The Persico equation for minimum uncertainty states

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We present an important and forgotten result of fundamental quantum mechanics obtained in 1930 by Enrico Persico, consisting in an eigenvalue equation for minimum uncertainty states.

I. INTRODUCTION

The purpose of this work is to bring to light an important and forgotten result on the Heisenberg principle obtained in the early years of quantum mechanics by the Italian physicist Enrico Persico (1900–1969), a close friend and collaborator of Enrico Fermi and a leading figure in the dissemination of quantum mechanics in Italy (he was professor of Theoretical Physics at the Universities of Florence, Turin and Rome, and at Laval University in Québec). The result uses a variational method for determining minimum uncertainty wavefunctions for any pair of non-commuting physical observables, to derive a simple and elegant eigenvalue equation.

Persico's findings were communicated to an Italian scientific meeting in 1930 and never reported elsewhere. At about the same time, the Irish physicist Robert Ditchburn presented another procedure for minimizing the uncertainty product, which is widely used in modern textbooks but is less general than Persico's method, as we will see.

Persico's equation was rederived in 1968 by Roman Jackiw. We believe, however, that it is important to situate it historically, as an important episode in the early developments of the uncertainty principle.

II. THE DAWN OF THE UNCERTAINTY PRINCIPLE

Soon after Heisenberg's work on the position–momentum uncertainty relation [1] (called "indetermination principle" by Ruark [2] and "uncertainty principle" by Condon [3]), the American physicist Earle Hesse Kennard, who was spending in 1927 a sabbatical leave in Göttingen and

Copenhagen, published a remarkable paper [4] in which he presented for the first time the Heisenberg relation in the form of an inequality [5], i.e. in modern notation

$$\Delta q\, \Delta p \geq \frac{\hbar}{2}\,. \tag{1}$$

Kennard defined the expectation value $\langle q \rangle$ and the uncertainty Δq in a state described by a wavefunction $\psi(q)$ as

$$\langle q \rangle \equiv \int q\, \psi^*(q)\, \psi(q)\, \mathrm{d}q\,, \quad (\Delta q)^2 \equiv \int (q - \langle q \rangle)^2\, \psi^*(q)\, \psi(q)\, \mathrm{d}q\,. \tag{2}$$

Analogously, introducing the wavefunction $\psi(p)$ in momentum space, i.e. the Fourier transform of $\psi(q)$, he defined the uncertainty Δp and the expectation value $\langle p \rangle$ as

$$\langle p \rangle \equiv \int p\, \psi^*(p)\, \psi(p) \mathrm{d}p\,, \quad (\Delta p)^2 \equiv \int (p - \langle p \rangle)^2\, \psi^*(p)\, \psi(p)\, \mathrm{d}p\,. \tag{3}$$

Taking a wavefunction of the form

$$\psi(q) = f(q)\, \psi_0(q)\,, \tag{4}$$

where ψ_0 is a Gaussian function and f is any function of q, Kennard showed by a direct calculation that the product of the two uncertainties $\Delta q \Delta p$ is always greater than, or equal to, $\hbar/2$, and takes the minimum value when f is constant; that is, when the wavefunction has a Gaussian form.

The formal aspects of the uncertainty principle were discussed in the following years by Hermann Weyl [6], Howard P. Robertson [7] and Erwin Schrödinger [8]. These authors introduced the use of the Schwarz inequality to prove Heisenberg's relations, but did not discuss the problem of minimum uncertainty wavefunctions. After Kennard's work, the only paper dealing with this problem was published in 1930 by the Irish physicist Robert W. Ditchburn [9]. Following a suggestion by J.L. Synge, Ditchburn considered the non-negative quantity

$$\int \left| q\, \psi + \gamma\, \frac{\mathrm{d}\psi}{\mathrm{d}q} \right|^2 \mathrm{d}q \geq 0\,, \tag{5}$$

where γ is any real constant. Explicitly, the inequality (5) reads

$$\gamma^2 \int \frac{\mathrm{d}\psi^*}{\mathrm{d}q} \frac{\mathrm{d}\psi}{\mathrm{d}q}\, \mathrm{d}q + \gamma \int q \left(\frac{\mathrm{d}\psi}{\mathrm{d}q}\, \psi^* + \frac{\mathrm{d}\psi^*}{\mathrm{d}q}\, \psi \right) \mathrm{d}q + \int q^2\, \psi^* \psi\, \mathrm{d}q \geq 0\,. \tag{6}$$

Performing partial integrations in the first two integrals of (6) and assuming $\langle q \rangle = \langle p \rangle = 0$ yields

$$\int \frac{\mathrm{d}\psi^*}{\mathrm{d}q} \frac{\mathrm{d}\psi}{\mathrm{d}q}\, \mathrm{d}q = - \int \psi^*\, \frac{\mathrm{d}^2\psi}{\mathrm{d}q^2}\, \mathrm{d}q = \frac{1}{\hbar^2}\, (\Delta p)^2\,, \tag{7}$$

$$\int q \left(\frac{\mathrm{d}\psi}{\mathrm{d}q}\, \psi^* + \frac{\mathrm{d}\psi^*}{\mathrm{d}q}\, \psi \right) \mathrm{d}q = - \int \psi^* \psi\, \mathrm{d}q = -1 \tag{8}$$

$$\int q^2\, \psi^* \psi\, \mathrm{d}q = (\Delta q)^2\,. \tag{9}$$

Equation (6) then becomes

$$\frac{(\Delta p)^2}{\hbar^2}\,\gamma^2 - \gamma + (\Delta q)^2 \geq 0\,. \tag{10}$$

In general, an inequality of the type $a\gamma^2 + b\gamma + c \geq 0$ holds for all real values of γ if the discriminant $b^2 - 4ac$ is negative or zero, i.e. in our case

$$1 - \frac{4}{\hbar^2}\,(\Delta p)^2(\Delta q)^2 \geq 0\,, \tag{11}$$

from which we immediately obtain the momentum-position uncertainty relation, $\Delta q\,\Delta p \geq \hbar/2$.

The minimum uncertainty condition $\Delta q\,\Delta p = \hbar/2$, that is $1 - 4(\Delta p)^2(\Delta q)^2/\hbar^2 = 0$, implies that the second-order algebraic equation

$$\frac{(\Delta p)^2}{\hbar^2}\,\gamma^2 - \gamma + (\Delta q)^2 = 0 \tag{12}$$

has a double root γ_0 given by

$$\gamma_0 = \frac{\hbar^2}{2(\Delta p)^2} = 2(\Delta q)^2\,. \tag{13}$$

From (5) one then sees that the wavefunctions which minimize the uncertainty product satisfy the equation

$$q\,\psi + \gamma_0\,\frac{\mathrm{d}\psi}{\mathrm{d}q} = 0\,, \tag{14}$$

the solution of which is easily found to be a Gaussian,

$$\begin{aligned}\psi(q) &= C\,\mathrm{e}^{-q^2/2\gamma_0} \\ &= C\,\mathrm{e}^{-q^2/4(\Delta q)^2}\,.\end{aligned} \tag{15}$$

Let us now depart from the historical development of the subject to notice that Ditchburn's procedure can be extended to any pair of observables A and B with non vanishing expectation values, starting from the inequality

$$\int \left|(\hat{A} + \mathrm{i}\,\gamma\,\hat{B})\,\psi\right|^2 \mathrm{d}\tau \geq 0\,, \tag{16}$$

where $\hat{A} \equiv A - \langle A\rangle$, $\hat{B} \equiv B - \langle B\rangle$ and $\mathrm{d}\tau$ is the volume element in configuration space. Proceeding as before, one obtains the generalized uncertainty relation

$$\Delta A\,\Delta B \geq \frac{1}{2}\,|\langle C\rangle|\,, \tag{17}$$

where C is related to the commutator of A and B by $[A, B] = \mathrm{i}C$.

The equivalent of Eq. (12) is now

$$(\Delta B)^2\,\gamma^2 - \langle C\rangle\gamma + (\Delta A)^2 = 0\,, \tag{18}$$

with a double root γ_0 given by

$$\gamma_0 = \frac{\langle C\rangle}{2\,(\Delta B)^2} = \frac{2\,(\Delta A)^2}{\langle C\rangle} = \pm\sqrt{\frac{(\Delta A)^2}{(\Delta B)^2}}\,. \tag{19}$$

Thus, the minimum uncertainty wavefunctions satisfy the equation

$$\left(\hat{A} + \mathrm{i}\gamma_0\hat{B}\right)\psi = 0\,. \tag{20}$$

Ditchburn's method for determining the minimum uncertainty states is adopted in many modern textbooks (see, e.g., [10]). However, it is not usually recognized that this method cannot in general be applied when the commutator of A and B (i.e. $\mathrm{i}C$) is not a c-number. In fact, if C is an operator, the parameter γ_0 in (20) depends on the wavefunction ψ via the expectation value $\langle C\rangle$. Therefore, it is not guaranteed that Eq. (20) provides the wavefunction that really minimizes the uncertainty product.

III. THE PERSICO EQUATION

In September 1930, right before the publication of Ditchburn's paper, Persico communicated a brief note titled *Considerazioni sul principio di indeterminazione* (*Remarks on the indetermination principle*) to the 19th Annual Meeting of the Società Italiana per il Progresso delle Scienze (Italian Society for the Advancement of Sciences) (see the Appendix for an English translation of this note) [11].

Let us try to reconstruct the genesis of Persico's interest in the uncertainty principle. In an interview given in 1963 to Thomas Kuhn [12], Persico said: "What I remember about the uncertainty principle is that I was much impressed by Heisenberg's paper and when I spoke to Fermi about it, I was surprised to find that he was not so enthusiastic about it. I had the impression that he did not think it was so important as I believed. Probably Fermi was not much interested in philosophical aspects of physics and that was too philosophical."

Heisenberg's work gave rise in Italy to many discussions among scientists and philosophers, often based on "a rather foggy knowledge of the question", as Persico recalled [13]. Stimulated by a comment made by the eminent mathematician Guido Castelnuovo, Fermi decided to contribute to the question in his typical way: by formulating and solving a well defined problem. In a brief paper presented to the Accademia dei Lincei in May 1930 and then published in "Il Nuovo Cimento" [14], he explored the conditions under which one can predict the outcome of a measurement made at a certain time by means of another measurement at an earlier time [15].

Persico discussed this subject with Fermi [16] and it was probably in that context that he conceived his work on minimum-uncertainty states. An instrumental role was played by Kennard's paper, which was one of the two references quoted by Fermi in his note and the starting point of Persico's study.

Persico's idea was to extend Kennard's result (1) – "the quantitative expression of Heisenberg indetermination principle", in his own words – to the case of a generic pair of physical quantities, not necessarily conjugate to each other. By a canonical transformation one of the quantities can be taken as a coordinate q, while the other will be a function $P(q,p)$ of q and its conjugate momentum $p = -\mathrm{i}\,\hbar\,\mathrm{d}/\mathrm{d}q$. Let us introduce the expectation values

$$\langle q\rangle \equiv \int \psi^*\, q\, \psi\, \mathrm{d}q\,, \quad \langle P\rangle \equiv \int \psi^* P(q, -\mathrm{i}\hbar \mathrm{d}/\mathrm{d}q)\, \psi\, \mathrm{d}q\,. \tag{21}$$

and the uncertainties Δq and ΔP:

$$(\Delta q)^2 \equiv \int \psi^*(q - \langle q\rangle)^2 \psi\, \mathrm{d}q\,, \quad (\Delta P)^2 \equiv \int \psi^*\left(P(q, -\mathrm{i}\hbar\,\mathrm{d}/\mathrm{d}q) - \langle P\rangle\right)^2 \psi\, \mathrm{d}q\,. \tag{22}$$

In order to find the wavefunction $\Psi(q)$ which minimizes the product $\Pi(\psi,\psi^*) \equiv (\Delta q)^2(\Delta P)^2$, Persico adopted a variational method [17]. He did not give the details of the procedure, but the derivation goes essentially as follows.

Since we have to extremize $\Pi(\psi,\psi^*)$ with the normalization constraint $\int \psi^*\psi\,\mathrm{d}q = 1$, we introduce a Lagrangian multiplier λ and impose that the variation of the quantity $\Pi(\psi,\psi^*) - \lambda \int \psi^*\psi\,\mathrm{d}q$ be zero:

$$\delta\left[(\Delta q)^2(\Delta P)^2 - \lambda \int \psi^*\psi\,\mathrm{d}q\right] = 0\,. \tag{23}$$

Variational techniques are commonly used in analytical mechanics [18] and in field theory [19]. In quantum mechanics (where they allow obtaining an upper bound for the ground state energy) one has to recall that ψ and ψ^* must be varied independently of each other [20].

Let us vary $\Pi(\psi,\psi^*) - \lambda \int \psi^*\psi\,\mathrm{d}q$ with respect to ψ^*:

$$\frac{\delta}{\delta\psi^*}\left[\int \psi^*(q - \langle q\rangle)^2\psi\,\mathrm{d}q \int \psi^*(P - \langle P\rangle)^2\psi\,\mathrm{d}q - \lambda \int \psi^*\psi\,\mathrm{d}q\right] = 0\,. \tag{24}$$

From this we get

$$\frac{\delta}{\delta\psi^*}\left[\int \psi^*(q-\langle q\rangle)^2\psi\,\mathrm{d}q\right](\Delta P)^2+(\Delta q)^2\frac{\delta}{\delta\psi^*}\left[\int \psi^*(P-\langle P\rangle)^2\psi\,\mathrm{d}q\right]-\lambda\frac{\delta}{\delta\psi^*}\int \psi^*\psi\,\mathrm{d}q=0\,, \tag{25}$$

and, performing the functional derivatives,

$$(q-\langle q\rangle)^2\psi\,(\Delta P)^2+(\Delta q)^2\,(P-\langle P\rangle)^2\psi=\lambda\,\psi\,. \tag{26}$$

Multiplying this equation on the left by ψ^* and integrating over q gives

$$\int \psi^*(q-\langle q\rangle)^2\psi\,\mathrm{d}q\;(\Delta P)^2+(\Delta q)^2\int \psi^*(P-\langle P\rangle)^2\psi\,\mathrm{d}q=\lambda\int \psi^*\psi\,\mathrm{d}q\,, \tag{27}$$

which implies

$$\lambda=2\,(\Delta q)^2(\Delta P)^2\,. \tag{28}$$

By inserting this result into Eq. (26) and dividing by $(\Delta q)^2(\Delta P)^2$ we find

$$\frac{1}{2}\left[\frac{(q-\langle q\rangle)^2}{(\Delta q)^2}+\frac{(P-\langle P\rangle)^2}{(\Delta P)^2}\right]\psi=\psi\,. \tag{29}$$

This is the equation that must be satisfied in order to minimize $\Delta q\,\Delta P$. It was explicitly written down by Persico in his 1930 note, and we will call it the Persico equation. Notice that it actually provides the states for which the product $\Pi(\psi,\psi^*)=(\Delta q)^2\,(\Delta P)^2$ is stationary: its solutions may in general represent not only minima, but also maxima or inflection points of $\Pi(\psi,\psi^*)$. Therefore, one has to choose among them the wavefunctions that give the smallest value of $\Delta q\,\Delta P$.

As also pointed out by Persico, Eq. (29) shows that the minimum uncertainty states are those for which the dimensionless observable

$$\Lambda(q,p)\equiv\frac{1}{2}\left[\frac{(q-\langle q\rangle)^2}{(\Delta q)^2}+\frac{(P-\langle P\rangle)^2}{(\Delta P)^2}\right] \tag{30}$$

takes the value 1.

Persico's result was never published in a physics journal and is not reported in his textbook *Fondamenti della meccanica atomica*, translated in English in 1950 [21]. Therefore it has been completely ignored by physicists, historians and teachers. In 1968, the same result was reobtained by Roman Jackiw, in the context of his work on number-phase uncertainty products [22]. Jackiw derived by variational techniques the eigenvalue equation [23]

$$\frac{1}{2}\left[\frac{(A-\langle A\rangle)^2}{(\Delta A)^2}+\frac{(B-\langle B\rangle)^2}{(\Delta B)^2}\right]\psi=\psi\,, \tag{31}$$

which generalizes Eq. (29), and pointed out that this method for obtaining minimum uncertainty states is more general than others, such as Ditchburn's one. He noticed in particular that by acting on Eq. (20) by $\hat{A} - \mathrm{i}\gamma_0 \hat{B}$ one gets

$$\left(\hat{A}^2 + \gamma_0^2 \hat{B}^2 - \gamma_0 C\right) \psi = 0\,. \tag{32}$$

Using (19), this equation can be written as

$$\left[\hat{A}^2 + \frac{(\Delta A)^2}{(\Delta B)^2}\, \hat{B}^2 - \frac{2(\Delta A)^2}{\langle C\rangle}\, C\right] \psi = 0\,, \tag{33}$$

that is

$$\left[\frac{(A - \langle A\rangle)^2}{(\Delta A)^2} + \frac{(B - \langle B\rangle)^2}{(\Delta B)^2} - 2\,\frac{C}{\langle C\rangle}\right] \psi = 0\,, \tag{34}$$

which is equivalent to Eq. (31) if ψ is an eigenfunction of C. Thus Ditchburn's method determines a minimum (or, at least, a stationary) value of the product $\Delta A\, \Delta B$ if and only if ψ is an eigenfunction of the commutator $[A, B]$ [22].

IV. A SIMPLE EXAMPLE

In practice, the Persico equation (29) takes the form

$$\frac{1}{2}\left[\frac{(q-\alpha)^2}{a^2} + \frac{\left(P(q, -\mathrm{i}\hbar\, \mathrm{d}/\mathrm{d}q) - \beta\right)^2}{b^2}\right] \psi = \psi\,, \tag{35}$$

with the four parameters α, a, β, b to be determined consistently with

$$\alpha = \int \psi^*\, q\, \psi\, \mathrm{d}q\,, \quad a^2 = \int \psi^* (q - \langle q\rangle)^2 \psi\, \mathrm{d}q\,, \tag{36}$$

$$\beta = \int \psi^*\, P\, \psi\, \mathrm{d}q\,, \quad b^2 = \int \psi^* (P - \langle P\rangle)^2 \psi\, \mathrm{d}q\,. \tag{37}$$

As an example of application of Eq. (35), let us explicitly work out the simplest case: $A = x$ and $B = p_x = -\mathrm{i}\hbar \mathrm{d}/\mathrm{d}x$. Then we have

$$\left[\frac{(x-\alpha)^2}{a^2} + \frac{(-\mathrm{i}\hbar\, \mathrm{d}/\mathrm{d}x - \beta)^2}{b^2}\right] \psi(x) = 2\, \psi(x) \tag{38}$$

where $\alpha = \langle x\rangle$, $a = \Delta x$, $\beta = \langle p_x\rangle$, $b = \Delta p_x$. More explicitly Eq. (38) becomes (apices denote differentiation with respect to x)

$$\psi'' - \frac{2\mathrm{i}\beta}{\hbar}\, \psi' - \frac{b^2}{\hbar^2}\left[\frac{(x-\alpha)^2}{a^2} + \frac{\beta^2}{b^2} - 2\right] \psi = 0\,. \tag{39}$$

Following Persico's suggestion [11] we transform this equation into a generalized Riccati equation [24–26]. To this end we introduce the function $u(x)$ defined by

$$u(x) = \frac{\mathrm{d}\ln\psi}{\mathrm{d}x} = \frac{\psi'}{\psi}\,, \tag{40}$$

from which ψ is obtained as

$$\psi(x) = C\,\exp\int^{x} u(\xi)\,\mathrm{d}\xi\,, \tag{41}$$

where C is an arbitrary constant of integration. In terms of u, Eq. (39) becomes

$$u' = -u^2 + \frac{2\mathrm{i}\beta}{\hbar}\,u + \frac{b^2}{\hbar^2}\left[\frac{(x-\alpha)^2}{a^2} + \frac{\beta^2}{b^2} - 2\right], \tag{42}$$

which is a nonlinear first-order equation of the Riccati type. Its general solution can be easily constructed once a particular solution is known [24, 25]. For our purposes finding a particular solution will be sufficient. By inspection we see that Eq. (42) admits a solution of the form

$$u(x) = C_1\,x + C_2\,, \tag{43}$$

where C_1 and C_2 are in general complex constants. By inserting the function (43) into Eq. (42) and equating the coefficients of the x^2 and x terms on the two sides, one gets

$$C_1 = -\frac{b}{\hbar a}\,, \quad C_2 = \frac{\alpha b}{\hbar a} + \frac{\mathrm{i}\beta}{\hbar}\,. \tag{44}$$

With these conditions, Eq. (42) turns out to be satisfied if $ab = \hbar/2$, that is if $\Delta x\,\Delta p_x = \hbar/2$, which is indeed the minimum uncertainty value. The solution of the Riccati equation we were searching for is therefore (with $b = \hbar/2a$)

$$u(x) = -\frac{1}{2a^2}\,(x-\alpha) + \frac{\mathrm{i}\beta}{\hbar}\,. \tag{45}$$

Going back to Eq. (41) and using the solution (45) yields

$$\psi(x) = C\,\exp\left[-\frac{(x-\alpha)^2}{4\,a^2} + \frac{\mathrm{i}\beta x}{\hbar}\right], \tag{46}$$

which is the well-known Gaussian wave packet minimizing the position-momentum uncertainty product.

V. CONCLUSIONS

Minimum–uncertainty states play an important role in many areas of physics: not only in quantum optics, where they take the well-known form of coherent and squeezed states [27], but also in modern research in quantum information and computation [28], and in various generalized approaches to uncertainty relations [29].

In most presentations and textbooks the determination of minimum-uncertainty states is based on a technique devised by Robert Ditchburn, which has some limitations. We have shown that in 1930 the Italian theorist Enrico Persico proposed a different approach, based on an eigenvalue equation, which is the most general method for calculating minimum–uncertainty states and represents a significant contribution to the foundations of the uncertainty principle. Persico's equation is a remarkable result that should at last find its place in pedagogical presentations and historical accounts of quantum mechanics.

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Appendix: Remarks on the indetermination principle (by E. Persico)

[This is the English translation of E. Persico, "Considerazioni sul principio di indeterminazione" [11]. We will adhere to the original notation, using overlines to denote expectation values and introducing a factor 2 in the definition of the uncertainties.]

Kennard has proven that the uncertainties q_i and p_i in the measurement of a coordinate q and of the corresponding conjugate momentum p, defined as

$$q_i^2 = \overline{2(q-\bar{q})^2}, \quad p_i^2 = \overline{2(p-\bar{p})^2}\,, \tag{A1}$$

are related by

$$p_i q_i \geq \frac{h}{2\pi}\,, \tag{A2}$$

which represents the quantitative expression of Heisenberg indetermination principle. Then, in order for $p_i q_i$ to have the minimum possible value, that is $h/2\pi$, the probability amplitude for q must have the form

$$\Psi(q) = C_0\, \mathrm{e}^{-\frac{(q-\bar{q})^2}{2q_i^2} + \mathrm{i}\left[b^2(q-\bar{q})^2 + c(q-\bar{q})\right]}\,, \tag{A3}$$

and therefore the probability curve for q must be the well-known Gauss curve [Actually Kennard shows that the minimum uncertainty wavefunction is (A3) with $b = 0$ (author's note)].

I decided to study the analogous problem for any pair of physical variables, not necessarily conjugate to each other.

Since we can always take, by a canonical transformation, one of the two variables as a coordinate, we can refer, with no loss of generality, to the case where the variables are q and a function $P(q,p)$, and determine the conditions under which one has

$$q_i P_i = \text{min.}, \tag{A4}$$

where P_i is defined analogously to Eq. (A1).

The method adopted by Kennard cannot be generalized to the present case. On the other hand, treating the problem by the ordinary methods of the calculus of variations (with some caution due to the presence of complex quantities), one finds the differential equation that must be satisfied by $\Psi(q)$ so that $q_i P_i$ is minimum. This equation, remarkably simple in form, is the following

$$\left[\left(\frac{q-\bar{q}}{q_i}\right)^2 + \left(\frac{P\left(q, \frac{h}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}q}\right) - \bar{P}}{P_i}\right)^2\right]\Psi = \Psi\,, \tag{A5}$$

which must be obviously associated to the equations defining $\bar{q}$, q_i, $\bar{P}$, P_i, namely

$$\bar{q} = \int \Psi^* q \Psi \,\mathrm{d}q\,, q_i = 2\int \Psi^* \left(q-\bar{q}\right)^2 \Psi \,\mathrm{d}q\,,$$

and to the similar ones for $\bar{P}$ and P_i.

In the case of $P = p$, Eq. (A5) can be transformed into a Riccati equation, from which one gets for Ψ the expression (A3), hence reobtaining Kennard's result.

In the other cases, from (A5) one finds instead that Ψ vanishes everywhere except for $q = q_i$, hence $q_i = 0$, and similarly $P_i = 0$: which means that P and q can be both measured with absolute precision.

Equation (A5) lends itself to the following suggestive interpretation. If one measures exactly the quantity

$$\Lambda(q,p) = \left(\frac{q-\bar{q}}{q_i}\right)^2 + \left(\frac{P-\bar{P}}{P_i}\right)^2\,,$$

and finds $\Lambda = 1$, then the equation which determines the Schrödinger function $\Psi(q)$ is indeed

Eq. (A5), and therefore the state of the system satisfying the condition (A4) is the one defined by $\Lambda = 1$.

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[1] W.K. Heisenberg, "Über den anschaulichen Inhalt der quantentheoretischen Kinematik and Mechanik", Zeit. f. Phys. **43**, 172–198 (1927).

[2] A.E. Ruark, "Heisenberg's indetermination principle and the motion of free particles", Phys. Rev. **31**, 311–312 (1928).

[3] E.U. Condon, "Remarks on Uncertainty Principles", Science, **69**, 573–574 (1929).

[4] E.H. Kennard, "Zur Quantenmechanik einfacher Bewegungstypen", Zeit. f. Phys. **44**, 326–352 (1927).

[5] For a historical account of the early work on the uncertainty principle see for instance M. Jammer, *The Conceptual Development of Quantum Mechanics*, McGraw-Hill, New York, 1966.

[6] H. Weyl, *Gruppentheorie und Quantenmechanik*, Hirzel, Leipzig, 1928 (2nd edition, 1931). English translation (by H.P. Robertson), *The Theory of Groups and Quantum Mechanics*, Methuen, London, 1931.

[7] H.P. Robertson, "The Uncertainty Principle", Phys. Rev. **34**, 163–164 (1929).

[8] E. Schrödinger, "Zum Heisenbergschen Unschärfeprinzip", Sitzungsber. Preuss. Akad. Wiss. **19**, 296–303 (1930).

[9] R.W. Ditchburn, "The Uncertainty Principle in Quantum Mechanics", Proc. Royal Irish Academy, Sect. A, **39**, 73–81 (1930).

[10] C. Cohen-Tannoudji, D. Diu, F. Laloë, *Quantum Mechanics*, vol. I, John Wiley & Sons, New York, 1977, p. 286-289.

[11] E. Persico, "Considerazioni sul principio di indeterminazione", in *Atti della* 19^a *Riunione della Società Italiana per il Progresso delle Scienze*, Bolzano-Trento (Italy), vol. II, p. 94-95 (1930).

[12] Interview with Franco Rasetti and Enrico Persico by Thomas S. Kuhn, Rome, April 8, 1963, Niels Bohr Library & Archives, American Istitute of Physics (www.aip.org/history-programs/niels-bohr-library/oral-histories/4995).

[13] E. Persico, review of E. Fermi, *Collected Papers*, Scientific American, **207**, n. 5, 181–187 (1962).

[14] E. Fermi, "L'interpretazione del principio di causalità nella meccanica quantistica", Nuovo Cimento, **7**, 361–366 (1930).

[15] Fermi's paper has been recently recognized as the earliest example of a phase–space approach to quantum dynamics. See G. Benenti and G. Strini, "Gaussian wave packets in phase space: the Fermi g_F function", Am. J. Phys. **77**, 546–551 (2009).

[16] This emerges from some handwritten notes by Persico conserved in Rome (Archivio Persico, Department of Physics, Sapienza University of Rome, box 28, folder 189).

[17] Persico was very familiar with the techniques of mathematical physics and theoretical mechanics. He had studied advanced mechanics with Vito Volterra in Rome and, as a young professor, he lectured on rational mechanics and mathematical physics in Florence and Turin. See E. Amaldi and F. Rasetti, "Ricordo di Enrico Persico", Giornale di Fisica, **20**, 236–260 (1979).

[18] See, *e.g.*, H. Goldstein, *Classical Mechanics*, 2nd edition, Addison–Wesley, Reading (MA), 1980, Chapter 2.

[19] See, *e.g.*, P.M. Morse and H. Feshbach, *Methods of Theoretical Physics*, vol. I, McGraw-Hill, New York, 1953, Chapter 3.

[20] A discussion of variational methods in quantum mechanics can be found, for instance, in E. Merzbacher, *Quantum Mechanics*, 3rd edition, John Wiley & Sons, New York, 1998, Chapter 8.

[21] E. Persico, *Fondamenti della meccanica atomica*, Zanichelli, Bologna, 1936 (English translation, *Fundamentals of Quantum Mechanics*, Prentice-Hall, New York, 1950). It is curious to note that in his book Persico uses a method similar to Ditchburn's (perhaps developed independently) to show that a Gaussian wavepacket minimizes the product of the position and wave number half-widths (see p. 110-114 of the English edition).

[22] R. Jackiw, "Minimum Uncertainty Product, Number–Phase Uncertainty Product, and Coherent States", J. Math. Phys. **9**, 339–346 (1968).

[23] Jackiw's paper is well known especially for its discussion of the number–phase problem. The variational equation for minimum uncertainty states has received a more limited attention. For some applications, see: S. Kechrimparis, S. Weigert, "Universality in uncertainty relations for a quantum particle", J. Phys. A **49**, 355303 (2016); M. Alonso, T. Setälä, A.T. Friberg, "Minimum uncertainty solutions for partally coherent fields and quantum mixed states", New. J. Phys. **16**, 123023 (2016).

[24] G.M. Murphy, *Ordinary Differential Equations and Their Solutions*, Van Nostrand, New York, 1960.

[25] W.T. Reid, *Riccati Differential Equations*, Academic Press, New York, 1972.

[26] S.B. Haley, "An underrated entanglement: Riccati and Schrödinger equations", Am. J. Phys. **65**, 237–243 (1997).

[27] For a recent introduction to this subject and an extensive list of references, see O. Rosas-Ortiz, "Coherent and squeezed states: introductory review of basic notions, properties and generalizatios", arXiv:1812.07523 [quant-ph].

[28] See, *e.g.*: K.-W. Bong *et al.*, "Strong unitary and overlap uncertainty relations: theory and experiment", Phys. Rev. Lett. **120**, 230402 (2018); T. Hashimoto, A. Hayashi and M. Horibe, "Minimum–uncertainty states and completeness of non-negative quasiprobability of finite-dimensional quantum systems", Phys. Rev. **A99**, 022126 (2019); H. Zhu, "Minimum uncertainty states in the presence of quantum memory", arXiv:2003.02103 [quant-ph].

[29] See, *e.g.*: S. Kechrimparis, S. Weigert, "Heisenberg uncertainty relations for three canonical ob-

servables", Phys. Rev. A **90**, 062118 (2014); A. Hertz and N.J. Cerf, "Continuos-variable entropic uncertainty relations", J. Phys. A **52**, 173001 (2019); S. Fu, S. Luo and Y. Zhang, "Gaussian states as minimum uncertainty states", Phys. Lett. **A384**, 126037 (2020).