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Asymptotic estimates for the torsional rigidity of rods with thinning cross sections

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Abstract
We study the torsional rigidity for rods with thinning cross sections. The main purpose of the paper is to prove rigorous asymptotic formulas for the torsional rigidity as the thickness of the cross section tends to zero. These asymptotic formulas are empirically known and are widely used in the field of Mechanics of Materials. From a more theoretical point of view, thinning domains are considered when studying optimal inequalities for suitable classes of functionals depending on domains. We recall as an example of this kind of inequalities the celebrated *Saint Venant inequality* stating that, among planar domains with fixed Lebesgue measure, the disk is the cross section corresponding to a maximal torsional rigidity. It is well known that this statement is equivalent to say that disks are maximizer of a suitable functional, see for example (Amato et al. in On the optimal sets in Pólya and Makai type inequalities, 2025) and the references therein. Actually, in the present paper other kinds of functionals are more relevant when considering thinning domains. We refer in particular to the so-called Pólya and Makai functionals, see the papers (Makai in On the principal frequency of a membrane and the torsional rigidity of a beam, Stanford Univ. Press, Stanford, 1962; Pólya, J Indian Math Soc (N.S.) 24(1960):413–419, 1961; Pólya and Szegö, Isoperimetric Inequalities in Mathematical Physics, Princeton University Press, Princeton, 1951) for more details and (1.14) in the present paper for the precise definitions.

Keywords Torsional rigidity of rods · Thinning cross sections · Asymptotic formulas · Functional inequalities

Mathematics Subject Classification 35Q74 · 41A60 · 49Q10 · 74B05 · 74K10

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1 Introduction and main results

The main purpose of this paper is to show the validity of some asymptotic formulas for the torsional rigidity of rods with cross sections represented by *thinning planar domains*, possibly nonconvex. More precisely, we consider cross sections made of one or more thin portions and we look at the behavior of the torsional rigidity when their thickness tends to zero.

The study of this problem has two different motivations: the first one is more applicative and it comes from the necessity of providing a rigorous proof of already known formulas used in the field of Mechanics of Materials for determining approximate values of the torsional rigidity of rods with composite cross sections; the second one is more theoretical and it comes from some classical estimates obtained by Makai [16], Pólya [22] and Pólya & Szegö [23] in the class of planar convex domains. For more historical notes and more recent results about estimates for torsion of convex planar sets, see [1, 10] and the references therein. For more references on the problem of torsion of rods with convex cross sections, see [2, 9, 11, 21, 25, 26, 29, 30].

Let us describe in more detail the questions that we would like to investigate. Denote by $\Omega_d \subset \mathbb{R}^2$ the section of a rod of thickness d.

In the field of Mechanics of Materials, see for example [28], [7, p. 210, (3.41)] and [31], it is empirically known that if the cross section of the rod Ω_d is made of the union of several straight thin parts of thickness d, then the total torsional rigidity of the rod is given approximately by

$$K_d \approx \sum_{i=1}^{N} \frac{1}{3} L_i \, d^3 \, , \tag{1.1}$$

provided that $\partial\Omega_d$ is connected, see Fig. 1. In (1.1), N denotes the number of straight parts, L_i denotes the length of each part and d their thickness. In general, thickness could not necessarily be always the same for every single part but, assuming that it is essentially of the same order of magnitude, just to simplify the notation, we may assume that it equals d for all of them.

The main purpose of this paper is to prove in a rigorous way the validity of (1.1) by providing an asymptotic formula when the common thickness d of each part tends to zero.

We also prove that this kind of result can be extended in a suitable way to a wider class of domains made up of subdomains that can also be *curved*.

Before giving the rigorous statements of the main results, we briefly want to recall some well known facts from the theory of elasticity for the torsion of elastic rods, see for example [15, Chapter 2, Paragraph 16] for more details. Assuming that the rod is parallel to the z-axis, we can describe it as a set of the form $\Omega_d \times I$ where $\Omega_d \subset \mathbb{R}^2$ denotes the section of

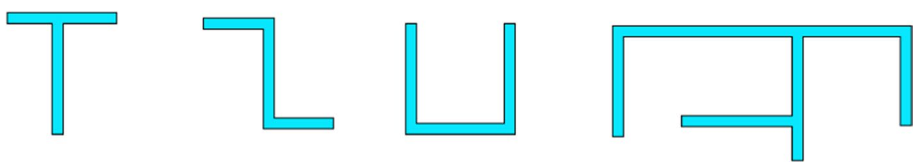

Fig. 1 Examples of domains with connected boundary made up of straight strips

the rod and I is a bounded interval. Denoting by U_x, U_y, U_z the components of the displacement vector and by τ the torsion angle per unit of length, then

$$U_x = -\tau\, yz\,, \qquad U_y = \tau\, xz\,, \qquad U_z = \tau\, \psi(x,y)\,,$$

where $\psi = \psi(x,y)$ is known as the torsion function. Assuming τ to be constant, for the components of the linearized strain tensor, we have

$$U_{xx} = U_{yy} = U_{xy} = U_{zz} = 0\,, \quad U_{xz} = \frac{1}{2}\tau\left(\frac{\partial\psi}{\partial x} - y\right)\,, \quad U_{yz} = \frac{1}{2}\tau\left(\frac{\partial\psi}{\partial y} + x\right)\,.$$

Moreover, by Hooke's law, for the components of the stress tensor, we have

$$\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{zz} = 0\,, \quad \sigma_{xz} = 2\mu U_{xz} = \mu\tau\left(\frac{\partial\psi}{\partial x} - y\right)\,, \quad \sigma_{yz} = 2\mu U_{yz} = \mu\tau\left(\frac{\partial\psi}{\partial y} + x\right)\,,$$

where μ denotes the modulus of rigidity.

From the above representation of the stress tensor it follows that the equations of equilibrium reduce to the single equation

$$0 = \frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} = \mu\tau\frac{\partial^2\psi}{\partial x^2} + \mu\tau\frac{\partial^2\psi}{\partial y^2}\,,$$

which shows that ψ is a harmonic function in Ω_d. In place of the torsion function, one can introduce an auxiliary function that we denote here by u defined implicitly by

$$\frac{\partial\psi}{\partial x} = y + 2\frac{\partial u}{\partial y}\,, \qquad \frac{\partial\psi}{\partial y} = -x - 2\frac{\partial u}{\partial x}\,.$$

Recalling that ψ is harmonic, it turns out that u solves the equation $-\Delta u = 1$ in Ω_d.

The advantage of introducing such a function comes from the fact that assuming $\sigma\,\mathrm{n} = 0$ on the lateral surface $\partial\Omega_d \times I$ of the rod, where σ denotes the stress tensor and n the outer unit normal vector to $\partial\Omega_d \times I$, then it turns out that ∇u vanishes identically on $\partial\Omega_d$, whence the function u is constant on each connected component of $\partial\Omega_d$.

In the case of a connected boundary, up to an additive constant, we may assume that u solves the Dirichlet problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega_d\,, \\ u = 0 & \text{on } \partial\Omega_d\,. \end{cases} \tag{1.2}$$

In the case of a non connected boundary, which essentially means that the cross section Ω_d has one or more holes, we may always assume that $u = 0$ on the external boundary of Ω_d and, denoting by $S_1, \ldots, S_M$ the regions enclosed by the internal connected components of $\partial\Omega_d$, namely the holes, from the definition of u and the fact that $\oint_{\partial S_j} \mathrm{d}\psi = 0$, we may recover the following integral boundary conditions

$$\oint_{\partial S_j} \frac{\partial u}{\partial \mathrm{n}}\, d\ell = |S_j| \qquad \text{for any } j \in \{1, \dots, M\}\,,$$

where n is the outer unit normal to $\partial\Omega_d$ (and hence inner with respect to the hole S_j) and $d\ell$ the arc length differential. Hence, in the case of a non connected boundary, denoting by Γ_0 the external boundary of the cross section, up to an additive constant, the solution u solves the problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega_d\,, \\ u = 0 & \text{on } \Gamma_0\,, \\ \oint_{\partial S_j} \frac{\partial u}{\partial \mathrm{n}}\, d\ell = |S_j| \qquad \text{for any } j \in \{1, \dots, M\}\,. & \end{cases} \tag{1.3}$$

The elastic energy per unit of volume is given by $2\mu\tau^2|\nabla u|^2$ and the elastic energy per unit length is given by $\frac{1}{2}\mu K_d\,\tau^2$ where the constant K_d is given by

$$K_d = 4\int_{\Omega_d} |\nabla u|^2\, dxdy\,. \tag{1.4}$$

In the present paper, we refer directly to K_d as the torsional rigidity of the rod, even though the real torsional rigidity also depends on the modulus of rigidity μ and is given by μK_d.

The present paper deals only with the case of domains with a connected boundary. For more details on the case of the non connected boundary see the open questions at the end of this section.

In the sequel, in order to emphasize the dependence on *d*, the solution of (1.2) will be denoted by u_d.

We start by stating the first main result in the case of a rod with an H-shaped cross section. This specific geometry of the cross section is typical, in good approximation, of some real structures like the deck of a bridge, see for example Fig. 2 taken from [24].

The approximation that can be obtained by (1.1) was used, for example, in [12] to derive a realistic value for the torsional rigidity of the deck of the Tacoma Narrows Bridge, which has become very famous for its collapse occurred in 1940, see [27].

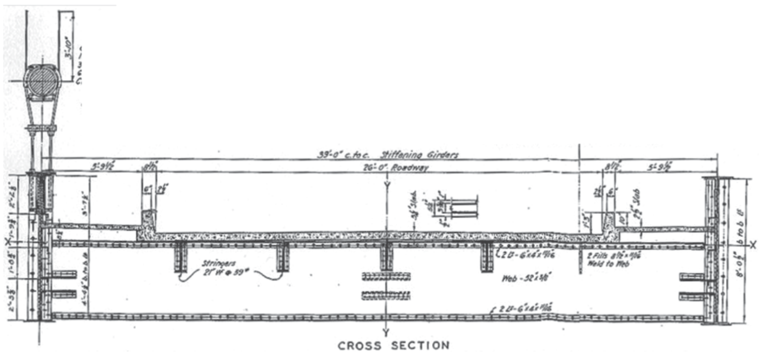

Fig. 2 Cross section of a suspension bridge (Tacoma Narrows Bridge)

That bridge was known to be extremely flexible and probably this was one of the main reasons for its failure. For more information about the problem of suspension bridges, see among others, the following papers [3–6, 8, 13, 14, 17–20] and the references therein.

We now define the open H-shaped domain Ω_d as the union of the three open rectangles defined by

$$U_{1,d} := (-\ell_1, \ell_1) \times \left(-\tfrac{d}{2}, \tfrac{d}{2}\right) \tag{1.5}$$

which, roughly speaking, represents the central horizontal part of Ω_d with $\ell_1 = \frac{L_1}{2}$, and

$$U_{2,d} := \left(\ell_1 - \tfrac{d}{2}, \ell_1 + \tfrac{d}{2}\right) \times (-\ell_2, \ell_2)\ , \quad U_{3,d} := \left(-\ell_1 - \tfrac{d}{2}, -\ell_1 + \tfrac{d}{2}\right) \times (-\ell_2, \ell_2)\ , \tag{1.6}$$

which represent the lateral vertical parts of Ω_d with $\ell_2 = \frac{L_2}{2}$, see Fig. 3.

For this kind of H-shaped domain Ω_d, the torsional rigidity K_d is not exactly the sum of the following three terms

$$K_{1,d} := 4\int_{U_{1,d}} |\nabla u_d|^2\, dxdy\,, \qquad K_{2,d} := 4\int_{U_{2,d}} |\nabla u_d|^2\, dxdy\,, \qquad K_{3,d} := 4\int_{U_{3,d}} |\nabla u_d|^2\, dxdy\,,$$

because the three sets $U_{1,d}, U_{2,d}, U_{3,d}$ are not disjoint and the intersection of $U_{1,d}$ with the other two sets has a positive measure. However, the asymptotic formula for K_d can be obtained by summing together the asymptotic estimates for $K_{1,d}, K_{2,d}, K_{3,d}$.

This splitting is what is usually done in the field of Mechanics of Materials to obtain the torsional rigidity of the entire structure. However, in the field of Mathematical Analysis, this has to be proved rigorously.

We are ready to give the statement of the following asymptotic estimate.

Theorem 1.1 *Let Ω_d be a H-shaped domain as described above. Then we have*

$$K_{1,d} \sim \frac{1}{3}\, L_1\, d^3\,, \qquad K_{2,d} = K_{3,d} \sim \frac{1}{3}\, L_2\, d^3 \qquad \text{as } d \to 0^+,$$

and, moreover, for the torsional rigidity we have

$$K_d \sim \frac{1}{3}\, L_1\, d^3 + \frac{2}{3}\, L_2\, d^3 \qquad \text{as } d \to 0^+.$$

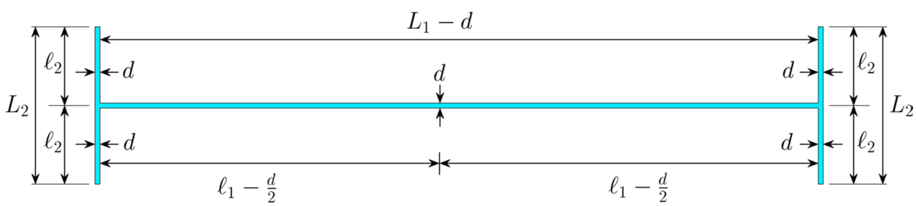

Fig. 3 An example of H-shaped domain

In the proof of Theorem 1.1 we benefit from the particular geometry of the three components of Ω_d and from their orthogonality. The resulting proof becomes technically simpler, so that the reader can focus the attention on the main ideas contained in it.

However, as one can guess, these quite restrictive assumptions are not so fundamental and the result of Theorem 1.1 can be extended to a more general setting. Having this in mind, we now introduce a wider class of domains that are not necessarily H-shaped and that could possibly be composed of curved subdomains.

More precisely, we define a class of domains that can be considered as the union of a finite number of tubular neighborhoods of regular curves. For this purpose, for $i = 1, \dots, N$, let $\boldsymbol{\gamma}_i : [a_i, b_i] \to \mathbb{R}^2$ be regular non closed simple curves of class C^3, i.e.

$$\boldsymbol{\gamma}_i \in C^3([a_i, b_i]; \mathbb{R}^2), \quad \boldsymbol{\gamma}_i : [a_i, b_i] \to \mathbb{R}^2 \text{ is injective}, \quad |\boldsymbol{\gamma}_i'(t)| \neq 0 \text{ for any } t \in [a_i, b_i]. \tag{1.7}$$

It is not restrictive to assume that

$$\boldsymbol{\gamma}_i((a_i, b_i)) \cap \boldsymbol{\gamma}_j((a_j, b_j)) = \emptyset \quad \text{for } i \neq j \qquad \text{and} \qquad \bigcup_{i=1}^{N} \boldsymbol{\gamma}_i([a_i, b_i]) \quad \text{is connected}. \tag{1.8}$$

Moreover, we also assume that

$$\text{either } \boldsymbol{\gamma}_i([a_i, b_i]) \cap \boldsymbol{\gamma}_j([a_j, b_j]) \text{ is empty or it contains exactly one point}, \tag{1.9}$$

and if P_{ij} is such a point and $t_i \in [a_i, b_i]$, $t_j \in [a_j, b_j]$ are such that $\boldsymbol{\gamma}_i(t_i) = P_{ij}$ and $\boldsymbol{\gamma}_j(t_j) = P_{ij}$, then

$$\boldsymbol{\gamma}_i'(t_i) \text{ and } \boldsymbol{\gamma}_j'(t_j) \text{ are not parallel vectors.} \tag{1.10}$$

Then, for any $i = 1, \dots, N$, we can define the *tubular neighborhood* of thickness d of $\boldsymbol{\gamma}_i([a_i, b_i])$ as the set

$$U_{i,d} := \{\boldsymbol{\gamma}_i(t) + s\mathrm{n}(t) : t \in [a_i, b_i], s \in (-d/2, d/2)\}, \tag{1.11}$$

where n is the unit normal vector to the curve $\boldsymbol{\gamma}_i$ such that the couple of vectors $\{\boldsymbol{\gamma}_i', \mathrm{n}\}$ is oriented as the canonical basis of $\mathbb{R}^2$ at any point of the curve.

The domain Ω_d can be expected to be given by the union of all the sets $U_{i,d}$ but this is not exactly what we would like to do for the problem shown in Fig. 4.

To overcome this problem, to construct our domain we add the colored green set in Fig. 5.

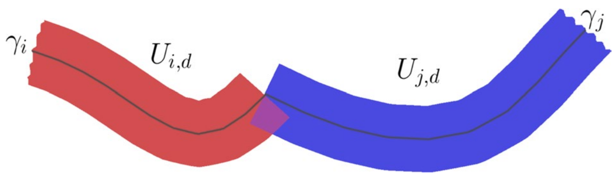

Fig. 4 Two adjacent sets $U_{i,d}$, $U_{j,d}$

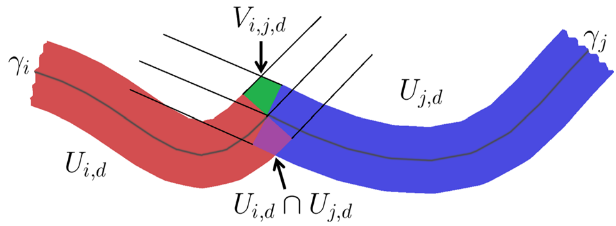

Fig. 5 Two adjacent sets $U_{i,d}$, $U_{j,d}$

To do so, for any $i \in \{1, \dots, N\}$, we introduce the extended curve $\boldsymbol{\eta}_i : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\boldsymbol{\eta}_i(t) = \begin{cases} \boldsymbol{\gamma}_i(t) & \text{if } t \in [a_i, b_i], \\ \boldsymbol{\gamma}_i(a_i) + (t - a_i)\, \boldsymbol{\gamma}_i'(a_i) & \text{if } t < a_i, \\ \boldsymbol{\gamma}_i(b_i) + (t - b_i)\, \boldsymbol{\gamma}_i'(b_i) & \text{if } t > b_i, \end{cases} \tag{1.12}$$

and the corresponding tubular neighborhood defined by

$$V_{i,d} := \{\boldsymbol{\eta}_i(t) + s\mathrm{n}(t) : t \in \mathbb{R}, s \in (-d/2, d/2)\}.$$

Notice that the curve $\boldsymbol{\eta}_i \in C^1(\mathbb{R}; \mathbb{R}^2)$ and it is piecewise C^2 with $\boldsymbol{\eta}_i'' \in L^\infty(\mathbb{R}; \mathbb{R}^2)$.

Then, if $U_{i,d}$ and $U_{j,d}$ are two adjacent sets, i.e. the intersection between $\boldsymbol{\gamma}_i([a_i, b_i])$ and $\boldsymbol{\gamma}_j([a_j, b_j])$ is an extreme point for both curves (see Fig. 5), we define $V_{i,j,d}$ as the connected component of the set $\big(V_{i,d} \backslash (U_{i,d} \cup U_{j,d})\big) \cap \big(V_{j,d} \backslash (U_{i,d} \cup U_{j,d})\big)$ which is in between $U_{i,d}$ and $U_{j,d}$, see the green set in Fig. 5. Note that the extended tubular neighborhoods $V_{i,d}$ and $V_{j,d}$ may have a non empty intersection on the other side with respect to the unique point of intersection between $\boldsymbol{\gamma}_i([a_i, b_i])$ and $\boldsymbol{\gamma}_j([a_j, b_j])$, see (1.9).

We define $\mathcal{I}$ as the set of all the couples (i, j) of indices such that $U_{i,d}$ and $U_{j,d}$ are adjacent sets in the sense explained above.

Our domain Ω_d will then be defined by

$$\Omega_d := \left(\bigcup_{i=1}^{N} U_{i,d} \right) \cup \left(\bigcup_{(i,j) \in \mathcal{I}} V_{i,j,d} \right). \tag{1.13}$$

Theorem 1.2 *Assume the validity of (1.7)-(1.10) and let Ω_d be the domain defined in (1.13). Then for the torsional rigidity K_d the following asymptotic estimate holds true*

$$K_d \sim \sum_{i=1}^{N} \frac{1}{3} \operatorname{length}(\boldsymbol{\gamma}_i)\, d^3 \qquad \text{as } d \to 0^+ .$$

Remark 1.3 One may wonder why in Theorem 1.2 we required a C^3-regularity for the curves $\boldsymbol{\gamma}_i$ and not only a C^2-regularity, the last one appearing more natural. The reason is that, in some sense, we want to make the curves straight as in the case of Theorem 1.1. In doing this, for each curve $\boldsymbol{\gamma}_i$, we have to construct a specific diffeomorphism of class C^2 in such a way that the other curves $\boldsymbol{\gamma}_j$ still remain at least of class C^2 after the deformation. As one can see in more detail in the proof of Theorem 1.2, the construction of the diffeomorphism makes use of the unit normal vector n defined in (1.11), which clearly depends on first order derivatives of the two components of $\boldsymbol{\gamma}_i$. For this reason, the necessity of having a C^2 regularity for the diffeomorphism requires a C^3 regularity on $\boldsymbol{\gamma}_i$.

The second remark that could be made deals with the condition (1.10) in which it is assumed that two different curves do not have a tangential intersection. This prevents the possibility of having a too large superposition of the two corresponding tubular neighborhoods and allows us to estimate the contribution to the torsional rigidity of such a superposition. As partially anticipated at the beginning of this introduction, our second purpose is to study the behavior of Makai and Pólya type functionals when applied to nonconvex thinning domains. See (1.14) below for their precise definition. The study of these functionals led the authors of the recent paper [1] to consider convex thinning domains to prove optimality results in inequalities (1.15). As a byproduct of the results in [1, Theorem 1.2, Theorem 1.4], we can recover the asymptotic estimate in Theorem 1.2 in the special case where Ω_d is convex. However, the only convex domain that satisfies the assumptions of Theorem 1.2 is when Ω_d is a rectangle. In other words, to our knowledge, the estimate of Theorem 1.2 is completely new for the class of domains that satisfy (1.7)-(1.10), which clearly also contains nonconvex domains.

Recalling the notation used in [1], we introduce the following functionals

$$T(\Omega) := \max_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\left(\int_\Omega \varphi\, dxdy\right)^2}{\int_\Omega |\nabla \varphi|^2 dxdy}, \quad \mathcal{P}(\Omega) := \frac{T(\Omega)\,(\mathrm{Per}(\Omega))^2}{|\Omega|^3}, \quad \mathcal{M}(\Omega) := \frac{T(\Omega)}{R_\Omega^2\,|\Omega|} \tag{1.14}$$

see also [16, 22]. We denoted by $\mathrm{Per}(\Omega)$ the perimeter of Ω, by $|\Omega|$ its area, and by R_Ω its inner radius defined as $R_\Omega := \sup\{r > 0 : B_r(P) \subset \Omega, P \in \Omega\}$. The functional $\mathcal{P}$ is known as *Pólya torsion functional* and $\mathcal{M}$ as *Makai functional*. We observe that for our domains Ω_d we have $T(\Omega_d) = K_d/4$ and $R_{\Omega_d} = d/2$.

It is well known that by combining the results of [16, 22, 23] we have the validity of the following sharp estimates for the functionals $\mathcal{P}$ and $\mathcal{M}$:

$$\frac{1}{3} \le \mathcal{P}(\Omega) \le \frac{2}{3}, \qquad \frac{1}{8} \le \mathcal{M}(\Omega) \le \frac{1}{3}, \tag{1.15}$$

valid for every bounded convex domain $\Omega \subset \mathbb{R}^2$. We recall that for $\mathcal{P}$, the lower bound is asymptotically achieved by a sequence of thinning rectangles and the upper bound by a sequence of thinning triangles, while for the functional $\mathcal{M}$, the lower bound is achieved if and only if Ω is a circle and the upper bound is asymptotically achieved by a sequence of thinning rectangles.

One may ask what happens to the functionals $\mathcal{P}$ and $\mathcal{M}$ when they are extended to the class of bounded nonconvex planar domains.

When considering thinning domains, we can show, by exploiting the result of Theorem 1.2, that the domains satisfying the assumptions (1.7)-(1.10), roughly speaking, behave like thinning rectangles in the sense stated in the following corollary.

Corollary 1.4 *Suppose that all the assumptions of Theorem 1.2 hold true and let $\mathcal{P}$ and $\mathcal{R}$ be the Pólya torsion functional and the Makai functional defined in (1.14). Then we have*

$$\mathcal{P}(\Omega_d) \to \frac{1}{3}, \qquad \mathcal{M}(\Omega_d) \to \frac{1}{3}, \qquad as\ d \to 0^+ .$$

Concluding remarks and open questions.

We conclude the introduction with a couple of open questions that could be worthy of future investigation.

- We wonder if the inequalities (1.15) still hold true on nonconvex simply connected planar domains. A possible way to investigate on this question could be to test the functionals $\mathcal{P}$ and $\mathcal{M}$ on domains of the type

$$A_d := \left\{(\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2 : \rho \in (1, 1+d)\,,\ \theta \in (0,\alpha)\right\}$$

  with $\alpha \in (0, 2\pi)$, where some explicit computations could be performed. The precise answer to this question has not been contemplated in the present paper and it could be addressed in future works.
- Another problem that could be investigated in future works is the study of torsional rigidity in the case of thinning domains with holes. In such cases, the boundary of the cross section Ω_d is no longer connected so that, as we have already observed, the quantity K_d is still given by (1.4) but this time u satisfies (1.3) and not (1.2).
  This drastically changes the behavior of K_d as $d \to 0^+$. Indeed, it is well known that for thin-walled shafts the torsional rigidity goes to zero much slower as $d \to 0^+$, thus suggesting, as expected, that similar cross sections confer a larger rigidity to the structure. Quoting the book [7] about *Mechanics of Materials*, one can recover the following empirical formula for K_d in the case of thinning domains with a single hole (see (3.51), p. 214 in [7]):

$$K_d \approx \frac{4\mathcal{A}^2}{\oint_\gamma \frac{1}{d}\, d\ell} = \frac{4\mathcal{A}^2 d}{\mathrm{length}(\gamma)}\,, \tag{1.16}$$

  where $\mathcal{A}$ denotes the area of the region bounded by the center line γ of the wall cross section, d is the thickness of the wall and $d\ell$ is the arc length differential, see Fig. 6. In our setting, we may assume that d is constant along the center line γ so that the second identity in (1.16) follows.
  We observe from (1.16) that K_d is an infinitesimal quantity of the first order with respect to d, while it was of the third order for the domains considered in Theorems 1.1-1.2.
  We point out that (1.16) is widely used in the field of Mechanics of Materials and one can also find empirical justifications for its validity, see again [7], but they cannot be considered as rigorous proofs in the sense of Mathematical Analysis. It can be considered an open problem to provide a rigorous proof of (1.16).
  Now, assuming the validity of (1.16), one may ask which could be the consequences of it when dealing with the Pólya and Makai functionals. For domains Ω_d as in Fig. 6 we would have

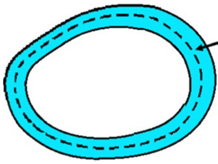

Fig. 6 An example of thin walled shaft

$$T(\Omega_d) \sim \frac{\mathcal{A}^2\, d}{\mathrm{length}(\gamma)}, \quad \mathrm{Per}(\Omega_d) \sim 2\mathrm{length}(\gamma), \quad |\Omega_d| \sim \mathrm{length}(\gamma) d, \quad \text{as } d \to 0^+,$$

from which we obtain

$$\mathcal{P}(\Omega_d) \sim \frac{4\mathcal{A}^2}{(\mathrm{length}(\gamma))^2} d^{-2}, \quad \mathcal{M}(\Omega_d) \sim \frac{4\mathcal{A}^2}{(\mathrm{length}(\gamma))^2} d^{-2}, \quad \text{as } d \to 0^+.$$

We see that both functionals would diverge to infinity like d^{-2}, thus showing that the upper estimates contained in (1.15) are no longer true in the class of domains with holes.

The paper is organized in the following way. We first give in Sect. 2 the proof of Theorem 1.1 which is actually a particular case of Theorem 1.2 but it shows, as already pointed out, which are the main ideas in the proofs, like the use of specific Poincaré-type inequalities and scaling arguments. Then in Sect. 3 we give the proof of Theorem 1.2 which is clearly more involved since suitable deformations of the domain have to be introduced to straighten the different subdomains of which the whole domain is composed. Also in Sect. 1.2, the crucial part is to obtain a very specific Poincaré-type inequality in which the dependence on the parameter *d* is well emphasized, see Proposition 3.7 for more details. The final part of Sect. 1.2 is devoted to the end of the proof of Theorem 1.2 and to the proof of Corollary 1.4.

2 Proof of theorem 1.1

We start by stating the following Poincaré-type inequality.

Proposition 2.1 *Let us consider the open rectangle $R = (a_1, b_1) \times (a_2, b_2)$ and denote by*

$$\Gamma_{x,1} := (a_1, b_1) \times \{a_2\}, \quad \Gamma_{x,2} := (a_1, b_1) \times \{b_2\}, \quad \Gamma_{y,1} := \{a_1\} \times (a_2, b_2), \quad \Gamma_{y,2} := \{b_1\} \times (a_2, b_2),$$

the four edges of the rectangle R, where the indices x and y stand for horizontal edge and vertical edge respectively.

For $i \in \{1, 2\}$, let us denote by $C^\infty_{0,x,i}(\overline{R})$ and $C^\infty_{0,y,i}(\overline{R})$ the space of all $C^\infty(\overline{R})$-functions vanishing on $\Gamma_{x,i}$ and $\Gamma_{y,i}$, respectively. Then, for $i \in \{1, 2\}$, the following inequalities hold true

$$\int_R |v|^2\, dxdy \le (b_2 - a_2)^2 \int_R \left|\frac{\partial v}{\partial y}\right|^2 dxdy \qquad for\ any\ v \in C^\infty_{0,x,i}(\overline{R}), \tag{2.1}$$

$$\int_R |v|^2\, dxdy \leq (b_1 - a_1)^2 \int_R \left|\frac{\partial v}{\partial x}\right|^2 dxdy \qquad for\ any\ v \in C^\infty_{0,y,i}(\overline{R})\,. \tag{2.2}$$

In particular, the maps $v \mapsto \left(\int_R \left|\frac{\partial v}{\partial y}\right|^2 dxdy\right)^{\frac{1}{2}}$ *and* $v \mapsto \left(\int_R \left|\frac{\partial v}{\partial x}\right|^2 dxdy\right)^{\frac{1}{2}}$ *are norms on the spaces* $C^\infty_{0,x,i}(\overline{R})$ *and* $C^\infty_{0,y,i}(\overline{R})$ *respectively, so that we may define the Hilbert spaces* $H_{0,x,i}(R) \subset L^2(R)$ *and* $H_{0,y,i}(R) \subset L^2(R)$ *as the completion of* $C^\infty_{0,x,i}(\overline{R})$ *and* $C^\infty_{0,y,i}(\overline{R})$ *with respect to the corresponding norms.*

As a consequence, the two inequalities in (2.1)-(2.2) hold true for any $v \in H_{0,x,i}(R)$ *and for any* $v \in H_{0,y,i}(R)$ *respectively, for* $i \in \{1, 2\}$ *in both cases.*

Proof Let us prove (2.1) being the proof of (2.2) completely equivalent.

For any function $v \in C^\infty_{0,x,i}(\overline{R})$ we have $v(x, a_2) = 0$ if $i = 1$ and $v(x, b_2) = 0$ if $i = 2$. However, in both cases, by the Hölder inequality, we have

$$|v(x, y)|^2 \leq (b_2 - a_2) \int_{a_2}^{b_2} \left|\frac{\partial v}{\partial y}(x, t)\right|^2 dt\,.$$

Integrating the last estimate first with respect to *x* and then with respect to *y*, we obtain the desired inequality. □

2.1 Asymptotic estimate of $K_{1,d}$

In this subsection, we define $\widetilde{\Omega}_d := \{(x, y) \in \mathbb{R}^2 : (x, d\,y) \in \Omega_d\}$ and $\widetilde{u}_d(x, y) = \frac{1}{d^2}\, u_d(x, dy)$. Then by (1.2), the function $\widetilde{u}_d$ solves the boundary value problem

$$\begin{cases} -\left(\frac{\partial^2 \widetilde{u}_d}{\partial x^2} + \frac{1}{d^2}\,\frac{\partial^2 \widetilde{u}_d}{\partial y^2}\right) = \frac{1}{d^2} & \text{in } \widetilde{\Omega}_d\,, \\ \widetilde{u}_d = 0 & \text{on } \partial\widetilde{\Omega}_d \end{cases} \tag{2.3}$$

that admits the following variational formulation

$$d^2 \int_{\widetilde{\Omega}_d} \frac{\partial \widetilde{u}_d}{\partial x}\,\frac{\partial v}{\partial x}\, dxy + \int_{\widetilde{\Omega}_d} \frac{\partial \widetilde{u}_d}{\partial y}\,\frac{\partial v}{\partial y}\, dxdy = \int_{\widetilde{\Omega}_d} v\, dxdy \qquad \text{for any } v \in H^1_0(\widetilde{\Omega}_d)\,. \tag{2.4}$$

Recalling the definitions of $U_{1,d}$, $U_{2,d}$, $U_{3,d}$ given in (1.5)-(1.6), we define here the three disjoint rectangles by $S_{1,d} := U_{1,d} \backslash (\overline{U}_{2,d} \cup \overline{U}_{3,d})$, $S_{2,d} := U_{2,d}$ and $S_{3,d} := U_{3,d}$. For each of those sets, we define the corresponding rescaled sets

$$\widetilde{U}_{1,d} := T_{y,d}(U_{1,d})\,, \quad \widetilde{S}_{1,d} := T_{y,d}(S_{1,d})\,, \quad \widetilde{U}_{2,d} := \widetilde{S}_{2,d} := T_{y,d}(S_{2,d})\,, \quad \widetilde{U}_{3,d} := \widetilde{S}_{3,d} := T_{y,d}(S_{3,d})\,, \tag{2.5}$$

where the operator $T_{y,d}$ is given by $T_{y,d}(x, y) := (x, d^{-1}\, y)$. We observe that the set $\widetilde{U}_{1,d}$ is actually independent of *d* and coincides with $\widetilde{U}_{1,0} := (-\ell_1, \ell_1) \times (-1/2, 1/2)$.

Let us prove in the next lemma that $\|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)}$ remains bounded as $d \to 0^+$.

Lemma 2.2 *Let $\widetilde{u}_d$ be as in (2.3). Then $\|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)}$ remains bounded as $d \to 0^+$. Moreover, we also find that $\left\|\frac{\partial \widetilde{u}_d}{\partial y}\right\|_{L^2(\widetilde{\Omega}_d)}$ and $\left\|d\,\frac{\partial \widetilde{u}_d}{\partial x}\right\|_{L^2(\widetilde{\Omega}_d)}$ remain bounded as $d \to 0^+$.*

Proof Suppose by contradiction that along a sequence $d_n \downarrow 0$ we have $M_n := \|\widetilde{u}_{d_n}\|_{L^2(\widetilde{\Omega}_{d_n})} \to +\infty$. Let us define $w_n := \frac{\widetilde{u}_{d_n}}{M_n}$ so that $\|w_n\|_{L^2(\widetilde{\Omega}_{d_n})} = 1$ and

$$\int_{\widetilde{\Omega}_{d_n}} \left|\frac{\partial w_n}{\partial x}\right|^2 dxdy + \frac{1}{d_n^2}\int_{\widetilde{\Omega}_{d_n}} \left|\frac{\partial w_n}{\partial y}\right|^2 dxdy = \frac{1}{d_n^2 M_n}\int_{\widetilde{\Omega}_{d_n}} w_n\, dxdy\,. \tag{2.6}$$

In particular, by the Hölder inequality, we obtain

$$\int_{\widetilde{\Omega}_{d_n}} \left|\frac{\partial w_n}{\partial y}\right|^2 dxdy \le \frac{|\widetilde{\Omega}_{d_n}|^{\frac12}}{M_n}\,\|w_n\|_{L^2(\widetilde{\Omega}_{d_n})} = \frac{(L_1+2L_2-d_n)^{\frac12}}{M_n} \to 0 \quad \text{as } n \to +\infty\,.$$

We observe that $w_n \in H_{0,x,1}(\widetilde{S}_{1,d_n}) \cap H_{0,x,2}(\widetilde{S}_{1,d_n})$, being a function of $H_0^1(\widetilde{\Omega}_{d_n})$, so that by (2.1) we have

$$\|w_n\|_{L^2(\widetilde{S}_{1,d_n})} \le \left(\int_{\widetilde{S}_{1,d_n}} \left|\frac{\partial w_n}{\partial y}\right|^2 dxdy\right)^{\frac12} \le \left(\int_{\widetilde{\Omega}_{d_n}} \left|\frac{\partial w_n}{\partial y}\right|^2 dxdy\right)^{\frac12} \to 0 \quad \text{as } n \to +\infty\,. \tag{2.7}$$

On the other hand, by (2.6) and the Hölder inequality, we also deduce that

$$d_n^2\int_{\widetilde{\Omega}_{d_n}} \left|\frac{\partial w_n}{\partial x}\right|^2 dxdy \le \frac{|\widetilde{\Omega}_{d_n}|^{\frac12}}{M_n}\,\|w_n\|_{L^2(\widetilde{\Omega}_{d_n})} = \frac{(L_1+2L_2-d_n)^{\frac12}}{M_n} \to 0 \quad \text{as } n \to +\infty\,.$$

We observe that $w_n \in H_{0,y,2}(\widetilde{S}_{2,d_n}) \cap H_{0,y,1}(\widetilde{S}_{3,d_n})$ being a function in $H_0^1(\widetilde{\Omega}_{d_n})$, so that by (2.2), we have for $i = 2, 3$

$$\|w_n\|_{L^2(\widetilde{S}_{i,d_n})} \le d_n\left(\int_{\widetilde{S}_{i,d_n}} \left|\frac{\partial w_n}{\partial x}\right|^2 dxdy\right)^{\frac12} \le \left(d_n^2\int_{\widetilde{\Omega}_{d_n}} \left|\frac{\partial w_n}{\partial x}\right|^2 dxdy\right)^{\frac12} \to 0 \quad \text{as } n \to +\infty\,. \tag{2.8}$$

Combining (2.7) and (2.8) we conclude that $\|w_n\|_{L^2(\widetilde{\Omega}_{d_n})} \to 0$ as $n \to +\infty$, contradicting the fact that $\|w_n\|_{L^2(\widetilde{\Omega}_{d_n})} = 1$. This completes the proof of the fact that $\|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)}$ remains bounded as $d \to 0^+$. The remaining part of the proof follows from the estimate

$$d^2\int_{\widetilde{\Omega}_d} \left|\frac{\partial \widetilde{u}_d}{\partial x}\right|^2 dxdy + \int_{\widetilde{\Omega}_d} \left|\frac{\partial \widetilde{u}_d}{\partial y}\right|^2 dxdy = \int_{\widetilde{\Omega}_d} \widetilde{u}_d\, dxdy \le (L_1+2L_2-d)^{\frac12}\,\|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)}\,.$$

This completes the proof of the lemma. □

In the next two lemmas we prove two convergence results for the sequence $\{\widetilde{u}_d\}$ as $d \to 0^+$ and for the L^2-norms of its derivatives.

Lemma 2.3 *Let $\widetilde{u}_d \in H_0^1(\widetilde{\Omega}_d)$ be as in (2.3). Then, $\widetilde{u}_d \rightharpoonup w$ and $\frac{\partial \widetilde{u}_d}{\partial y} \rightharpoonup \frac{\partial w}{\partial y}$ weakly in $L^2(\widetilde{U}_{1,0})$ as $d \to 0^+$ for some function $w \in L^2(\widetilde{U}_{1,0})$ with $\frac{\partial w}{\partial y} \in L^2(\widetilde{U}_{1,0})$. Moreover, w is independent of the variable x and is given by $w(x,y) = -\frac{1}{2}\left(y^2 - \frac{1}{4}\right)$.*

Proof By Lemma 2.2 along a sequence $d_n \downarrow 0$, we deduce that there exist two functions $w, \phi \in L^2(\widetilde{U}_{1,0})$ such that

$$\widetilde{u}_{d_n} \rightharpoonup w \quad \text{and} \quad \frac{\partial \widetilde{u}_{d_n}}{\partial y} \rightharpoonup \phi \quad \text{weakly in } L^2(\widetilde{U}_{1,0}) \quad \text{as } n \to +\infty. \tag{2.9}$$

On the other hand, for any $\overline{d} > 0$ small enough, we see that $\widetilde{u}_{d_n} \rightharpoonup w$ in both spaces $H_{0,x,1}(\widetilde{S}_{1,\overline{d}})$ and $H_{0,x,2}(\widetilde{S}_{1,\overline{d}})$.

Now, let *v* be a test function in $C_c^\infty(\widetilde{U}_{1,0}) \subset H_0^1(\widetilde{\Omega}_d)$, so that

$$d \int_{\widetilde{U}_{1,0}} d\, \frac{\partial \widetilde{u}_d}{\partial x} \frac{\partial v}{\partial x}\, dxdy + \int_{\widetilde{U}_{1,0}} \frac{\partial \widetilde{u}_d}{\partial y} \frac{\partial v}{\partial y}\, dxdy = \int_{\widetilde{U}_{1,0}} v\, dxdy\,. \tag{2.10}$$

Then, passing to the limit in (2.10), by Lemma 3.8, we obtain

$$\int_{\widetilde{U}_{1,0}} \frac{\partial w}{\partial y} \frac{\partial v}{\partial y}\, dxdy = \int_{\widetilde{U}_{1,0}} v\, dxdy \qquad \text{for any } v \in C_c^\infty(\widetilde{U}_{1,0})$$

which means that $\frac{\partial^2 w}{\partial y^2} = -1$ in the sense of distributions in $\widetilde{U}_{1,0}$.

It is a standard fact that a function with a zero partial distributional derivative with respect to some variable is necessarily constant in that variable. In particular, if the distributional derivative is -1, as in the case of $\frac{\partial w}{\partial y}$, then we have $\frac{\partial w}{\partial y}(x,y) = -y + a(x)$ for some function $a \in L^2(-\ell_1, \ell_1)$.

Recalling that $w \in H_{0,x,1}(\widetilde{S}_{1,\overline{d}})$, we infer the existence of a sequence $\{w_n\} \subset C^\infty_{0,x,1}\left(\overline{\widetilde{S}_{1,\overline{d}}}\right)$ such that $w_n \to w$ in $L^2(\widetilde{S}_{1,\overline{d}})$ and $\frac{\partial w_n}{\partial y} \to \frac{\partial w}{\partial y}$ in $L^2(\widetilde{S}_{1,\overline{d}})$. In particular, we have that

$$w_n(x,y) = \int_{-\frac{1}{2}}^{y} \frac{\partial w_n}{\partial y}(x,t)\, dt \xrightarrow{L^2(\widetilde{S}_{1,\overline{d}})} \int_{-\frac{1}{2}}^{y} \frac{\partial w}{\partial y}(x,t)\, dt = -\frac{1}{2}\left(y^2 - \frac{1}{4}\right) + a(x)\left(y + \frac{1}{2}\right)$$

which shows that $w(x,y) = -\frac{1}{2}\left(y^2 - \frac{1}{4}\right) + a(x)\left(y + \frac{1}{2}\right)$. On the other hand, $w \in H_{0,x,2}(\widetilde{S}_{1,\overline{d}})$ and with a similar procedure we infer $w(x,y) = -\frac{1}{2}\left(y^2 - \frac{1}{4}\right) + a(x)\left(y - \frac{1}{2}\right)$.

Comparing the two expressions of *w* we conclude that $a \equiv 0$ and hence $w(x,y) = -\frac{1}{2}\left(y^2 - \frac{1}{4}\right)$.

The weak convergence $\widetilde{u}_{d_n} \rightharpoonup w$ in the two spaces $H_{0,x,1}(\widetilde{S}_{1,\overline{d}})$, $H_{0,x,2}(\widetilde{S}_{1,\overline{d}})$ then implies that $\widetilde{u}_{d_n} \rightharpoonup w$ and $\frac{\partial \widetilde{u}_{d_n}}{\partial y} \rightharpoonup \frac{\partial w}{\partial y}$ in $L^2(\widetilde{S}_{1,\overline{d}})$ for any $\overline{d} > 0$ small. In particular, this shows that $\phi = \frac{\partial w}{\partial y}$.

We have shown that the weak limits in (2.9) do not depend on the sequence $d_n \downarrow 0$, thus completing the proof of the lemma. $\square$

Lemma 2.4 *Let $\widetilde{u}_d \in H_0^1(\widetilde{\Omega}_d)$ be as in (2.3) and $\widetilde{U}_{1,d}, \widetilde{S}_{1,d}$ be as in (2.5). Then*

$$\lim_{d\to 0^+}\left[d^2\int_{\widetilde{U}_{1,0}}\left|\frac{\partial \widetilde{u}_d}{\partial x}\right|^2 dxdy+\int_{\widetilde{U}_{1,0}}\left|\frac{\partial \widetilde{u}_d}{\partial y}\right|^2 dxdy\right]=\frac{L_1}{12}\,, \tag{2.11}$$

and

$$\lim_{d\to 0^+}\left[d^2\int_{\widetilde{U}_{1,0}\setminus \widetilde{S}_{1,d}}\left|\frac{\partial \widetilde{u}_d}{\partial x}\right|^2 dxdy+\int_{\widetilde{U}_{1,0}\setminus \widetilde{S}_{1,d}}\left|\frac{\partial \widetilde{u}_d}{\partial y}\right|^2 dxdy\right]=0\,. \tag{2.12}$$

Proof By direct computation, we find that w satisfies the identity

$$\int_{\widetilde{U}_{1,0}}\left|\frac{\partial w}{\partial y}\right|^2 dxdy=\int_{\widetilde{U}_{1,0}} w\,dxdy\,.$$

Let $\Sigma_d:=\widetilde{\Omega}_d\cap\left(\mathbb{R}\times\left(-\frac{1}{\sqrt{d}},\frac{1}{\sqrt{d}}\right)\right)$, let $\eta\in C_c^\infty(\mathbb{R})$ be a cut-off function satisfying $0<\eta\le 1$ in $(-1,1)$, $\eta\equiv 1$ in $\left(-\frac{1}{2},\frac{1}{2}\right)$, $\eta\equiv 0$ outside $(-1,1)$, and let $\eta_d(y):=\eta(\sqrt{d}\,y)$.

Then, by Lemma 2.2, Lemma 2.3, weak lower semicontinuity of the L^2-norm, (2.4) with $v=\eta_d\,\widetilde{u}_d$ and Hölder inequality, we obtain

$$\begin{aligned}
\int_{\widetilde{U}_{1,0}} w\,dxdy&=\int_{\widetilde{U}_{1,0}}\left|\frac{\partial w}{\partial y}\right|^2 dxdy\le \liminf_{d\to 0^+}\int_{\widetilde{U}_{1,0}}\left|\frac{\partial \widetilde{u}_d}{\partial y}\right|^2 dxdy\le \limsup_{d\to 0^+}\int_{\widetilde{U}_{1,0}}\left|\frac{\partial \widetilde{u}_d}{\partial y}\right|^2 dxdy\\
&\le \limsup_{d\to 0^+}\left[d^2\int_{\Sigma_d}\eta_d(y)\left|\frac{\partial \widetilde{u}_d}{\partial x}\right|^2 dxdy+\int_{\Sigma_d}\eta_d(y)\left|\frac{\partial \widetilde{u}_d}{\partial y}\right|^2 dxdy\right]\\
&=\limsup_{d\to 0^+}\left[\int_{\Sigma_d}\eta_d(y)\widetilde{u}_d\,dxdy-\int_{\Sigma_d}\eta_d'(y)\frac{\partial \widetilde{u}_d}{\partial y}\,\widetilde{u}_d\,dxdy\right]\\
&\le \limsup_{d\to 0^+}\left[\int_{\widetilde{U}_{1,0}}\widetilde{u}_d\,dxdy+\int_{\Sigma_d\setminus\widetilde{U}_{1,0}}\eta_d(y)\widetilde{u}_d\,dxdy+\sqrt{d}\,\|\eta'\|_{L^\infty(\mathbb{R})}\left\|\frac{\partial \widetilde{u}_d}{\partial y}\right\|_{L^2(\widetilde{\Omega}_d)}\|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)}\right]\\
&\le \limsup_{d\to 0^+}\left[\int_{\widetilde{U}_{1,0}}\widetilde{u}_d\,dxdy+|\Sigma_d\setminus\widetilde{U}_{1,0}|^{\frac12}\,\|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)}+O\left(\sqrt{d}\right)\right]\\
&=\lim_{d\to 0^+}\left[\int_{\widetilde{U}_{1,0}}\widetilde{u}_d\,dxdy+O\left(\sqrt[4]{d}\right)\right]=\int_{\widetilde{U}_{1,0}} w\,dxdy\,.
\end{aligned}$$

We observe that the first and the last terms in the previous chain of inequalities coincide, thus showing that

$$\lim_{d\to 0^+} d^2\int_{\widetilde{U}_{1,0}}\left|\frac{\partial \widetilde{u}_d}{\partial x}\right|^2 dxdy=0\,,\quad \lim_{d\to 0^+}\int_{\widetilde{U}_{1,0}}\left|\frac{\partial \widetilde{u}_d}{\partial y}\right|^2 dxdy=\int_{\widetilde{U}_{1,0}}\left|\frac{\partial w}{\partial y}\right|^2 dxdy\,, \tag{2.13}$$

which, combined with Lemma 2.3, shows that

$$\frac{\partial \widetilde{u}_d}{\partial y} \to \frac{\partial w}{\partial y} \quad \text{strongly in } L^2(\widetilde{U}_{1,0}). \tag{2.14}$$

The proof of (2.11) follows immediately by (2.13) and the explicit expression of the function w obtained in Lemma 2.3.

The proof of (2.12) follows by (2.13), (2.14) and by observing that $\widetilde{S}_{1,d} \subset \widetilde{U}_{1,0}$, $|\widetilde{U}_{1,0} \setminus \widetilde{S}_{1,d}| \to 0$. This completes the proof of the lemma. □

As an immediate consequence of Lemma 2.4 we obtain the following asymptotic estimate for $K_{1,d}$:

$$K_{1,d} = 4\int_{U_{1,d}} |\nabla u_d|^2\, dxdy = 4d^3 \left[d^2 \int_{\widetilde{U}_{1,0}} \left|\frac{\partial \widetilde{u}_d}{\partial x}\right|^2 dxdy + \int_{\widetilde{U}_{1,0}} \left|\frac{\partial \widetilde{u}_d}{\partial y}\right|^2 dxdy \right] \sim \frac{1}{3}\, L_1 d^3$$

as $d \to 0^+$ which combined with (2.12) also gives

$$4\int_{S_{1,d}} |\nabla u_d|^2\, dxdy = 4d^3 \left[d^2 \int_{\widetilde{S}_{1,d}} \left|\frac{\partial \widetilde{u}_d}{\partial x}\right|^2 dxdy + \int_{\widetilde{S}_{1,d}} \left|\frac{\partial \widetilde{u}_d}{\partial y}\right|^2 dxdy \right] = \frac{1}{3}\, L_1 d^3 + o(d^3)\,. \tag{2.15}$$

2.2 Asymptotic estimate of $K_{2,d} = K_{3,d}$

Similarly to Subsect. 2.1, we define a family of rescaled domains adapted this time to provide an asymptotic estimate for the quantities $K_{2,d}$ and $K_{3,d}$.

Let $T_{y,d}(x, y) := (x, d^{-1}\, y)$ be as in Subsect. 2.1. We define by T_d the transformation obtained by rotating of ninety degrees counterclockwise, translating of ℓ_1 downwards and applying the rescaling $T_{y,d}$.

In this way, the domain $T_d(\Omega_d)$, which will be denoted in this subsection by $\widetilde{\Omega}_d$, up to a set of zero measure, will be the union of the three rectangles $\widetilde{S}_{i,d} := T_d(S_{i,d})$, $i = 1, 2, 3$. Those three rectangles take the form

$$\begin{aligned} &\widetilde{S}_{1,d} = \left(-\tfrac{d}{2}, \tfrac{d}{2}\right) \times \left(-\tfrac{2\ell_1}{d} + \tfrac{1}{2}, -\tfrac{1}{2}\right)\,, \\ &\widetilde{S}_{2,d} = (-\ell_2, \ell_2) \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)\,, \qquad \widetilde{S}_{3,d} = (-\ell_2, \ell_2) \times \left(-\tfrac{2\ell_1}{d} - \tfrac{1}{2}, -\tfrac{2\ell_1}{d} + \tfrac{1}{2}\right)\,. \end{aligned} \tag{2.16}$$

As one can see from (2.16), the transformation T_d has the advantage of making independent of d the rectangles $\widetilde{S}_{2,d}$, which for this reason will be simply denoted by $\widetilde{S}_{2,0}$.

Similarly to Subsect. 2.1, we define a rescaled function $\widetilde{u}_d(x, y) := u_d(T_d^{-1}(x, y))$ for any $(x, y) \in \widetilde{\Omega}_d$. Then $\widetilde{u}_d$ solves the problem

$$\begin{cases} -\left(\frac{\partial^2 \widetilde{u}_d}{\partial x^2} + \frac{1}{d^2}\, \frac{\partial^2 \widetilde{u}_d}{\partial y^2}\right) = \frac{1}{d^2} & \text{in } \widetilde{\Omega}_d\,, \\ \widetilde{u}_d = 0 & \text{on } \partial\widetilde{\Omega}_d\,. \end{cases}$$

The first step is to prove that $\|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)}$, $\left\|\frac{\partial \widetilde{u}_d}{\partial y}\right\|_{L^2(\widetilde{\Omega}_d)}$ and $\left\|d\,\frac{\partial \widetilde{u}_d}{\partial x}\right\|_{L^2(\widetilde{\Omega}_d)}$ remain bounded as $d \to 0^+$.

This can be proved by following, with the appropriate adaptations, the same argument used in the proof of Lemma 2.2 based on the Poincaré-type inequalities stated in Proposition 2.1.

We can now state the following lemma.

Lemma 2.5 *Let $\widetilde{u}_d \in H_0^1(\widetilde{\Omega}_d)$ be defined as in the present subsection. Then $\widetilde{u}_d \rightharpoonup w$ weakly in the Hilbert space $H_{0,x,2}(\widetilde{S}_{2,0})$ as $d \to 0^+$ where $w \in H_{0,x,1}(\widetilde{S}_{2,0}) \cap H_{0,x,2}(\widetilde{S}_{2,0})$ is independent of the variable x and it is given by $w(x,y) = -\frac{1}{2}\left(y^2 - \frac{1}{4}\right)$.*

Proof Proceeding as in the proof of Lemma 2.3 we infer that along a sequence $d_n \downarrow 0$, $\widetilde{u}_{d_n} \rightharpoonup w$ in $H_{0,x,2}(\widetilde{S}_{2,0})$ for some function *w* satisfying the equation $\frac{\partial^2 w}{\partial y^2} = -1$ in the sense of distributions. We cannot say directly that we also have $w \in H_{0,x,1}(\widetilde{S}_{2,0})$ since $\widetilde{u}_{d_n} \notin H_{0,x,1}(\widetilde{S}_{2,0})$ as one can see by looking at the edges definitions in Proposition 2.1 and at the decomposition of $\widetilde{\Omega}_d$ in (2.16). This is due to the fact that $\widetilde{u}_{d_n}$ is not necessarily zero on the small portion of $\partial\widetilde{S}_{2,0}$ where $\widetilde{S}_{2,0}$ is attached to $\widetilde{S}_{1,d}$. However, for any $0 < a < \ell_2$, when d_n has become sufficiently small, we have

$$\widetilde{u}_{d_n} \in H_{0,x,1}\left((a,\ell_2)\times\left(-\tfrac{1}{2},\tfrac{1}{2}\right)\right) \cap H_{0,x,1}\left((-\ell_2,-a)\times\left(-\tfrac{1}{2},\tfrac{1}{2}\right)\right) .$$

One can verify that *w* is also the weak limit of $\{\widetilde{u}_{d_n}\}$ in both Hilbert spaces $H_{0,x,1}\left((a,\ell_2)\times\left(-\frac{1}{2},\frac{1}{2}\right)\right)$ and $H_{0,x,1}\left((-\ell_2,-a)\times\left(-\frac{1}{2},\frac{1}{2}\right)\right)$.

Combining this with the fact that we already know that $w \in H_{0,x,2}(\widetilde{S}_{2,0})$, proceeding as in the proof of Lemma 2.3, we may conclude that $w(x,y) = -\frac{1}{2}(y^2 - \frac{1}{4})$ in $(a,\ell_2)\times\left(-\frac{1}{2},\frac{1}{2}\right)$ and in $(-\ell_2,-a)\times\left(-\frac{1}{2},\frac{1}{2}\right)$. Since *a* can be chosen arbitrarily small, the conclusion of the lemma then follows. □

Once we have proved Lemma 2.5 we can repeat the argument used in the proof of Lemma 2.4 and show that

$$\lim_{d\to 0^+}\left[d^2\int_{\widetilde{S}_{2,0}}\left|\frac{\partial \widetilde{u}_d}{\partial x}\right|^2 dxdy + \int_{\widetilde{S}_{2,0}}\left|\frac{\partial \widetilde{u}_d}{\partial y}\right|^2 dxdy\right] = \int_{\widetilde{S}_{2,0}}\left|\frac{\partial w}{\partial y}\right|^2 dxdy = \frac{L_2}{12}\,.$$

Finally, we have

$$K_{2,d} = 4\int_{S_{2,d}}|\nabla u_d|^2\,dxdy = 4d^3\left[d^2\int_{\widetilde{S}_{2,0}}\left|\frac{\partial \widetilde{u}_d}{\partial x}\right|^2 dxdy + \int_{\widetilde{S}_{2,0}}\left|\frac{\partial \widetilde{u}_d}{\partial y}\right|^2 dxdy\right] \sim \frac{1}{3}\,L_2\,d^3$$

as $d \to 0^+$.

By the symmetry of the solution u_d we easily deduce that $K_{3,d} = K_{2,d}$ and hence also the asymptotic estimate for $K_{3,d}$ is done.

End of the proof of Theorem 1.1. By the definition of K_d, (2.15) and Subsections 2.1-2.2, we may conclude that

$$K_d = 4\int_{S_{1,d}} |\nabla u_d|^2 dxdy + K_{2,d} + K_{3,d} = \frac{1}{3}\, L_1\, d^3 + \frac{2}{3}\, L_2\, d^3 + o(d^3)$$

and hence, the proof of the theorem is complete.

3 Proof of Theorem 1.2 and of Corollary 1.4

As a first step, we have to construct a family of C^2-diffeomorphisms with the purpose of straightening local portions of the curves γ_i and their corresponding tubular neighborhoods. The construction of these diffeomorphisms is performed in the next lemma.

Lemma 3.1 *Let $i \in \{1, \dots, N\}$ and let γ_i be the corresponding curve satisfying conditions (1.7)-(1.10). Then there exist $\overline{d}_i$, M_i, $C^2(\mathbb{R}^2;\mathbb{R}^2)$ global diffeomorphisms $\Phi_{i,1}, \dots, \Phi_{i,M_i}$, a subdivision $t_{i,0} = a_i < t_{i,1} < \cdots < t_{i,M_i-1} < t_{i,M_i} = b_i$ of the interval $[a_i, b_i]$, such that for any $d \in (0, \overline{d}_i/2)$*

$$\Phi_{i,m}^{-1}(t,s) = \gamma_i(t) + s\mathrm{n}(t) \qquad for\ any\ (t,s) \in I_{i,m} \times \left(-\tfrac{d}{2}, \tfrac{d}{2}\right) \qquad for\ any\ m \in \{1, \dots, M_i\},$$

where we put $I_{i,m} = (t_{i,m-1}, t_{i,m})$ for any $m \in \{1, \dots, M_i\}$. We recall that n denotes the unit normal vector to the curve γ_i according to the definition given in (1.11).

In particular, the sets defined by

$$Z_{i,m,d} = \Phi_{i,m}^{-1}\left(I_{i,m} \times \left(-\tfrac{d}{2}, \tfrac{d}{2}\right)\right), \qquad m \in \{1, \dots, M_i\}, \tag{3.1}$$

are disjoint sets contained in $U_{i,d}$ such that $U_{i,d} \setminus \bigcup_{m=1}^{M_i} Z_{i,m,d}$ has zero measure.

Moreover, the Jacobian matrices of $\Phi_{i,m}$ and $\Phi_{i,m}^{-1}$ and the Hessian matrices of their components have bounded coefficients in the whole $\mathbb{R}^2$.

Proof Since the index $i \in \{1, \dots, N\}$ remains fixed from the beginning, it will sometimes be omitted in the definitions appearing in the present proof, unless required. With the purpose of making the proof of this lemma easier to read, we divide it into several steps.

Step 1. In this first step, we show that the curve γ_i can be split into the graphs of M_i functions defined on compact intervals $I_{i,1}, \dots, I_{i,M_i}$. Each of these functions is then extended as a function defined on the whole real line. Moreover, for a sufficiently small $\overline{d}_i$, we construct local diffeomorphisms between the tubular neighborhoods of these extended graphs and a strip of the type $\mathbb{R} \times (-\overline{d}_i/2, \overline{d}_i/2)$.

By (1.7), we deduce that there exist M_i and $t_{i,0} = a_i < t_{i,1} < \cdots < t_{i,M_i-1} < t_{i,M_i} = b_i$ such that for any $m \in \{1, \dots, M_i\}$, the curve $\gamma_i([t_{i,m-1}, t_{i,m}])$ is the graph of a real function of one real variable. We put $I_{i,m} = (t_{i,m-1}, t_{i,m})$ for any $m \in \{1, \dots, M_i\}$ according to the notation present in the statement of the lemma.

In the remaining part of the proof we fix an arbitrary $m \in \{1, \dots, M_i\}$ so that we may omit the index *m* unless strictly necessary. First of all, we denote by *g* the function whose graph is given by $\gamma_i(\overline{I}_{i,m})$. Up to a rotation, we may assume that the graph is in the form $y = g(x)$ and that the unit normal vector n to $\gamma_i(\overline{I}_{i,m})$ always has a positive *y*-component. Being $g \in C^2(\overline{I}_{i,m})$, it admits a $C^2_c(\mathbb{R})$ extension that will still be denoted for simplicity by *g*. We point out that with $C^2_c(\mathbb{R})$ we mean the class of compactly supported $C^2(\mathbb{R})$ functions. At the same time, we can parameterize the entire graph of *g* by extending $\gamma_{i|\overline{I}_{i,m}}$ to the entire real line and denoting this new curve by $\boldsymbol{\eta} \in C^2(\mathbb{R}; \mathbb{R}^2)$. By a polynomial interpolation argument, we may choose the extended parametrization $\boldsymbol{\eta}$ in such a way that the derivative of the *x*-component of $\boldsymbol{\eta}$ becomes constant outside a compact set.

Resuming, writing $\boldsymbol{\eta}(t) = (\eta_1(t), \eta_2(t))$, we may assume that η_1' admits a positive minimum, $|\boldsymbol{\eta}'|$ and $|\kappa|$ admits a maximum where we denoted by $\kappa = \kappa(t)$ the curvature of $\boldsymbol{\eta}$:

$$\mu_1 := \min_{t \in \mathbb{R}} \eta_1'(t) > 0\,, \qquad \mu_2 := \max_{t \in \mathbb{R}} |\boldsymbol{\eta}'(t)| > 0\,, \qquad K := \max_{t \in \mathbb{R}} |\kappa(t)| \geq 0\,. \tag{3.2}$$

Let us define the map

$$\Psi : \mathbb{R}^2 \to \mathbb{R}^2\,, \qquad \Psi(t, s) = \boldsymbol{\eta}(t) + s\mathrm{n}(t) \quad \forall\, (t, s) \in \mathbb{R}^2\,.$$

Applying the Local Inversion Theorem and exploiting a compactness argument, it follows that there exists $\overline{d}_i$ such that the following restriction of Ψ, which for simplicity we still denote by Ψ,

$$\Psi : \mathbb{R} \times \left(-\tfrac{\overline{d}_i}{2}, \tfrac{\overline{d}_i}{2}\right) \to V_{i,\overline{d}_i}\,, \tag{3.3}$$

is a C^2-diffeomorphism, where we put

$$V_{i,\overline{d}_i} := \{\boldsymbol{\eta}(t) + s\mathrm{n}(t) : t \in \mathbb{R},\ s \in (-\overline{d}_i/2, \overline{d}_i/2)\}\,.$$

We show this as a consequence of the following two claims.

Claim 1. We first observe that by (1.7), the map Ψ is of class C^2 and $\det(J\Psi) \neq 0$ at any point (*t*, 0) with $t \in \mathbb{R}$. Then, by a compactness argument we can show that there exists $\overline{d}_i > 0$ such that $\det(J\Psi(t, s)) \neq 0$ for any $t \in \mathbb{R}$ and $s \in \left(-\frac{\overline{d}_i}{2}, \frac{\overline{d}_i}{2}\right)$. Indeed, suppose by contradiction that for any integer $n \geq 1$, we can find (t_n, s_n) with $s_n \in \left(-\frac{1}{n}, \frac{1}{n}\right)$ such that $\det(J\Psi(t_n, s_n)) = 0$. We observe that for *t* outside a sufficiently large compact set $[-R, R]$, $\det(J\Psi(t, s)) \equiv C_\pm$, for suitable positive constants C_+ and C_-, in the cases $t > R$ or $t < -R$ respectively, as one can deduce from the construction of the extended curve $\boldsymbol{\eta}$. For this reason, the sequence $\{t_n\}$ is bounded and hence we may suppose that, up to a subsequence, $t_n \to \bar{t}$ for some $\bar{t} \in \mathbb{R}$, so that by continuity $\det(J\Psi(\bar{t}, 0)) = 0$, a contradiction.

Claim 2. Now, we need to prove that, up to shrink $\overline{d}_i$ if necessary, the map Ψ is injective in $\mathbb{R} \times \left(-\frac{\overline{d}_i}{2}, \frac{\overline{d}_i}{2}\right)$. Suppose by contradiction that no such $\overline{d}_i$ can be found to make Ψ injective. Then for any integer $n \geq 1$ we can find $(t_n, s_n) \neq (\tau_n, \sigma_n)$ with $s_n, \sigma_n \in \left(-\frac{1}{n}, \frac{1}{n}\right)$, such that $\Psi(t_n, s_n) = \Psi(\tau_n, \sigma_n)$. It is easy to see $t_n, \tau_n \in [-R, R]$ since otherwise we would have $(t_n, s_n) = (\tau_n, \sigma_n)$, being $\Psi(t, s) = (C_\pm t + D_\pm, s)$ for $t \notin [-R, R]$.

On the other hand, a contradiction can be easily reached when, among the two sequences $\{t_n\}$ and $\{\tau_n\}$, one is bounded and the other one diverges to infinity. Indeed, η_1 remains bounded on the bounded sequence and diverges on the divergent sequence.

We have just shown that both sequences $\{t_n\}$ and $\{\tau_n\}$ are bounded. Up to subsequences, we may suppose that $t_n \to \bar{t}$ and $\tau_n \to \bar{\tau}$, for some $\bar{t}, \bar{\tau} \in \mathbb{R}$, so that by continuity we obtain $\Psi(\bar{t}, 0) = \Psi(\bar{\tau}, 0)$ and, in turn, $\boldsymbol{\eta}(\bar{t}) = \boldsymbol{\eta}(\bar{\tau})$. This implies $\bar{t} = \bar{\tau}$ being η_1 strictly increasing as a consequence of (3.2). Therefore, the restriction of Ψ to any neighborhood of the point $(\bar{t}, 0)$ is never injective, thus contradicting the Local Inversion Theorem.

Collecting what we have proved in Claim 1 and Claim 2, we may conclude that for a small $\overline{d}_i$, the map in (3.3) is invertible and its inverse is still of class C^2.

Step 2. In this second step, after having fixed an arbitrary $m \in \{1, \dots, M_i\}$, with M_i as in Step 1, we define a map Γ, coinciding with Ψ in the domain $\mathbb{R} \times \left(-\frac{\overline{d}_i}{4}, \frac{\overline{d}_i}{4}\right)$, which will be shown to be a global diffeomorphism in the subsequent steps.

First of all, if necessary, we may shrink the number $\overline{d}_i$ found in Step 1 in such a way that the following inequality holds true

$$\mu_1 - K\mu_2\,\overline{d}_i > 0\,, \tag{3.4}$$

where μ_1, μ_2, K are the constants defined in (3.2).

We now introduce a suitable cut-off function $\alpha \in C^1(\mathbb{R})$ such that α is even, $0 \le \alpha \le 1$, $\alpha \equiv 1$ in $[-\overline{d}_i/4, \overline{d}_i/4]$, $\alpha \equiv 0$ outside $(-\overline{d}_i/2, \overline{d}_i/2)$, and α decreasing in $[\overline{d}_i/4, \overline{d}_i/2]$.

Then we define the map $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Gamma(t, s) := \boldsymbol{\eta}(t) + \int_0^s [\,\alpha(\sigma)\mathrm{n}(t) + (1 - \alpha(\sigma))\mathrm{j}\,]\, d\sigma \qquad \text{for any } (t, s) \in \mathbb{R}^2\,,$$

where $\mathrm{j} = (0, 1)$ is as usual the unit vector of the *y*-axis.

We observe that Γ actually coincides with Ψ in $\mathbb{R} \times \left(-\frac{\overline{d}_i}{4}, \frac{\overline{d}_i}{4}\right)$, as claimed at the beginning of this step.

Step 3. We show that for any $s \in \mathbb{R}$ the curve $t \mapsto \Gamma(t, s)$ is the graph of a function in the form $y = g_s(x)$ for some $g_s : \mathbb{R} \to \mathbb{R}$.

We start by considering the case $s \in \left[-\overline{d}_i/2, \overline{d}_i/2\right]$. Using the notation $\Gamma = (\Gamma_1, \Gamma_2)$ for denoting the two components of the vector valued function Γ, by (3.2), (3.4) and the choice of α, if $|s| \le \overline{d}_i/2$, for any $t \in \mathbb{R}$ we have that

$$\begin{aligned} \frac{\partial \Gamma_1}{\partial t}(t, s) &= \eta_1'(t) + \left(\int_0^s \alpha(\sigma) d\sigma\right) \mathrm{n}_1'(t) \ge \mu_1 - |s|\,|\mathrm{n}'(t)| \\ &= \mu_1 - |s|\,|\kappa(t)|\,|\boldsymbol{\eta}'(t)| \ge \mu_1 - \tfrac{\overline{d}_i}{2}\,K\mu_2 \ge \mu_1 - K\mu_2\,\overline{d}_i > 0\,. \end{aligned} \tag{3.5}$$

As a consequence of the previous estimate, we deduce that for any $s \in \left[-\overline{d}_i/2, \overline{d}_i/2\right]$ the map $t \mapsto \Gamma_1(t, s)$ is invertible as a function from $\mathbb{R}$ to $\mathbb{R}$ and hence the curve $t \mapsto \Gamma(t, s)$ is the graph of a function in the form $y = g_s(x)$ for some function $g_s : \mathbb{R} \to \mathbb{R}$ depending on *s*.

We observe that from the definition of α, for $s > \overline{d}_i/2$ we have that $\Gamma(t,s) = \Gamma(t,\overline{d}_i/2) + (s - \overline{d}_i/2)\mathrm{j}$ which means that the curve $t \mapsto \Gamma(t,s)$ is a vertical translation of the graph of $g_{\overline{d}_i/2}$ so that it is, in turn, the graph of a function that we can still call g_s.

We can proceed similarly in the case $s < -\overline{d}_i/2$.

Step 4. We prove that for any $\bar{s} \in \mathbb{R}$, $g_s \to g_{\bar{s}}$ pointwise as $s \to \bar{s}$. Let $x \in \mathbb{R}$ and for any $s \in \mathbb{R}$ denote by t_s the unique value for which $\Gamma_1(t_s, s) = x$. Such a t_s exists and is unique thanks to Step 3. Moreover, from the definition of Γ we deduce that the map $s \mapsto \eta_1(t_s)$ is bounded, and by (3.2) we deduce that also $s \mapsto t_s$ is bounded. Along any sequence $s_n \to \bar{s}$ we can find a subsequence $\{s_{n_k}\}$ such that $t_{s_{n_k}} \to \bar{t}$ for some $\bar{t} \in \mathbb{R}$. From the continuity of Γ it follows that $\Gamma(t_{s_{n_k}}, s_{n_k}) \to \Gamma(\bar{t}, \bar{s})$ and in particular

$$x = \Gamma_1(t_{s_{n_k}}, s_{n_k}) \to \Gamma_1(\bar{t}, \bar{s}), \qquad g_{s_{n_k}}(x) = \Gamma_2(t_{s_{n_k}}, s_{n_k}) \to \Gamma_2(\bar{t}, \bar{s}) \qquad \text{as } k \to +\infty .$$

Then, the fact that $x = \Gamma_1(\bar{t}, \bar{s})$ implies that $g_{\bar{s}}(x) = \Gamma_2(\bar{t}, \bar{s})$ and hence $g_{s_{n_k}}(x) \to g_{\bar{s}}(x)$ as $k \to +\infty$. The independence of the limit proves that the last convergence holds true for the sequence $\{s_n\}$ and, in turn, as $s \to \bar{s}$ in the usual sense.

Step 5. We prove that the restriction $\Gamma_{|\mathbb{R}\times\left[-\frac{\overline{d}_i}{2}, \frac{\overline{d}_i}{2}\right]}$ is injective.

Let us start by defining the set

$$\mathcal{S}_+ := \left\{ S \in \left(0, \frac{\overline{d}_i}{2}\right] : \Gamma_{|\mathbb{R}\times[0,S]} \text{ is injective} \right\} .$$

From Step 1 we deduce that $\mathcal{S}_+$ is a nonempty bounded interval since, as already observed, Γ coincides with Ψ when $|s|$ is small enough. We now define $\overline{S} := \sup \mathcal{S}_+ \le \overline{d}_i/2$.

We claim that $\overline{S} \in \mathcal{S}_+$. We have to prove that $\Gamma_{|\mathbb{R}\times[0,\overline{S}]}$ is injective. By definition, for any $S < \overline{S}$ the map $\Gamma_{\mathbb{R}\times[0,S]}$ is injective and hence for any $0 \le s_1 < s_2 < \overline{S}$ the graphs of g_{s_1} and g_{s_2} have no intersections. Moreover, we have that

$$\frac{\partial \Gamma_1}{\partial s}(t,s) = -\frac{\alpha(s)}{|\boldsymbol{\eta}'(t)|}\,\eta_2'(t)\,, \qquad \frac{\partial \Gamma_2}{\partial s}(t,s) = \frac{\alpha(s)}{|\boldsymbol{\eta}'(t)|}\,\eta_1'(t) + (1-\alpha(s)) > 0\,,$$

which implies that $\Gamma_2(t,s_1) < \Gamma_2(t,s_2)$ for any $t \in \mathbb{R}$ and in particular the graph of g_{s_1} is below the graph of g_{s_2}. This can be easily seen by considering those values of t for which $\eta_2'(t) = 0$ on which the identity $\Gamma_1(t,s_1) = \Gamma_1(t,s_2)$ holds true.

We have just shown that

$$g_{s_1}(x) < g_{s_2}(x) \qquad \text{for any } x \in \mathbb{R} \text{ and } 0 \le s_1 < s_2 < \overline{S}\,. \tag{3.6}$$

Combining Step 4 and (3.6) we deduce that $g_{\overline{S}}$ is strictly larger than g_s for any $0 < s < \overline{S}$.

Resuming, we have so shown that for any $s_1, s_2 \in [0, \overline{S}]$ with $s_1 < s_2$ we have $g_{s_1} < g_{s_2}$, differently from (3.6) where the value $\overline{S}$ was not included.

From this and from(3.5) we easily deduce that for any $(t_1, s_1), (t_2, s_2) \in \mathbb{R}\times[0,\overline{S}]$ such that $\Gamma(t_1,s_1) = \Gamma(t_2,s_2)$, we necessarily have $(t_1,s_1) = (t_2,s_2)$. We have completed the proof of the claim $\overline{S} \in \mathcal{S}$.

We now claim that $\overline{S} = \overline{d}_i/2$. Suppose by contradiction that $\overline{S} < \overline{d}_i/2$.

By (3.4) we have that for any $(t,s) \in \mathbb{R} \times [0, \overline{d}_i/2]$

$$\begin{aligned} \det(J\Gamma(t,s)) &= \alpha(s)|\boldsymbol{\eta}'(t)| + (1-\alpha(s))\eta_1'(t) + \left(\int_0^s \alpha(\sigma)\, d\sigma\right)\left[\frac{\alpha(s)}{|\boldsymbol{\eta}'(t)|}\, \mathrm{n}'(t)\cdot\boldsymbol{\eta}'(t) + \mathrm{n}_1'(t)(1-\alpha(s))\right] \\ &\geq \eta_1'(t) - 2s|\mathrm{n}'(t)| = \eta_1'(t) - 2s|\kappa(t)|\,|\boldsymbol{\eta}'(t)| \geq \mu_1 - 2sK\mu_2 \geq \mu_1 - K\mu_2\,\overline{d}_i > 0\,. \end{aligned} \tag{3.7}$$

Using the same argument introduced in Step 1 with $s = \overline{S}$ in place of $s = 0$, we can prove that there exists $\delta > 0$ with $\overline{S} + \delta < \overline{d}_i/2$ such that Γ is injective in $\mathbb{R} \times (\overline{S} - \delta, \overline{S} + \delta)$.

Proceeding as above, we then deduce that the graphs of the functions g_s are ordered for $s \in (\overline{S} - \delta, \overline{S} + \delta)$ and combining this with (3.6) we obtain the ordering of the graphs for any $s \in [0, \overline{S} + \delta)$. This, combined with (3.5), proves that Γ is injective in $\mathbb{R} \times [0, \overline{S} + \delta)$, thus contradicting the maximality of $\overline{S}$. This completes the proof of the claim $\overline{S} = \overline{d}_i/2$.

With the very same kind of procedure we can prove that Γ is injective in $\mathbb{R} \times [-\overline{d}_i/2, 0]$ and that (3.7) is also true for $s \in [-\overline{d}_i/2, 0]$. We have completed the proof of this step.

Step 6. In this step, we prove that $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$ is bijective. Let us start with injectivity. We observe that

$$\Gamma(t,s) = \Gamma\left(t, \tfrac{\overline{d}_i}{2}\right) + \left(s - \tfrac{\overline{d}_i}{2}\right)\mathrm{j} \qquad \text{and} \qquad \Gamma(t,s) = \Gamma\left(t, -\tfrac{\overline{d}_i}{2}\right) + \left(s + \tfrac{\overline{d}_i}{2}\right)\mathrm{j} \tag{3.8}$$

in the cases $s \geq \overline{d}_i/2$ and $s \leq -\overline{d}_i/2$, respectively. This means that for $s > \overline{d}_i/2$ or $s < -\overline{d}_i/2$ the curve $t \mapsto \Gamma(t,s)$ is still the graph of a function g_s which is a vertical translation of the graphs of $g_{\overline{d}_i/2}$ and $g_{-\overline{d}_i/2}$, respectively in the two cases.

This proves that the graphs of g_s are strictly ordered for $s \in \mathbb{R}$. Combining this with (3.5) we finally proved that Γ is injective in the whole $\mathbb{R}^2$.

To prove that Γ is surjective, we observe that by (3.8), the constructions of g and $\boldsymbol{\eta}$ in Step 1 and the definition of Γ in Step 2, we have

$$\lim_{s\to+\infty}\min_{x\in\mathbb{R}} g_s(x) = +\infty \qquad \text{and} \qquad \lim_{s\to-\infty}\max_{x\in\mathbb{R}} g_s(x) = -\infty\,.$$

Therefore, letting (x, y) be an arbitrary point of $\mathbb{R}^2$, we see that there exist $s_0 < S_0$ such that $g_{s_0}(x) < y < g_{S_0}(x)$. Finally, performing a bisection procedure on the parameter s, we find two monotone sequences

$$s_0 \leq s_1 \leq \cdots \leq s_{n-1} \leq s_n \leq S_n \leq S_{n-1} \leq \cdots \leq S_1 \leq S_0$$

such that $S_n - s_n = (S_0 - s_0)/2^n$ and $g_{s_n}(x) \leq y \leq g_{S_n}(x)$, for any n. Either we find n for which $y = g_{s_n}(x)$ or $y = g_{S_n}(x)$, or for any $n \geq 0$ we have $g_{s_n}(x) < y < g_{S_n}(x)$. In the second case, passing to the limit as $n \to +\infty$, by Step 4, we obtain $y = g_{\bar{s}}(x)$ where $\bar{s} = \lim_{n\to+\infty} s_n = \lim_{n\to+\infty} S_n$.

In both cases, we have shown that (x, y) belongs to the graph of a function g_s for some s and hence there exists a corresponding t such that $\Gamma(t,s) = (x,y)$, thus completing the proof that Γ is surjective.

Step 7. We complete the proof of the lemma. In Step 6 we have proved that $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$ is bijective and, therefore, recalling the dependence on the indices *i* and *m*, we may define $\Phi_{i,m} := \Gamma^{-1}$.

We have to prove that $\Phi_{i,m}$ is a C^2 diffeomorphism and for showing this, it is enough to observe that $\Gamma \in C^2(\mathbb{R}^2; \mathbb{R}^2)$ thanks to (1.7), and by (3.7), (3.5) and (3.8), we have $\det(J\Gamma(t,s)) \neq 0$ for any $(t,s) \in \mathbb{R}^2$.

The conclusion about the sets $Z_{i,m,d}$ defined in (3.1) follows by recalling the definitions of the intervals $I_{i,m}$, of the sets $U_{i,d}$ and of the map Γ.

Finally, the conclusion on boundedness of coefficients of $J\Phi_{i,m}$ and $J\Phi_{i,m}^{-1}$ and of the Hessian matrices of the components of $\Phi_{i,m}$ and $\Phi_{i,m}^{-1}$, follows from the definition of Γ, (3.7), (3.5) and (3.8). □

Let u_d be a solution of (1.2) and following the notations of Lemma 3.1, for $m \in \{1, \dots, M_i\}$, let w_d be the function defined by

$$w_d(t,s) := u_d(\Phi_{i,m}^{-1}(t,s)) \qquad \text{for any } (t,s) \in \Phi_{i,m}(\Omega_d)\,. \tag{3.9}$$

Then w_d solves the following boundary value problem

$$\begin{cases} -\mathrm{div}(A\nabla w_d) = f & \text{in } \Phi_{i,m}(\Omega_d)\,, \\ w_d = 0 & \text{on } \partial(\Phi_{i,m}(\Omega_d))\,, \end{cases}$$

where, after defining $\Psi_{i,m} := \Phi_{i,m}^{-1}$,

$$A(t,s) := |\det(J\Psi_{i,m}(t,s))|\,(J\Psi_{i,m}(t,s))^{-1}\,[(J\Psi_{i,m}(t,s))^{-1}]^T \quad \text{and} \quad f(t,s) := |\det(J\Psi_{i,m}(t,s))|\,. \tag{3.10}$$

For simplicity, we omitted the dependence on the indices *i* and *m* in the notation used for *A* and *f*.

Since $\Phi_{i,m}$ and in particular its inverse function $\Psi_{i,m}$ are C^1 diffeomorphisms globally defined in $\mathbb{R}^2$, then the matrix A is positive definite and, moreover, since Ω_d and $\Phi_{i,m}(\Omega_d)$ are bounded domains, we see that there exists a constant $\alpha_0 > 0$, independent of $d \in (0, \overline{d}_i)$, such that

$$(A(t,s)\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq \alpha_0\,|\boldsymbol{\xi}|^2 \qquad \text{for any } \ (t,s) \in \Phi_{i,m}(\Omega_d) \ \text{ and } \boldsymbol{\xi} \in \mathbb{R}^2, \tag{3.11}$$

which shows that the linear differential operator $v \mapsto -\mathrm{div}(A\nabla v)$ is uniformly elliptic in $\Phi_{i,m}(\Omega_d)$.

Moreover, letting $A = (a_{ij})$, we also have for $d \to 0^+$

$$\|A\|_{L^\infty(\Phi_{i,m}(\Omega_d))} := \max_{i,j\in\{1,2\}} \sup_{(t,s)\in\Phi_{i,m}(\Omega_d)} |a_{ij}(t,s)| = O(1) \quad \text{and} \quad \|f\|_{L^\infty(\Phi_{i,m}(\Omega_d))} = O(1)\,. \tag{3.12}$$

For any $j \in \{1, \dots, N\}$, we also define the curves $\widetilde{\boldsymbol{\gamma}}_j := \Phi_{i,m} \circ \boldsymbol{\gamma}_j$, $\widetilde{\boldsymbol{\eta}}_j := \Phi_{i,m} \circ \boldsymbol{\eta}_j$, where we omitted for simplicity the dependence on the indices *i* and *m*, and the relative tubular neighborhoods of thickness *d*:

$$V(\widetilde{\boldsymbol{\eta}}_j, d) := \left\{\widetilde{\boldsymbol{\eta}}_j(t) + s\mathrm{n}(t) : t \in \mathbb{R}, s \in \left(-\tfrac{d}{2}, \tfrac{d}{2}\right)\right\}\,,$$

with n unit normal vector to $\widetilde{\boldsymbol{\eta}}_j$. We recall that $\boldsymbol{\eta}_j$ are the curves defined in (1.12).

It now follows a sequence of technical lemmas that have the purpose of obtaining some Poincaré-type inequalities of fundamental importance in the subsequent convergence arguments.

From the Lipschitz continuity of the function $\Phi_{i,m}$, we immediately have the following result.

Lemma 3.2 *Assume the validity of (1.7). For any $i, j \in \{1, \ldots, N\}$ and $m \in \{1, \ldots, M_i\}$ there exists a constant L depending on i and m such that*

$$\Phi_{i,m}(V_{j,d}) \subseteq V(\widetilde{\boldsymbol{\eta}}_j, Ld) \qquad \textit{for any } d > 0\,,$$

where we recall that $V_{j,d}$ was defined after (1.12).

Proof From Lemma 3.1 we deduce that $\Phi_{i,m}$ is Lipschitz continuous, so that we can denote by L its Lipschitz constant. Moreover, by a minimization argument, one can easily verify that a tubular neighborhood of thickness *d* of unbounded curves such as $\boldsymbol{\eta}_j$ and $\widetilde{\boldsymbol{\eta}}_j$ can be characterized as the set of points whose distance from the corresponding curve is less than *d*/2. The inclusion then follows immediately from the Lipschitz continuity of $\Phi_{i,m}$. □

Lemma 3.3 *Assume the validity of conditions (1.7)-(1.10). Let $i \in \{1, \ldots, N\}$ and $m \in \{1, \ldots, M_i\}$. For any $j \in \{1, \ldots, N\}$, let $J_j \supset [a_j, b_j]$ be the maximal open interval on which the function $\widetilde{\boldsymbol{\eta}}_j$ is injective.*

Then the following assertions hold true:

(*i*) there exists $\overline{d} > 0$ such that for any $j \in \{1, \ldots, N\}$ and $d \in (0, \overline{d})$, the map defined by

$$\begin{aligned}
&\Gamma_{j,d} : J_j \times (-d/2, d/2) \to \Gamma_{j,d}\Big(J_j \times (-d/2, d/2)\Big)\,, \\
&\Gamma_{j,d}(t, s) := \widetilde{\boldsymbol{\eta}}_j(t) + s\mathrm{n}(t)\,,
\end{aligned}$$

is a C^1 diffeomorphism, where as usual n *denotes the unit normal vector to the curve.*

(*ii*) For $j \in \{1, \ldots, N\}$ and $\alpha, \beta \in J_j$, with $\alpha < \beta$, define

$$W(j, \alpha, \beta, d) := \Gamma_{j,d}\Big([\alpha, \beta] \times (-d/2, d/2)\Big)\,.$$

Then for some $\overline{d}$, possibly smaller than the one found in (i), we have that there exists a projection map

$$\Pi_{j,d} : W(j, \alpha, \beta, d) \to \widetilde{\boldsymbol{\eta}}_j([\alpha, \beta])\,,$$

such that for any point $P \in W(j, \alpha, \beta, d)$, $\Pi_{j,d}(P)$ is the unique point in $\widetilde{\boldsymbol{\eta}}_j([\alpha, \beta])$ satisfying

$$|P - \Pi_{j,d}(P)| = \mathrm{dist}(P, \widetilde{\boldsymbol{\eta}}_j([\alpha, \beta]))\,.$$

(*iii*) For $j, k \in \{1, \ldots, N\}$, recalling the notation of (1.9)-(1.10), suppose that the intersection between the curves $\boldsymbol{\gamma}_j([a_j, b_j])$ and $\boldsymbol{\gamma}_k([a_k, b_k])$ is given by the singleton $\{P_{jk}\}$; then there exists $\delta > 0$ such that

$$\operatorname{diam}\Big(\partial V(\widetilde{\boldsymbol{\eta}}_j, d) \cap V(\widetilde{\boldsymbol{\eta}}_k, d) \cap B_\delta(\Phi_{i,m}(P_{jk}))\Big) = O(d) \qquad \text{as } d \to 0^+ ,$$

where $B_\delta(\Phi_{i,m}(P_{jk}))$ denotes the open disk centered at $\Phi_{i,m}(P_{jk})$ of radius δ.

(*iv*) For $j, k \in \{1, \ldots, N\}$ and δ as in (*iii*), up to shrink δ, we have for any $d \in (0, \overline{d})$

$$\Gamma_{j,d}^{-1}\Big(\partial V(\widetilde{\boldsymbol{\eta}}_j, d) \cap V(\widetilde{\boldsymbol{\eta}}_k, d) \cap B_\delta(\Phi_{i,m}(P_{jk}))\Big) \subset \big(K^+_{j,k,d} \times \{\tfrac{d}{2}\}\big) \cup \big(K^-_{j,k,d} \times \{-\tfrac{d}{2}\}\big) \quad (3.13)$$

for some $K^+_{j,k,d}$ and $K^-_{j,k,d}$ compact intervals satisfying

$$\operatorname{length}(K^+_{j,k,d}) = O(d)\, , \quad \operatorname{length}(K^-_{j,k,d}) = O(d) \qquad \text{as } d \to 0^+ . \quad (3.14)$$

Proof We prove separately the different items of the lemma.

Proof of (i). Proceeding as in the proof of Lemma 3.1, using (1.7) and the construction of $\boldsymbol{\eta}_j$ in (1.12), we deduce that the map $\Gamma_{j,d}$, defined in the statement of the present lemma, is a C^1 diffeomorphism for $d \in (0, \overline{d})$, for some $\overline{d}$ sufficiently small.

Proof of (ii). Given P, by a minimization argument, it can easily be proved that there exists a point $Q \in \widetilde{\boldsymbol{\eta}}_j([\alpha, \beta])$ such that $|P - Q|$ equals $\operatorname{dist}(P, \widetilde{\boldsymbol{\eta}}_j([\alpha, \beta])$. Moreover P lies on the normal direction to the curve passing through Q. We preliminarily observe that if $P \in W(j, \alpha, \beta, d)$, for stationary points $t_0 \in [\alpha, \beta]$ of the function $g(t) := |\widetilde{\boldsymbol{\eta}}_j(t) - P|^2$, we see that P lies on the normal direction to $\widetilde{\boldsymbol{\eta}}_j$ passing through $\widetilde{\boldsymbol{\eta}}_j(t_0)$.

For $P \in W(j, \alpha, \beta, d)$ let $t \in [\alpha, \beta]$ and $s \in (-d/2, d/2)$ be such that $\Gamma_{j,d}(t, s) = P$.

If $t \in \{\alpha, \beta\}$ we can verify that P is on the normal direction to the curve passing through Q with $Q = \widetilde{\boldsymbol{\eta}}_j(\alpha)$ if $t = \alpha$ and $Q = \widetilde{\boldsymbol{\eta}}_j(\beta)$ if $t = \beta$ since otherwise, for d small enough, $Q = \widetilde{\boldsymbol{\eta}}_j(t_0)$ for some $t_0 \in (\alpha, \beta)$, so that $g'(t_0) = 0$. Therefore, P lies on the normal direction to $\widetilde{\boldsymbol{\eta}}_j$ passing through $Q = \widetilde{\boldsymbol{\eta}}_j(t_0)$. This implies $\Gamma_{j,d}(t, s) = P = \Gamma_{j,d}(t_0, \sigma)$ for some $\sigma \in \mathbb{R}$ that necessarily belongs to $(-d/2, d/2)$ being $|Q - P| = \operatorname{dist}(P, \widetilde{\boldsymbol{\eta}}_j([a, b]) < d/2$. Since $\Gamma_{j,d}$ is injective in $J_j \times (-d/2, d/2)$ and $(t, s), (t_0, \sigma) \in J_j \times (-d/2, d/2)$ then $t_0 = t \in \{\alpha, \beta\}$ and $\sigma = s$, a contradiction.

On the other hand, if $t \in (\alpha, \beta)$ suppose by contradiction that $Q = \widetilde{\boldsymbol{\eta}}_{j,d}(\tau)$ for $\tau \in \{\alpha, \beta\}$ and $|Q - P| < |\widetilde{\boldsymbol{\eta}}_j(t) - P|$. Just to fix ideas, suppose that $\tau = \beta$. Fix $a, b \in J_j$ such that $\alpha < a < \beta < b$. Then for d small enough, the minimum of g on $[a, b]$ cannot be achieved at a or b and therefore it is achieved at some point $t_0 \in (a, b)$ where $g'(t_0) = 0$. In this way, $|\widetilde{\boldsymbol{\eta}}_j(t_0) - P| \le |Q - P| < |\widetilde{\boldsymbol{\eta}}_j(t) - P|$.

With the same argument used above, based on the injectivity of $\Gamma_{j,d}$ we conclude that $t_0 = t$, a contradiction. We have proved that $Q = \widetilde{\boldsymbol{\eta}}_{j,d}(\tau)$ for some $\tau \in (\alpha, \beta)$. Then, we have $g'(\tau) = 0$ and, hence, repeating the above argument, we conclude that $\tau = t$.

Summarizing, we have proved that given $P \in W(j, \alpha, \beta, d)$, Q is uniquely determined by P and, moreover, $Q = \widetilde{\boldsymbol{\eta}}_j(t)$ where $(t, s) = \Gamma_{j,d}^{-1}(P)$.

This completes the proof of the existence of the projection map $\Pi_{j,d}$.

Proof of (iii). Letting P_{jk} be as in (1.10), we now claim that there exists $\delta > 0$ such that

$$\sup_{P\in\partial V(\widetilde{\boldsymbol{\eta}}_j,d)\cap V(\widetilde{\boldsymbol{\eta}}_k,d)\cap B_\delta(\Phi_{i,m}(P_{jk}))} |P-\Phi_{i,m}(P_{jk})| \to 0 \qquad \text{as } d\to 0^+ . \tag{3.15}$$

Proceeding by contradiction, for any $\delta > 0$ there exist $\varepsilon > 0$ and a sequence $\{P_n\} \subset \partial V(\widetilde{\boldsymbol{\eta}}_j, 1/n) \cap V(\widetilde{\boldsymbol{\eta}}_k, 1/n) \cap B_\delta(\Phi_{i,m}(P_{jk}))$ such that $|P_n - \Phi_{i,m}(P_{jk})| \geq \varepsilon$. Up to subsequences, we may assume that $P_n \to \overline{P}$ for some $\overline{P}$ that necessarily belongs to $\widetilde{\boldsymbol{\eta}}_j(\mathbb{R})$ and to $\widetilde{\boldsymbol{\eta}}_k(\mathbb{R})$, being the distance of P_n from the two curves less or equal than $1/n$. But, if δ is small enough, the intersection of the two curves in $B_\delta(\Phi_{i,m}(P_{jk}))$ is the singleton $\{\Phi_{i,m}(P_{jk})\}$ thus proving that $\overline{P} = \Phi_{i,m}(P_{jk})$, a contradiction. This gives the proof of (3.15).

To complete the proof of (*iii*), it will suffice to prove that there exists a constant $C > 0$ such that

$$\sup_{P\in\partial V(\widetilde{\boldsymbol{\eta}}_j,d)\cap V(\widetilde{\boldsymbol{\eta}}_k,d)\cap B_\delta(\Phi_{i,m}(P_{jk}))} |P-\Phi_{i,m}(P_{jk})| \leq C\, d\, . \tag{3.16}$$

Let $t_j \in [a_j, b_j]$ and $t_k \in [a_k, b_k]$ be such that $\widetilde{\boldsymbol{\eta}}_j(t_j) = P_{jk}$ and $\widetilde{\boldsymbol{\eta}}_k(t_k) = P_{jk}$. Let $[\alpha_j, \beta_j]$ and $[\alpha_k, \beta_k]$ be as in (*ii*) so that $t_j \in [\alpha_j, \beta_j]$ and $t_k \in [\alpha_k, \beta_k]$, where the indices *j* and *k* refer to the curves $\widetilde{\boldsymbol{\eta}}_j$ and $\widetilde{\boldsymbol{\eta}}_k$, respectively. Choosing δ small enough, if *P* is as in (3.16), then $P \in W(k, \alpha_k, \beta_k, d)$ and

$$\operatorname{dist}(\Pi_{k,d}(P), \widetilde{\boldsymbol{\eta}}_j([\alpha_j,\beta_j]) \leq |\Pi_{k,d}(P) - P| + \operatorname{dist}(P, \widetilde{\boldsymbol{\eta}}_j([\alpha_j,\beta_j]) < \tfrac{d}{2} + \tfrac{d}{2} = d$$

so that $\Pi_{k,d}(P) \in W(j, \alpha_j, \beta_j, 2d)$ and, moreover,

$$|\Pi_{j,2d}(\Pi_{k,d}(P)) - \Pi_{k,d}(P)| < d\, . \tag{3.17}$$

Combining (3.15) with the fact that $|\Pi_{k,d}(P) - P| < d/2$, we have

$$\sup_{P\in\partial V(\widetilde{\boldsymbol{\eta}}_j,d)\cap V(\widetilde{\boldsymbol{\eta}}_k,d)\cap B_\delta(\Phi_{i,m}(P_{jk}))} |\Pi_{k,d}(P)-\Phi_{i,m}(P_{jk})| \to 0 \qquad \text{as } d\to 0^+ \tag{3.18}$$

and by (3.17) we also have

$$\sup_{P\in\partial V(\widetilde{\boldsymbol{\eta}}_j,d)\cap V(\widetilde{\boldsymbol{\eta}}_k,d)\cap B_\delta(\Phi_{i,m}(P_{jk}))} |\Pi_{j,2d}(\Pi_{k,d}(P))-\Phi_{i,m}(P_{jk})| \to 0 \qquad \text{as } d\to 0^+ . \tag{3.19}$$

Now, let $t(d, P) \in [\alpha_k, \beta_k]$ and $\tau(d, P) \in [\alpha_j, \beta_j]$ be such that $\widetilde{\boldsymbol{\eta}}_k(t(d, P)) = \Pi_{k,d}(P)$, $\widetilde{\boldsymbol{\eta}}_j(\tau(d, P)) = \Pi_{j,2d}(\Pi_{k,d}(P))$. By Steps (*i*)-(*ii*) and (3.18)-(3.19), we have

$$\sup_{P\in\partial V(\widetilde{\boldsymbol{\eta}}_j,d)\cap V(\widetilde{\boldsymbol{\eta}}_k,d)\cap B_\delta(\Phi_{i,m}(P_{jk}))} |t(d,P)-t_k| \to 0 \quad \text{as } d\to 0^+ , \tag{3.20}$$

and

$$\sup_{P\in\partial V(\widetilde{\boldsymbol{\eta}}_j,d)\cap V(\widetilde{\boldsymbol{\eta}}_k,d)\cap B_\delta(\Phi_{i,m}(P_{jk}))} |\tau(d,P)-t_j| \to 0 \quad \text{as } d\to 0^+ . \tag{3.21}$$

Then we can use Taylor expansions to write

$$\Pi_{k,d}(P) = \Phi_{i,m}(P_{jk}) + (t(d,P)-t_k)[\widetilde{\boldsymbol{\eta}}'_k(t_k) + \mathrm{R}_k(d,P)] ,$$
$$\Pi_{j,2d}(\Pi_{k,d}(P)) = \Phi_{i,m}(P_{jk}) + (\tau(d,P)-t_j)[\widetilde{\boldsymbol{\eta}}'_j(t_j) + \mathrm{R}_j(d,P)] ,$$

as $d\to 0^+$, where and R_k, R_j are two vector functions that, thanks to (3.20) and (3.21), satisfy

$$\begin{aligned} &\sup_{P\in\partial V(\widetilde{\boldsymbol{\eta}}_j,d)\cap V(\widetilde{\boldsymbol{\eta}}_k,d)\cap B_\delta(\Phi_{i,m}(P_{jk}))} |\mathrm{R}_k(d,P)| \to 0 ,\\ &\sup_{P\in\partial V(\widetilde{\boldsymbol{\eta}}_j,d)\cap V(\widetilde{\boldsymbol{\eta}}_k,d)\cap B_\delta(\Phi_{i,m}(P_{jk}))} |\mathrm{R}_j(d,P)| \to 0 , \end{aligned} \tag{3.22}$$

as $d\to 0^+$.

Combining the two expansions with (3.17), we obtain

$$\Big|(t(d,P)-t_k)[\widetilde{\boldsymbol{\eta}}'_k(t_k)+\mathrm{R}_k(d,P)] - (\tau(d,P)-t_j)[\widetilde{\boldsymbol{\eta}}'_j(t_j)+\mathrm{R}_j(d,P)]\Big| < d . \tag{3.23}$$

Now we can apply in (3.23) the following general inequality valid for non parallel vectors W_1, W_2 with angle between them given by $\theta\in(0,\pi)$

$$|a\mathrm{W}_1 + b\mathrm{W}_2|^2 \ge (1-|\cos\theta|)|\mathrm{W}_1|^2\, a^2 + (1-|\cos\theta|)|\mathrm{W}_2|^2\, b^2 \qquad \text{for any } a,b\in\mathbb{R} . \tag{3.24}$$

For d small enough, using (3.22), we estimate the modulus of the vectors $\widetilde{\boldsymbol{\eta}}'_k(t_k)+\mathrm{R}_k(d,P), \widetilde{\boldsymbol{\eta}}'_j(t_j)+\mathrm{R}_j(d,P)$ and the cosine of the angle θ_d between them in the following way

$$|\widetilde{\boldsymbol{\eta}}'_k(t_k)+\mathrm{R}_k(d,P)| \ge \tfrac{|\widetilde{\boldsymbol{\eta}}'_k(t_k)|}{2} , \quad |\widetilde{\boldsymbol{\eta}}'_j(t_j)+\mathrm{R}_j(d,P)| \ge \tfrac{|\widetilde{\boldsymbol{\eta}}'_j(t_j)|}{2} , \quad 1-|\cos\theta_d| \ge \tfrac{1-|\cos\theta|}{2} ,$$

with θ the angle between the vectors $\widetilde{\boldsymbol{\eta}}'_k(t_k), \widetilde{\boldsymbol{\eta}}'_j(t_j)$.

Then, by (3.23) and (3.24), we obtain

$$d^2 > \tfrac{(1-|\cos\theta|)|\widetilde{\boldsymbol{\eta}}'_k(t_k)|}{8}\,[t(d,P)-t_k]^2 + \tfrac{(1-|\cos\theta|)|\widetilde{\boldsymbol{\eta}}'_j(t_j)|}{8}\,[\tau(d,P)-t_j]^2$$

from which it follows

$$|t(d,P)-t_k| < \sqrt{\tfrac{8}{(1-|\cos\theta|)|\widetilde{\boldsymbol{\eta}}'_k(t_k)|}}\, d \quad \text{and} \quad |\tau(d,P)-t_j| < \sqrt{\tfrac{8}{(1-|\cos\theta|)|\widetilde{\boldsymbol{\eta}}'_j(t_j)|}}\, d .$$

Inserting the estimate on $|t(d,P)-t_k|$ in the first Taylor expansion above, by (3.22), we deduce that

$$\sup_{P\in\partial V(\widetilde{\boldsymbol{\eta}}_j,d)\cap V(\widetilde{\boldsymbol{\eta}}_k,d)\cap B_\delta(\Phi_{i,m}(P_{jk}))} |\Pi_{k,d}(P)-\Phi_{i,m}(P_{jk})| \le C\,d$$

for some positive constant C independent of d and this combined with the estimate $|\Pi_{k,d}(P)-P|<d/2$ proves (3.16) and completes the proof of (*iii*).

Proof of (iv). It is enough to observe that for d small enough $\Gamma_{j,d}^{-1}$ is Lipschitz continuous in $\partial V(\widetilde{\boldsymbol{\eta}}_j,d)\cap V(\widetilde{\boldsymbol{\eta}}_k,d)\cap B_\delta(\Phi_{i,m}(P_{jk}))$. Then (3.13) follows from the definition of $V(\widetilde{\boldsymbol{\eta}}_j,d)$. Finally, (3.14) follows from (*iii*) and the Lipschitz continuity of $\Gamma_{j,d}^{-1}$. □

We now need to prove a couple of Poincaré-type inequality for domains depending on the parameter d. We start with following inequality for domains independent of d.

Lemma 3.4 *Let $I\subseteq\mathbb{R}$ be an open interval, let $G^+,G^-\subseteq I$ be finite unions of open intervals such that the sets $K^+:=I\backslash G^+$ and $K^-:=I\backslash G^-$ are finite unions of compact disjoint intervals, i.e. $K^+=\bigcup_{j=1}^{N^+}K_j^+$ and $K^-=\bigcup_{j=1}^{N^-}K_j^-$.*

Let $R>0$ and $D>0$ be such that

$$R\ge\max_{i\in\{1,\dots,N^+\},\,j\in\{1,\dots,N^-\}} R_{ij}:=\max_{i\in\{1,\dots,N^+\},\,j\in\{1,\dots,N^-\}}\operatorname{length}(K_i^+\cap K_j^-)\,, \tag{3.25}$$

$$D\le\min_{i_1,i_2\in\{1,\dots,N^+\},\ j_1,j_2\in\{1,\dots,N^-\},\ (i_1,j_1)\neq(i_2,j_2)}\ \inf_{P\in K_{i_1}^+\cap K_{j_1}^-,\,Q\in K_{i_2}^+\cap K_{j_2}^-}|P-Q| \tag{3.26}$$

where the infimum in the above definition equals $+\infty$ whenever at least one of the two sets $K_{i_1}^+\cap K_{j_1}^-$, $K_{i_2}^+\cap K_{j_2}^-$ is empty.

Letting $G:=\left(G^+\times\left\{\frac{1}{2}\right\}\right)\cup\left(G^-\times\left\{-\frac{1}{2}\right\}\right)$, we denote by $C_{0,G}^\infty\left(I\times\left(-\frac{1}{2},\frac{1}{2}\right)\right)$ the space of functions belonging to $C^\infty\left(\overline{I}\times\left[-\frac{1}{2},\frac{1}{2}\right]\right)$ and vanishing on G, and by $H_{0,G}^1\left(I\times\left(-\frac{1}{2},\frac{1}{2}\right)\right)$ the closure of $C_{0,G}^\infty\left(I\times\left(-\frac{1}{2},\frac{1}{2}\right)\right)$ in $H^1\left(I\times\left(-\frac{1}{2},\frac{1}{2}\right)\right)$.

Then there exists a positive constant C(R, D) depending only on R and D such that

$$\int_{I\times\left(-\frac{1}{2},\frac{1}{2}\right)} w^2\,dtds\le C(R,D)\int_{I\times\left(-\frac{1}{2},\frac{1}{2}\right)}|\nabla w|^2\,dtds \tag{3.27}$$

for any $w\in H_{0,G}^1\left(I\times\left(-\frac{1}{2},\frac{1}{2}\right)\right)$.

Proof Up to sets of zero measure, we can split $I\times\left(-\frac{1}{2},\frac{1}{2}\right)$ into the union of the following sets:

$$(G^+\cap G^-)\times\left(-\tfrac{1}{2},\tfrac{1}{2}\right),\quad (G^+\setminus\overline{G^-})\times\left(-\tfrac{1}{2},\tfrac{1}{2}\right),\quad (G^-\setminus\overline{G^+})\times\left(-\tfrac{1}{2},\tfrac{1}{2}\right),\quad (\mathring{K}^+\cap\mathring{K}^-)\times\left(-\tfrac{1}{2},\tfrac{1}{2}\right). \tag{3.28}$$

Proceeding as in the proof of Proposition 2.1, if $\mathcal{A}$ is one of the first three domains in (3.28) then we have

$$\int_{\mathcal{A}} w^2 dtds \le \int_{\mathcal{A}} |\nabla w|^2 dtds \tag{3.29}$$

for any $w \in H^1_{0,G}\left(I \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$, being sets of width 1 and being *w* functions vanishing identically on $\overline{\mathcal{A}} \cap \left(I \times \left\{\frac{1}{2}\right\}\right)$ or on $\overline{\mathcal{A}} \cap \left(I \times \left\{-\frac{1}{2}\right\}\right)$.

Concerning the subdomain $(\mathring{K}^+ \cap \mathring{K}^-) \times (-1/2, 1/2)$, we can split the set $\mathring{K}^+ \cap \mathring{K}^-$ as

$$\bigcup_{i \in \{1,\dots,N^+\}\,,\, j \in \{1,\dots,N^-\}} (\mathring{K}_i^+ \cap \mathring{K}_j^-)\,.$$

Whenever $\mathring{K}_i^+ \cap \mathring{K}_j^- \neq \emptyset$, we can define the open rectangles $\mathcal{R}_{ij} := [(\mathring{K}_i^+ \cap \mathring{K}_j^-) + (0, D)] \times (-1/2, 1/2)$.

Our purpose is to prove a Poincaré-type inequality in $\mathcal{R}_{ij}$ for the restriction to $\mathcal{R}_{ij}$ of functions belonging to the space $H^1_{0,G}\left(I \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$.

After a horizontal translation, $\mathcal{R}_{ij}$ becomes $(0, R_{ij} + D) \times (-1/2, 1/2)$, with R_{ij} as in the statement of the lemma, and restrictions of functions in $H^1_{0,G}\left(I \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ become functions that belong to the space $H^1\left((0, R_{ij} + D) \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ and vanish at least on one of the two sets

$$(R_{ij}, R_{ij} + D) \times \{1/2\}\,, \quad (R_{ij}, R_{ij} + D) \times \{-1/2\}\,.$$

By an extension procedure based on symmetric reflections, we can extend such functions to the set $(0, R + D) \times (-1/2, 1/2)$: let $w \in H^1\left((0, R_{ij} + D) \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$, just for fixing ideas, with a zero trace on $(R_{ij}, R_{ij} + D) \times \{1/2\}$, being the other case with $(R_{ij}, R_{ij} + D) \times \{-1/2\}$ completely equivalent. After a symmetric reflection with respect to the vertical edge $\{R_{ij} + D\} \times (-1/2, 1/2)$ of $w_{|(R_{ij}, R_{ij}+D)\times(-1/2,1/2)}$, we obtain a function $\tilde{w} \in H^1\left((0, R_{ij} + 2D) \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ with zero trace on $(R_{ij}, R_{ij} + 2D) \times \{1/2\}$. After a number of steps less or equal than $R/D + 1$, we can construct an extended function, that we can still denote by $\tilde{w} \in H^1\left((0, R + D) \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$, with zero trace on $(R_{ij}, R + D) \times \{1/2\}$ such that

$$\int_{(0,R+D)\times\left(-\frac{1}{2},\frac{1}{2}\right)} |\nabla \tilde{w}|^2\, dtds \le \left(\tfrac{R}{D} + 1\right) \int_{(0,R_{ij}+D)\times\left(-\frac{1}{2},\frac{1}{2}\right)} |\nabla w|^2\, dtds\,. \tag{3.30}$$

Thanks to (3.30), in order to obtain a Poincaré-type inequality with a constant depending only on *R* and *D*, it will suffice to prove a Poincaré-type inequality for functions $w \in H^1_{0,\Sigma}(\mathcal{R})$ where $\mathcal{R} := (0, R + D) \times (-1/2, 1/2)$ and Σ is indifferently one among the sets

$$(R, R + D) \times \left\{\tfrac{1}{2}\right\}\,, \quad (R, R + D) \times \left\{-\tfrac{1}{2}\right\}\,.$$

We claim that there exists a positive constant $C_1(R, D)$ depending only on *R* and *D* such that

$$\int_{\mathcal{R}} w^2 dtds \leq C_1(R,D) \int_{\mathcal{R}} |\nabla w|^2 dtds \qquad \text{for any } w \in H^1_{0,\Sigma}(\mathcal{R})\,. \tag{3.31}$$

In order to prove the claim, we define in $H^1(\mathcal{R})$ the norm

$$\|w\|_* := \left(\int_{\mathcal{R}} |\nabla w|^2 dtds + \int_{\Sigma} w^2 \, d\sigma \right)^{\frac{1}{2}}$$

where $d\sigma$ denotes the arc length differential. We now prove that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|w\|_{L^2(\mathcal{R})} \leq \varepsilon \|w\|_{H^1(\mathcal{R})} + C_\varepsilon \|w\|_* \qquad \text{for any } w \in H^1(\mathcal{R})\,. \tag{3.32}$$

Suppose by contradiction that there exist $\varepsilon > 0$ and a sequence $\{w_n\}$ such that

$$\|w_n\|_{L^2(\mathcal{R})} > \varepsilon \|w_n\|_{H^1(\mathcal{R})} + n\|w_n\|_*\,. \tag{3.33}$$

It is not restrictive, after normalization, to assume that $\|w_n\|_{H^1(\mathcal{R})} = 1$ and up to a subsequence that $w_n \rightharpoonup w$ weakly in $H^1(\mathcal{R})$ for some $w \in H^1(\mathcal{R})$.

Hence, by continuous embedding $H^1(\mathcal{R}) \subset L^2(\mathcal{R})$, $\|w_n\|_{L^2(\mathcal{R})}$ is bounded and, by (3.33), we obtain $\|w_n\|_* \to 0$ as $n \to +\infty$.

Since by trace inequality, the linear functional $f_w : H^1(\mathcal{R}) \to \mathbb{R}$ defined by

$$f_w(v) := \int_{\mathcal{R}} \nabla w \cdot \nabla v \, dtds + \int_{\Sigma} wv \, d\sigma$$

is continuous, then by weak convergence $f_w(w_n) \to f_w(w) = \|w\|_*^2$. On the other hand, by the Schwarz inequality $|f_w(w_n)| \leq \|w\|_* \, \|w_n\|_* \to 0$ as $n \to +\infty$, thus proving that $w = 0$ and $w_n \rightharpoonup 0$ weakly in $H^1(\mathcal{R})$ and, by compact embedding of $H^1(\mathcal{R})$ into $L^2(\mathcal{R})$, $w_n \to 0$ strongly in $L^2(\mathcal{R})$. This contradicts (3.33) since $\|w_n\|_{L^2(\mathcal{R})} > \varepsilon$. This completes the proof of (3.32).

Choosing $\varepsilon < 1$ in (3.32), from the definition of $\|\cdot\|_*$, it follows the existence of a constant $C_0(R,D)$ depending only on R and D such that

$$\|w\|_{L^2(\mathcal{R})} \leq C_0(R,D) \left[\left(\int_{\mathcal{R}} |\nabla w|^2 dtds \right)^{\frac{1}{2}} + \left(\int_{\Sigma} w^2 \, d\sigma \right)^{\frac{1}{2}} \right] \qquad \text{for any } w \in H^1(\mathcal{R})\,.$$

Choosing $w \in H^1_{0,\Sigma}(\mathcal{R})$ and putting $C_1(R,D) = (C_0(R,D))^2$, the last estimate proves the claim (3.31).

Combining (3.30) and (3.31), we obtain the following Poincaré-type inequality

$$\int_{\mathcal{R}_{ij}} w^2 \, dtds \leq C_1(R,D) \left(\tfrac{R}{D} + 1 \right) \int_{\mathcal{R}_{ij}} |\nabla w|^2 \, dtds \qquad \text{for any } w \in H^1_{0,G}\left(I \times \left(-\tfrac{1}{2}, \tfrac{1}{2} \right) \right). \tag{3.34}$$

We may conclude the proof of the lemma by observing that the rectangle $I \times (-1/2, 1/2)$ may be split into a finite number of rectangles including the rectangles of the type $\mathcal{R}_{ij}$

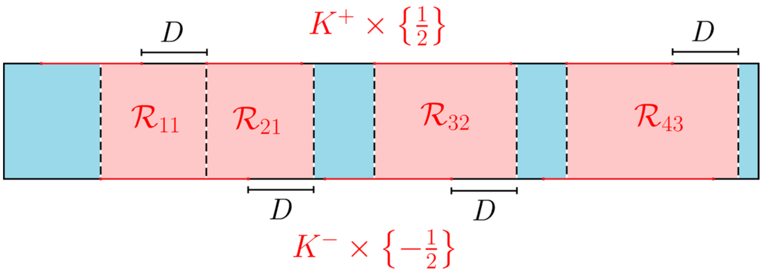

Fig. 7 Example of decomposition of the set $I \times (-1/2, 1/2)$: in red the boundary sets $K^+ \times \{1/2\}$ and $K^- \times \{-1/2\}$, in light red the sets $\mathcal{R}_{ij}$ and in light blue the sets where we apply Proposition 2.1

defined above and the remaining rectangles on which functions in $H^1_{0,G}\left(I \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ either have a zero trace on the upper horizontal edge or they have a zero trace on the lower horizontal edge. For this last class of rectangles, one may proceed as in (3.29) by applying Proposition 2.1.

On all these rectangles, we have a Poincaré inequality of the type (3.29) or of the type (3.34) and hence combining all of them we arrive to the conclusion.

For an example about the decomposition of the set $I \times (-1/2, 1/2)$, see Fig. 7. $\square$

As an immediate consequence of Lemma 3.4 we have the following

Corollary 3.5 *Let $I \subseteq \mathbb{R}$ be an open interval, let $G_d^+, G_d^- \subseteq I$ be unions of open intervals depending on a parameter $d > 0$ such that the sets $K_d^+ := I \setminus G_d^+$ and $K_d^- := I \setminus G_d^-$ are unions of N^+ and N^- compact disjoint intervals, with N^+ and N^- independent of d, and such that, writing $K_d^+ = \bigcup_{j=1}^{N^+} K_{j,d}^+$, $K_d^- = \bigcup_{j=1}^{N^-} K_{j,d}^-$, we also have*

$$\mathrm{length}(K_{j,d}^+) = O(d)\,, \quad \mathrm{length}(K_{j,d}^-) = O(d)\,, \qquad \text{as } d \to 0^+\,, \tag{3.35}$$

for any $j \in \{1, \dots, N^+\}$ and $j \in \{1, \dots, N^-\}$ respectively.

Letting $G_d := \left(G_d^+ \times \left\{\frac{d}{2}\right\}\right) \cup \left(G_d^- \times \left\{-\frac{d}{2}\right\}\right)$, consider the space $H^1_{0,G_d}\left(I \times \left(-\frac{d}{2}, \frac{d}{2}\right)\right)$ according to the definition in Lemma 3.4.

Then there exist $\overline{d}$ and a positive constant C independent of d such that for any $d \in (0, \overline{d})$

$$\int_{I \times \left(-\frac{d}{2}, \frac{d}{2}\right)} w^2 dtds \le Cd^2 \int_{I \times \left(-\frac{d}{2}, \frac{d}{2}\right)} |\nabla w|^2 dtds\,,$$

for any $w \in H^1_{0,G_d}\left(I \times \left(-\frac{d}{2}, \frac{d}{2}\right)\right)$.

Proof The proof of the corollary follows immediately from Lemma 3.4 and a rescaling argument. More precisely if $w \in H^1_{0,G_d}\left(I \times \left(-\frac{d}{2}, \frac{d}{2}\right)\right)$, we define a function $\widetilde{w}(t,s) := w(dt, ds)$ for any $(t,s) \in \widetilde{I}_d \times (-1/2, 1/2)$ where we put $\widetilde{I}_d := \frac{1}{d} I$. In this way, letting $\widetilde{G}_d := \frac{1}{d} G_d$, $\widetilde{w} \in H^1_{0,\widetilde{G}_d}\left(\widetilde{I}_d \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$, we can apply to it Lemma 3.4. We finally observe that the interval $\widetilde{I}_d$ becomes larger and larger as $d \to 0^+$. However, we point out that the constant $C(R, D)$ defined in Lemma 3.4 does not depend on the length of the interval I defined there.

Moreover, by (3.35), we can find R independent of d such that (3.25) holds for d small when calculated on the corresponding sets $\frac{1}{d} K^+_{j,d}, \frac{1}{d} K^-_{j,d}$. We can also find $D > 0$ independent of d such that (3.26) holds for d small, again when calculated on the corresponding sets $\frac{1}{d} K^+_{j,d}, \frac{1}{d} K^-_{j,d}$.

Finally, it is enough to apply (3.27) to the function $\widetilde{w}$ and proceed with a change of variables. □

Lemma 3.6 *Let $I \subseteq \mathbb{R}$ be an open interval and let $\boldsymbol{\gamma} : \overline{I} \to \mathbb{R}^2$ be a curve of class $C^1(\overline{I}; \mathbb{R}^2)$ and piecewise C^2 in the sense that there exist $-\infty \le \inf I = t_0 < t_1 < t_2 < \cdots < t_{M-1} < t_M = \sup I \le +\infty$ such that $\boldsymbol{\gamma}_{|\, \overline{(t_{j-1}, t_j)}} \in C^2\left(\overline{(t_{j-1}, t_j)}\right)$ for any $j \in \{1, \ldots, M\}$. We also assume that $|\boldsymbol{\gamma}'(t)| \neq 0$ for any $t \in \overline{I}$ and that $\boldsymbol{\gamma}', \boldsymbol{\gamma}'' \in L^\infty(I; \mathbb{R}^2)$. Letting $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ and letting $\mathrm{n} = (\mathrm{n}_1, \mathrm{n}_2)$ be the unit normal vector to $\boldsymbol{\gamma}$, for any $d > 0$, let U_d be the domain defined by*

$$U_d := \left\{ \left(\gamma_1(t) + s\mathrm{n}_1(t), \tfrac{1}{d}\gamma_2(t) + \tfrac{s}{d}\mathrm{n}_2(t)\right) : t \in I,\ s \in \left(-\tfrac{d}{2}, \tfrac{d}{2}\right) \right\} .$$

Let $\Upsilon_d : I \times (-d/2, d/2) \to U_d$ be the map defined by

$$\Upsilon_d(t, s) := \left(\gamma_1(t) + s\mathrm{n}_1(t), \tfrac{1}{d}\gamma_2(t) + \tfrac{s}{d}\mathrm{n}_2(t)\right) .$$

(*i*) Then there exists $\overline{d} > 0$ such that for any $d \in (0, \overline{d})$ the map Υ_d is a homeomorphism and, for any $j \in \{1, \ldots, M\}$, $\Upsilon_{d\,|\,\overline{(t_{j-1}, t_j)} \times (-d/2, d/2)} \in C^1\left(\overline{(t_{j-1}, t_j)} \times (-d/2, d/2)\right)$.

(*ii*) For $d \in (0, \overline{d})$, let $G^+_d, G^-_d, K^+_d, K^-_d, G_d$ be sets satisfying all conditions of Corollary 3.5 and let $\Gamma_d := \Upsilon_d(G_d)$ where we observe that Υ_d is naturally defined also in $I \times [-d/2, d/2]$. Then there exists a positive constant C independent of d such that for any $d \in (0, \overline{d})$, the following inequality holds true

$$\int_{U_d} v^2\, dtds \le C \left[d^2 \int_{U_d} \left| \frac{\partial v}{\partial t} \right|^2 dtds + \int_{U_d} \left| \frac{\partial v}{\partial s} \right|^2 dtds \right] \qquad \text{for any } v \in H^1_{0,\Gamma_d}(U_d) ,$$

where $H^1_{0,\Gamma_d}(U_d)$, accordingly with Lemma 3.4 and Corollary 3.5, is the closure in $H^1(U_d)$ of the space of smooth functions vanishing on Γ_d.

Proof Part (*i*) of the lemma can be proved proceeding as in the proof of Lemma 3.1.

The proof of (*ii*) follows by applying Corollary 3.5 to the function $w(t, s) := v(\Upsilon_d(t, s))$ and using a change of variables. We first observe that, thanks to the assumptions of the lemma, there exist three positive constants c_1, c_2, c_3, independent of $d \in (0, \overline{d})$ and of $(t, s) \in I \times (-d/2, d/2)$, such that

$$c_1 \le d\, |\det(J\Upsilon_d(t, s))| \le c_2 \qquad \text{for any } (t, s) \in I \times \left(-\tfrac{d}{2}, \tfrac{d}{2}\right) \tag{3.36}$$

and

$$|\nabla w(t,s)|^2 \le c_3 \left[\left|\tfrac{\partial v}{\partial t}(\Upsilon_d(t,s))\right|^2 + \tfrac{1}{d^2}\left|\tfrac{\partial v}{\partial s}(\Upsilon_d(t,s))\right|^2\right] \qquad \text{for any } (t,s) \in I \times \left(-\tfrac{d}{2},\tfrac{d}{2}\right). \tag{3.37}$$

By (3.36), (3.37) and Corollary 3.5 we have

$$\begin{aligned}
\int_{U_d} v^2\,dtds &\le \frac{c_2}{d}\int_{I\times(-\frac{d}{2},\frac{d}{2})} w^2\,dtds \le c_2\,C\,d\int_{I\times(-\frac{d}{2},\frac{d}{2})} |\nabla w|^2\,dtds \\
&\le c_2\,c_3\,C\,d\int_{I\times(-\frac{d}{2},\frac{d}{2})}\left[\left|\frac{\partial v}{\partial t}(\Upsilon_d(t,s))\right|^2 + \tfrac{1}{d^2}\left|\frac{\partial v}{\partial s}(\Upsilon_d(t,s))\right|^2\right] dtds \\
&\le \frac{c_2\,c_3\,C}{c_1}\,d^2\int_{U_d}\left(\left|\frac{\partial v}{\partial t}\right|^2 + \frac{1}{d^2}\left|\frac{\partial v}{\partial s}\right|^2\right) dtds\,.
\end{aligned}$$

This completes the proof of the lemma with a new constant C given by $(c_2\,c_3\,C)/c_1$. □

Similarly to what we did in Subsect. 2.1, here we define a rescaled domain that we denote again by $\widetilde{\Omega}_d$. This time, $\widetilde{\Omega}_d$ is defined by rescaling in the vertical direction the deformed domain $\Phi_{i,m}(\Omega_d)$, with $i \in \{1,\ldots,N\}$ and $m \in \{1,\ldots,M_i\}$, and not directly the domain Ω_d. In this way, the domain $\widetilde{\Omega}_d$ depends on the indices i and m but, since we keep them fixed, to simplify notation, we omit them. In other words, we have

$$\widetilde{\Omega}_d := \{(t,s) \in \mathbb{R}^2 : (t, d\,s) \in \Phi_{i,m}(\Omega_d)\}\,.$$

We are in a position to combine together the previous lemmas to obtain the crucial Poincaré-type inequality for functions in $H^1_0(\widetilde{\Omega}_d)$. This inequality will be fundamental to obtain the desired estimates on the rescaled solutions $\widetilde{u}_d$ defined below.

The main difficulty in proving the Poincaré-type inequality is that the domain $\widetilde{\Omega}_d$ is made up of many subdomains and the functions of $H^1_0(\widetilde{\Omega}_d)$ vanish only on a portion of their boundaries. Indeed, those functions do not vanish on the junctions where each subdomain is connected to all the other subdomains adjacent to it. This difficulty justifies the presence of Lemmas 3.2-3.6.

Once we have fixed the indices i and m, we can define on the domain $\widetilde{\Omega}_d$ the function $\widetilde{u}_d(t,s) := \frac{1}{d^2}\,w_d(t,d\,s)$ with w_d as in (3.9). Then the function $\widetilde{u}_d$ solves the boundary value problem

$$\begin{cases} -\nabla_d\cdot(\widetilde{A}_d\,\nabla_d\,\widetilde{u}_d) = \frac{1}{d^2}\,\widetilde{f}_d & \text{in } \widetilde{\Omega}_d\,,\\ \widetilde{u}_d = 0 & \text{on } \partial\widetilde{\Omega}_d\,, \end{cases} \tag{3.38}$$

where $\widetilde{A}_d(t,s) := A(t,d\,s)$, $\widetilde{f}_d(t,s) := f(t,d\,s)$ and $\nabla_d := (\partial_t, d^{-1}\partial_s)$ is the vector operator which acts on scalar functions v and vector fields $V = (v_1, v_2)$ in the following way

$$\nabla_d\,v := \frac{\partial v}{\partial t}\,\mathrm{i} + \frac{1}{d}\,\frac{\partial v}{\partial s}\,\mathrm{j}\,, \qquad \nabla_d\cdot V := \frac{\partial v_1}{\partial t} + \frac{1}{d}\,\frac{\partial v_2}{\partial s}\,.$$

We now prove the following

Proposition 3.7 *Suppose that all the assumptions of Theorem 1.2 hold true. Then there exist a positive constant C and $\overline{d} > 0$ such that for any $d \in (0, \overline{d})$, the following Poincaré-type inequality holds true*

$$\int_{\widetilde{\Omega}_d} w^2\, dtds \le C\, d^2 \int_{\widetilde{\Omega}_d} |\nabla_d\, w|^2\, dtds \qquad for\ any\ w \in H^1_0(\widetilde{\Omega}_d)\,.$$

Moreover, we also have $|\widetilde{\Omega}_d| = O(1)$ as $d \to 0^+$.

Proof From (1.7)-(1.10), from the construction of the curves $\boldsymbol{\eta}_i$ in (1.12) and the properties of the deformation $\Phi_{i,m}$ stated in Lemma 3.1, we deduce that for any $j \in \{1, \dots, N\}$, the curves $\widetilde{\boldsymbol{\eta}}_j = \Phi_{i,m} \circ \boldsymbol{\eta}_j$ satisfy the conditions assumed in Lemma 3.6 on the curve $\boldsymbol{\gamma}$. Now, in view of Lemma 3.2, every function $w \in H^1_0(\Phi_{i,m}(\Omega_d))$ can be trivially extended outside $\Phi_{i,m}(\Omega_d)$ and then restricted to $V(\widetilde{\boldsymbol{\eta}}_j, L\,d)$.

Keeping *j* fixed, let us consider the values of *k* as in Lemma 3.3 (*iv*) and the corresponding sets $K^+_{j,k,Ld}$ and $K^-_{j,k,Ld}$. Moreover, let us define

$$G^+_{j,d} := \mathbb{R} \setminus \bigcup_k K^+_{j,k,Ld} \quad \text{and} \quad G^-_{j,d} := \mathbb{R} \setminus \bigcup_k K^-_{j,k,Ld}\,,$$

where the unions are extended to the values of *k* described above, and the set

$$G_{j,d} := \left(G^+_{j,d} \times \left\{\tfrac{Ld}{2}\right\}\right) \cup \left(G^-_{j,d} \times \left\{-\tfrac{Ld}{2}\right\}\right)\,.$$

Then functions $w \in H^1_0(\Phi_{i,m}(\Omega_d))$ can be interpreted, in the sense explained above, as functions in $H^1_{0,\Gamma_{j,\overline{d}}(G_{j,d})}(V(\widetilde{\boldsymbol{\eta}}_j, L\,d))$ with $\Gamma_{j,\overline{d}}(t,s) = \widetilde{\boldsymbol{\eta}}_j(t) + s\mathrm{n}(t)$ defined as in Lemma 3.3. We observe that the set $G_{j,d}$ defined here satisfies the assumptions of the set G_d defined in Corollary 3.5 and Lemma 3.6, due to Lemma 3.3 (*iv*). Then, according to the notation introduced in Lemma 3.6, we can define the map $\Upsilon_{j,d}$ associated with $\widetilde{\boldsymbol{\eta}}_j = (\widetilde{\eta}_{j,1}, \widetilde{\eta}_{j,2})$:

$$\Upsilon_{j,d}(t,s) := \left(\widetilde{\eta}_{j,1}(t) + s\mathrm{n}_1(t), \tfrac{1}{d}\,\widetilde{\eta}_{j,2}(t) + \tfrac{s}{d}\,\mathrm{n}_2(t)\right) \qquad \text{for any } (t,s) \in \mathbb{R} \times \left(-\tfrac{Ld}{2}, \tfrac{Ld}{2}\right)\,.$$

For *d* small the map $\Upsilon_{j,d}$ defined here is still a homeomorphism despite the fact that the variable s may vary in the set $\left(-\frac{Ld}{2}, \frac{Ld}{2}\right)$ and not in $\left(-\frac{d}{2}, \frac{d}{2}\right)$ as in Lemma 3.6. Then we define the rescaling map $T_{s,d}(t,s) := (t, d^{-1}\, s)$ and the sets $\Gamma_{j,d} := \Upsilon_{j,d}(G_{j,d})$, $U^j_d := T_{s,d}(V(\widetilde{\boldsymbol{\eta}}_j, L\,d))$ in such a way that a function $w \in H^1_0(\widetilde{\Omega}_d)$, when extended trivially outside $\widetilde{\Omega}_d$ and then restricted to U^j_d, becomes a function of the space $H^1_{0,\Gamma_{j,d}}(U^j_d)$.

The proof of the lemma then follows applying Lemma 3.6 to the space $H^1_{0,\Gamma_{j,d}}(U^j_d)$, then restricting this functional inequality to functions vanishing outside $\widetilde{\Omega}_d$. Indeed, for $w \in H^1_0(\widetilde{\Omega}_d)$ we have

$$
\begin{aligned}
\int_{\widetilde{\Omega}_d} w^2\,dtds &\le \sum_{j=1}^{N} \int_{\widetilde{\Omega}_d\cap U_d^j} w^2\,dtds \le \sum_{j=1}^{N} C_j\,d^2 \int_{U_d^j} |\nabla_d w|^2\,dtds \\
&= \sum_{j=1}^{N} C_j\,d^2 \int_{\widetilde{\Omega}_d\cap U_d^j} |\nabla_d w|^2\,dtds \le d^2 \left(\sum_{j=1}^{N} C_j\right) \int_{\widetilde{\Omega}_d} |\nabla_d w|^2\,dtds\,.
\end{aligned}
$$

Concerning the last statement of the proposition about $|\widetilde{\Omega}_d|$, we first estimate $|V(\widetilde{\boldsymbol{\eta}}_j, L\,d)\cap\Phi_{i,m}(\Omega_{\overline{d}})|$ with $d\in(0,\overline{d})$ for some $\overline{d}$ small enough. Indeed, for some $R>0$ large enough, we have

$$
|V(\widetilde{\boldsymbol{\eta}}_j, L\,d)\cap\Phi_{i,m}(\Omega_{\overline{d}})| \le \int_{(-R,R)\times\left(-\frac{Ld}{2},\frac{Ld}{2}\right)} |\det(J\Gamma_{j,\overline{d}}(t,s)|\,dtds = O(d) \qquad \text{as } d\to 0^+\,. \tag{3.39}
$$

It remains to apply the scaling $T_{s,d}$ and observe that

$$
\widetilde{\Omega}_d \subseteq \bigcup_{j=1}^{N} T_{s,d}\Big(V(\widetilde{\boldsymbol{\eta}}_j, L\,d)\cap\Phi_{i,m}(\Omega_{\overline{d}})\Big)
$$

□

We now have the tools to apply again the procedure used in Subsection 2.1 in the case of the H-shaped domain. The next lemma is the analogous of Lemma 2.2 adapted to the present case.

Lemma 3.8 *Let $\widetilde{u}_d$ be as in (3.38). Then $\|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)}$ remains bounded as $d\to 0^+$. In particular, we also have $\left\|\frac{\partial\widetilde{u}_d}{\partial s}\right\|_{L^2(\widetilde{\Omega}_d)}$ and $\left\|d\,\frac{\partial\widetilde{u}_d}{\partial t}\right\|_{L^2(\widetilde{\Omega}_d)}$ remain bounded as $d\to 0^+$.*

Proof Testing (3.38) with $\widetilde{u}_d$ and using (3.11), (3.12), Proposition 3.7 and the Hölder inequality, we obtain

$$
\begin{aligned}
\int_{\widetilde{\Omega}_d} |\widetilde{u}_d|^2\,dtds &\le Cd^2 \int_{\widetilde{\Omega}_d} |\nabla_d\,\widetilde{u}_d|^2\,dtds \le \frac{C}{\alpha_0}\,d^2 \int_{\widetilde{\Omega}_d} (\widetilde{A}_d\nabla_d\,\widetilde{u}_d)\cdot\nabla_d\,\widetilde{u}_d\,dtds = \frac{C}{\alpha_0}\int_{\widetilde{\Omega}_d} \widetilde{f}_d\,\widetilde{u}_d\,dtds \\
&\le \frac{C}{\alpha_0}\,\|f\|_{L^\infty(\Phi_{i,m}(\Omega_d))}\,|\widetilde{\Omega}_d|^{\frac{1}{2}}\,\|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)}
\end{aligned}
$$

and hence, considering the first and the last term, we immediately infer $\|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)} = O(1)$ as $d\to 0^+$. Once we have $\|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)} = O(1)$, we again use the previous estimates to obtain $d^2\int_{\widetilde{\Omega}_d}|\nabla_d\widetilde{u}_d|^2\,dtds = O(1)$ as $d\to 0^+$ and hence the conclusion. □

We now prove an extended version of Lemma 2.3.

Lemma 3.9 *Let $\widetilde{u}_d\in H_0^1(\widetilde{\Omega}_d)$ be as in (3.38).*

Then, letting $I_{i,m}$ be as in Lemma 3.1 and $\mathcal{R}_{i,m} := I_{i,m}\times\left(-\frac{1}{2},\frac{1}{2}\right)$, we have

$$\widetilde{u}_d \rightharpoonup w \; weakly\ in\ L^2\left(\mathcal{R}_{i,m}\right), \quad \frac{\partial \widetilde{u}_d}{\partial y} \rightharpoonup \frac{\partial w}{\partial y} \; weakly\ in\ L^2\left(\mathcal{R}_{i,m}\right), \quad as\ d \to 0^+,$$

for some function $w \in L^2\left(\mathcal{R}_{i,m}\right)$ *with* $\frac{\partial w}{\partial y} \in L^2\left(\mathcal{R}_{i,m}\right)$.

Moreover, w is independent of the variable t and it is given by $w(t,s) = -\frac{1}{2}\left(s^2 - \frac{1}{4}\right)$.

Proof Let v be a test function in $C_c^\infty(\mathcal{R}_{i,m}) \subset H_0^1(\widetilde{\Omega}_d)$ in such a way that

$$\int_{\mathcal{R}_{i,m}} \left[\widetilde{A}_d \left(d\,\frac{\partial \widetilde{u}_d}{\partial t}, \frac{\partial \widetilde{u}_d}{\partial s}\right)^T\right] \cdot \left(d\,\frac{\partial v}{\partial t}, \frac{\partial v}{\partial s}\right) dtds = \int_{\mathcal{R}_{i,m}} \widetilde{f}_d\, v\, dtds \tag{3.40}$$

and by Lemma 3.8, along a sequence $d_n \downarrow 0$, we infer that there exist two functions $\psi \in L^2(\mathcal{R}_{i,m})$ and $w \in L^2(\mathcal{R}_{i,m})$, with $\frac{\partial w}{\partial s} \in L^2(\mathcal{R}_{i,m})$, such that

$$\widetilde{u}_{d_n} \rightharpoonup w \text{ weakly in } L^2(\mathcal{R}_{i,m}), \frac{\partial \widetilde{u}_{d_n}}{\partial s} \rightharpoonup \frac{\partial w}{\partial s} \text{ weakly in } L^2(\mathcal{R}_{i,m}), d_n\frac{\partial \widetilde{u}_{d_n}}{\partial t} \rightharpoonup \psi \text{ weakly in } L^2(\mathcal{R}_{i,m}).$$

On the other hand, by the definitions of $\widetilde{f}_d$ and $\widetilde{A}_d$ appearing below (3.38), and by (3.10), we deduce that, as $d \to 0^+$,

$$\widetilde{f}_d \to f(t,0) = |\gamma_i'(t)| \qquad \text{uniformly in } \mathcal{R}_{i,m}\,. \tag{3.41}$$

and

$$\widetilde{A}_d \to A(t,0) = \begin{pmatrix} |\gamma_i'(t)|^{-1} & 0 \\ 0 & |\gamma_i'(t)| \end{pmatrix} \qquad \text{uniformly in } \mathcal{R}_{i,m}\,. \tag{3.42}$$

Then, passing to the limit in (3.40) as $n \to +\infty$, we obtain

$$\int_{\mathcal{R}_{i,m}} |\gamma_i'(t)| \frac{\partial w}{\partial s}\,\frac{\partial v}{\partial s}\, dsdt = \int_{\mathcal{R}_{i,m}} |\gamma_i'(t)|\, v\, dtds \qquad \text{for any } v \in C_c^\infty(\mathcal{R}_{i,m})$$

which means that $\frac{\partial^2 w}{\partial s^2} = -1$ in the sense of distributions.

Then one can proceed similarly to the proof of Lemma 2.5 by exploiting the fact that $\widetilde{u}_{d_n}$ vanishes on the two horizontal edges of the rectangle $\mathcal{R}_{i,m}$ expect for a finite number of regions whose diameters vanish as $n \to +\infty$. More precisely, according to the notation of Lemma 3.3, we introduce the sets $K^+_{i,k,d}$ and $K^-_{i,k,d}$ for all values of k as in Lemma 3.3 (*iii*)-(*iv*) and we define $G^+_{i,m,d} := I_{i,m} \setminus \bigcup_k K^+_{i,k,d}$, $G^-_{i,m,d} := I_{i,m} \setminus \bigcup_k K^-_{i,k,d}$ where the unions are extended to the values of k described above. We also define the set $G_{i,m,d} := \left(G^+_{i,m,d} \times \left\{\frac{1}{2}\right\}\right) \cup \left(G^-_{i,m,d} \times \left\{-\frac{1}{2}\right\}\right) \subseteq \partial\mathcal{R}_{i,m}$. Thanks to Lemmas 3.2-3.3, it is not restrictive to assume that the set valued maps $d \mapsto G^+_{i,m,d}$ and $d \mapsto G^-_{i,m,d}$ are monotone decreasing with respect to the inclusion when d is small enough, and that their 1-dimensional Lebesgue measures $|I_{i,m} \setminus G^+_{i,m,d}|$ and $|I_{i,m} \setminus G^-_{i,m,d}|$ tend to zero as $d \to 0^+$.

Then, we can define $C^\infty_{0,G_{i,m,d}}(\overline{\mathcal{R}}_{i,m})$ as the space of $C^\infty(\overline{\mathcal{R}}_{i,m})$-functions vanishing on $G_{i,m,d}$ and the space $H_{0,G_{i,m,d}}(\mathcal{R}_{i,m}) \subset L^2(\mathcal{R}_{i,m})$ as the completion of $C^\infty_{0,G_{i,m,d}}(\overline{\mathcal{R}}_{i,m})$ with respect to the norm

$$v \mapsto \left(\|v\|^2_{L^2(\mathcal{R}_{i,m})} + \left\| \tfrac{\partial v}{\partial s} \right\|^2_{L^2(\mathcal{R}_{i,m})} \right)^{\frac{1}{2}} .$$

In this way, once we fix an arbitrarily small $\overline{d}$, we see that $\{\widetilde{u}_{d_n}\}$ is a bounded sequence in $H_{0,G_{i,m,\overline{d}}}(\mathcal{R}_{i,m})$ and also weakly convergent to *w* in $H_{0,G_{i,m,\overline{d}}}(\mathcal{R}_{i,m})$ as $n \to +\infty$.

This shows that $w \in H_{0,G_{i,m,\overline{d}}}(\mathcal{R}_{i,m})$ for any $\overline{d}$ small and therefore proceeding as in the proof of Lemma 2.5, we may conclude that $w(t,s) = -\frac{1}{2}\left(s^2 - \frac{1}{4}\right)$ for any $(t,s) \in (G^+_{i,m,\overline{d}} \cup G^-_{i,m,\overline{d}}) \times (-1/2, 1/2)$. Letting $\overline{d} \to 0^+$ we deduce that $w(t,s) = -\frac{1}{2}\left(s^2 - \frac{1}{4}\right)$ a.e. in $\mathcal{R}_{i,m}$.

This shows that the weak limit does not depend on the sequence $d_n \downarrow 0$, thus proving that the convergence $\widetilde{u}_d \rightharpoonup w$ holds true as $d \to 0^+$ and completing the proof of the lemma. □

We now proceed with the statement of the last lemma which is the natural generalization of Lemma 2.4.

Lemma 3.10 *Let $\widetilde{u}_d \in H^1_0(\widetilde{\Omega}_d)$ be as in (3.38) and let $\mathcal{R}_{i,m}$ be as in Lemma 3.9. Then*

$$\lim_{d\to 0^+} \int_{\mathcal{R}_{i,m}} d^2 \left(\widetilde{A}_d \nabla_d \widetilde{u}_d \right) \cdot \nabla_d \widetilde{u}_d \, dtds = \frac{\mathrm{length}(\gamma_{i|I_{i,m}})}{12} , \tag{3.43}$$

and, recalling the notation of Lemma 3.1, for any $j \in \{1, \ldots, N\} \setminus \{i\}$ and $m \in \{1, \ldots, M_i\}$ such that the set $Z_{i,m,d} \cap U_{j,d} \neq \emptyset$, we have

$$\lim_{d\to 0^+} \int_{\mathcal{R}_{i,m} \cap \widetilde{U}_{i,m,j,d}} d^2 \left(\widetilde{A}_d \nabla_d \widetilde{u}_d \right) \cdot \nabla_d \widetilde{u}_d \, dtds = 0 , \tag{3.44}$$

where $\widetilde{U}_{i,m,j,d} := T_{s,d}(\Phi_{i,m}(U_{j,d}))$ and $T_{s,d}(t,s) = (t, d^{-1}s)$, and for any $(i,j) \in \mathcal{I}$ and $m \in \{1, M_i\}$, we also have

$$\lim_{d\to 0^+} \int_{\widetilde{V}_{i,m,j,d}} d^2 \left(\widetilde{A}_d \nabla_d \widetilde{u}_d \right) \cdot \nabla_d \widetilde{u}_d \, dtds = 0 , \tag{3.45}$$

where $V_{i,m,j,d} := T_{s,d}(\Phi_{i,m}(V_{i,j,d}))$.

Proof Recalling from Lemma 3.1 the definition $I_{i,m} = (t_{i,m-1}, t_{i,m})$, let

$$\Sigma_d := \widetilde{\Omega}_d \cap \left((t_{i,m-1} - \sqrt{d}, t_{i,m} + \sqrt{d}) \times \left(-\tfrac{1}{\sqrt{d}}, \tfrac{1}{\sqrt{d}} \right) \right) .$$

We claim that

$$|\Sigma_d \setminus \mathcal{R}_{i,m}| = O(\sqrt{d}) \qquad \text{as } d \to 0^+ . \tag{3.46}$$

For this purpose, let $U_{j,d}$ be such that $Z_{i,m,d} \cap U_{j,d} \neq \emptyset$, where we recall that the sets $Z_{i,m,d}$ were defined in Lemma 3.1. Thanks to (1.7) and (1.10) we deduce that for points of the portion of the curve $\widetilde{\boldsymbol{\eta}}_j = \Phi_{i,m} \circ \boldsymbol{\eta}_j$ contained in the strip $\mathbb{R} \times (-\sqrt{d}, \sqrt{d})$, the tangent vector to the curve itself at those points has a nontrivial vertical component and therefore for the same points, the parameter t of the curve varies in an interval J_d whose length is $O(\sqrt{d})$ as $d \to 0^+$.

Recalling (3.39), we deduce that

$$\left|\Phi_{i,m}(U_{j,d}) \cap \left(\mathbb{R} \times (-\sqrt{d}, \sqrt{d})\right)\right| \le \int_{J_d \times \left(-\frac{Ld}{2}, \frac{Ld}{2}\right)} |\det(J\Gamma_{j,\overline{d}}(t,s)|\, dtds = O\left(d^{\frac{3}{2}}\right) \qquad \text{as } d \to 0^+ .$$

When we consider the image of the last set after applying the map $T_{s,d}$, we infer that

$$\left|\widetilde{U}_{i,m,j,d} \cap \left(\mathbb{R} \times \left(-\tfrac{1}{\sqrt{d}}, \tfrac{1}{\sqrt{d}}\right)\right)\right| = O(\sqrt{d}) \qquad \text{as } d \to 0^+ . \tag{3.47}$$

Concerning the intersection between the rectangle $\left((t_{i,m-1} - \sqrt{d}, t_{i,m} + \sqrt{d}) \times \left(-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}\right)\right)$ and the sets $T_{s,d}(\Phi_{i,m}(U_{i,m-1}))$ and $T_{s,d}(\Phi_{i,m}(U_{i,m+1}))$, we see that for points of the portion of the curve $\widetilde{\boldsymbol{\eta}}_i = \Phi_{i,m} \circ \boldsymbol{\eta}_i$ contained in the rectangle $\left((t_{i,m-1} - \sqrt{d}, t_{i,m} + \sqrt{d}) \times (\sqrt{d}, \sqrt{d})\right)$, the parameter t of the curve varies in intervals whose length is $O(\sqrt{d})$ as $d \to 0^+$ thanks to (1.7). Proceeding as above, we then have

$$\left|(\Phi_{i,m}(U_{i,m-1}) \cup \Phi_{i,m}(U_{i,m+1})) \cap \left((t_{i,m-1} - \sqrt{d}, t_{i,m} + \sqrt{d}) \times (-\sqrt{d}, \sqrt{d})\right)\right| = O\left(d^{\frac{3}{2}}\right) \quad \text{as } d \to 0^+$$

and, in turn,

$$\left|T_{s,d}\left(\Phi_{i,m}(U_{i,m-1}) \cup \Phi_{i,m}(U_{i,m+1})\right) \cap \left((t_{i,m-1} - \sqrt{d}, t_{i,m} + \sqrt{d}) \times \left(-\tfrac{1}{\sqrt{d}}, \tfrac{1}{\sqrt{d}}\right)\right)\right| = O(\sqrt{d}) \tag{3.48}$$

as $d \to 0^+$. Combining (3.47) and (3.48) we obtain the proof of (3.46).

In the argument below, we also need the following estimates:

$$|\mathcal{R}_{i,m} \cap \widetilde{U}_{i,m,j,d})| = O(d) \qquad \text{as } d \to 0^+ , \tag{3.49}$$

and when $(i,j) \in \mathcal{I}$ and $m \in \{1, M_i\}$

$$|\widetilde{V}_{i,m,j,d}| = O(d) \quad \text{as } d \to 0^+ \quad \text{and} \quad \widetilde{V}_{i,m,j,d} \subset \Sigma_d \setminus \mathcal{R}_{i,m} \quad \text{for } d \text{ small enough}, \tag{3.50}$$

that both follows from the arguments used in Lemmas 3.2-3.3 and the definition of Σ_d.

By direct computation, we see that w satisfies the identity

$$\int_{\mathcal{R}_{i,m}} |\gamma_i'(t)| \left|\frac{\partial w}{\partial s}\right|^2 dtds = \int_{\mathcal{R}_{i,m}} |\gamma_i'(t)| w\, dtds .$$

Let $\eta, \phi \in C_c^\infty(\mathbb{R})$ be two cut-off functions such that $\eta > 0$ in $(-1, 1)$, $\eta \equiv 1$ in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, $\eta \equiv 0$ outside $(-1, 1)$, $\phi > 0$ in $(-\infty, 1)$, $\phi \equiv 1$ in $(-\infty, 0]$, $\phi \equiv 0$ in $[1, +\infty)$. Then we define $\eta_d(s) := \eta(\sqrt{d}\, s)$ and ϕ_d so that $\phi_d \equiv 1$ in $\overline{I}_{i,m}$, $\phi_d(t) := \phi((t - t_{i,m})/\sqrt{d})$ if $t > t_{i,m}$, $\phi_d(t) := \phi((t_{i,m-1} - t)/\sqrt{d})$ if $t < t_{i,m-1}$.

Then, by Lemmas 3.8-3.9, weak lower semicontinuity of weighted L^2-norms, (3.40) with $v(t,s) = \phi_d(t)\eta_d(s)\,\widetilde{u}_d(t,s)$, (3.42), Lemma 3.1, (3.11), (3.12), (3.46), (3.41), the Hölder inequality and Lemma 3.9, we obtain

$$
\begin{aligned}
&\int_{\mathcal{R}_{i,m}} |\gamma_i'(t)| w\, dtds = \int_{\mathcal{R}_{i,m}} |\gamma_i'(t)| \left|\frac{\partial w}{\partial s}\right|^2 dtds \le \liminf_{d\to 0^+} \int_{\mathcal{R}_{i,m}} |\gamma_i'(t)| \left|\frac{\partial \widetilde{u}_d}{\partial s}\right|^2 dtds \\
&\le \liminf_{d\to 0^+} \int_{\mathcal{R}_{i,m}} d^2\left(A(t,0)\nabla_d \widetilde{u}_d\right)\cdot \nabla_d \widetilde{u}_d\, dtds = \liminf_{d\to 0^+} \int_{\mathcal{R}_{i,m}} d^2 \left(\widetilde{A}_d \nabla_d \widetilde{u}_d\right)\cdot \nabla_d \widetilde{u}_d\, dtds \\
&\le \limsup_{d\to 0^+} \int_{\mathcal{R}_{i,m}} d^2 \left(\widetilde{A}_d \nabla_d \widetilde{u}_d\right)\cdot \nabla_d \widetilde{u}_d\, dtds \le \limsup_{d\to 0^+} \int_{\Sigma_d} d^2\, \phi_d(t)\eta_d(s) \left(\widetilde{A}_d \nabla_d \widetilde{u}_d\right)\cdot \nabla_d \widetilde{u}_d\, dtds \\
&= \limsup_{d\to 0^+} \Bigg[\int_{\Sigma_d} \phi_d(t)\eta_d(s)\widetilde{f}_d\,\widetilde{u}_d\, dtds - d^2 \int_{\Sigma_d} \left(\widetilde{A}_d\,\nabla_d \widetilde{u}_d\right)\cdot\left(0, \frac{1}{d}\,\phi_d(t)\,\eta_d'(s)\,\widetilde{u}_d\right) dtds \\
&\qquad\qquad - d^2 \int_{\Sigma_d} \left(\widetilde{A}_d\,\nabla_d \widetilde{u}_d\right)\cdot\left(\phi_d'(t)\,\eta_d(s)\,\widetilde{u}_d, 0\right)\, dtds\Bigg] \\
&\le \limsup_{d\to 0^+} \Bigg[\int_{\mathcal{R}_{i,m}} \widetilde{f}_d\,\widetilde{u}_d\, dtds + \int_{\Sigma_d\setminus\mathcal{R}_{i,m}} \phi_d(t)\eta_d(s)\widetilde{f}_d\,\widetilde{u}_d\, dtds \\
&\qquad + \sqrt{d}\, \|A\|_{L^\infty(\mathbb{R}^2)} \left(\|\phi'\|_{L^\infty(\mathbb{R})} + \|\eta'\|_{L^\infty(\mathbb{R})}\right) \left(\left\|d\,\frac{\partial \widetilde{u}_d}{\partial t}\right\|^2_{L^2(\widetilde{\Omega}_d)} + \left\|\frac{\partial \widetilde{u}_d}{\partial s}\right\|^2_{L^2(\widetilde{\Omega}_d)}\right)^{\frac{1}{2}} \|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)}\Bigg] \\
&\le \limsup_{d\to 0^+} \left[\int_{\mathcal{R}_{i,m}} \widetilde{f}_d\,\widetilde{u}_d\, dtds + |\Sigma_d \setminus \mathcal{R}_{i,m}|^{\frac{1}{2}}\, \|f\|_{L^\infty(\mathbb{R}^2)} \|\widetilde{u}_d\|_{L^2(\widetilde{\Omega}_d)} + O\left(\sqrt{d}\right)\right] \\
&= \lim_{d\to 0^+} \left[\int_{\mathcal{R}_{i,m}} \widetilde{f}_d\,\widetilde{u}_d\, dtds + O\left(\sqrt[4]{d}\right)\right] = \int_{\mathcal{R}_{i,m}} |\gamma'(t)| w\, dtds = \frac{\mathrm{length}(\gamma_{i|I_{i,m}})}{12}\,.
\end{aligned}
\tag{3.51}
$$

We observe that the first term and the last term in the previous chain of inequalities coincide, thus showing the validity of (3.43) and the strong convergences

$$
\frac{\partial \widetilde{u}_d}{\partial s} \to \frac{\partial w}{\partial s} \quad \text{strongly in } L^2(\mathcal{R}_{i,m}), \quad \widetilde{u}_d \to w \quad \text{strongly in } L^2(\mathcal{R}_{i,m}), \tag{3.52}
$$

where for the second convergence, we also use Proposition 3.7.

Combining (3.52) with (3.49) we also obtain (3.44). To show the validity of (3.45), we use (3.50) and we observe that the cut-off functions η and ϕ are 1 in $\widetilde{V}_{i,m,j,d}$ so that

$$
\int_{\widetilde{V}_{i,m,j,d}} d^2 \left(\widetilde{A}_d \nabla_d \widetilde{u}_d\right)\cdot \nabla_d \widetilde{u}_d\, dtds \le \int_{\Sigma_d\setminus\mathcal{R}_{i,m}} d^2\, \phi_d(t)\eta_d(s) \left(\widetilde{A}_d \nabla_d \widetilde{u}_d\right)\cdot \nabla_d \widetilde{u}_d\, dtds \to 0
$$

as $d \to 0^+$, where the convergence to zero of the last term can be deduced from (3.51).

This completes the proof of the lemma. □

End of the proof of Theorem 1.2. For any $i \in \{1, \ldots, N\}$ and $m \in \{1, \ldots, M_i\}$, let us define the sets

$$S_{i,d} := U_{i,d} \setminus \bigcup_{j\in\{1,\dots,N\}\setminus\{i\}} U_{j,d}\,, \qquad \widetilde{S}_{i,m,d} := T_{s,d}(\Phi_{i,m}(S_{i,d}))$$

so that recalling the notation used in this section, recalling that $\widetilde{u}_d$ depends on the indices *i* and *m*, by (3.43)-(3.45), we obtain

$$\begin{aligned}
K_d &= 4\int_{\Omega_d} |\nabla u_d|^2 dxdy = 4\sum_{i=1}^{N}\int_{S_{i,d}} |\nabla u_d|^2 dxdy + 4\sum_{(i,j)\in\mathcal{I},\, i<j}\int_{V_{i,j,d}} |\nabla u_d|^2 dxdy\\
&= 4d^3\sum_{i=1}^{N}\sum_{m=1}^{M_i}\int_{\widetilde{S}_{i,m,d}} d^2\left(\widetilde{A}_d\nabla_d\,\widetilde{u}_d\right)\cdot\nabla_d\,\widetilde{u}_d\,dtds\\
&\quad + 4d^3\sum_{(i,j)\in\mathcal{I},\, i<j}\ \sum_{m\in\{1,M_i\}}\int_{\widetilde{V}_{i,m,j,d}} d^2\left(\widetilde{A}_d\nabla_d\,\widetilde{u}_d\right)\cdot\nabla_d\,\widetilde{u}_d\,dtds\\
&= 4d^3\sum_{i=1}^{N}\sum_{m=1}^{M_i}\int_{\mathcal{R}_{i,m}} d^2\left(\widetilde{A}_d\nabla_d\,\widetilde{u}_d\right)\cdot\nabla_d\,\widetilde{u}_d\,dtds\\
&\quad - 4d^3\sum_{i=1}^{N}\sum_{m=1}^{M_i}\sum_{j\neq i}\int_{\mathcal{R}_{i,m}\cap\widetilde{U}_{i,m,j,d}} d^2\left(\widetilde{A}_d\nabla_d\,\widetilde{u}_d\right)\cdot\nabla_d\,\widetilde{u}_d\,dtds\\
&\quad + 4d^3\sum_{(i,j)\in\mathcal{I},\, i<j}\ \sum_{m\in\{1,M_i\}}\int_{\widetilde{V}_{i,m,j,d}} d^2\left(\widetilde{A}_d\nabla_d\,\widetilde{u}_d\right)\cdot\nabla_d\,\widetilde{u}_d\,dtds\\
&= 4d^3\left(\sum_{i=1}^{N}\sum_{m=1}^{M_i}\frac{\mathrm{length}(\boldsymbol{\gamma}_{i|I_{i,m}})}{12} + o(1)\right) - 4d^3\cdot o(1) + 4d^3\cdot o(1)\\
&= \sum_{i=1}^{N}\frac{1}{3}\,\mathrm{length}(\boldsymbol{\gamma}_i)\,d^3 + o(d^3)\,.
\end{aligned}$$

This completes the proof of the theorem.

Proof of Corollary 1.4 In order to prove the corollary we need the asymptotic estimates on $T(\Omega_d)$, $\mathrm{Per}(\Omega_d)$ and $|\Omega_d|$. For the first one it is enough to use Theorem 1.2 and recall that $T(\Omega_d) = \frac{1}{4}\,K_d$. Let us proceed with the perimeter.

Let us define the map $\Gamma_i : [a_i, b_i]\times\mathbb{R}\to\mathbb{R}^2$ defined by $\Gamma_i(t,s) := \boldsymbol{\gamma}_i(t) + s\mathrm{n}(t)$ where, as usual, n denotes the unit normal vector to $\boldsymbol{\gamma}_i$, see (1.11) for more details. Let us estimate the length of the curves $t\mapsto\boldsymbol{\gamma}_i(t, d/2)$ and $t\mapsto\boldsymbol{\gamma}_i(t,-d/2)$, for $t\in[a_i,b_i]$ in both cases. For simplicity, we only consider the case $s = d/2$ being the other one completely equivalent:

$$\int_{a_i}^{b_i}\left|\frac{\partial\Gamma_i}{\partial t}\left(t,\frac{d}{2}\right)\right| dt = \int_{a_i}^{b_i}\left|\boldsymbol{\gamma}_i'(t) + \frac{d}{2}\,\mathrm{n}'(t)\right| dt = \mathrm{length}(\boldsymbol{\gamma}_i) + O(d) \qquad \text{as } d\to 0^+\,. \tag{3.53}$$

Actually, only a portion of the two curves $t \mapsto \boldsymbol{\gamma}_i(t, d/2)$ and $t \mapsto \boldsymbol{\gamma}_i(t, -d/2)$ belongs to $\partial\Omega_d$ since we have to exclude the small portions that may belong to the interior of adjacent sets of the type $U_{j,d}$ for some $j \in \{1, \dots, N\}$. But, looking for example at Lemma 3.3 (*iii*), we see that those small portions can be parametrized by values of *t* belonging to sets contained in intervals of length converging to 0 like *O*(*d*) as $d \to 0^+$. Hence, the lengths of these portions behave like *O*(*d*) as $d \to 0^+$. In our setting, the situation is even simpler since we do not have to do with the deformation $\Phi_{i,m}$ used in that lemma.

On the other hand, we also observe that $\partial\Omega_d$ also contains portions of the boundary of the sets $V_{i,j,d}$ that can be found in (1.13). Invoking again Lemma 3.3 (*iii*), we may conclude that the lengths of these small portions also behave like *O*(*d*) as $d \to 0^+$. A contribution to the perimeter could also be given by the sets in the form $\{\boldsymbol{\gamma}_i(a_i) + s\,\mathrm{n} : s \in (-d/2, d/2)\}$ and $\{\boldsymbol{\gamma}_i(b_i) + s\,\mathrm{n} : s \in (-d/2, d/2)\}$ that, however, have a length which is *d*.

Collecting the above arguments, we may conclude that

$$\mathrm{Per}(\Omega_d) = 2\sum_{i=1}^{N} \mathrm{length}(\boldsymbol{\gamma}_i) + O(d) \qquad \text{as } d \to 0^+ . \tag{3.54}$$

For the area of Ω_d we can proceed by estimating the areas of the sets $U_{i,d}$. Indeed, it is enough to use again the map Γ_i employed for the estimate in (3.53):

$$\begin{aligned} |U_{i,d}| &= \int_{(a_i,b_i)\times\left(-\frac{d}{2},\frac{d}{2}\right)} |\det(J\Gamma_i(t,s))|\, dtds = \int_{(a_i,b_i)\times\left(-\frac{d}{2},\frac{d}{2}\right)} (1 - s\kappa(t))|\boldsymbol{\gamma}_i'(t)|\, dtds \\ &= d\int_{a_i}^{b_i} |\boldsymbol{\gamma}_i'(t)|\, dt - \int_{-\frac{d}{2}}^{\frac{d}{2}} s\, ds \cdot \int_{a_i}^{b_i} \kappa(t)|\boldsymbol{\gamma}_i(t)|\, dt = \mathrm{length}(\boldsymbol{\gamma}_i)\, d \end{aligned}$$

where κ is the curvature of $\boldsymbol{\gamma}_i$ in such a way that n' satisfies $\mathrm{n}' = -\kappa\boldsymbol{\gamma}_i'$.

Invoking Lemma 3.3 (*ii*), we observe that the diameter of the intersection of two possibly intersecting sets $U_{i,d}$ and $U_{j,d}$ behaves like *O*(*d*) and so it does a set of the type $V_{i,j,d}$, hence we deduce that the area of all of them behaves like $O(d^2)$ as $d \to 0^+$.

Therefore, we may deduce that

$$|\Omega_d| = \sum_{i=1}^{N} \mathrm{length}(\boldsymbol{\gamma}_i)\, d + O(d^2) \qquad \text{as } d \to 0^+ . \tag{3.55}$$

Finally, for the inner radius we have that $R_{\Omega_d} = d/2$ for every *d* small enough being the curvature of every curve $\boldsymbol{\gamma}_i$ bounded.

The proof of the corollary then follows immediately from (3.54), (3.55), the estimate on the inner radius, the definitions (1.14) and the estimate on $T(\Omega_d)$ coming from Theorem 1.2.

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