitly, using a pair of dual bases of Γ as in (2.30),

$$\begin{aligned} \langle \cdot, \cdot \rangle : (\Omega^*(A) \otimes_A \Gamma) \otimes_{\Omega^*(A)} (*\Gamma \otimes_A \Omega^*(A)) &\longrightarrow \Omega^*(A), \quad \langle \theta \otimes_A s, *s \otimes_A \eta \rangle = \theta \wedge \langle s, *s \rangle \wedge \eta \\ \text{coev} : \Omega^*(A) &\longrightarrow (*\Gamma \otimes_A \Omega^*(A)) \otimes_{\Omega^*(A)} (\Omega^*(A) \otimes_A \Gamma), \quad \text{coev}(\theta) = \theta \wedge *s^i \otimes_{\Omega^*(A)} s_i, \end{aligned} \tag{5.1}$$

where, recall (4.9), $\theta \wedge *s^i \otimes_{\Omega^*(A)} s_i = \theta \wedge (*s^i \otimes_A 1_{\Omega^*(A)}) \otimes_{\Omega^*(A)} (1_{\Omega^*(A)} \otimes_A s_i) = (\bar{R}^\alpha \triangleright *s^i) \otimes_A (\bar{R}_\alpha \triangleright^{\text{cop}} \theta) \otimes_{\Omega^*(A)} s_i$.

We study the duals of \mathbb{k} -linear maps, of morphisms in $H_{\Omega^*(A)} \mathcal{M}_{\Omega^*(A)}^{\text{sym}}$ (\mathbb{k} -linear and H -equivariant maps) and of internal morphisms in $H_{\Omega^*(A)} \mathcal{M}_{\Omega^*(A)}^{\text{sym}}$ (graded left $\Omega^*(A)$ -linear or right $\Omega^*(A)$ -linear maps).

Definition 5.1. The dual (or adjoint) $*\tilde{L}$ of a \mathbb{k} -linear map $\tilde{L} : \Omega^*(A) \otimes_A \Gamma \rightarrow \Omega^{\bullet+|\tilde{L}|}(A) \otimes_A \Gamma$, of degree $|\tilde{L}|$, is the right $\Omega^*(A)$ -linear map $*\tilde{L} : *\Gamma \otimes_A \Omega^*(A) \rightarrow *\Gamma \otimes \Omega^{\bullet+|\tilde{L}|}(A)$, of degree $|\tilde{L}|$, defined by

$$\langle \sigma, *\tilde{L}(*\sigma) \rangle = (-1)^{|\tilde{L}||\sigma|} \langle \tilde{L}(\sigma), *\sigma \rangle, \tag{5.2}$$

for all $\sigma \in \Omega^*(A) \otimes_A \Gamma$ of homogeneous form degree $|\sigma|$ and $*\sigma \in *\Gamma \otimes_A \Omega^*(A)$.

Vice versa, the dual L^* of a \mathbb{k} -linear map $L : *\Gamma \otimes_A \Omega^*(A) \rightarrow *\Gamma \otimes_A \Omega^{\bullet+|L|}(A)$ of degree $|L|$ is the graded left $\Omega^*(A)$ -linear map $L^* : \Omega^*(A) \otimes_A \Gamma \rightarrow \Omega^{\bullet+|L|}(A) \otimes_A \Gamma$ of degree $|L^*| = |L|$ defined by

$$\langle L^*(\sigma), *\sigma \rangle = (-1)^{|L||\sigma|} \langle \sigma, L(*\sigma) \rangle, \tag{5.3}$$

for all $\sigma \in \Omega^*(A) \otimes_A \Gamma$ of homogeneous form degree $|\sigma|$ and $*\sigma \in *\Gamma \otimes_A \Omega^*(A)$.

Notice that while $*\tilde{L}$ is a right $\Omega^*(A)$ -linear map, L^* is a graded left $\Omega^*(A)$ -linear map, cf. (2.19).

By \mathbb{k} -linearity the map $(\cdot)^* : \tilde{L} \mapsto *\tilde{L}$, defined in (5.2) when \tilde{L} has homogeneous degree, extends to arbitrary \mathbb{k} -module maps $\tilde{L} : \Omega^*(A) \otimes_A \Gamma \rightarrow \Omega^*(A) \otimes_A \Gamma$. Similarly we have the \mathbb{k} -linear map $(\cdot)^* : L \mapsto L^*$ defined on arbitrary \mathbb{k} -module maps $L : *\Gamma \otimes_A \Omega^*(A) \rightarrow *\Gamma \otimes_A \Omega^*(A)$.

Proposition 5.2. The \mathbb{k} -linear and grade preserving maps $(\cdot)^*$ and $(\cdot)^*$ restrict to H -equivariant maps

$$(\cdot)^* : \mathbb{k}\text{-hom}(\Omega^*(A) \otimes_A \Gamma, \Omega^*(A) \otimes_A \Gamma) \longrightarrow \text{hom}_{\Omega^*(A)}(*\Gamma \otimes_A \Omega^*(A), *\Gamma \otimes \Omega^*(A)), \quad \tilde{L} \mapsto *\tilde{L}, \tag{5.4}$$

$$(\cdot)^* : \text{hom}_{\mathbb{k}}(*\Gamma \otimes_A \Omega^*(A), *\Gamma \otimes \Omega^*(A)) \longrightarrow \text{hom}_{\Omega^*(A)}(\Omega^*(A) \otimes_A \Gamma, \Omega^*(A) \otimes_A \Gamma), \quad L \mapsto L^*. \tag{5.5}$$

The map in (5.4) further restricts to an isomorphism in $H_{\Omega^*(A)} \mathcal{M}_{\Omega^*(A)}^{\text{sym,fp}}$,

$$(\cdot)^* : \text{hom}_{\Omega^*(A)}(\Omega^*(A) \otimes_A \Gamma, \Omega^*(A) \otimes_A \Gamma) \longrightarrow \text{hom}_{\Omega^*(A)}(*\Gamma \otimes_A \Omega^*(A), *\Gamma \otimes \Omega^*(A)), \tag{5.6}$$

with inverse $(\cdot)^* : \text{hom}_{\Omega^*(A)}(*\Gamma \otimes_A \Omega^*(A), *\Gamma \otimes \Omega^*(A)) \longrightarrow \text{hom}_{\Omega^*(A)}(\Omega^*(A) \otimes_A \Gamma, \Omega^*(A) \otimes_A \Gamma)$, which is the restriction of the map in (5.5).

Proof. We show that the map $(\cdot)^*$ in (5.4) is H -equivariant: for all $h \in H$, $h \triangleright *\tilde{L} = *(h \triangleright^{\text{cop}} \tilde{L})$. By \mathbb{k} -linearity it is enough to prove H -equivariance on elements \tilde{L} of homogenous degree $|\tilde{L}|$. This is indeed the case because for all $\sigma \in \Omega^*(A) \otimes_A \Gamma$ of homogeneous form degree $|\sigma|$ and all $*\sigma \in *\Gamma \otimes_A \Omega^*(A)$, we have

$$\begin{aligned} \langle \sigma, (h \triangleright *\tilde{L})(*\sigma) \rangle &= \langle \sigma, h_{(1)} \triangleright (*\tilde{L}(S(h_{(2)})) \triangleright *\sigma) \rangle = h_{(2)} \triangleright \langle S^{-1}(h_{(1)}) \triangleright \sigma, *\tilde{L}(S(h_{(3)})) \triangleright *\sigma \rangle \\ &= (-1)^{|\tilde{L}||\sigma|} \langle h_{(2)} \triangleright (\tilde{L}(S^{-1}(h_{(1)})) \triangleright \sigma), S(h_{(3)}) \triangleright *\sigma \rangle \\ &= (-1)^{|\tilde{L}||\sigma|} \langle h_{(2)} \triangleright \tilde{L}(S^{-1}(h_{(1)})) \triangleright \sigma, *\sigma \rangle \\ &= (-1)^{|\tilde{L}||\sigma|} \langle (h \triangleright^{\text{cop}} \tilde{L})(\sigma), *\sigma \rangle \\ &= \langle \sigma, *(h \triangleright^{\text{cop}} \tilde{L})(*\sigma) \rangle. \end{aligned} \tag{5.7}$$

H -equivariance of $(\cdot)^*$ is proven substituting in (5.7) $*\tilde{L}$ and \tilde{L} with L and L^* , respectively.

The restricted map $(\cdot)^*$ in (5.6) is a morphism in $H_{\Omega^*(A)} \mathcal{M}_{\Omega^*(A)}^{\text{sym}}$ since for all internal morphisms $\tilde{L} \in \text{hom}_{\Omega^*(A)}(\Omega^*(A) \otimes_A \Gamma, \Omega^{\bullet+|\tilde{L}|}(A) \otimes_A \Gamma)$ and forms $\theta \in \Omega^*(A)$ of homogeneous degree $|\theta|$, we have $(\theta \tilde{L}) = \theta * \tilde{L}$ and $(\tilde{L} \theta) = *\tilde{L} \theta$. We prove for example the first relation, for all $\sigma \in \Omega^*(A) \otimes_A \Gamma$ of homogeneous form degree $|\sigma|$ and $*\sigma \in *\Gamma \otimes_A \Omega^*(A)$, we have

$$\begin{aligned} \langle \sigma, *(\theta \tilde{L})(*\sigma) \rangle &= (-1)^{|\theta \tilde{L}||\sigma|} \langle (\theta \tilde{L})(\sigma), *\sigma \rangle = (-1)^{|\theta||\tilde{L}|} (-1)^{|\theta||\sigma|} (-1)^{|\theta \tilde{L}||\sigma|} \langle \tilde{L}(\sigma \wedge \theta), *\sigma \rangle \\ &= (-1)^{|\tilde{L}||\theta \wedge \sigma|} \langle \tilde{L}(\sigma \wedge \theta), *\sigma \rangle = \langle \sigma, \theta *\tilde{L}(*\sigma) \rangle = \langle \sigma, (\theta *\tilde{L})(*\sigma) \rangle \end{aligned}$$

where we used the definition (5.2), the bimodule structure of internal morphisms in ${}^H_{\Omega^\bullet(A)}\mathcal{M}^{\text{sym}}_{\Omega^\bullet(A)}$ given in (2.20), then the definition (5.3) and again (2.20).

Finally, the morphism ${}^*(\)$ in (5.6) is an isomorphism, with inverse $(\)^*$, since for all L and \tilde{L} , $({}^*\tilde{L})^* = \tilde{L}$ and ${}^*(L^*) = L$. This immediately follows from the definitions (5.2) and (5.3). \square

We have explicit expressions for the isomorphisms ${}^*(\)$ and $(\)^*$ in ${}^H_{\Omega^\bullet(A)}\mathcal{M}^{\text{sym}}_{\Omega^\bullet(A)}$ in terms of a dual basis of Γ , i.e., of the coevaluation map $\text{coev}(1_{\Omega^\bullet(A)}) = {}^*s^i \otimes_A s_i$. For all internal morphisms $\tilde{L} \in \Omega^\bullet(A)\text{hom}(\Omega^\bullet(A) \otimes_A \Gamma, \Omega^{\bullet+|\tilde{L}|}(A) \otimes_A \Gamma)$, $L \in \text{hom}_{\Omega^\bullet(A)}({}^*\Gamma \otimes_A \Omega^\bullet(A), {}^*\Gamma \otimes_A \Omega^{\bullet+|L|}(A))$ and $\sigma \in \Omega^\bullet(A) \otimes_A \Gamma$ of homogeneous form degree $|\sigma|$ and ${}^*\sigma \in {}^*\Gamma \otimes_A \Omega^\bullet(A)$, we have

$${}^*\tilde{L}({}^*\sigma) = {}^*s^i \langle \tilde{L}(s_i), {}^*\sigma \rangle, \quad L^*(\sigma) = (-1)^{|L||\sigma|} \langle \sigma, L({}^*s^i) \rangle s_i. \tag{5.8}$$

This is due to the identities $\langle \sigma, {}^*s^i \langle \tilde{L}(s_i), {}^*\sigma \rangle \rangle = \langle \sigma, {}^*s^i \rangle \langle \tilde{L}(s_i), {}^*\sigma \rangle = (-1)^{|\tilde{L}||\sigma|} \langle \tilde{L}(\langle \sigma, {}^*s^i \rangle s_i), {}^*\sigma \rangle = (-1)^{|\tilde{L}||\sigma|} \langle \tilde{L}(\sigma), {}^*\sigma \rangle = \langle \sigma, {}^*L({}^*\sigma) \rangle$, where we first used right $\Omega^\bullet(A)$ -linearity of the pairing, then its left $\Omega^\bullet(A)$ -linearity and that of \tilde{L} . The expression for L^* is similarly proven.

Definition 5.3. Let $\nabla \in {}_A\text{Con}(\Gamma)$. The dual of the left connection d_∇ is the \mathbb{k} -linear map ${}^*(d_\nabla) : {}^*\Gamma \otimes_A \Omega^\bullet(A) \rightarrow {}^*\Gamma \otimes_A \Omega^{\bullet+1}(A)$, defined by

$$d\langle \sigma, {}^*\sigma \rangle = \langle d_\nabla \sigma, {}^*\sigma \rangle + (-1)^{|\sigma|} \langle \sigma, {}^*(d_\nabla) {}^*\sigma \rangle, \tag{5.9}$$

for all $\sigma \in \Omega^\bullet(A) \otimes_A \Gamma$ of homogeneous form degree $|\sigma|$ and ${}^*\sigma \in {}^*\Gamma \otimes_A \Omega^\bullet(A)$.

From the definition it follows that ${}^*(d_\nabla)$ is a right connection. Indeed, for all $\sigma, {}^*\sigma$ of homogeneous degree $|\sigma|$ and $\vartheta \in \Omega^\bullet(A)$, we have the identity $\langle \sigma, {}^*(d_\nabla)({}^*\sigma \wedge \vartheta) \rangle = \langle \sigma, {}^*(d_\nabla) {}^*\sigma \wedge \vartheta \rangle + (-1)^{|\sigma|} \langle \sigma, {}^*\sigma \wedge d\vartheta \rangle$.

Vice versa, given a right connection $\nabla \in {}_A\text{Con}({}^*\Gamma)$, equation (5.9), rewritten as

$$d\langle \sigma, {}^*\sigma \rangle = \langle (d_\nabla) {}^*\sigma, {}^*\sigma \rangle + (-1)^{|\sigma|} \langle \sigma, d_\nabla {}^*\sigma \rangle, \tag{5.10}$$

defines a left connection $(d_\nabla)^*$. Obviously, the dual of a dual connection is the initial connection.

If $\sigma = s \in \Gamma \subset \Gamma \otimes_A \Omega^\bullet(A)$ and similarly, if $\sigma' = s' \in \Gamma' \subset \Omega^\bullet(A) \otimes_A \Gamma'$, equations (5.9) and (5.10) read

$$d\langle s, {}^*s \rangle = \langle \nabla s, {}^*s \rangle + \langle s, {}^*\nabla {}^*s \rangle, \quad d\langle s', {}^*s' \rangle = \langle \nabla' s', {}^*s' \rangle + \langle s', {}^*\nabla' {}^*s' \rangle \tag{5.11}$$

and define ${}^*\nabla \in \text{Con}_A({}^*\Gamma)$ in terms of $\nabla \in {}_A\text{Con}(\Gamma)$, and $\nabla'^* \in {}_A\text{Con}({}^*\Gamma')$ in terms of $\nabla' \in \text{Con}_A(\Gamma')$. Since the extensions d_∇ and $d_{\nabla'}$ of the connections ∇ and ∇' are uniquely determined by the Leibniz rule, we have ${}^*(d_\nabla) = d_{{}^*\nabla}$ and $(d_{\nabla'})^* = d_{\nabla'^*}$.

Using a dual basis we have the explicit expressions, for all $s \in \Gamma, {}^*s \in {}^*\Gamma$,

$${}^*\nabla {}^*s = {}^*s^i \otimes_A d\langle s^i, {}^*s \rangle - {}^*s^i \otimes_A \langle \nabla s^i, {}^*s \rangle, \quad \nabla'^* s' = d\langle s', {}^*s' \rangle \otimes_A s_i - \langle s', \nabla'^* s' \rangle \otimes_A s_i.$$

For example, pairing the first expression with $s \in \Gamma$ and using $\langle s, {}^*s^i \rangle \langle \nabla s^i, {}^*s \rangle = \langle \nabla(\langle s, {}^*s^i \rangle s_i), {}^*s \rangle - \langle d\langle s, {}^*s^i \rangle \rangle \langle s_i, {}^*s \rangle = \langle \nabla s, {}^*s \rangle - \langle d\langle s, {}^*s^i \rangle \rangle \langle s_i, {}^*s \rangle$ we obtain the first expression in (5.11).

The difference of two connections $\nabla, \nabla' \in {}_A\text{Con}(\Gamma)$ is a left A -linear map, the difference of their duals is the right A -linear map ${}^*\nabla - {}^*\nabla' = -{}^*(\nabla - \nabla')$ where on the right hand side we used the restriction to ${}^H_A\mathcal{M}^{\text{sym,fp}}_A$ of the isomorphism ${}^*(\)$ of Proposition 5.2. Since also $-{}^*(\)$ is an isomorphism in ${}^H_A\mathcal{M}^{\text{sym,fp}}_A$ we immediately have

Corollary 5.4. ${}^*(\) : {}_A\text{Con}(\Gamma) \rightarrow \text{Con}_A({}^*\Gamma)$ with $-{}^*(\) : {}_A\text{hom}(\Gamma, \Omega(A) \otimes_A \Gamma) \rightarrow \text{hom}_A({}^*\Gamma, \Gamma \otimes_A \Omega(A))$ is an isomorphism of affine spaces over modules in ${}^H_A\mathcal{M}^{\text{sym,fp}}_A$.

There is a unique way of inducing connections on duals of tensor product modules, indeed, the sum of dual connections is the dual of the sum of connections.

Proposition 5.5. Consider the connections $\nabla \in {}_A\text{Con}(\Gamma), \widehat{\nabla} \in {}_A\text{Con}(\widehat{\Gamma})$ and $\nabla \in \text{Con}_A(\Gamma), \widehat{\nabla} \in \text{Con}_A(\widehat{\Gamma})$ on the modules $\Gamma, \widehat{\Gamma}$ in ${}^H_A\mathcal{M}^{\text{sym,fp}}_A$. Let ${}^*\nabla \in \text{Con}_A({}^*\Gamma), {}^*\widehat{\nabla} \in \text{Con}_A({}^*\widehat{\Gamma})$ and $\nabla^* \in {}_A\text{Con}(\Gamma^*), \widehat{\nabla}^* \in {}_A\text{Con}(\widehat{\Gamma}^*)$ be the connections on the dual modules. We have

$${}^*(\nabla \widehat{\otimes}_R \widehat{\nabla}) = {}^*\widehat{\nabla} \widehat{\otimes}_R {}^*\nabla, \quad (\widehat{\nabla} \widehat{\otimes}_R \nabla)^* = \nabla^* \widehat{\otimes}_R \widehat{\nabla}^*,$$

as connections in $\text{Con}_A({}^*\widehat{\Gamma} \otimes_A {}^*\Gamma)$ and in ${}_A\text{Con}(\Gamma^* \otimes_A \widehat{\Gamma}^*)$, respectively.

We leave the proof of this proposition to the reader. It follows from triangularity of the \mathcal{R} -matrix, including the compatibility (2.37) of the braiding with the dual braiding. The second equality follows from the first recalling that the dual of a dual connection is the initial connection.

5.2. Cartan structure equations and Bianchi identities

According to Corollary 5.4 we have

Lemma 5.6. *Let Γ be a module in $H_A \mathcal{M}_A^{\text{sym,fp}}$. The dual of the curvature 2-form of a left connection $\nabla \in {}_A \text{Con}(\Gamma)$ is minus the curvature two form of the dual connection ${}^* \nabla \in \text{Con}_A({}^* \Gamma)$, i.e., ${}^*(d_{\nabla}^2) = -d_{{}^* \nabla}^2$. Similarly, for a right connection $\nabla \in \text{Con}_A(\Gamma)$, $(d_{\nabla}^2)^* = -d_{{}^* \nabla}^2$.*

Proof. Use twice (5.9) and ${}^*(d_{\nabla}) = d_{{}^* \nabla}$ to obtain, for all $\sigma \in \Omega^*(A) \otimes_A \Gamma$ of homogeneous form degree $|\sigma|$ and ${}^* \sigma \in {}^* \Gamma \otimes_A \Omega^*(A)$,

$$0 = d^2 \langle \sigma, {}^* \sigma \rangle = d(\langle d_{\nabla} \sigma, {}^* \sigma \rangle + (-1)^{|\sigma|} \langle \sigma, d_{{}^* \nabla} {}^* \sigma \rangle) = \langle d_{\nabla}^2 \sigma, {}^* \sigma \rangle + \langle \sigma, d_{{}^* \nabla}^2 {}^* \sigma \rangle. \tag{5.12}$$

By definition, the dual of the curvature 2-form satisfies, cf. (5.2), $\langle \sigma, {}^*(d_{\nabla}^2) {}^* \sigma \rangle = \langle d_{\nabla}^2 \sigma, {}^* \sigma \rangle$, hence ${}^*(d_{\nabla}^2) = -d_{{}^* \nabla}^2$. The second equality, $(d_{\nabla}^2)^* = -d_{{}^* \nabla}^2$, then follows setting $\nabla = \nabla^*$ and recalling that the double dual of a connection (curvature) is the original connection (curvature). \square

Let Γ and $\mathfrak{X}(A)$ to be in $H_A \mathcal{M}_A^{\text{sym,fp}}$. Their duals ${}^* \Gamma$ and $\Omega(A)$ are also in $H_A \mathcal{M}_A^{\text{sym,fp}}$. According to Remark 2.6 the right dual of $\mathfrak{X}(A) \otimes_A \Gamma$ is ${}^* \Gamma \otimes_A \Omega(A)$. Similarly (cf. also Proposition 3.5) the right dual of $\mathcal{T}^{0,r} \otimes_A \Gamma$ is ${}^* \Gamma \otimes_A \mathcal{T}^{r,0}$, $r \in \mathbb{N}$. In the next theorem we use the associated exact pairing $\langle \cdot, \cdot \rangle : \mathcal{T}^{0,r} \otimes_A \Gamma \otimes_A {}^* \Gamma \otimes_A \mathcal{T}^{r,0} \rightarrow A$ with $r = 2$, and also the exact pairing in $H_{\Omega^*(A)} \mathcal{M}_{\Omega^*(A)}^{\text{sym,fp}}$ defined in (5.1).

Theorem 5.7 (Second Cartan structure equation). *Let Γ and $\mathfrak{X}(A)$ be modules in $H_A \mathcal{M}_A^{\text{sym,fp}}$ and let $\nabla \in {}_A \text{Con}(\Gamma)$. For all $u, v \in \mathfrak{X}(A)$, $s \in \Gamma$, ${}^* s \in {}^* \Gamma$ we have*

$$\langle R_{\nabla}(u, v, s), {}^* s \rangle = \langle u \otimes_A v \otimes_A s, d_{{}^* \nabla}^2 {}^* s \rangle$$

or, equivalently, $\langle R_{\nabla}(u, v, s), {}^* s \rangle = \frac{1}{2} \langle u \wedge v \otimes_A s, d_{{}^* \nabla}^2 {}^* s \rangle$.

Proof. First notice that for all $\vartheta \in \Omega^2 \subset \mathcal{T}^{2,0}(A)$, $s \in \Gamma$, ${}^* s \in {}^* \Gamma$, considering the pairing of $\Omega^*(A) \otimes_A \Gamma$ with ${}^* \Gamma \otimes_A \Omega^*(A)$ defined in equation (5.1) we have

$$\begin{aligned} \langle i_u \circ i_v(\vartheta \otimes_A s), {}^* s \rangle &= \langle \langle u \otimes_A v, \vartheta \rangle s, {}^* s \rangle = \langle u \otimes_A v, \vartheta \rangle \langle s, {}^* s \rangle \\ &= \langle u \otimes_A v, \vartheta \langle s, {}^* s \rangle \rangle = \langle u \otimes_A v, \langle \vartheta \otimes_A s, {}^* s \rangle \rangle \\ &= i_u \circ i_v \langle \vartheta \otimes_A s, {}^* s \rangle \end{aligned} \tag{5.13}$$

where in the second line we first used right A -linearity of $\langle \cdot, \cdot \rangle : \mathfrak{X}^{\otimes 2} \otimes_A \Omega^{\otimes 2} \rightarrow A$ and then left $\Omega^*(A)$ -linearity of the pairing in (5.1). Then, recalling also Theorem 4.6 and Lemma 5.6, we have

$$\begin{aligned} \langle R_{\nabla}(u, v, s), {}^* s \rangle &= -\langle i_u \circ i_v \circ d_{\nabla}^2(s), {}^* s \rangle = -i_u \circ i_v \langle d_{\nabla}^2(s), {}^* s \rangle = i_u \circ i_v \langle s, d_{{}^* \nabla}^2 {}^* s \rangle \\ &= \langle u \otimes_A v, \langle s, d_{{}^* \nabla}^2 {}^* s \rangle \rangle = \langle u \otimes_A v \otimes_A s, d_{{}^* \nabla}^2 {}^* s \rangle \end{aligned} \tag{5.14}$$

where in the last equality we used right $\Omega^*(A)$ -linearity of the pairing in (5.1). The equivalent expression $\langle R_{\nabla}(u, v, s), {}^* s \rangle = \frac{1}{2} \langle u \wedge v \otimes_A s, d_{{}^* \nabla}^2 {}^* s \rangle$ trivially follows from $u \wedge v = u \otimes_A v - \bar{R}^\alpha \triangleright v \otimes_A \bar{R}_\alpha \triangleright u$. \square

Theorem 5.8 (First Cartan structure equation). *Let $\mathfrak{X}(A)$ be in $H_A \mathcal{M}_A^{\text{sym,fp}}$ and $\nabla \in {}_A \text{Con}(\mathfrak{X}(A))$. For all $u, v \in \mathfrak{X}(A)$, $\theta \in \Omega(A)$, we have*

$$\langle T_{\nabla}(u, v), \theta \rangle = -\langle u \otimes_A v, (d + \wedge \circ {}^* \nabla) \theta \rangle$$

or, equivalently, $\langle T_{\nabla}(u, v), \theta \rangle = -\frac{1}{2} \langle u \wedge v, (d + \wedge \circ {}^* \nabla) \theta \rangle$.

Proof.

$$\begin{aligned} \langle T_{\nabla}(u, v), \theta \rangle &= -\langle i_u \circ i_v \circ d_{\nabla}(I), \theta \rangle = -i_u \circ i_v \langle d_{\nabla}(I), \theta \rangle = -i_u \circ i_v \langle d(I, \theta) + \langle I, * \nabla \theta \rangle \rangle \\ &= -i_u \circ i_v \langle d\theta + \wedge \circ * \nabla \theta \rangle = -\langle u \otimes_A v, d\theta + \wedge \circ * \nabla \theta \rangle \end{aligned}$$

where we used Theorem 4.7, then (5.13) with $\Gamma = \mathfrak{X}(A)$, next the Definition 5.3 of dual connection with $*\Gamma = \Omega(A)$, and in the second line $\Omega^*(A)$ -bilinearity of the pairing $\langle \cdot, \cdot \rangle : \Omega^*(A) \otimes_A \mathfrak{X}(A) \otimes_A \Omega(A) \otimes_A \Omega^*(A) \rightarrow \Omega^*(A)$, so that $\langle I, \theta \rangle = \langle \text{coev}(1_{\Omega^*(A)}), \theta \rangle = \langle \omega^i \otimes_A e_i, \theta \rangle = \theta$ and $\langle I, \omega \otimes_A \eta \rangle = \langle \omega^i \otimes_A e_i, \omega \otimes_A \eta \rangle = \omega^i \wedge \langle e_i, \omega \rangle \wedge \eta = \omega \wedge \eta$ for all $\omega \in \Omega(A)$, $\eta \in \Omega^*(A)$. The equivalent expression $\langle T_{\nabla}(u, v), \theta \rangle = -\frac{1}{2} \langle u \wedge v, (\wedge \circ * \nabla + d)\theta \rangle$ trivially follows from $u \wedge v = u \otimes v - \bar{R}^\alpha \triangleright v \otimes \bar{R}_\alpha \triangleright u$. \square

In the proof we have shown that for all $\theta \in \Omega(A)$, $\langle d_{\nabla}(I), \theta \rangle = (d + \wedge \circ * \nabla)\theta$. This defines the *torsion on one forms*

$$(d + \wedge \circ * \nabla) : \Omega(A) \rightarrow \Omega^2(A) \tag{5.15}$$

of an arbitrary right connection $*\nabla \in \text{Con}_A(\Omega(A))$ and shows that it is an internal morphism in $\text{hom}_A(\Omega, \Omega^2(A))$. As for the curvature $d_{*\nabla}^2$, there is a unique lift to the torsion $(d \circ \wedge + \wedge \circ d_{*\nabla}) \in \text{hom}_{\Omega^*(A)}(\Omega(A) \otimes_A \Omega^*(A), \Omega^{*+2})$.

Remark 5.9. In Theorems 5.7 and 5.8 we use the pairing $\langle \cdot, \cdot \rangle : \mathcal{T}^{0,2} \otimes_A \Gamma \otimes_A *\Gamma \otimes_A \mathcal{T}^{2,0} \rightarrow A$. The pairing between 2-vector fields and 2-forms $\langle \cdot, \cdot \rangle_\wedge : \mathfrak{X}^2(A) \otimes_A \Gamma \otimes_A *\Gamma \otimes_A \Omega^2(A) \rightarrow A$ is half the value of the restriction of the first pairing to $\mathfrak{X}^2(A) \subset \mathcal{T}^{0,2}$ and $\Omega^2(A) \subset \mathcal{T}^{2,0}$, so that the Cartan structure equations read $\langle R_{\nabla}(u, v, s), *s \rangle = \langle u \wedge v \otimes_A s, d_{*\nabla}^2 *s \rangle_\wedge$ and $\langle T_{\nabla}(u, v), \theta \rangle = -\langle u \wedge v, (d + \wedge \circ * \nabla)\theta \rangle_\wedge$.

Using a dual basis $\{e_i, \omega^i : i = 1 \dots, n\}$ of $\mathfrak{X}(A)$ we define the curvature and torsion coefficients of a connection $\nabla \in {}_A\text{Con}(\mathfrak{X}(A))$,

$$R_{ijk}^l := \langle R_{\nabla}(e_i, e_j, e_k), \omega^l \rangle, \quad T_{ij}^l := \langle T_{\nabla}(e_i, e_j), \omega^l \rangle \tag{5.16}$$

and the curvature and torsion two forms (the signs are chosen to match the commutative differential geometry case)

$$R_k^l := -\frac{1}{2} \omega^j \wedge \omega^i R_{ijk}^l, \quad T^l := -\frac{1}{2} \omega^j \wedge \omega^i T_{ij}^l. \tag{5.17}$$

Since $\omega^j \otimes_A \omega^i \otimes_A \langle e_i \otimes_A e_j, \cdot \rangle$ is the identity map on $\Omega(A) \otimes_A \Omega(A)$ we have the equality $\omega^j \otimes \omega^i \langle e_i \otimes_A e_j \otimes_A e_k, d_{*\nabla}^2 \omega^l \rangle = \omega^j \otimes \omega^i \langle e_i \otimes_A e_j, \langle e_k, d_{*\nabla}^2 \omega^l \rangle \rangle = \langle e_k, d_{*\nabla}^2 \omega^l \rangle$ and similarly $\omega^j \otimes \omega^i \langle e_i \otimes_A e_j, (d + \wedge \circ * \nabla)\omega^l \rangle = (d + \wedge \circ * \nabla)\omega^l$. This leads to the coefficient expression of the Cartan structure equations

$$\begin{aligned} \langle e_k, d_{*\nabla}^2 \omega^l \rangle &= \frac{1}{2} \omega^j \wedge \omega^i R_{ijk}^l = -R_k^l, \\ (d + \wedge \circ * \nabla)\omega^l &= -\frac{1}{2} \omega^j \wedge \omega^i T_{ij}^l = T^l. \end{aligned}$$

We similarly define the coefficients one forms of the connection $*\nabla \in \text{Con}_A(\Omega(A))$, dual to $\nabla \in {}_A\text{Con}(\mathfrak{X}(A))$,

$$\omega_k^l := \langle e_k, * \nabla \omega^l \rangle,$$

so that, since $\omega^k \otimes_A \langle e_k, \cdot \rangle$ is the identity map on $\Omega(A)$,

$$* \nabla \omega^l = \omega^k \otimes_A \omega_k^l.$$

In terms of these coefficients we obtain

$$\begin{aligned} d_{*\nabla}^2 \omega^l &= \omega^k \otimes_A (d\omega_k^l + \omega_k^j \wedge \omega_j^l) \\ (d + \wedge \circ * \nabla)\omega^l &= d\omega^l + \omega^j \wedge \omega_j^l \end{aligned}$$

which, together with the identity $\omega^k \otimes_A \langle e_k, d_{*\nabla}^2 \omega^l \rangle = d_{*\nabla}^2 \omega^l$, give the full coefficient expression of the Cartan structure equations

$$\begin{aligned} \omega^k \otimes_A (d\omega_k^l + \omega_k^j \wedge \omega_j^l) &= \omega^k \otimes_A (-R_k^l), \\ d\omega^l + \omega^j \wedge \omega_j^l &= T^l. \end{aligned}$$

As in commutative differential geometry, applying $\text{id}_{\Omega(A)} \otimes_A d$ to the first equation and differentiating the second we readily obtain the Bianchi identities,

$$\begin{aligned} \omega^k \otimes_A (dR_k^l + \omega_k^j \wedge R_j^l - R_k^j \wedge \omega_j^l) &= 0, \\ dT^l - T^j \wedge \omega_j^l &= \omega^j \wedge R_j^l. \end{aligned} \tag{5.18}$$

Notice that the commutator $[\omega, R]_k^l := \omega_k^j \wedge R_j^l - R_k^j \wedge \omega_j^l$ in the first identity is not a braided commutator.

Similarly, if we consider a connection $\nabla \in {}_A\text{Con}(\Gamma)$ on a module Γ in ${}^H_A\mathcal{M}_A^{\text{sym,fp}}$, by setting $\omega_k^l := \langle s_k, {}^*\nabla {}^*s^l \rangle$ (with $\{s_i, {}^*s^i : i = 1 \dots, m\}$ a dual basis) we have the Bianchi identity ${}^*s^k \otimes_A (dR_k^l + \omega_k^j \wedge R_j^l - R_k^j \wedge \omega_j^l) = 0$, where R_k^l is defined in (5.17) and where now $R_{ijk}^l := \langle R_{\nabla}(e_i, e_j, s_k), {}^*s^l \rangle$. Equivalently, $\langle s_p, {}^*s^k \rangle dR_k^l + \omega_p^j \wedge R_j^l - R_p^j \wedge \omega_j^l = 0$; this last expression is obtained observing that if $P \in {}^*\Gamma \otimes_A \Omega^*(A)$ then $\langle s_l, {}^*s^k \rangle P_k = P_l$, where $P_k := \langle s_k, P \rangle$, and using that $\langle s_k, d_{\nabla}^2 {}^*s^l \rangle = -R_k^l$.

6. Riemannian geometry

We use the sum of connections (based on the tensor product of internal morphisms in \mathcal{M}^H), the notion of dual connection and the Cartan calculus results for the torsion in order to determine the Koszul formula for the Levi-Civita connection associated with a pseudo-Riemannian metric tensor on the algebra A . Hence its uniqueness, existence and explicit expression. The Ricci tensor is canonically introduced leading to the notion of noncommutative Einstein manifold.

6.1. Metric tensor

Let A be a braided commutative H -module algebra with H triangular and A -bimodule of braided derivations $\mathfrak{X}(A)$ that is finitely generated and projective, hence in ${}^H_A\mathcal{M}_A^{\text{sym,fp}}$. Let $\Omega(A) = {}^*\mathfrak{X}(A) = {}_A\text{hom}(\mathfrak{X}(A), A)$ be the dual A -bimodule of forms. Proposition 2.3 and Theorem 2.5 give two isomorphisms in ${}^H_A\mathcal{M}_A^{\text{sym,fp}}$, that with slight abuse of notation are both denoted by b ,

$$\begin{aligned} \Omega(A) \otimes_A \Omega(A) &\xrightarrow{b} {}_A\text{hom}(\mathfrak{X}(A), \Omega(A)) \xrightarrow{b} {}_A\text{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A), A) \\ g &= g^a \otimes_A g_a \mapsto g^b := \langle \cdot, g^a \rangle \otimes_A g_a \mapsto \langle - \otimes_A - , g \rangle = \langle \cdot, \langle \cdot, g^a \rangle g_a \rangle \end{aligned} \tag{6.1}$$

(sum on the index a understood).

Definition 6.1. A pseudo-Riemannian metric on $\mathfrak{X}(A)$ is a braided symmetric element $g \in \Omega(A) \otimes_A \Omega(A)$, i.e., $g = \tau(g)$, with associated internal morphism $g^b \in {}_A\text{hom}(\mathfrak{X}(A), \Omega(A))$ that is an isomorphism. We also simply say that g is a metric on A .

Similarly to (6.1), consider $b' : \mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \rightarrow {}_A\text{hom}(\Omega(A), \mathfrak{X}(A))$, $f^a \otimes_A f_a \mapsto \langle \cdot, f^a \rangle' \otimes_A f_a$, (cf. Proposition 2.8 and Remark 2.9) and let b'^{-1} be the inverse isomorphism. The metric tensor $g \in \Omega(A) \otimes_A \Omega(A)$ induces the metric on $\Omega(A)$ (or inverse metric tensor) \bar{g} , defined by

$$\bar{g} := (g^b)^{-1 b'} \in \mathfrak{X}(A) \otimes_A \mathfrak{X}(A). \tag{6.2}$$

Setting $\bar{g} = \bar{g}^b \otimes_A \bar{g}_b$ we have $\bar{g}^{b'} = \langle \cdot, \bar{g}^b \rangle' \otimes_A \bar{g}_b$, and the equality $\bar{g}^{b'} = (g^b)^{-1}$ reads

$$\langle \cdot, g^a \rangle \langle g_a, \bar{g}^b \rangle' \otimes_A \bar{g}_b = \text{id}_{\mathfrak{X}(A)}, \quad \langle \cdot, \bar{g}^b \rangle' \langle \bar{g}_b, g^a \rangle \otimes_A g_a = \text{id}_{\Omega(A)}.$$

Using dual bases for $\mathfrak{X}(A)$ as a right and as a left rigid A -module in ${}^H_A\mathcal{M}_A^{\text{sym,fp}}$, that is $\text{coev}(1_A) = \omega^i \otimes_A e_i$ and $\text{coev}'(1_A) = {}^\alpha e_i \otimes_A \alpha \omega^i := \bar{R}^\alpha \triangleright e_i \otimes_A \bar{R}_{\alpha} \triangleright \omega^i$ we have $\text{id}_{\mathfrak{X}(A)} = \langle \cdot, \omega^i \rangle \otimes_A e_i$ and $\text{id}_{\Omega(A)} = \langle \cdot, {}^\alpha e_i \rangle \otimes_A \alpha \omega^i$. Then, recalling that $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ induce the isomorphisms $\Omega(A) \otimes_A \mathfrak{X}(A) \simeq {}_A\text{hom}(\mathfrak{X}(A), \mathfrak{X}(A))$ and $\mathfrak{X}(A) \otimes_A \Omega(A) \simeq {}_A\text{hom}(\Omega(A), \Omega(A))$, we see that \bar{g} is the inverse metric of g if and only if

$$g^a \langle g_a, \bar{g}^b \rangle' \otimes_A \bar{g}_b = \omega^i \otimes_A e_i, \quad \bar{g}^b \otimes_A \langle \bar{g}_b, g^a \rangle \otimes_A g_a = {}^\alpha e_i \otimes_A \alpha \omega^i.$$

6.2. Levi-Civita connection

We prove existence and uniqueness of a metric compatible and torsion free connection establishing a noncommutative Koszul formula.

In this section we simplify the notation of the braiding via the \mathcal{R} -matrix action and set, for any $w \in W$ with W module in ${}^H\mathcal{M}, {}^H_A\mathcal{M}_A^{\text{sym}}$ or ${}^H_{\Omega^*(A)}\mathcal{M}_{\Omega^*(A)}^{\text{sym}}$, ${}^\alpha w = \bar{R}^\alpha \triangleright w$ and ${}_\alpha w = \bar{R}_\alpha \triangleright w$ (for any α). Hence, for example, for all $u, v \in \mathfrak{X}(A)$, $\theta \in \Omega(A)$,

$$\begin{aligned} \alpha v \otimes_A \alpha u &= \bar{R}^\alpha \triangleright v \otimes_A \bar{R}_\alpha \triangleright u = \tau(u \otimes_A v), \\ \beta \theta \otimes (\beta^* \nabla) &= \bar{R}^\beta \triangleright \theta \otimes (\bar{R}_\beta \triangleright^* \nabla), \quad (\alpha \nabla) \otimes \alpha u = (\bar{R}^\alpha \triangleright^{cop} \nabla) \otimes \alpha u. \end{aligned}$$

Recall that considering sums of connections (connections on tensor products) and dual connections a left connection $\nabla \in {}_A\text{Con}(\mathfrak{X}(A))$ uniquely lifts to a left connection on $\mathcal{T}^{0,\bullet}(A)$ and to a dual right connection on $\mathcal{T}^{\bullet,0}(A)$, cf. Corollary 4.9 and Proposition 5.5. For example on the metric tensor we have (cf. Theorem 4.8)

$${}^* \nabla(g) = {}^* \nabla(g^a \otimes_A g_a) = \tau_{23} \circ ({}^* \nabla(g^a) \otimes_A g_a) + \beta g^a \otimes_A (\beta^* \nabla)(g_a).$$

Similarly for a right connection $\nabla \in \text{Con}_A(\Omega(A))$. Moreover, this latter is torsion free if its dual $\nabla^* \in {}_A\text{Con}(\mathfrak{X}(A))$ is torsion free, cf. Theorem 5.8 and (5.15).

Definition 6.2. Let $g \in \Omega(A) \otimes_A \Omega(A)$ be a pseudo-Riemannian metric on A . A right connection $\nabla \in \text{Con}_A(\Omega(A))$ is metric compatible if it satisfies $\nabla(g) = 0$. A left connection $\nabla \in {}_A\text{Con}(\mathfrak{X}(A))$ is metric compatible if its dual ${}^* \nabla \in \text{Con}_A(\Omega(A))$ is metric compatible.

A Levi-Civita connection is a metric compatible and torsion free connection.

For ease of the reader in the statements of the next two theorems we spell out the condition that $\mathfrak{X}(A)$ is a module in ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$.

Theorem 6.3 (Uniqueness of Levi-Civita connection). Let H be a triangular Hopf algebra, A a braided commutative H -module algebra, let the associated module in ${}^H_A \mathcal{M}_A^{\text{sym}}$ of braided derivations $\mathfrak{X}(A)$ be finitely generated and projective, and let $g \in \Omega(A) \otimes_A \Omega(A)$ be a metric on A , where $\Omega(A) = {}^* \mathfrak{X}(A)$. If a torsion free metric compatible left connection $\nabla \in {}_A\text{Con}(\mathfrak{X}(A))$ exists, it is unique.

Proof. Assume $\nabla \in {}_A\text{Con}(\mathfrak{X}(A))$ is a torsion free metric compatible connection. Applying the contraction operator to the identity

$$d\langle v \otimes_A z, g \rangle = \langle \nabla(v \otimes_A z), g \rangle + \langle v \otimes_A z, {}^* \nabla g \rangle = \langle \nabla(v \otimes_A z), g \rangle$$

we have, for all $u, v, z \in \mathfrak{X}(A)$,

$$\begin{aligned} \mathcal{L}_u \langle v \otimes_A z, g \rangle &= \langle \nabla_u(v \otimes_A z), g \rangle \\ &= \langle \alpha \nabla_{\beta u} \gamma v \rangle \otimes_A \alpha^{\beta\gamma} z + \langle \alpha v \otimes_A \nabla_{\alpha u} z, g \rangle \\ &= \langle \alpha \nabla_{\eta_\gamma v} \eta_\beta u + [\beta u, \gamma v] \rangle \otimes_A \alpha^{\beta\gamma} z, g \rangle + \langle \alpha v \otimes_A \nabla_{\alpha u} z, g \rangle \tag{6.3} \\ &= \langle \beta\gamma z \otimes_A \nabla_{\eta_\gamma v} \eta_\beta u, g \rangle + \langle \alpha [\beta u, \gamma v] \otimes_A \alpha^{\beta\gamma} z, g \rangle + \langle \alpha v \otimes_A \nabla_{\alpha u} z, g \rangle \\ &= \langle \beta\gamma z \otimes_A \nabla_{\beta \eta_\gamma v} \eta u, g \rangle + \langle [u, v] \otimes_A z, g \rangle + \langle \alpha v \otimes_A \nabla_{\alpha u} z, g \rangle \end{aligned}$$

where in the second line we used Corollary 4.10 for the braided derivation rule of the covariant derivative; in the third line we used the torsion free condition $T(u, v) = \nabla_u v - \nabla_{\eta_v \eta u} - [u, v] = 0$; in the fourth the braided symmetry of the metric and that the adjoint of the braiding on forms is the braiding on vector fields, cf. equation (2.37); in the last line we used the Yang–Baxter equation (in the form $\mathcal{R}_{23}^{-1} \mathcal{R}_{13}^{-1} \mathcal{R}_{12}^{-1} = \mathcal{R}_{12}^{-1} \mathcal{R}_{13}^{-1} \mathcal{R}_{23}^{-1}$) and that the braided bracket $[\ , \]$ is H -equivariant.

The identity in (6.3) is \mathbb{k} -linear in u, v, z and we rewrite it for the cyclically permuted elements $u \otimes v \otimes z \mapsto \alpha\beta z \otimes_\alpha u \otimes_\beta v$ and $u \otimes v \otimes z \mapsto \eta v \otimes_\gamma z \otimes_\gamma \eta u$. We then subtract the second identity from the first and add the third thus obtaining (after using the Yang–Baxter equation, the braided symmetry of the metric and the braided antisymmetry of the Lie bracket of vector fields)

$$\begin{aligned} 2\langle \alpha v \otimes_A \nabla_{\alpha u} z, g \rangle &= \mathcal{L}_u \langle v \otimes_A z, g \rangle - \mathcal{L}_{\alpha v} \langle \alpha u \otimes_A z, g \rangle + \mathcal{L}_{\alpha\beta z} \langle \alpha u \otimes_A \beta v, g \rangle \\ &\quad - \langle [u, v] \otimes_A z, g \rangle + \langle u \otimes_A [v, z], g \rangle + \langle [u, \beta z] \otimes_A \beta v, g \rangle. \end{aligned} \tag{6.4}$$

The right hand side of this identity uniquely determines the left hand side, that, in turn, because of the exactness of the pairing $\langle \ , \ \rangle : \mathfrak{X}(A) \otimes_A \Omega(A) \rightarrow A$ and the invertibility of g^\flat , uniquely determines the covariant derivative $\nabla_u : \mathfrak{X}(A) \rightarrow \mathfrak{X}(A)$ for all $u \in \mathfrak{X}(A)$, i.e., $\nabla^{\text{cd}} : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \rightarrow \mathfrak{X}(A)$, $\nabla^{\text{cd}}(u \otimes z) = \nabla_u(z)$. Recalling Remark 4.3 this proves uniqueness of the metric compatible and torsion free connection ∇ . \square

Remark 6.4. If g is H -equivariant, i.e., if $h \triangleright g = \varepsilon(h)g$ for all $h \in H$ then ∇ is H -equivariant, $h \triangleright \nabla = \varepsilon(h)\nabla$. This can be show by acting with h on (6.4). Due to H -equivariance of $\mathcal{L}, \langle \ , \ \rangle, g$ and $[\ , \]$ the right hand side is obtained replacing $u \otimes v \otimes z$ in (6.4) with $h_{(1)} \triangleright u \otimes h_{(2)} \triangleright v \otimes h_{(3)} \triangleright z$ which equals $2\langle \alpha(h_{(2)} \triangleright v) \otimes_A \nabla_{\alpha(h_{(1)} \triangleright u)}(h_{(3)} \triangleright z), g \rangle$. On the other hand the h action on the left hand side gives $2\langle (h_{(1)} \triangleright \alpha v) \otimes_A (h_{(4)} \triangleright^{cop} \nabla)_{h_{(2)} \triangleright \alpha u}(h_{(3)} \triangleright z), g \rangle$, cf. (2.6). Comparison of these two expressions shows that $h \triangleright \nabla = \varepsilon(h)\nabla$ for any $h \in H$.

Remark 6.5. Using the braided symmetry of the metric and that the adjoint of the braiding on forms is the braiding on vector fields we can rewrite the left hand side of (6.4) as

$$\langle \alpha v \otimes_A \nabla_{\alpha u} z, g \rangle = \langle \beta (\nabla_{\alpha u} z) \otimes_A \beta^\alpha v, g \rangle = \langle (\delta \nabla)_{\beta^\alpha u} \gamma z \otimes_A \delta \gamma \beta^\alpha v, g \rangle = \langle (\delta \nabla)_u \gamma z \otimes_A \delta \gamma v, g \rangle$$

where $(\delta \nabla)_u z = \langle u, (\bar{R}^\delta \triangleright \nabla z) \rangle$ and in the second equality we used that $h \triangleright (\nabla_{\alpha u} z) = h \triangleright \langle \alpha u, \nabla z \rangle = \langle h_{(1)} \triangleright \alpha u, h_{(2)} \triangleright (\nabla z) \rangle$, for all $h \in H$, the braid property $(\Delta \otimes \text{id})\mathcal{R}^{-1} = \mathcal{R}_{23}^{-1}\mathcal{R}_{13}^{-1}$ and equation (2.6). By \mathbb{k} -linearity in u, v, z and their arbitrariness, equation (6.4) holds also when rewritten for the element $u \otimes \beta v \otimes \beta z$ instead of $u \otimes v \otimes z$, it reads

$$2 \langle (\delta \nabla)_u z \otimes_A \delta v, g \rangle = \mathcal{L}_u \langle z \otimes_A v, g \rangle - \mathcal{L}_{\alpha \beta v} \langle \alpha u \otimes_A \beta z, g \rangle + \mathcal{L}_{\alpha z} \langle \alpha u \otimes_A v, g \rangle - \langle [u, \beta v] \otimes_A \beta z, g \rangle - \langle u \otimes_A [z, v], g \rangle + \langle [u, z] \otimes_A v, g \rangle,$$

where we simplified $\mathcal{L}_u \langle \beta v \otimes_A \beta z, g \rangle = \mathcal{L}_u \langle z \otimes_A v, g \rangle$ and $[\beta v, \beta z] = -[z, v]$. The right hand side of this expression equals the right hand side of the Koszul formula in [33, eq. (6.65)] (use $\langle [u, \beta v] \otimes_A \beta z, g \rangle = \langle \gamma z \otimes_A [\gamma u, v], g \rangle = -\langle \gamma z \otimes_A [\beta v, \beta \gamma u], g \rangle$) and rename u, v, z as X, Z, Y . We remark that the left hand side however differs, it equals that in [33, eq. (6.65)], namely $2 \langle \nabla_u z \otimes_A v, g \rangle$, only when the action of the braiding on the connection is trivial, this is indeed the case considered there, where the metric is H -equivariant (hence in particular central, for all $a \in A, ga = \alpha_a g$ simply reads $ga = ag$) and therefore the connection too is H -equivariant, cf. Remark 6.4. Thus the present result, where we consider an arbitrary metric g , generalizes to not necessarily H -equivariant metrics the Koszul formula obtained in [33].

Let $K_g : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \otimes \mathfrak{X}(A) \rightarrow A, u \otimes v \otimes z \mapsto K_g(u \otimes v \otimes z)$ be the \mathbb{k} -linear map defined by the right hand side of equation (6.4). The map K_g is a \mathbb{k} -linear combination of compositions of the map $\langle \cdot, g \rangle : \mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \rightarrow A, v \otimes_A z \mapsto \langle \cdot, g \rangle(v \otimes_A z) = \langle v \otimes_A z, g \rangle$, with the maps $\mathcal{L} : \mathfrak{X}(A) \otimes A \rightarrow A, [\cdot, \cdot] : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \rightarrow \mathfrak{X}(A)$, the braiding $\tau : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \rightarrow \mathfrak{X}(A) \otimes \mathfrak{X}(A)$ and the projection $\pi : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \rightarrow \mathfrak{X}(A) \otimes_A \mathfrak{X}(A)$ to the balanced tensor product over A . For example the first addend on the right hand side of (6.4) reads $\mathcal{L}_u \langle v \otimes_A z, g \rangle = \mathcal{L} \circ (\text{id} \otimes \langle \cdot, g \rangle \circ \pi)(u \otimes v \otimes z)$, where id stands for $\text{id}_{\mathfrak{X}(A)}$; the explicit expression of K_g in terms of these maps is

$$K_g = \mathcal{L} \circ (\text{id} \otimes \langle \cdot, g \rangle \circ \pi) \circ (\text{id} - \tau_{12} + \tau_{12} \circ \tau_{23}) - \langle \cdot, g \rangle \circ \pi \circ ([\cdot, \cdot] \otimes \text{id} - \text{id} \otimes [\cdot, \cdot] - ([\cdot, \cdot] \otimes \text{id}) \circ \tau_{23}). \tag{6.5}$$

Existence of the Levi-Civita connection is proven by studying the properties of this map.

Recall that ${}_A \text{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \otimes \mathfrak{X}(A), A) \subset {}_{\mathbb{k}} \text{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \otimes \mathfrak{X}(A), A)$ is the submodule in ${}^H_A \mathcal{M}_A$ of left A -linear maps; it is not a module in ${}^H_A \mathcal{M}_A^{\text{sym}}$ because $\mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \otimes \mathfrak{X}(A)$ is not in ${}^H_A \mathcal{M}_A^{\text{sym}}$.

Lemma 6.6. *The \mathbb{k} -linear map K_g is a left A -linear map in ${}_A \text{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \otimes \mathfrak{X}(A), A)$ and satisfies the derivation property, for all $u, v, z \in \mathfrak{X}(A), a \in A$,*

$$K_g(u \otimes_A v \otimes az) = K_g(u \otimes_A va \otimes z) + 2 \langle \alpha v \cdot \mathcal{L}_u(a) \otimes_A z, g \rangle. \tag{6.6}$$

Proof. We first show that $K_g \in {}_{\mathbb{k}} \text{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \otimes \mathfrak{X}(A), A)$, i.e., that H acts on K_g via the $\triangleright^{\text{cop}}$ adjoint action. Recall from (2.10) that the composition of internal morphisms is an internal morphism. We prove that K_g carries the $\triangleright^{\text{cop}}$ adjoint action by showing that its components in (6.5) carry the $\triangleright^{\text{cop}}$ adjoint action. The map $\langle \cdot, g \rangle : \mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \rightarrow A, v \otimes_A z \mapsto \langle \cdot, g \rangle(v \otimes_A z) = \langle v \otimes_A z, g \rangle$, is easily seen to be in ${}_{\mathbb{k}} \text{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A), A)$, indeed, for all $h \in H$ and $v, z \in \mathfrak{X}(A)$,

$$h \triangleright (\langle \cdot, g \rangle(v \otimes_A z)) = h \triangleright \langle v \otimes_A z, g \rangle = \langle h_{(1)} \triangleright (v \otimes_A z), h_{(2)} \triangleright g \rangle = (h_{(2)} \triangleright^{\text{cop}} \langle \cdot, g \rangle)(h_{(1)} \triangleright (v \otimes_A z)).$$

The maps $\mathcal{L}, [\cdot, \cdot], \tau$ and π are all H -equivariant and hence can be seen as internal morphisms with trivial $\triangleright^{\text{cop}}$ adjoint action (recall end of Section 2.1.1). Thus K_g is a composition of maps that carry the $\triangleright^{\text{cop}}$ adjoint action and therefore $K_g \in {}_{\mathbb{k}} \text{hom}(\mathfrak{X}(A) \otimes \mathfrak{X}(A) \otimes \mathfrak{X}(A), A)$.

Next we show that the map K_g is well-defined on the balanced tensor product $\mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \otimes \mathfrak{X}(A)$. The third addend in the right hand side of (6.4) is well-defined because $\mathcal{L}_{\alpha \beta z} \langle \alpha u \otimes_A \beta v, g \rangle = \mathcal{L}_{\alpha z} \langle \alpha (u \otimes_A v), g \rangle$. From the identity $\mathcal{L}_u(v) = [u, v] = -[\alpha v, \alpha u] = -\mathcal{L}_{\alpha v}(\alpha u)$ and the braided Leibniz rule of the Lie derivative we have

$$u \otimes_A [v, z] + [u, \beta z] \otimes_A \beta v = -\mathcal{L}_{\alpha \beta z}(\alpha u \otimes_A \beta v) = -\mathcal{L}_{\alpha z}(\alpha (u \otimes_A v))$$

that implies that also the last two addends of K_g are well-defined on $\mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \otimes \mathfrak{X}(A)$. We are left to prove the equality $K_g(ua \otimes v \otimes z) = K_g(u \otimes av \otimes z)$ for the sum of the first, second and fourth addend in K_g . This is directly checked recalling that $\mathcal{L}_{av} = av = a \cdot \mathcal{L}_v$ on A and using the compatibility between the Lie bracket and the A -module structure of vector fields

$$[u, av] = \mathcal{L}_u(av) = \mathcal{L}_u(a)v + \alpha_a \mathcal{L}_{\alpha u}(v) = \mathcal{L}_u(a)v + \alpha_a [\alpha u, v],$$

$$[au, v] = -[\alpha^\beta v, \alpha a \beta u] = -\mathcal{L}_{\alpha^\beta v}(\alpha a) \beta u - a \mathcal{L}_{\beta v} \alpha u = a[u, v] - \mathcal{L}_{\alpha^\beta v}(\alpha a) \beta u .$$

In order to show that $K_g \in {}_A \text{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \otimes \mathfrak{X}(A), A)$ we are left to prove left A -linearity of K_g . This follows from left A -linearity of the second plus fourth addend and of the third plus sixth addend in the right hand side of (6.4).

The derivation property is equivalent to

$$K_g(u \otimes_A v \otimes az) = \delta^\eta a K_g(\delta u \otimes_A \eta v \otimes z) + 2\langle \alpha v, \mathcal{L}_{\alpha u}(a) \otimes_A z, g \rangle . \tag{6.7}$$

We use the braided Leibniz rule of the Lie derivative on covariant and contravariant tensors (cf. (3.20)) in the first two addends of K_g in (6.4) and, since the metric is braided symmetric, we also write the last addend of K_g as $\langle [u, \beta z] \otimes_A \beta v, g \rangle = \langle \alpha v \otimes_A [\alpha u, z], g \rangle = \langle \alpha v \otimes_A \mathcal{L}_{\alpha u} z, g \rangle$, thus obtaining the following expression,

$$K_g(u \otimes_A v \otimes z) = \langle \alpha v \otimes_A \beta z, \mathcal{L}_{\beta \alpha u} g \rangle + \langle [u, v] \otimes_A z, g \rangle - \langle u \otimes_A \alpha z, \mathcal{L}_{\alpha v} g \rangle + \mathcal{L}_{\alpha z} \langle \alpha(u \otimes_A v), g \rangle + 2\langle \alpha v \otimes_A \mathcal{L}_{\alpha u} z, g \rangle . \tag{6.8}$$

We use this expression to compute the left hand side and the right hand side of (6.7); each of the first four addends in (6.8) satisfies the homogeneous version of equation (6.7), for example we have

$$\langle \alpha v \otimes_A \beta(az), \mathcal{L}_{\beta \alpha u} g \rangle = \langle \alpha v \otimes_A \eta a^\gamma z, \mathcal{L}_{\gamma \eta \alpha u} g \rangle = \delta^\eta a \langle \delta^\alpha v \otimes_A \gamma z, \mathcal{L}_{\gamma \eta \alpha u} g \rangle = \delta^\eta a \langle \delta^\alpha v \otimes_A \gamma z, \mathcal{L}_{\gamma \alpha \delta u} g \rangle$$

where in the last equality we used the Yang–Baxter equation (in the form $\mathcal{R}_{23}^{-1} \mathcal{R}_{13}^{-1} \mathcal{R}_{12}^{-1} = \mathcal{R}_{12}^{-1} \mathcal{R}_{13}^{-1} \mathcal{R}_{23}^{-1}$). The last addend in (6.8) gives also an inhomogeneous term: $2\langle \alpha v \otimes_A \mathcal{L}_{\alpha u}(az), g \rangle = 2\langle \alpha v \otimes_A \mathcal{L}_{\alpha u}(a) z, g \rangle + \delta^\eta a 2\langle \delta^\alpha v \otimes_A \mathcal{L}_{\eta \alpha u} z, g \rangle$, thus proving that K_g in (6.8) satisfies (6.7). \square

Theorem 6.7 (Levi-Civita connection). *Let H be a triangular Hopf algebra, A a braided commutative H -module algebra, let the associated module in ${}^H_A \mathcal{M}_A^{\text{sym}}$ of braided derivations $\mathfrak{X}(A)$ be finitely generated and projective, $\Omega(A) := {}^* \mathfrak{X}(A)$ be the right dual module of one-forms and $g \in \Omega(A) \otimes_A \Omega(A)$ be a metric on A . There exists a unique torsion free metric compatible left connection $\nabla \in {}_A \text{Con}(\mathfrak{X}(A))$.*

The explicit expression of the Levi-Civita connection in terms of the inverse metric tensor $\bar{g} = \bar{g}^b \otimes_A \bar{g}_b$ (cf. (6.2)), the dual basis $\{e_i, \omega^i : i = 1 \dots, n\}$ of $\mathfrak{X}(A)$ and the exact pairings $\langle \cdot, \cdot \rangle : \mathfrak{X}(A) \otimes_A \Omega(A) \rightarrow A$ and $\langle \cdot, \cdot \rangle' : \Omega(A) \otimes_A \mathfrak{X}(A) \rightarrow A$ (cf. (2.32)) is, for all $v, z \in \mathfrak{X}(A)$,

$$\nabla_v(z) = \frac{1}{2} \langle \omega^i K_g(\alpha v \otimes_A \alpha e_i \otimes z), \bar{g}^b \rangle' \bar{g}_b \tag{6.9}$$

where $K_g(u \otimes v \otimes z) \in A$ is given by the right hand side of equation (6.4) for all $u, v, z \in \mathfrak{X}(A)$, (i.e., by (6.5)).

Proof. Uniqueness has been proven in Theorem 6.3. We are left to prove existence. Since g is nondegenerate we define the map $\nabla^{\text{cd}} : \mathfrak{X}(A) \otimes \mathfrak{X}(A) \rightarrow \mathfrak{X}(A)$ implicitly by, for all $u, v, z \in \mathfrak{X}(A)$,

$$2\langle u \otimes_A \nabla^{\text{cd}}(v \otimes z), g \rangle = K_g(\alpha v \otimes_A \alpha u \otimes z) . \tag{6.10}$$

Recalling that $\langle u \otimes_A \nabla^{\text{cd}}(v \otimes z), g \rangle = \langle u, g^b(\nabla^{\text{cd}}(v \otimes z)) \rangle$, cf. (6.1), the explicit expression is $\nabla^{\text{cd}} = \frac{1}{2}(g^b)^{-1} \circ (K_g \circ (\tau \text{id}_{\mathfrak{X}(A)}))^\sharp$, where $K_g \circ (\tau \otimes \text{id}_{\mathfrak{X}(A)}) \in {}_A \text{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \otimes \mathfrak{X}(A), A)$,

$$\sharp : {}_A \text{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \otimes \mathfrak{X}(A), A) \rightarrow {}_A \text{hom}(\mathfrak{X}(A)^{\otimes 2}, \Omega(A)), \tilde{L} \mapsto \tilde{L}^\sharp; \tilde{L}^\sharp(v \otimes z) = \omega^i \tilde{L}(e_i \otimes_A v \otimes z)$$

is the isomorphism of Theorem 2.5 (equations (2.29), (2.31)), $(g^b)^{-1} \in {}_A \text{hom}(\Omega(A), \mathfrak{X}(A))$, and

$$(g^b)^{-1} \circ : {}_A \text{hom}(\mathfrak{X}(A)^{\otimes 2}, \Omega(A)) \rightarrow {}_A \text{hom}(\mathfrak{X}(A)^{\otimes 2}, \mathfrak{X}(A)), \tilde{L} \mapsto (g^b)^{-1} \circ \tilde{L} .$$

This shows that $\nabla^{\text{cd}} \in {}_A \text{hom}(\mathfrak{X}(A) \otimes \mathfrak{X}(A), \mathfrak{X}(A))$. The map ∇^{cd} is furthermore a covariant derivative (as defined in Remark 4.3) because the derivation property (6.6) of K_g implies the Leibniz rule $\nabla^{\text{cd}}(v \otimes az) = \nabla^{\text{cd}}(va \otimes z) + \mathcal{L}_v(a)z$,

$$\begin{aligned} \langle u \otimes_A \nabla^{\text{cd}}(v \otimes az), g \rangle &= \frac{1}{2} K_g(\alpha v \otimes_A \alpha u \otimes az) = \frac{1}{2} K_g(\alpha v \beta a \otimes_A \beta \alpha u \otimes z) + \langle u, \mathcal{L}_v(a) \otimes_A z, g \rangle \\ &= \frac{1}{2} K_g(\alpha(va) \otimes_A \alpha u \otimes z) + \langle u \otimes_A \mathcal{L}_v(a)z, g \rangle \\ &= \langle u \otimes_A \nabla^{\text{cd}}(va \otimes z), g \rangle + \langle u \otimes_A \mathcal{L}_v(a)z, g \rangle . \end{aligned}$$

From Remark 4.3 it then follows that there exists a unique connection $\nabla \in {}_A \text{Con}(\mathfrak{X}(A))$ with the property $\nabla_v(z) = \nabla^{\text{cd}}(v \otimes z)$, for all $v, z \in \mathfrak{X}(A)$.

We now show that this connection ∇ is torsion free. For all $u, v, z \in \mathfrak{X}(A)$ consider the permutation $u \otimes v \otimes z \mapsto \gamma^\delta z \otimes \eta_\delta v \otimes \eta_\gamma u$. On the one hand, from the Yang–Baxter equation and (6.10),

$$\begin{aligned} K_g(u \otimes_A v \otimes z) - K_g(\gamma^\delta z \otimes \eta_\delta v \otimes \eta_\gamma u) &= K_g(u \otimes_A v \otimes z) - K_g(\gamma^\delta z \otimes \gamma^\eta v \otimes \delta_\eta u) \\ &= 2\langle \eta v \otimes_A (\nabla_{\eta u} z - \nabla_{\delta z} \delta_\eta u), g \rangle. \end{aligned}$$

On the other hand, recalling the definition of K_g , and using the braided symmetry of the metric, the left hand side of the above expression equals $2\langle [u, \beta z] \otimes_A \beta v, g \rangle = 2\langle \eta v \otimes_A [\eta u, z], g \rangle$, thus showing that ∇ is torsionless.

We similarly show the metric compatibility of ∇ . From (6.10) and Corollary 4.10 we have

$$K_g(u \otimes_A v \otimes z) + K_g(u \otimes_A \gamma z \otimes \gamma v) = 2\langle \nabla_u (v \otimes z), g \rangle = 2\mathcal{L}_u \langle v \otimes_A z, g \rangle - 2\langle u \otimes_A v \otimes z, {}^* \nabla g \rangle.$$

From the definition of K_g , cf. (6.4), it easily follows that the left hand side of this expression simplifies to $2\mathcal{L}_u \langle v \otimes_A z, g \rangle$, thus proving metric compatibility of the connection ∇ .

The explicit expression (6.9) of the Levi-Civita connection follows from $\nabla_v(z) = \nabla^{\text{cd}}(v \otimes z) = \frac{1}{2}(g^b)^{-1}((K_g \circ (\tau \otimes \text{id}_{\mathfrak{X}(A)}))^{\sharp}(v \otimes z))^{b'}$ recalling that $(\bar{g})^{b'} = (g^b)^{-1}$. \square

6.3. Ricci tensor, scalar curvature and Einstein manifolds

There is a canonical notion of trace in a ribbon category and a fortiori in a compact closed category. For an internal morphism $\tilde{L} \in {}_A \text{hom}(\mathfrak{X}(A), \mathfrak{X}(A))$ in ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$ we have $\text{tr}(\tilde{L}) = \langle \cdot, \cdot \rangle' \circ (\text{id}_{\Omega(A)} \otimes_A \tilde{L}) \circ \text{coev}$ that belongs to ${}_A \text{hom}(A, A)$ in ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$ because the coevaluation map $\text{coev} : A \rightarrow \Omega(A) \otimes_A \mathfrak{X}(A)$ and the exact pairing $\langle \cdot, \cdot \rangle' : \Omega(A) \otimes_A \mathfrak{X}(A) \rightarrow A$ are morphisms in ${}^H_A \mathcal{M}_A^{\text{sym,fp}}$. The trace is determined by its value on $1_A \in A$, which is $\text{tr}(\tilde{L}) = (\omega^i, \tilde{L}(e_i))'$, where we used a dual basis of $\mathfrak{X}(A)$. The Ricci tensor is the trace of the Riemann tensor given by (use $A \otimes_A \mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \simeq \mathfrak{X}(A) \otimes_A \mathfrak{X}(A)$),

$$\begin{aligned} Ric &:= \langle \cdot, \cdot \rangle' \circ (\text{id}_{\Omega(A)} \otimes_A R_{\nabla}) \circ (\text{coev} \otimes_A \text{id}_{\mathfrak{X}(A) \otimes_A \mathfrak{X}(A)}) : \mathfrak{X}(A) \otimes_A \mathfrak{X}(A) \rightarrow A, \\ Ric(u, v) &= \langle \omega^i, R_{\nabla}(e_i, u, v) \rangle'. \end{aligned} \tag{6.11}$$

Since the curvature R_{∇} is left A -linear it follows that $Ric \in {}_A \text{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A), A)$. Similarly, the scalar curvature tensor S is given by $S = Ric(\bar{g})$, where the inverse metric $\bar{g} \in \mathfrak{X}(A) \otimes_A \mathfrak{X}(A)$ has been defined in (6.2).

We also define an Einstein metric on A to be a metric g proportional to its Ricci tensor,

$$Ric = \lambda \langle \cdot, \cdot \rangle, \quad (\lambda \in \mathbb{k}). \tag{6.12}$$

This equation in ${}_A \text{hom}(\mathfrak{X}(A) \otimes_A \mathfrak{X}(A), A)$ allows to study noncommutative Einstein manifolds.

7. Examples

Consider any of the examples discussed in Section 3.3 (cotriangular Hopf algebras, noncommutative manifolds via Drinfeld twists) together with a metric g . Then there is a unique Levi-Civita connection associated with g , with Ricci and scalar curvature as studied above. In particular we can consider Einstein in vacuum equations on a cotriangular Hopf algebra A for any metric g on A .

In the following we first briefly show how to recover previous results on noncommutative tori and (hyper)planes. Then we study the Riemannian geometry of the Hopf algebra $A = K \otimes K$ with K the four dimensional Sweedler Hopf algebra with its basic cotriangular structure. We solve the noncommutative Einstein manifold equations with a metric that is non-central and non-equivariant, its scalar curvature is $S = 12$.

7.1. Noncommutative torus

Let $\mathbb{k} = \mathbb{C}$ and $\Theta = (\theta_{jk})_{j,k=1,\dots,n}$ be a real valued skew symmetric matrix and A the noncommutative n -dimensional torus generated by the unitaries $U_j, j = 1, \dots, n$ with commutation relations $U_j U_k = e^{2\pi i \theta_{jk}} U_k U_j$. Consider the n derivations $\partial_j : A \rightarrow A$, defined by $\partial_j(U_k) = 2\pi i \delta_{jk} U_k$ and $\partial_j(aa') = \partial_j(a)a' + a\partial_j(a')$, for any $a, a' \in A$. They generate the commutative and cocommutative Hopf algebra H , the universal enveloping algebra of the Lie algebra of the (commutative) n -torus. Considering the triangular structure $R = e^{2\pi i \theta_{jk} \partial_j \otimes \partial_k}$ of H (belonging to a suitable topological completion of $H \otimes H$) we have that A becomes a braided commutative H -comodule algebra. The derivations ∂_j generate the free A -module of vector fields $\mathfrak{X}(A)$. Since the action of H on ∂_i is trivial, Koszul formula (6.4) for the Levi-Civita connection simplifies to $2\langle \partial_j \otimes_A \nabla_{\partial_i} \partial_k, g \rangle = \partial_i \langle \partial_j \otimes_A \partial_k, g \rangle - \partial_j \langle \partial_i \otimes_A \partial_k, g \rangle + \partial_k \langle \partial_i \otimes_A \partial_j, g \rangle$ and equals that in [31]. There, however, the connection ∇_u and the curvature tensor $R(u, v, z)$ were defined only for u and v derivations of the algebra A (such as elements of the \mathbb{C} -linear span of the ∂_j 's), not for arbitrary vector fields.

One can proceed similarly with the quantum (hyper)plane with generators $x_j, j = 1, \dots, n$ and commutation relations $x_j x_k - x_k x_j = 2\pi i \theta_{jk}$. This is also an H -comodule algebra with $\partial_j(x_k) = \delta_{jk}$. The expression of the Levi-Civita connection is as above and equals that in [3].

7.2. Sweedler Hopf algebra and its tensor product

We first study a differential calculus on the Sweedler four dimensional Hopf algebra K as an explicit case of Example 3.9, it is uniquely defined by the choice of the cotriangular structure. There is a whole family of cotriangular structures on K (cf. e.g. [24, Ex. 2 §10.1.1]), among the different ones we choose the basic one, defined in equation (7.1) below. There is no metric tensor on this differential geometry with one dimensional tangent and cotangent space since the tensor product of forms is automatically antisymmetric. We then exemplify the previous section by studying the Riemannian geometry of the tensor product $K \otimes K$. Differently from the noncommutative torus and (hyper)plane cases, the free module of vector fields is not generated by derivations of the algebra $K \otimes K$, we then use the general Koszul formula (6.4) to determine the Levi-Civita connection.

7.2.1. Differential calculus on Sweedler Hopf algebra K

Let K be the Sweedler Hopf algebra over a field \mathbb{k} of characteristic 0. It is the algebra generated by x and g and defining relations $g^2 = 1, x^2 = 0, xg = -gx$. A vector space basis is given by $(1, g, x, gx)$. The coalgebra structure is determined by $\Delta(g) = g \otimes g, \Delta x = x \otimes g + 1 \otimes x$, with antipode $S(g) = g, S(x) = gx$. Sweedler Hopf algebra is self dual. Let U be a second copy of K , we rename the generators g, x as γ, ψ (with the same costructures). The relations $\langle \gamma, 1 \rangle = 1, \langle \gamma, g \rangle = -1, \langle 1, g \rangle = 1, \langle \psi, x \rangle = 1$, all other relations among the generators vanishing, extend to a Hopf algebra pairing $\langle \cdot, \cdot \rangle : U \otimes K \rightarrow \mathbb{k}$ proving self duality of K .

We consider the following triangular structure on U ,

$$\mathcal{R} = \mathcal{R}^\alpha \otimes \mathcal{R}_\alpha = \frac{1}{2}(1 \otimes 1 + \gamma \otimes 1 + 1 \otimes \gamma - \gamma \otimes \gamma) \tag{7.1}$$

so that K is cotriangular (cf. Example 3.9). The (U, \mathcal{R}) -Lie algebra ${}_{\text{inv}}\mathfrak{X}(K) \simeq \mathfrak{g}$ of left invariant vector fields is $\mathfrak{g} = \{\chi \in U; \Delta\chi = \chi \otimes 1 + \overline{\mathcal{R}}^\alpha \otimes \overline{\mathcal{R}}_\alpha \triangleright \chi\}$, cf. (3.28). It is the linear span of

$$\chi := \gamma \psi .$$

Indeed we have $\Delta\chi = \chi \otimes 1 + \gamma \otimes \chi = \chi \otimes 1 + \overline{\mathcal{R}}^\alpha \otimes \overline{\mathcal{R}}_\alpha \triangleright \chi$, while $\overline{\mathcal{R}}^\alpha \otimes \overline{\mathcal{R}}_\alpha \triangleright \gamma = 1 \otimes \gamma$ and $\overline{\mathcal{R}}^\alpha \otimes \overline{\mathcal{R}}_\alpha \triangleright \psi = \gamma \otimes \psi$ imply that the subspace $\mathfrak{g} \subset U$ is one dimensional.

The bicovariant differential calculus associated with the cotriangular structure of K is therefore determined by the left invariant vector field $u \in {}_{\text{inv}}\mathfrak{X}(K), u(k) := k_1 \langle \chi, k_2 \rangle$ for all $k \in K$. Explicitly, from $\langle \chi, 1 \rangle = 0, \langle \chi, g \rangle = 0, \langle \chi, x \rangle = 1, \langle \chi, gx \rangle = -1$ we obtain $u(1) = 0, u(g) = 0, u(x) = 1, u(xg) = g$. Accordingly with the non-cocommutative coproduct of χ , the left-invariant vector field u is a braided derivation, e.g., $u(gx) = u(g)x - gu(x), u(xg) = u(x)g + xgu(x)$. The braided commutator vanishes, $[u, u] = 0$, since $\overline{\mathcal{R}}^\alpha \triangleright \chi \otimes \overline{\mathcal{R}}_\alpha \triangleright \chi = -\chi \otimes \chi$ and $\chi \chi = 0$. Let $\omega \in {}_{\text{inv}}\Omega^1(K)$ be the dual left invariant one-form:

$$\langle u, \omega \rangle = 1 . \tag{7.2}$$

The K -bimodule of one-forms is the free right module $\Omega^1(K) \simeq {}_{\text{inv}}\Omega^1(K) \otimes K$. From $dk = \omega u(k)$ for all $k \in K$ (cf. (3.26)) we have

$$dg = 0, dx = \omega, d(gx) = g\omega .$$

The bimodule commutation relations $\omega g = -g\omega, \omega x = -x\omega$ then read $(dx)g = -gdx, (dx)x = -xdx$ and are easily obtained from the exterior derivative on $x^2 = 0, gx = -xg$. The space of right invariant one forms is the \mathbb{k} -linear span of $\eta = \omega g = dxg$. Dually, from (7.2) the free bimodule of vector fields is determined by $u g = -gu, u x = -xu$, it is the liner span of u, gu, xu ; only xu is a derivation, it does not generate the K -bimodule of vector fields.

Recall from Example 2.10 and 3.3 that K is a $U^{op} \otimes U$ -module algebra with R -matrix $\mathcal{R} = R^\alpha \otimes R_\alpha = (\text{id} \otimes \text{flip} \otimes \text{id})(\mathcal{R}^{-1} \otimes \mathcal{R})$. The braiding τ in $\Omega(K) \otimes \Omega(K)$ (a representation of the permutation group since it squares to the identity) is with respect to this triangular structure. We have $\tau(\omega \otimes_K \eta) = \eta \otimes_K \omega$ since the left-invariant one-form ω trivialises the R -matrix of U^{op} while the right-invariant one η that of U (we thus recover Woronowicz braiding). By K -linearity it then follows that $\tau(\omega \otimes_K \omega) = \tau(\omega \otimes_K \eta)g = \eta \otimes_K \omega g = -\omega \otimes_K \omega$. The tensor product is therefore antisymmetric. This shows the first of the following K -bimodule isomorphisms

$$\Omega(K) \wedge \Omega(K) \simeq \Omega(K) \otimes_K \Omega(K) \simeq K . \tag{7.3}$$

The last follows from the \mathbb{Z}_2 -graded structure of the bimodule relations $\omega g = -g\omega, \omega x = -x\omega$.

More generally, the contravariant tensor algebra is isomorphic to the exterior algebra, $\mathcal{T}^{\bullet,0} = \bigoplus_{r \in \mathbb{N}} \mathcal{T}^{r,0} \simeq \Omega^\bullet(A)$, with bimodule isomorphisms $\Omega^{2r}(K) \simeq K, \Omega^{2r+1}(K) \simeq \Omega(K)$.

Form (7.3) we see that there is no symmetric two tensor and hence no metric tensor on this differential geometry with one dimensional tangent and cotangent space and no top form.

7.2.2. Riemannian geometry on $K \otimes K$

Define the tensor product algebra $A := K \otimes K$, where the two copies of K mutually commute. Let g, x denote the generators of $K \simeq K \otimes 1 \subset A$, and g', x' those of $K \simeq 1 \otimes K \subset A$, similarly for $\gamma, \psi \in U \otimes 1$ and $\gamma', \psi' \in 1 \otimes U$. The triangular structure of $\mathcal{U} := U \otimes U$ is canonically given as $\mathfrak{R} = \mathfrak{R}^\alpha \otimes \mathfrak{R}_\alpha = \mathcal{R}^\alpha \mathcal{R}'^\beta \otimes \mathcal{R}_\alpha \mathcal{R}'_\beta$, whence the cotriangular one of A . Then A is an example of braided commutative H -module algebra, where $H = \mathcal{U}^{op} \otimes \mathcal{U}$. The H -Lie algebra is the direct sum of one-dimensional Lie subalgebras: $\mathfrak{g}_A = \mathfrak{g} \oplus \mathfrak{g}' \subset U \otimes U$, with $\chi := \gamma\psi \in \mathfrak{g} \subset U \otimes 1$, $\chi' := \gamma'\psi' \in \mathfrak{g}' \subset 1 \otimes U$ linear generators of $\mathfrak{g}, \mathfrak{g}'$. Since $\overline{\mathfrak{R}}_\alpha \otimes \overline{\mathfrak{R}}^\alpha \triangleright \chi = \overline{\mathcal{R}}_\alpha \otimes \overline{\mathcal{R}}^\alpha \triangleright \chi$, and similarly for χ' , we have $[\chi, \chi'] = \chi\chi' - \overline{\mathfrak{R}}_\alpha \triangleright \chi' \overline{\mathfrak{R}}^\alpha \triangleright \chi = \chi\chi' - \chi'\chi = 0$. The associated left invariant vector fields are u, u' and hence have vanishing braided brackets. The dual left-invariant one-forms ω, ω' commute with $1 \otimes K$ and $K \otimes 1$ respectively and generate the bimodule of one-forms $\Omega(A) = \Omega(K) \otimes K \oplus K \otimes \Omega(K)$ as a free right (left) A -module. The exterior derivative is the sum $d \otimes 1 + 1 \otimes d$. The bimodule of braided symmetric contravariant 2-tensors is freely generated (as a right module) by $\omega \vee \omega' := \omega \otimes_A \omega' + \omega' \otimes_A \omega$. The most general metric on A is $g = \omega \vee \omega' t$ with $t \in A$ invertible, this is generally neither central nor H -equivariant.

We are now set to study the Levi-Civita connection on the Hopf algebra A . Since A is generated by x, g, x', g' , we write $a = a(x, g, x', g')$ for any $a \in A$ and define the algebra involutions

$$\widehat{a} := a(-x, -g, x', g'), \quad \widehat{a}' := a(x, g, -x', -g'), \quad \widetilde{a} := a(-x - g, -x' - g')$$

so that $\widehat{a}\omega = \omega a, \widehat{a}'\omega' = \omega' a, \widehat{a}u = ua, \widehat{a}'u' = u'a$. We apply Theorem 6.7 and obtain the following explicit expression of the Levi-Civita connection.

Theorem 7.1. *The Levi-Civita connection $\nabla \in {}_A\text{Con}(\mathfrak{X}(A))$ of the arbitrary metric tensor $g = \omega \vee \omega' t$ ($t \in A$ invertible) on the Hopf algebra A with bicovariant differential calculus defined above is given by*

$$\begin{aligned} \nabla : \mathfrak{X}(A) &\longrightarrow \Omega(A) \otimes_A \mathfrak{X}(A) \\ u &\longmapsto -\omega' \otimes_A (u u'(t) \widehat{t^{-1}} + u' u(t) \widetilde{t^{-1}}) \\ u' &\longmapsto -\omega \otimes_A (u u'(t) \widetilde{t^{-1}} + u' u(t) \widehat{t^{-1}}). \end{aligned} \tag{7.4}$$

Proof. The Koszul formula (6.4) gives 8 equations, for u, v, z respectively 1) $u, u, u, 2$) $u, u', u, 3$) $u, u, u', 4$) u, u', u' and the corresponding ones with $u \leftrightarrow u'$. The right hand side of the first equation vanishes since $\langle u, \omega' \rangle = 0$. The left hand side reads $-2\langle u \otimes_A \nabla_u u, g \rangle = -2\langle u, \langle \nabla_u u, \omega' \rangle \omega \rangle t = 2\langle \widehat{\nabla_u u}, \omega' \rangle t$, hence $\langle \nabla_u u, \omega' \rangle = 0$ that is $\nabla_u u = au$ for some $a \in A$. Similarly, the second equation implies $\nabla_u u = bu'$ for some $b \in A$. Together the first two equations therefore imply

$$\nabla_u u = 0.$$

Similarly, the third equation reads $-2\langle \widehat{\nabla_u u'}, \omega' \rangle t = 2u(t)$ that implies $\nabla_u u' = -\widehat{u(t)} \widehat{t^{-1}} u' + au$ for some $a \in A$. Together with the fourth, which implies $\nabla_u u' = -\widetilde{u'(t)} \widetilde{t^{-1}} u + bu$ for some $b \in A$, we then have $\nabla_u u' = -\widetilde{u'(t)} \widetilde{t^{-1}} u - \widehat{u(t)} \widehat{t^{-1}} u'$, i.e., $\nabla_u u' = -u u'(t) \widetilde{t^{-1}} - u' u(t) \widehat{t^{-1}}$. The remaining four equations, corresponding to $u \leftrightarrow u'$, give $\nabla_{u'} u' = 0$ and $\nabla_{u'} u = \nabla_u u'$. This proves expression (7.4). \square

Remark 7.2. The dual right connection on one-forms (readily obtained from (5.11)) reads: ${}^*\nabla : \Omega(A) \longrightarrow \Omega(A) \otimes_A \Omega(A)$, ${}^*\nabla(\omega) = \omega \vee \omega' \widetilde{u'(t)} \widetilde{t^{-1}}$, ${}^*\nabla(\omega') = \omega \vee \omega' \widehat{u(t)} \widehat{t^{-1}}$. The torsionless and the metric compatibility conditions can then be easily checked.

We next compute the curvature, Ricci tensor and scalar curvature. We set $b := -\widetilde{u'(t)} \widetilde{t^{-1}}$, $b' := -\widehat{u(t)} \widehat{t^{-1}}$, so that $\nabla_u u' = \nabla_{u'} u = bu - b'u'$. Then $\nabla_u(\nabla_u u') = u(b)b + u(b')u' + \widehat{b'}(bu + b'u')$, and similarly $\nabla_{u'}(\nabla_{u'} u) = u'(b')b' + u'(b)u + \widehat{b}(bu + b'u')$. Since $[u, u] = [u', u'] = [u, u'] = 0$ the expression of the Riemann tensor simplifies and the only nonvanishing components are:

$$\begin{aligned} \frac{1}{2}R(u, u, u') &= R(u, u', u) = -R(u', u, u) = u(b)u + u(b')b' + \widehat{b'}(bu + b'u') \\ \frac{1}{2}R(u', u', u) &= R(u', u, u') = -R(u, u', u') = u'(b')u' + u'(b)b + \widehat{b}(bu + b'u'). \end{aligned} \tag{7.5}$$

From (6.11) (and the pairing $\langle \omega, \omega' \rangle = -\langle u, \omega \rangle = -1$) the Ricci tensor is given by

$$\text{Ric}(u, u) = \widehat{u(b')} + \widetilde{b'b'}, \quad \text{Ric}(u, u') = -2\widehat{u(b)} - 2b'\widehat{b} - \widetilde{u'(b')} - b\widetilde{b}' \tag{7.6}$$

with $\text{Ric}(u', u')$ and $\text{Ric}(u', u)$ obtained from the above by exchanging $u \leftrightarrow u', b \leftrightarrow b', \widehat{\ } \leftrightarrow \widetilde{\ }$.

The inverse metric is $g^{-1} = -t^{-1}(u \otimes_A u' + u' \otimes_A u)$ (use $g^b(u) = \widehat{t}' \omega'$, $g^b(u') = \widehat{t} \omega$ and $\langle \cdot, g^{-1} \rangle' \otimes_A g_a^{-1} = g^{b^{-1}}$ where $g^{-1} = g^{-1a} \otimes_A g_a^{-1}$). Then by left A -linearity of the Ricci tensor the scalar curvature reads $S = Ric(g^{-1}) = 3t^{-1}(\widehat{u(b)} + \widehat{u'(b')} + b' \widehat{b} + b \widehat{b'})$.

The Einstein metric conditions (6.12) for $g = \omega \vee \omega' t$ are immediate from (7.6). A solution with proportionality factor $\lambda = -6$ is given by setting $t = 1 + x + x'$ (so that $b = -1 + x - x' + 2xx'$, $u(b) = 1 + 2x'$, $u'(b) = -1 + 2x$ and $b', u'(b'), u(b')$ are obtained by exchanging $x \leftrightarrow x'$).

The Einstein metric $g = \omega \vee \omega'(1 + x + x')$ is neither central nor equivariant. Its Levi-Civita connection is not a bimodule connection (there is no generalised braiding σ , cf. e.g. [13, Def. 3.66], that satisfies $\sigma(u \otimes dg) = \nabla(u)g - \nabla(u)g$, indeed $dg = 0$ while $\nabla(u)g - \nabla(u)g \neq 0$). The scalar curvature is $S = 12$.

Declaration of competing interest

The author states that there is no conflict of interest.

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Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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