International Journal of Quantum Information
(2023) 2350001 (14 pages)

DOI: 10.1142/S0219749923500016

Entropy of temporal entanglement

Leonardo Castellani

Dipartimento di Scienze e Innovazione Tecnologica,
Università del Piemonte Orientale,
viale T. Michel 11, 15121 Alessandria, Italy

INFN, Sezione di Torino, via P. Giuria 1, 10125 Torino, Italy

Arnold-Regge Center, via P. Giuria 1, 10125 Torino, Italy
leonardo.castellani@uniupo.it

Received 1 May 2022
Accepted 21 September 2022
Published

A recently proposed history formalism is used to define temporal entanglement in quantum systems, and compute its entropy. The procedure is based on the time-reduction of the history density operator, and allows a symmetrical treatment of space and time correlations. Temporal entanglement entropy is explicitly calculated in two simple quantum computation circuits.

Keywords: History formulation of quantum mechanics; entanglement entropy; reduced density matrix; temporal correlations.

1. Introduction

There are by now a number of proposals for defining and characterizing temporal entanglement.[1–7] Using the history formalism developed in Refs. 8 and 9, we introduce in this note a time-reduced history density matrix. This tool allows for a symmetrical treatment of spatial and temporal entanglement, much in the spirit of the approach of Refs. 1, 2 and 7, but within a different framework to describe quantum states over time.

Since the work of Feynman[10,11] (see also Dirac[12]), there have been various formulations of quantum mechanics based on histories, rather than on states at a given time. A very partial list of references, relevant for this paper, is given in Refs. 13–28.

Here, we use the history vector formalism introduced in Ref. 9, leading to a simple definition of history density operator for a quantum system. Taking "space" or "time" partial traces of this operator yields reduced density operators, and these can be used to characterize space or time entanglement between subsystems.

The history vector lives in a tensor space $\mathcal{H} \odot \mathcal{H} \cdots \odot \mathcal{H}$, where every $\mathcal{H}$ corresponds to a particular time t_i. The Born rules for probabilities and collapse are extended to history vectors in a straightforward way. Every history vector has a pictorial representation in terms of allowed histories, and its collapse after a measurement sequence entails the disappearance of some histories. As discussed in Ref. 9, this formalism is well suited to define entanglement of histories, and compute their density matrices and corresponding von Neumann entropies.

This approach is similar in spirit to the one advocated in Refs. 24–28, but with substantial differences. In Refs. 24–28, the scalar product between history states depends on chain operators containing information on evolution and measurements. In our framework, the algebraic structure does not depend on the dynamics, and *all* possible histories (not only "consistent" sets) correspond to orthonormal vectors in $\mathcal{H} \odot \mathcal{H} \cdots \odot \mathcal{H}$. The dynamical information is instead encoded in the coefficients (amplitudes) multiplying the basis vectors.

This paper is arranged as follows. We summarize the formalism in Sec. 2. In Sec. 3, the space-reduced history density operator is recalled, and in Sec. 4 we introduce its time-reduced analog. The corresponding von Neumann entropy, discussed in Sec. 5, can be used to detect time correlations. In Sec. 6, we derive temporal entanglement entropies in two examples taken from quantum computation circuits. Section 7 concludes.

2. History Vector Formalism

2.1. *History vector*

A quantum system over time, together with measuring devices that can be activated at times $t_1, \ldots, t_n$, is described by a *history vector* living in n-tensor space $\mathcal{H} \odot \cdots \odot \mathcal{H}$:

$$|\Psi\rangle = \sum_{\alpha} A(\psi, \alpha) |\alpha_1\rangle \odot \cdots \odot |\alpha_n\rangle, \tag{2.1}$$

where $\alpha = \alpha_1, \ldots, \alpha_n$ is a sequence of possible measurement results (a "history"), obtained at times $t_1, \ldots, t_n$, and $|\alpha_i\rangle$ are a basis of orthonormal vectors for $\mathcal{H}$ at each time t_i. If the α_i eigenvalues are nondegenerate, $|\alpha_i\rangle$ are just the eigenvectors of the observable(s) measured at time t_i. For simplicity, we assume here nondegenerate eigenvalues (for the general case see Ref. 9). The product $\odot$ has all the properties of a tensor product. The coefficients $A(\psi, \alpha)$ are the *history amplitudes*, computed as

$$A(\psi, \alpha) = \langle \alpha_n | U(t_n, t_{n-1}) P_{\alpha_{n-1}} U(t_{n-1}, t_{n-2}) \cdots P_{\alpha_1} U(t_1, t_0) |\psi\rangle \tag{2.2}$$

with $|\psi\rangle =$ initial state (at t_0). P_{α_i} is the projector on the eigensubspace of α_i, and $U(t_{i+1}, t_i)$ is the evolution operator between times t_i and t_{i+1}.

The data entering the history vector (2.1) are therefore:

– System data: evolution operator (or Hamiltonian), initial state $|\psi\rangle$.
– Measuring apparatus data: which observables are measured at different times t_i.

2.2. *Probabilities*

Using standard Born rules, it is straightforward to prove that the joint probability $p(\psi, \alpha)$ of obtaining the sequence $\alpha_1, \ldots, \alpha_n$ in measurements at times $t_1, \ldots, t_n$ is given by the square modulus of the amplitude $A(\psi, \alpha)$. If one defines the history projector

$$\mathbb{P}_\alpha = |\alpha_1\rangle\langle\alpha_1| \odot \cdots \odot |\alpha_n\rangle\langle\alpha_n| \tag{2.3}$$

the familiar formula holds

$$p(\psi, \alpha) = \langle\Psi|\mathbb{P}_\alpha|\Psi\rangle = |A(\psi, \alpha)|^2 \tag{2.4}$$

generalizing Born rule to measurement sequences. The probabilities $p(\psi, \alpha)$ satisfy

$$\sum_\alpha p(\psi, \alpha) = \sum_\alpha |A(\psi, \alpha)|^2 = 1 \tag{2.5}$$

due to completeness relations for the projectors P_{α_i} and unitarity of the evolution operators. As a consequence, the history vector is normalized:

$$\langle\Psi|\Psi\rangle = \sum_\alpha |A(\psi, \alpha)|^2 = 1. \tag{2.6}$$

Defining the chain operator:

$$C_{\psi,\alpha} = P_{\alpha_n} U(t_n, t_{n-1}) P_{\alpha_{n-1}} U(t_{n-1}, t_{n-2}) \cdots P_{\alpha_1} U(t_1, t_0) P_\psi \tag{2.7}$$

sequence probabilities can also be expressed as

$$p(\psi, \alpha) = \mathrm{Tr}(C_{\psi,\alpha} C^\dagger_{\psi,\alpha}). \tag{2.8}$$

Note. The operator $\mathbb{P}_\alpha$ defined in (2.3) projects onto the history state $|\alpha_1\rangle \odot \cdots \odot |\alpha_n\rangle$. One might wonder how to realize physically this basis history state. It must be such that a sequence of measurements will yield with certainty the values $\alpha_1, \ldots, \alpha_n$. This state can be realized by an appropriate choice of Hamiltonians governing the system in between measurement times. More precisely, the corresponding evolution operators $U(t_{i+1}, t_i)$ must be such as to connect the vectors $|\alpha_i\rangle$, $|\alpha_{i+1}\rangle$, i.e. $|\alpha_{i+1}\rangle = U(t_{i+1}, t_i)|\alpha_i\rangle$ (to obtain $|\phi\rangle = U|\chi\rangle$, it suffices to choose U of the form $\sum_k |u_k\rangle\langle v_k|$, where $|u_k\rangle$ and $|v_k\rangle$ are orthonormal bases, and $|\phi\rangle = |u_1\rangle$, $|\chi\rangle = |v_1\rangle$). These evolution operators have no relation with the U's entering the amplitudes (2.2), and only serve the purpose of *preparing* the basis history states.

2.3. *Sum rules*

Note that

$$\sum_{\alpha_n} p(\psi, \alpha_1, \alpha_2, \ldots, \alpha_n) = p(\psi, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}). \tag{2.9}$$

However, other standard sum rules for probabilities are not satisfied in general. For example, relations of the type

$$\sum_{\alpha_2} p(\psi, \alpha_1, \alpha_2, \alpha_3) = p(\psi, \alpha_1, \alpha_3) \tag{2.10}$$

hold only if the so-called *decoherence condition* is satisfied:

$$\mathrm{Tr}(C_{\psi,\alpha} C^{\dagger}_{\psi,\beta}) + c.c. = 0 \quad \text{when } \alpha \neq \beta \tag{2.11}$$

as can be checked on the example (2.10) written in terms of chain operators, and easily generalized. If all the histories we consider are such that the decoherence condition holds, they are said to form a *consistent* set,[13] and can be assigned probabilities satisfying all the standard sum rules.

Sec. 6 provides two examples of consistent histories,

Note. ~~An example of consistent histories is provided by the two examples of Sec. 6~~, while nonconsistent sets are necessary to describe e.g. the simplified Mach–Zehnder interferometer or the three box "paradox," as discussed in Ref. 8. Writing out condition (2.11) in detail, one sees that a consistent set is obtained if the projectors P_{α_i}, evolved to a common time t_j, do commute. This means

$$[P_{\alpha_{i+1}}, U(t_{i+1}, t_i) P_{\alpha_i} U(t_i, t_{i+1})] = 0, \tag{2.12}$$

implying that the observables measured at different times, once evolved to a common time, must commute. This is not the only criterion for a consistent set: for example if all histories contained in the history vector have different $|\alpha_n\rangle$ final state, it is immediate to check that the set is orthogonal in the sense of $\mathrm{Tr}(C_{\psi,\alpha} C^{\dagger}_{\psi,\beta}) = 0$ when $\alpha \neq \beta$. This happens in the two examples in Sec. 6. For detailed considerations on consistent sets, see Ref. 14.

However histories do not form in general a consistent set: interference effects between them can be important, as in the case of the double slit experiment. For this reason, in our history formalism, we do not limit ourselves to consistent sets. Formula (2.4) for the probability of successive measurement outcomes holds true in any case.

2.4. *Scalar and tensor products in history space*

Scalar and tensor products in history space, i.e. the vector space spanned by the basis vectors $|\alpha_1\rangle \odot \cdots \odot |\alpha_n\rangle$, can be defined as in ordinary tensor spaces.

Scalar product:

$$(\langle\alpha_1| \odot \cdots \odot \langle\alpha_n|)(|\beta_1\rangle \odot \cdots \odot |\beta_n\rangle) \equiv \langle\alpha_1|\beta_1\rangle \cdots \langle\alpha_n|\beta_n\rangle \tag{2.13}$$

and extended by (anti)linearity on all linear combinations of these vectors. This also defines bra vectors in history space.

Tensor product:

$$(|\alpha_1\rangle \odot \cdots \odot |\alpha_n\rangle)(|\beta_1\rangle \odot \cdots \odot |\beta_n\rangle) \equiv |\alpha_1\rangle|\beta_1\rangle \odot \cdots \odot |\alpha_n\rangle|\beta_n\rangle \tag{2.14}$$

and extended by bilinearity on all linear combinations of these vectors. No symbol is used for this tensor product to distinguish it from the tensor product $\odot$ involving different times t_k.

This tensor product allows a definition of *product history states*, which are defined to be expressible in the following form:

$$\left(\sum_{\alpha} A(\phi,\alpha)|\alpha_1\rangle \odot \cdots \odot |\alpha_n\rangle\right)\left(\sum_{\beta} A(\chi,\beta)|\beta_1\rangle \odot \cdots \odot |\beta_n\rangle\right) \tag{2.15}$$

or, using bilinearity:

$$\sum_{\alpha,\beta} A(\phi,\alpha)A(\chi,\beta)|\alpha_1\beta_1\rangle \odot \cdots \odot |\alpha_n\beta_n\rangle \tag{2.16}$$

with $|\alpha_i\beta_i\rangle \equiv |\alpha_i\rangle|\beta_i\rangle$ for short. A product history state is thus characterized by factorized amplitudes $A(\psi,\alpha,\beta) = A(\phi,\alpha)A(\chi,\beta)$. If the history state cannot be expressed as a product, we define it to be *history entangled*. In this case, results of measurements on system A are correlated with those on system B and viceversa.

2.5. *History density matrix*

A system in the history state $|\Psi\rangle$ can be described by the *history density matrix*:

$$\rho = |\Psi\rangle\langle\Psi| \tag{2.17}$$

a positive operator satisfying $\mathrm{Tr}(\rho) = 1$ (due to $\langle\Psi|\Psi\rangle = 1$). A mixed history state has density matrix

$$\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \tag{2.18}$$

with $\sum_i p_i = 1$, and $\{|\Psi_i\rangle\}$ an ensemble of history states. Probabilities of measuring sequences $\alpha = \alpha_1,\dots,\alpha_n$ in history state ρ are given by the standard formula:

$$p(\alpha_1,\dots,\alpha_n) = \mathrm{Tr}(\rho\mathbb{P}_\alpha) \tag{2.19}$$

cf. Eq. (2.4) for pure states.

3. Space-Reduced Density Matrix

Consider now a system AB composed by two subsystems A and B, and devices measuring observables $\mathbb{A}_i = A_i \otimes I$ and $\mathbb{B}_i = I \otimes B_i$ at each t_i. Its history state is

$$|\Psi^{AB}\rangle = \sum_{\alpha,\beta} A(\psi,\alpha,\beta)|\alpha_1\beta_1\rangle \odot \cdots \odot |\alpha_n\beta_n\rangle, \tag{3.1}$$

where α_i, β_i are the possible outcomes of a joint measurement at time t_i of $\mathbb{A}_i$ and $\mathbb{B}_i$. The amplitudes $A(\psi,\alpha,\beta)$ are computed using the general formula (2.2), with projectors

$$\mathbb{P}_{\alpha_i,\beta_i} = |\alpha_i,\beta_i\rangle\langle\alpha_i,\beta_i| = |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_i\rangle\langle\beta_i| \tag{3.2}$$

corresponding to the eigenvalues α_i, β_i. The density matrix of AB is

$$\begin{aligned}\rho^{AB} &= |\Psi^{AB}\rangle\langle\Psi^{AB}| \\ &= \sum_{\alpha,\beta,\alpha',\beta'} A(\psi,\alpha,\beta)A(\psi,\alpha',\beta')^*(|\alpha_1\beta_1\rangle \odot \cdots \odot |\alpha_n\beta_n\rangle)(\langle\alpha'_1\beta'_1| \odot \cdots \odot \langle\alpha'_n\beta'_n|).\end{aligned} \tag{3.3}$$

We define *space-reduced density matrices* by partially tracing on the subsystems:

$$\rho^A \equiv \mathrm{Tr}_B(\rho^{AB}), \quad \rho^B \equiv \mathrm{Tr}_A(\rho^{AB}). \tag{3.4}$$

In general, ρ^A and ρ^B will not describe pure history states anymore. These reduced density matrices can be used to compute statistics for measurement sequences on the subsystems. Taking for example the partial trace on B of (3.3) yields:

$$\rho^A = \sum_{\alpha,\alpha',\beta} A(\psi,\alpha,\beta)A^*(\psi,\alpha',\beta)(|\alpha_1\rangle \odot \cdots \odot |\alpha_n\rangle)(\langle\alpha'_1| \odot \cdots \odot \langle\alpha'_n|) \tag{3.5}$$

a positive operator with unit trace. The standard expression in terms of ρ^A for Alice's probability to obtain the sequence α is

$$p(\alpha) = \mathrm{Tr}(\rho^A \mathbb{P}_\alpha) \tag{3.6}$$

with

$$\mathbb{P}_\alpha = (P_{\alpha_1} \otimes I) \odot \cdots \odot (P_{\alpha_n} \otimes I), \quad P_{\alpha_i} = |\alpha_i\rangle\langle\alpha_i|. \tag{3.7}$$

The prescription (3.6) yields

$$p(\alpha) = \sum_\beta |A(\psi,\alpha,\beta)|^2 = \sum_\beta p(\alpha,\beta), \tag{3.8}$$

i.e. the probability for Alice to obtain the sequence α in measuring the observables A_i.

On the other hand, the probability for Alice to obtain the sequence $\alpha_1,\ldots,\alpha_n$ *with no measurements on Bob's part* is in general different from (3.8). Indeed, the history vector of the composite system is different, since only Alice's measuring device is activated, and reads

$$|\Psi^{AB}\rangle = \sum_\alpha A(\psi,\alpha)|\alpha_1\rangle \odot \cdots \odot |\alpha_n\rangle, \tag{3.9}$$

where the amplitudes $A(\psi,\alpha)$ are obtained from the general formula (2.2) using the projectors P_{α_i} of (3.7). Here, the reduced density operator ρ^A is simply

$$\rho^A = \sum_{\alpha,\alpha'} A(\psi,\alpha)A(\psi,\alpha')^*|\alpha_1\rangle \odot \cdots \odot |\alpha_n\rangle\langle\alpha'_1| \odot \cdots \odot \langle\alpha'_n| \tag{3.10}$$

(the trace on B has no effect, since history vectors contain only results of Alice), and the probability of Alice finding the sequence α is

$$p(\alpha) = \mathrm{Tr}(\rho^A \mathbb{P}_\alpha) = |A(\psi,\alpha)|^2 \tag{3.11}$$

differing in general from (3.8). Indeed $\sum_\beta A(\psi,\alpha,\beta) = A(\psi,\alpha)$ because of the completeness relation (at each time t_i) $\sum_\beta |\beta_i\rangle\langle\beta_i| = I$, so that

$$p(\alpha) = |A(\psi,\alpha)|^2 = \left|\sum_\beta A(\psi,\alpha,\beta)\right|^2 \tag{3.12}$$

differing, in general, from (3.8).

In fact, the probabilities (3.8) and (3.12) coincide only when the evolution operator is factorized $U = U^A \otimes U^B$, i.e. when A and B do not interact.[9] Thus, if there is no interaction, Bob cannot communicate with Alice by activating (or not activating) his measuring devices.

4. Time-Reduced Density Matrix

Partial traces of the history density matrix can be taken also on the Hilbert spaces $\mathcal{H}_i$ corresponding to different times $t_{\{k\}} = t_{k_1},\ldots,t_{k_p}$, $p < n$. We call the resulting density matrices, involving only the complementary times $t_{\{j\}} = t_{j_1},\ldots,t_{j_m}$ (i.e. with $j_1,\ldots,j_m$ and $k_1,\ldots,k_p$ having no intersection, and union coinciding with $1,\ldots,n$), *time-reduced density matrices*. They are used to compute sequence probabilities corresponding to measurements at times $t_{\{j\}}$, given that measurements are performed also at times $t_{\{k\}}$ without registering their result. Thus, they describe statistics for an experimenter that has access only to the measuring apparatus at times $t_{\{j\}}$, while the system gets measured at all times $t_i = t_1,\ldots,t_n$.

Consider a system described by the (pure) history vector (2.1). Its density matrix is

$$\rho = \sum_{\alpha,\alpha'} A(\psi,\alpha)A(\psi,\alpha')^*|\alpha_1\rangle \odot \cdots \odot |\alpha_n\rangle\langle\alpha'_1| \odot \cdots \odot \langle\alpha'_n|. \tag{4.1}$$

Dividing the set $\alpha = \alpha_1,\ldots,\alpha_n$ into the complementary sets $\alpha_{\{j\}} = \alpha_{j_1},\ldots,\alpha_{i_m}$ and $\alpha_{\{k\}} = \alpha_{k_1},\ldots,\alpha_{k_p}$, the $\{j\}$-time-reduced density matrix is defined by

$$\rho^{\{j\}} = \mathrm{Tr}_{\{k\}}\rho = \sum_{\alpha_{\{j\}},\alpha'_{\{j\}}} \sum_{\alpha_{\{k\}}} A(\psi,\alpha_{\{j\}},\alpha_{\{k\}})A^*(\psi,\alpha'_{\{j\}},\alpha_{\{k\}})|\alpha_{\{j\}}\rangle\langle\alpha'_{\{j\}}| \tag{4.2}$$

with $|\alpha_{\{j\}}\rangle \equiv |\alpha_{j_1}\rangle \odot \cdots \odot |\alpha_{j_m}\rangle$. Using the standard formula, we find the probability for the sequence $\alpha_{\{j\}}$

$$p(\alpha_{\{j\}}) = \mathrm{Tr}(\mathbb{P}_{\alpha_{\{j\}}}\rho^{\{j\}}) = \sum_{\alpha_{\{k\}}} |A(\psi,\alpha_{\{j\}},\alpha_{\{k\}})|^2, \tag{4.3}$$

where $\mathbb{P}_{\alpha_{\{j\}}}$ is the projector on the (sub)history $\alpha_{\{j\}}$, given by Eq. (2.3) with the α's in $\alpha_{\{j\}}$.

On the other hand, if no measurements are performed at complementary times $t_{\{k\}}$, the probability for the same sequence $\alpha_{\{j\}}$ is simply

$$p(\alpha_{\{j\}}) = |A(\psi,\alpha_{\{j\}}|^2 \tag{4.4}$$

in general differing from (4.3), see the discussion on sum rules after (2.9). Can this difference be used to violate causality? More precisely, can a future measurement by Alice be detected by herself in the past? The answer is of course negative, but the formal reason is interesting. It is based on the marginal rules of Sec. 2.3: no difference arises in the probabilities for an experimenter having access to measurement results up to time t, whether the system gets measured or not at times $t' > t$, due to the validity of formula (2.9) that reproduces a classical sum rule. When $t' < t$ this formula does not hold, and indeed past measurements have a verifiable impact on present statistics. This asymmetry in time is entirely due to the particular marginal rules for quantum probabilities of sequences.

5. Temporal Entanglement

5.1. *Time tensor product between histories*

The "time" tensor product $\odot$ introduced in Sec. 2.1 can be extended to a time tensor product between histories, in contradistinction with the product defined in Sec. 2.4, which could be referred to as a "space" tensor product.

The definition is given by the merging rule:

$$|\alpha_{\{j\}}\rangle \odot |\alpha_{\{k\}}\rangle \equiv |\alpha_{\{i\}}\rangle \tag{5.1}$$

with $\{j\}$ and $\{k\}$ having no intersection and union equal to $\{i\}$. For example

$$(|\alpha_1\rangle \odot |\alpha_3\rangle \odot |\alpha_5\rangle) \odot (|\alpha_2\rangle \odot |\alpha_6\rangle) = |\alpha_1\rangle \odot |\alpha_2\rangle \odot |\alpha_3\rangle \odot |\alpha_5\rangle \odot |\alpha_6\rangle. \tag{5.2}$$

We then denote $|\Psi\rangle$ as a *time-separable history state* if it can be expressed as a time product of two history states:

$$|\Psi\rangle = |\Psi_1\rangle \odot |\Psi_2\rangle \tag{5.3}$$

in analogy with the "space" product history state of Sec. 2.4. Similarly, we find here that a history state

$$|\Psi\rangle = \sum_{\alpha_{\{i\}}} A(\psi, \alpha_{\{i\}})|\alpha_{\{i\}}\rangle \tag{5.4}$$

is time-separable if and only if the amplitudes factorize

$$A(\psi, \alpha_{\{i\}}) = A(\psi, \alpha_{\{j\}})A(\psi, \alpha_{\{k\}}) \tag{5.5}$$

with $\{j\}$ and $\{k\}$ having no intersection and union equal to $\{i\}$.

As a consequence, probabilities factorize and there are no temporal correlations between measurement results $\alpha_{\{j\}}$ and $\alpha_{\{k\}}$. If the amplitudes do not factorize, we call the history state a *temporally entangled* history state. Note that a time-separable state can still contain entangled sub-histories, exactly as a (space) separable state in a composite system AB can still be entangled within the subsystems A and B.

5.2. *Temporal entanglement entropy*

History entropy has been defined in Ref. 9 as the von Neumann entropy associated to the history state ρ:

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho). \tag{5.6}$$

We have seen in Ref. 9 that when ρ describes a (pure) space entangled system, partial traces of ρ describe mixed history states. The same happens for (pure) time entangled systems: partial time-traces yield reduced density matrices describing mixed states. Examples taken from quantum computation circuits are discussed in the following section.

We call *temporal entanglement entropy* the von Neumann entropy corresponding to the time-reduced density matrix.

6. Examples

In this section, we examine two examples of quantum systems evolving from a given initial state, and subjected to successive measurements. They are taken from simple quantum computation circuits, where unitary gates determine the evolution between measurements. Only two gates are used: the Hadamard one-qubit gate H defined by

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{6.1}$$

and the two-qubit CNOT gate:

$$\begin{gathered} \mathrm{CNOT}|00\rangle = |00\rangle, \quad \mathrm{CNOT}|01\rangle = |01\rangle, \quad \mathrm{CNOT}|10\rangle = |11\rangle, \\ \mathrm{CNOT}|11\rangle = |10\rangle. \end{gathered} \tag{6.2}$$

(on the same line)

6.1. *Entangler*

The two-qubit entangler circuit of Fig. 1, with initial state $|00\rangle$, produces the entangled state $|\chi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$:

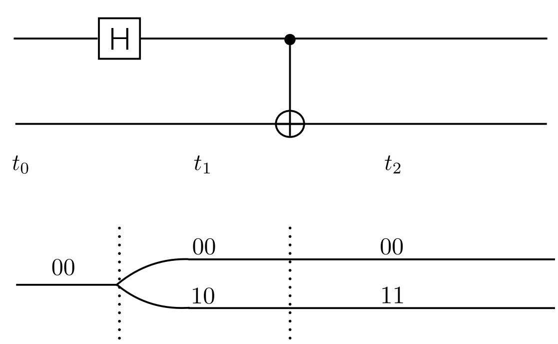

Fig. 1. Entangler circuit, and history graph for initial state $|00\rangle$.

The history state that describes the system together with its measuring devices[a] at times t_1 and t_2 is

$$|\Psi\rangle = A(00,00,00)|00\rangle \odot |00\rangle + A(00,10,11)|10\rangle \odot |11\rangle, \tag{6.3}$$

where

$$A(00,00,00) = \langle 00|\mathrm{CNOT}|00\rangle\langle 00|H \otimes I|00\rangle = \frac{1}{\sqrt{2}}, \tag{6.4}$$

$$A(00,10,11) = \langle 11|\mathrm{CNOT}|10\rangle\langle 10|H \otimes I|00\rangle = \frac{1}{\sqrt{2}} \tag{6.5}$$

are the only nonvanishing amplitudes. This simple system exhibits both space and time entanglement. Space entanglement is due to (ordinary) entanglement in the final state at t_2. Time entanglement is due to temporal correlations: the outcomes of measurements at t_1 are correlated with the outcomes at t_2. In other words, the history amplitudes $A(00,\alpha_1\beta_1,\alpha_2\beta_2)$ do not space-factorize as $A(0,\alpha_1,\alpha_2)A(0,\beta_1,\beta_2)$, and do not time-factorize as $A(00,\alpha_1\beta_1)A(00,\alpha_2\beta_2)$.

Note that the two histories in Fig. 1 are orthogonal, i.e. $\mathrm{Tr}(C^\dagger_{00,10,11}C_{00,00,00}) = 0$, and therefore form a consistent set.

The history density matrix of the system is given by

$$\rho = \frac{1}{2}(|00\rangle \odot |00\rangle + |10\rangle \odot |11\rangle)(\langle 00|| \odot \langle 00| + \langle 10| \odot \langle 11|). \tag{6.6}$$

The space-reduced density matrices are

$$\rho^A = \mathrm{Tr}_B(\rho) = \frac{1}{2}[(|0\rangle \odot |0\rangle)(\langle 0| \odot \langle 0|) + (|1\rangle \odot |1\rangle)(\langle 1| \odot \langle 1|)], \tag{6.7}$$

$$\rho^B = \mathrm{Tr}_A(\rho) = \frac{1}{2}[(|0\rangle \odot |0\rangle)(\langle 0| \odot \langle 0|) + (|0\rangle \odot |1\rangle)(\langle 0| \odot \langle 1|)], \tag{6.8}$$

i.e. mixed history states, to be expected since ρ is a pure space-entangled history state.[b]

The time-reduced density matrices are

$$\rho^{\{1\}} = \mathrm{Tr}_{\{2\}}\rho = \frac{1}{2}(|00\rangle\langle 00| + |10\rangle\langle 10|), \tag{6.9}$$

$$\rho^{\{2\}} = \mathrm{Tr}_{\{1\}}\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|), \tag{6.10}$$

i.e. mixed history states (in this case states corresponding to a single time), to be expected since ρ is a pure time-entangled history state.

The history entropy corresponding to ρ is $S(\rho) = 0$ since ρ is a pure history state, while the space and time entanglement entropies are $S(\rho^A) = S(\rho^B) = 1$ and $S(\rho^{\{1\}}) = S(\rho^{\{2\}}) = 1$ since they ~~all~~ have two eigenvalues equal to $\frac{1}{2}$. each

[a] We consider here measurements in the computational basis.

[b] These quantum mixtures would be called, in D' Espagnat's[29] terms, "improper" mixtures. See however Refs. 30 and 31 for a critique on the distinction between proper and improper mixtures.

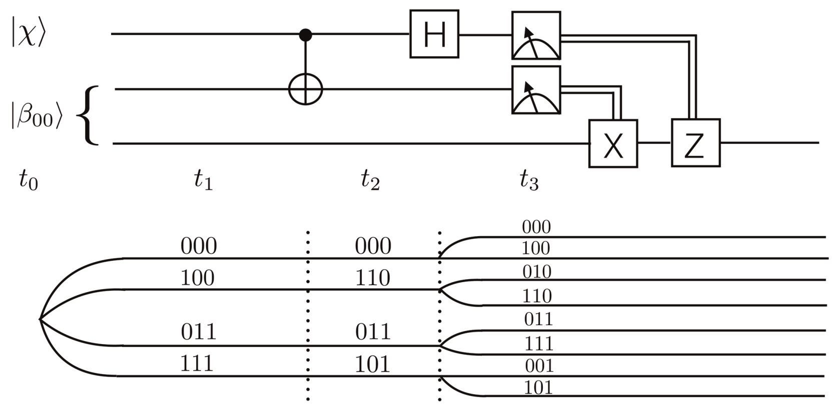

Fig. 2. Teleportation circuit, and history graph for initial state $(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

6.2. *Teleportation*

The teleportation circuit[32] is the three-qubit circuit given in Fig. 2, where the upper two qubits belong to Alice, and the lower one to Bob.

The initial state is a three-qubit state, tensor product of the single qubit $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$ to be teleported and the two-qubit entangled Bell state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The history vector contains 8 histories (a consistent set):

$$\begin{aligned}|\Psi\rangle = \frac{1}{2}(&\alpha|000\rangle \odot |000\rangle \odot |000\rangle + \alpha|000\rangle \odot |000\rangle \odot |100\rangle \\ &+ \beta|100\rangle \odot |110\rangle \odot |010\rangle - \beta|100\rangle \odot |110\rangle \odot |110\rangle \\ &+ \alpha|011\rangle \odot |011\rangle \odot |011\rangle + \alpha|011\rangle \odot |011\rangle \odot |111\rangle \\ &+ \beta|111\rangle \odot |101\rangle \odot |001\rangle - \beta|111\rangle \odot |101\rangle \odot |101\rangle\end{aligned} \tag{6.11}$$

the amplitudes being given by

$$A(\chi \otimes \beta_{00}, \alpha_1, \alpha_2, \alpha_3) = \langle \alpha_3 | H_1 P_{\alpha_2} \mathrm{CNOT}_{1,2} P_{\alpha_1} | \chi \otimes \beta_{00} \rangle. \tag{6.12}$$

For example

$$A(\chi \otimes \beta_{00}, 000, 000, 000) = \langle 000|H_1|000\rangle\langle 000|\mathrm{CNOT}_{1,2}|000\rangle\langle 000|\chi \otimes \beta_{00}\rangle = \alpha/2, \tag{6.13}$$

where $H_1 \equiv H \otimes I \otimes I$ and $\mathrm{CNOT}_{1,2} \equiv \mathrm{CNOT} \otimes I$.

As in the entangler example, here too the history state is space and time entangled. The partial traces of the history density matrix $\rho = |\Psi\rangle\langle\Psi|$ yield the density matrix for Alice:

$$\rho^A = \mathrm{Tr}_B(\rho) = \frac{1}{2}|\phi\rangle\langle\phi| + \frac{1}{2}|\chi\rangle\langle\chi| \tag{6.14}$$

with

$$\begin{aligned}|\phi\rangle &= \frac{1}{\sqrt{2}}(\alpha|00\rangle \odot |00\rangle \odot |00\rangle - \alpha|00\rangle \odot |00\rangle \odot |10\rangle + \beta|10\rangle \odot |11\rangle \odot |01\rangle \\ &\quad - \beta|10\rangle \odot |11\rangle \odot |11\rangle), \\ |\chi\rangle &= \frac{1}{\sqrt{2}}(\alpha|01\rangle \odot |01\rangle \odot |01\rangle - \alpha|01\rangle \odot |01\rangle \odot |11\rangle + \beta|11\rangle \odot |10\rangle \odot |00\rangle \\ &\quad - \beta|11\rangle \odot |10\rangle \odot |10\rangle)\end{aligned} \tag{6.15}$$

and the density matrix for Bob:

$$\rho^B = \mathrm{Tr}_A(\rho) = \frac{1}{2}(|0\rangle \odot |0\rangle \odot |0\rangle\langle 0| \odot \langle 0| \odot \langle 0| + |1\rangle \odot |1\rangle \odot |1\rangle\langle 1| \odot \langle 1| \odot \langle 1|). \tag{6.16}$$

As expected, both reduced history density operators describe mixed states. They both have two nonzero eigenvalues equal to 1/2, and the (space) entanglement entropy is therefore $S(\rho^A) = S(\rho^B) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1$.

Next, we compute the time-reduced density matrices. We can take partial traces of ρ over any combination of t_1, t_2, t_3. For example taking the partial trace over t_1 and t_2 yields the time-reduced density matrix for the system at time t_3:

$$\rho^{\{3\}} = \mathrm{Tr}_{t_1,t_2}(\rho) = \frac{|\alpha|^2}{2}([+00] + [+11]) + \frac{|\beta|^2}{2}([-10] + [-01]), \tag{6.17}$$

where $[\pm 00]$ indicates the projector on the vector $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)|00\rangle$, etc. This density matrix describes a mixed state. Its eigenvalues are $\frac{|\alpha|^2}{2}, \frac{|\alpha|^2}{2}, \frac{|\beta|^2}{2}, \frac{|\beta|^2}{2}$, and therefore the time entanglement entropy is

$$S(\rho^{\{3\}}) = -|\alpha|^2 \log|\alpha|^2 - |\beta|^2 \log|\beta|^2 + 1. \tag{6.18}$$

Setting $p = |\alpha|^2$, the entropy $S(p) = 1 - p\log p - (1-p)\log(1-p)$ is maximum and equal to $\log 2 + 1 = 2$ when $p = 1/2$, and is minimum and equal to 1 when $p = 0, 1$.

Taking the partial trace on the time complementary to t_1, t_2, i.e. on t_3, yields the time-reduced density matrix:

$$\rho^{\{1,2\}} = \mathrm{Tr}_{t_3}(\rho) = \frac{|\alpha|^2}{2}([000 \odot 000] + [011 \odot 011]) + \frac{|\beta|^2}{2}([100 \odot 110] + [111 \odot 101]), \tag{6.19}$$

where $[000 \odot 000]$ is the projector on the history vector $|000\rangle \odot |000\rangle$, etc. This reduced density matrix has, as expected, the same eigenvalues as $\rho^{\{3\}}$, and corresponds therefore to the same time entanglement entropy.

Finally, taking the partial trace of ρ on t_2, t_3 yields the time-reduced history density matrix:

$$\rho^{\{1\}} = \mathrm{Tr}_{t_2,t_3}(\rho) = \frac{|\alpha|^2}{2}([000] + [011]) + \frac{|\beta|^2}{2}([100] + [111]) \tag{6.20}$$

corresponding again to the same time entanglement entropy as in (6.18). Note that if no measurements are performed at t_2, t_3 the history density matrix at t_1 remains the same as in (6.20), due to quantum marginal rules of type (2.9). Indeed performing or not measurements at $t > t_1$ cannot change statistics at t_1.

7. Conclusions

The history formalism of Ref. 9 permits a symmetrical treatment of space and time correlations, based on the reduced history density operator. The same history density ρ can be partially traced both on space and time subsystems: in fact space and time partial tracings commute, so that the resulting reduced density does not depend on the order of the tracings, and describes the statistics of an observer having limited (in space and time) access to the system. Despite the similarity in computing space and time correlations, the history formalism is not Lorentz covariant since the evolution operators entering the history state vector ~~evolve~~ the system in time, and not in propagate space. On the other hand, a Lorentz covariant history formalism would be conceivable in the description of geometric theories like gravity, where indeed one has evolution operators both in time and space.

Acknowledgment

This work is supported by the research funds of the Eastern Piedmont University and INFN — Torino Section.

References

1. J. Fitzsimons, J. Jones and V. Vedral, *Sci. Rep.* **5** (2016) 18281.
2. Z. Zhao, R. Pisarczyk, J. Thompson, M. Gu, V. Vedral and J. F. Fitzsimons, *Phys. Rev. A* **98** (2018) 052312.
3. A. J. Leggett and A. Garg, *Phys. Rev. Lett.* **54** (1985) 857.
4. J. P. Paz and G. Mahler, *Phys. Rev. Lett.* **71** (1993) 3235.
5. C. Brukner, S. Taylor, S. Cheung and V. Vedral, arXiv:quant-ph/0402127.
6. D. Horsman, C. Heunen, M. F. Pusey, J. Barrett and R. W. Spekkens, arXiv:1607.03637.
7. C. Marletto, V. Vedral, S. Virz, A. Avella, F. Piacentini, M. Gramegna, I. P. Degiovanni and M. Genovese, arXiv:2103.12636 [quant-ph].
8. L. Castellani, *Int. J. Quantum Inf.* **17**(8) (2019) 1941001, arXiv:1810.03624 [quant-ph].
9. L. Castellani, *Phys. Scr.* **96**(5) (2021) 055217, arXiv:2009.02331 [quant-ph].
10. R. P. Feynman, *Rev. Mod. Phys.* **20** (1948) 367.
11. R. P. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
12. P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th edn, Chap. 32 (Oxford University Press, 1958).
13. R. B. Griffiths, *J. Stat. Phys.* **36** (1984) 219.
14. R. B. Griffiths, *Consistent Quantum Theory* (Cambridge University Press, Cambridge, 2002).

AQ: Please approve the edits in Ref. 12.

ok

15. R. B. Griffiths, The consistent histories approach to quantum mechanics, Stanford Encyclopedia of Philosophy (2014), http://plato.stanford.edu/entries/qm-consistent-histories/.
16. R. Omnès, *Rev. Mod. Phys.* **64** (1992) 339.
17. R. Omnès, *Interpretation of Quantum Mechanics* (Princeton University Press, Princeton, 1994).
18. M. Gell-Mann and J. B. Hartle, *Phys. Rev. D* **47** (1993) 3345, arXiv:gr-qc/9210010.
19. M. Gell-Mann and J. B. Hartle, arXiv:1803.04605 [gr-qc].
20. J. B. Hartle, Spacetime quantum mechanics and the quantum mechanics of spacetime, in *Gravitation and Quantizations, Proceedings of the 1992 Les Houches Summer School*, eds. by B. Julia and J. Zinn-Justin (North Holland, Amsterdam, 1995), arXiv:gr-qc/9304006.
21. J. B. Hartle, arXiv:1805.12246 [gr-qc].
22. H. Dowker and J. Halliwell, *Phys. Rev. D* **46** (1992) 1580.
23. C. J. Isham, *J. Math. Phys.* **35** (1994) 2157, arXiv:gr-qc/9308006.
24. J. Cotler and F. Wilczek, *Phys. Scr.* **2016**(T168) (2016) 014004.
25. J. Cotler and F. Wilczek, arXiv:1503.06458.
26. J. Cotler*et al.*, *Ann. Phys.* **387** (2017) 334.
27. J. Cotler and W. Frank, arXiv:1702.05838.
28. J. Cotler, C. M. Jian, X. L. Qi and F. Wilczek, *J. High. Energy Phys.* **1809** (2018) 093, arXiv:1711.03119 [quant-ph].
29. B. d'Espagnat, *Conceptual Foundations of Quantum Mechanics*, 2nd edn. (W. A. Benjamin, Menlo Park, CA, 1976).
30. L. Castellani, arXiv:2201.04143 [quant-ph].
31. L. Castellani, *Int. J. Theor. Phys.* **60**(6) (2021) 2084, arXiv:2011.08190 [quant-ph].
32. C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, *Phys. Rev. Lett.* **70** (1993) 1895.

∧ p. 285,

AQ: Please provide the page number in Ref. 20.