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The capital-on-capital cost in solvency II risk margin

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Abstract
This work contributes to the literature on time consistent valuation of insurance liabilities and to the ongoing discussion on revisions of risk margin (RM) calculation, by formally defining the concept of capital-on-capital cost. We describe the capital-on-capital as the amount required to cover unexpected variations in future regulatory capitals from the current time to liabilities maturity. That is, the capital-on-capital cost is the RM component dedicated to cover the risk of future RMs and not to cover variations of the best estimate of liabilities cash-flows. We mathematically formalize the capital-on-capital cost as the difference between two alternative time consistent liabilities valuation formula. The first is obtained through backward iteration of the one-period market-consistent valuation operator, by iterating the solvency capital requirement (SCR) risk measure. We propose a second alternative valuation formula for liabilities, based on a new time consistent dynamic formulation of the SCR risk measure, called additive-SCR (ASCR). The ASCR represents the expected total capital requirement from current time to liabilities maturity. We prove that the second valuation formula, based on ASCR, is time consistent, unless it is not based on iteration of the one-period SCR risk measure. Finally, we apply the proposed approach to a portfolio of long term equity linked life-insurance contracts. In particular, we estimate the capital-on-capital cost by calculating the difference between the two valuation formulas. Numerical results show that for liabilities with long maturities the capital-on-capital cost is a non negligible component of the RM.

Keywords Capital-on-capital cost · Risk margin · Solvency capital requirement · Technical provisions · Time consistency

JEL Classification C61

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1 Introduction

During the last decade, the European Community, in particular, the European Insurance and Occupational Pension Authority (EIOPA)[1] has introduced new prudential rules for insurance balance sheet evaluation. Its aim is to establish and maintain compatibility between accounting and regulatory standards for harmonization of requirements, transparency and avoidance of arbitrage, both among jurisdictions and across other financial sectors such as banking. This effort has led to the Solvency II directive and its technical standards [18]. The Solvency II directive adopts the cost of capital approach for the market consistent valuation of liabilities, also called technical provisions (TP), which should be equal to the sum of a Best Estimate (BE) of liabilities and a Risk Margin (RM), see [23] and [18, TP.1.1]. In recent times, practitioners have engaged in discussions about potential revisions to the calculation of the Solvency II risk margin. This issue has been highlighted in reports by the British Treasury Committee [10] and the Actuarial Association of Europe [1]. In particular, insurance companies have criticized the very high RM value for long-term life insurance portfolios. Life insurance companies and pension funds hold long-term liabilities on their books (from 30 to 50 years), as individuals begin saving at a relatively young age (between 30 and 50 years) and the average life expectancy at birth in the EU is about 80 years (Eurostat 2022). Roughly 20% of the net present value of the liabilities is attributed to cash-flows beyond 30 years. According to [1], EIOPA indicates that the total risk margin for the entire European insurance industry was 210 billion euros in the third quarter of 2016 of which 150 billions stemmed from life and composite insurance undertakings, representing more than 45% of the overall EU life insurance industry Solvency Capital Requirement (SCR). As highlighted by AAE [1], the size of the risk margin is an important issue, but merely because it is in aggregate a large number does not necessarily make it wrong. Rather, a determination on whether the risk margin is set too high or too low should ideally consider relevant financial and actuarial principles. In particular, a crucial point is the construction of risk measures and valuation formulas that combine time and market consistency with reasonable levels of prudence across different time scales.

As reported in [18, TP.5.3], the RM should be calculated by determining the cost of providing an amount of eligible own funds equal to the SCR necessary to support the insurance and reinsurance obligations over the lifetime thereof. Then, the RM calculation implicitly requires a dynamic definition of the SCR. In particular, [18, TP.5.8] reports the following formula for calculating the overall RM

$$RM = \delta \sum_{t \geq 0} \frac{SCR_{RU}(t)}{(1 + r_{t+1})^{t+1}}, \tag{1.1}$$

where δ is the Cost-of-Capital rate and it is fixed to 6% [18, TP.5.21], r_t is the basic risk free rate at time t and $SCR_{RU}(t)$ is the SCR for year t as calculated for the reference undertaking. The SCR for the reference undertaking is better specified in [18, TP.5.4] as the assets should be considered to be selected in such a way that

[1] All acronyms are reported in Table 1.

Table 1 Acronyms

AAE	Actuarial association of Europe
ASCR	Additive solvency capital requirement
BE	Best estimate
EIOPA	European insurance and occupational pension authority
ESCR	Expected solvency capital requirement
LSMC	Least square Monte Carlo
MCV	Market consistent valuation
RM	Risk margin
SCR	Solvency capital requirement
SEC	Separable expected conditional
TMCV	Time and market consistent valuation
TP	Technical provisions
VaR	Value at risk

they minimise the SCR for market risk that the reference undertaking is exposed to. This means that the SCR_{RU} is the result of a (possible) perfect hedging of the market risk component of liabilities. As highlighted also in [13], formula 1.1 has the following three deficiencies, which cannot be clarified from the official documents, since a mathematically rigorous definition is missing. Firstly, a precise mathematical definition of the SCR of a reference undertaking is missing. Second, the RM definition requires the calculation of the SCR at future points in time, but a dynamic definition of the SCR is also missing in the document. Third, formula 1.1 defines the RM only at time $t = 0$.

1.1 Literature discussion

From a mathematical point of view, the Solvency II directive and EIOPA technical documents open a lot of questions that have been addressed in literature in the last years. [13] analyse the SCR definition in the Solvency II directive and they propose and compare different mathematical interpretations of the regulators directive. Similarly, [21] discuss the adequacy with respect to the regulatory requirements of two different mathematical formulation for the SCR in a one period setting. In particular, we adopt the so-called mean value at risk formulas of the SCR. This definition is linked to the concept expressed in the directive that SCR shall cover only unexpected losses and it is widely used in practice. Additionally, [13] define the dynamic SCR using the conditional Value at Risk (VaR). Additionally, they obtain the SCR of the reference undertaking SCR_{RU} minimizing the conditional VaR with respect to the asset allocation (a sort of hedging strategy). The authors do not discuss the time consistency of their proposed approach, and it is well-established in the literature, that conditional risk measures (and then their optimal minimization policy) may not be consistent over time, see for instance, [2, 12].

1.1.1 Time consistency

In financial and insurance literature, an increasing attention is given to the time consistency property of dynamic risk measures and valuation operators, see for instance [2, 6, 9, 14, 15, 22, 25, 29, 33].

The concept of time consistency has to do with the consistency over time of the risk evaluation. This concept was firstly introduced by [36]. The main idea can be described as follows: if we know (almost) surely that an investment X is less risky than an investment Y in a future date, then the same order should apply today. Building upon this concept, subsequent works by [6, 32] focus on multi-period coherent risk measures and [32] develop different weaker notions of time consistency. In these time consistency definitions, the aim is to ensure that the ordering of risks remains consistent as we move across different time periods, in alignment with Wang's original concept.

In the above mentioned papers, the primary focus is on terminal wealth. However, in the context of multi-period investment problems, investors are interested in understanding their positions at various intermediate time points along the investment horizon. Recognizing this, [30] introduces the concept of dynamic time consistency, which can be defined as follows: for any two cash-flows processes X and Y, if they have the same value with respect to a given risk measure at future time T and X is the same as Y (almost surely) at all the stages between t and T, then X and Y have the same value with respect to the measure at present time t, see also Appendix A Definition A.3. Similar notions are also presented in [12, 33]. This second formulation, as introduced by Riedel [30], doesn't involve the property of monotonicity and, therefore, can be used to define time-consistent dynamic risk mappings,[2] as further elaborated by Chen et al. [11].

If the dynamic risk measure (or risk mapping) is dynamic time consistent, then, it can be equivalently expressed in a recursive form, see [11, 33] and Appendix A Theorem A.4. In light of this, some authors employ the recursive formulation to define the time consistency of dynamic risk mappings, see for instance definitions such as recursiveness or the tower property by [9, 29]. Furthermore, the equivalence between time consistency and recursive formulation is related to the application of dynamic programming for risk averse stochastic control problems. In particular, [6] prove the equivalence between time consistency of coherent risk measures and Bellman's optimality principle and [11] extend the results to dynamic risk mappings.

1.1.2 The cost of time-consistency

To obtain time consistent dynamic risk measures, the most prevalent method in the literature exploits the recursive formulation by backward iterating the corresponding single period conditional risk measure. Hardy and Wirch [22], for instance, propose the iterated conditional tail expectation to calculate the regulatory capital for an equity-linked life insurance contract. Likewise, [15] analyse a time consistent formulation of the SCR using iterated VaR. However, both papers highlight that the use of iterated

[2] A risk mapping differs from a risk measure, in that it does not require monotonicity, see Appendix A Definition A.1.

risk measures to achieve time consistency comes at the cost of substantial additional capital required throughout the duration of the contract. The construction of dynamic risk measures and valuation operators that combine time consistency with reasonable levels of prudence across different time scales remains a challenging task. This problem is also highlighted in [31], the authors argue that strong time consistency inhibits the construction of families of risk measures that maintain comparable standards of prudence on different time scales; for instance iterated VaR is likely to be very conservative. Then, [31, 35] use a weaker notion of time consistency called sequential consistency, based on the properties of weak acceptance and rejection, firstly proposed by [37]. In particular, [31] use the sequential consistency to construct consistent families of operators, which allow flexibility in the specification of prudence over time.

Another viable alternative to achieve strong time consistency[3] without using the backward iteration of the corresponding one-period operator is through the application of Separable Expected Conditional (SEC) risk mappings defined in [11, 14, 25]. SEC risk mappings have been recently introduced in the literature and have rarely been applied in the financial and insurance literature, see for instance [11] for a time consistent formulation of a dynamic mean–variance optimization problem. Time consistency and recursive formulation of SEC risk mappings are proved in literature, see [11, 14, 25]. We remark that dynamic SEC risk mappings are recursive, but are not self-recursive. This means that SEC are not iterated risk measures, more details are given in Appendix B. In particular, we apply SEC risk mappings to build our RM proposal.

1.1.3 Market consistent valuation

In recent years, a growing body of literature is aimed at extending, in a time consistent way the Market Consistent Valuation (MCV) of liabilities, obtained through the cost of capital approach. In this way, the calculation of TP, RM and SCR is tackled in a unitary and organic way. As remarked in the EIOPA technical documentation: the primary objective for valuation as set out in Article 75 of Directive 2009/138/EC requires an economic, market-consistent approach to the valuation of assets and liabilities, [18, V.1]. The concept of MCV of liabilities has been widely discussed in literature over the past few years. Gambaro et al. [20] examine the industry's standard approach to evaluating options embedded in life insurance contracts and propose a viable and reasonable proxy within the context of market consistent evaluations. Dhaene et al. [16] introduce a hedge-based valuation method for insurance liabilities, which involves two steps. In the first step, a portfolio of the best hedging assets is established for the liability based on the assets traded in the market. In the second step, the remaining part of the claim is evaluated through an actuarial valuation, such as the cost of capital approach proposed by [23]. Another approach is proposed in [16, 29], a two-steps valuation method, conditional on the perfect hedging of the market risk component of liabilities. Firstly, the liability claim is evaluated using an actuarial technique, conditionally to the knowledge of market risk factors evolution. In the second step, the expected value

[3] We refer the reader to Appendix A for details on the adopted concept of time consistency.

of the actuarial valuation is taken under a risk-neutral measure. In [24], the authors demonstrate the existence of a unique economic evaluation measure, such that the expectation with respect to this measure coincides with a two-steps valuation formula in the Black and Scholes framework. Additionally, [7] discuss the concept of market consistency in case of incomplete financial markets.

Barigou et al. [8, 9] extend the hedge-base valuation of [16] to a multi-period framework in a time consistent manner. Similarly, [34] propose a market and time consistent formulation of the EIOPA RM, extending the one-period two-steps valuation formula proposed in [29]. All these authors use a backward iteration procedure of the one-period MCV to achieve the time consistency of the dynamic formulation. [34] introduce the concept of the capital-on-capital effect resulting from iteration of risk measures, such as the VaR for the SCR. However, the authors do not provide a precise definition of the capital-on-capital effect and its cost. Instead, we decompose the RM into two components: the cost of capital to cover variations in the BE of liabilities and the cost of capital-on-capital.

The hedge-based valuation of [16] has the advantage of explicitly incorporating the asset allocation problem in the liability valuation. As any insurance company holds a real portfolio of assets to cover the liabilities, it is reasonable to consider it in the valuation. Moreover, using the hedge-based valuation the hypothesis of complete financial market can be relaxed, in fact, realistic market models are typically incomplete. However, different hedging strategies can lead to divergent definitions of the value of technical provisions. For example, [16] proposes mean–variance and convex loss strategies, while [8] suggest an hedging procedure based on mean–variance hedge and quantile regression. To maintain the linearity of the exposition of the capital-on-capital calculation, we adopt the two-steps approach proposed in [24, 29, 34]. It is worth noting that the procedures presented in Sects. 3.1 and 3.3 for extending the single period valuation formula to a multi-period framework apply similarly to both the two-steps or the hedge-based valuation. Future research could focus on analysing the effect of hedging strategies on the calculation of the capital-on-capital cost.

1.2 Our contributions

In this work, our contributions to the literature on time and market consistent valuation of insurance liabilities are mainly two. The first contribution is the proposal of a time consistent approach for the valuation of liabilities, without using backward iteration of the SCR risk measure. Our valuation approach is based on a new dynamic extension of the SCR, that we call additive-SCR (ASCR). We prove that the ASCR is a dynamic SEC risk mapping and then that is time consistent. The ASCR represents the expected total capital requirement for the period from the current time to the liabilities maturity. Furthermore, the ASCR can be decomposed into annual expected-SCRs (ESCRs), which represent the expected regulatory capitals for the next year, not only at the current time, but also at future payment dates. This allows us to build a term structure of expected annual SCRs. The ESCR represents the best estimate of the capital needed for a specific year of the product and is a risk measure particularly intuitive and economically meaningful. Our proposed formula for liability valuation is based on futures ESCRs, then it does not contains iterated risk measures, while maintaining time consistency.

The second contribution is the formal definition of the cost of capital-on-capital as the RM component that is dedicated to covering the risk of future RMs, rather than covering variations in the BE of liabilities cash-flows. This cost can be mathematically formulated and estimated as the difference between the value of RM based on backward iteration and the proposed valuation based on the ESCRs. Defining and estimating the cost of capital-on-capital brings a new element to the discussion about the possible revision of the RM calculation. Specifically, it sheds light on the amount of regulatory capital that is self-generated, growing from year to year and becoming a consistent source of cost for long-term liabilities.

Finally, we apply the two time consistent valuation formulas to a portfolio of equity-linked life-insurance contracts and we estimate the capital-on-capital cost by calculating the difference between the two time consistent valuation. Moreover, we implement also a non-time consistent valuation formula based on SCR projections, which is a widespread industry rule-of-thumb for the RM calculation. Then, we show that the capital-on-capital cost does not necessary coincide with a time consistent risk premium. In our numerical experiments, we note that the capital-on-capital cost is non negligible component of the RM value and it reaches the 5% and 12% of the RM value for a long term liabilities with 30 and 40 years time to maturity.

The work is organized as follows. In Sect. 2, we present the general framework of assets and liabilities and we describe the one period market consistent valuation formula of liabilities. In Sect. 3, we propose two alternative approaches for the time consistent calculus of liabilities. The first, presented in Sect. 3.1, is obtained by a backward iteration of the one-period MCV formula. The second in Sect. 3.3 is based on the definition of ASCR given in Sect. 3.2. Then, in Sect. 3.4 we define the capital-on-capital cost as the difference between the valuations obtained with the two alternative approaches. In Sect. 4, we present an illustrative example. We compare the two valuations of liabilities and we estimate the capital on capital cost by calculating the difference between them. Conclusive remarks are presented in the last section.

2 A market consistent valuation of liabilities

In this section, using the cost-of-capital approach, we present a market consistent valuation formula for the liabilities portfolio of an insurance or reinsurance company. We assume that the financial market is complete and arbitrage-free, which means that all claims only depend on financial risks and can be perfectly hedged. Then we apply the one-period two-steps valuation, using the physical measure and an actuarial valuation for purely insurance risk and expectation under the risk neutral measure for market risk, see [16, 24, 29].

2.1 Assets and liabilities

We consider a canonical probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a time horizon T with consecutive dates $\{0, 1, 2, \ldots, T\}$. For each $t = 0, 1, \ldots, T$, the σ-algebra $\mathcal{G}_t \subset \mathcal{G}$ denotes the set of all events, i.e., subsets of Ω, corresponding to information available at time t, with $\mathcal{G}_0 = \{\Omega, \emptyset\}$. The price processes of the traded assets are described by the $(n+1)$-dimensional stochastic process $X_{0,T} = \{X(t)\}_{t=0,1,\ldots,T}$. The process of actuarial risk factors is represented by the m-dimensional stochastic process $Y_{0,T} = \{Y(t)\}_{t=0,1,\ldots,T}$. At any time $t = 0, 1, \ldots, T$, the vector $X(t) = (X_0(t), X_1(t), \ldots, X_n(t))$ represents the market values of the risk free asset $X_0(t)$ and of n risky assets $(X_1(t), \ldots, X_n(t))$, while the vector $Y(t) = (Y_1(t), \ldots, Y_m(t))$ represents the value of actuarial risk factors not traded in the market. We assume that the processes $X_{0,T}$, $Y_{0,T}$ are adapted to the filtration $\mathcal{G}$, which means that $X(t)$ and $Y(t)$ are $\mathcal{G}_t$ measurable at any time t. Let $Z_{0,T}$ be the random process of liabilities cash-flows, at any fixed time t, the random variable $Z(t)$ is a function of the market and actuarial risk factors

$$Z(t) := f_t(X(t), Y(t)). \tag{2.1}$$

The cash-flow $Z(t)$ represents the sum of all payments that the insurer makes or receives on the interval $(t-1, t]$, including any premiums, costs and benefits from new and old business. Throughout the paper, we assume that the second moments of all random variables that we consider exist under $\mathbb{P}$. In particular, we assume that $Z(t) \in \mathcal{L}^2(\mathcal{G}_t)$ is $\mathcal{G}_t$-measurable, where $\mathcal{L}^2(\mathcal{G}_t) := \mathcal{L}_2(\Omega, \mathcal{G}_t, \mathbb{P})$ denotes the class of square integrable random functions on Ω. Then the stochastic process $Z_{0,T}$ belongs to $\mathcal{L}^2_{t,T}$, where $\mathcal{L}^2_{t,T} := \mathcal{L}^2(\mathcal{G}_t) \times \cdots \times \mathcal{L}^2(\mathcal{G}_T)$ is the corresponding product space. The assumption that financial and actuarial claims are square integrable is widespread in literature, see for instance [9, 24, 34], and it does not restrict the applicability of the proposed valuation formulas.

2.2 Two-step actuarial valuation

In this section we briefly present the two step actuarial valuation procedure, for further details we refer the reader to [29, 34]. Given the hybrid nature of insurance cash-flows, we model the information flow, using two separate filtrations: $\mathcal{F}^X$ generated by the process $X_{0,T}$ for the financial information and $\mathcal{F}^Y$ generated by the process $Y_{0,T}$ for the actuarial information. The information flow of both risk categories is given by $\mathcal{F} := \mathcal{F}^X \vee \mathcal{F}^Y$. We assume that the financial market is arbitrage free and complete, then any claim depending solely on market assets, $H(T) = f_T(X(T)) \in \mathcal{L}^2(\mathcal{F}^X_T)$, is perfectly replicable. As a numéraire asset, we select the money-market account $B(t) = e^{\int_0^t r(u)du}$, where the risk free interest rate process $r_{0,T}$ can be stochastic and is adapted to the filtration $\mathcal{F}^X$. Hence, there exists a unique martingale measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ on $\mathcal{F}^X$, such that the relative prices of all assets divided by the money market account $B(t)$ are martingales, that corresponds to the no-arbitrage risk neutral measure. Then, the market consistent value at time t is given by $\mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(u)du} H(T)|\mathcal{F}_t^X]$. As stated in [29], setting $\mathbb{Q}(E) := \mathbb{E}^{\mathbb{P}}\left[\mathrm{I}_E \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$ for any set $E \in \mathcal{F}_T$, the measure $\mathbb{Q}$ can be extended canonically to the whole filtration $\mathcal{F}$.

In general, the market given by all $\mathcal{F}$-measurable claims is incomplete. For instance, we could have an untraded insurance process Y which is correlated with the traded assets X but not perfectly replicable. Then, following [29], we adopt the following valuation procedure for financial and actuarial mixed payoffs.

Definition 2.1 A two-step market consistent actuarial valuation $A_t : \mathcal{L}^2(\mathcal{F}_T) \to \mathcal{L}^2(\mathcal{F}_t)$ is given by

$$A_t[Z(T)] = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(u)du}\, \Pi[Z(T)]|\mathcal{F}_t],$$

where $\Pi[Z(T)]$ is $\mathcal{F}_T^X$ conditional valuation from $\mathcal{L}^2(\mathcal{F}_T)$ to $\mathcal{L}^2(\mathcal{F}_T^X \vee \mathcal{F}_t^Y)$.

In literature, the $\mathcal{F}_T^X$ conditional valuation operator $\Pi[Z(T)]$ is defined using different actuarial principles, such as the mean–variance, the standard-deviation or different risk measures principles. The economic idea behind all these principles is to add a risk buffer to the best estimate of the claim, to cover the unhedgeable risks. In Sect. 2.3, we detailed the definition of $\Pi[Z(T)]$, using the cost of capital approach required by the Solvency II directive.

2.3 One-period market consistent valuation of liabilities

Solvency II directive states that the market consistent value of the liabilities should be defined using the cost-of-capital approach, see [18, 23]. In this approach, the insurance company has to hold a capital buffer, called SCR, against the unexpected losses. The technical provisions of liabilities are obtained adding the cost of capital related to SCR, called RM, to the expected value, called BE. For a single period from $T-1$ to T, the calculation of SCR and RM are stated in the Solvency II directive and its technical specifications. The SCR should correspond to the VaR of the basic own fund subject to a confidence level of 99.5%, see [18, SCR.1.9] and Article 101(3) of the Solvency II Directive.[4] The RM is the SCR multiplied by the cost-of-capital rate δ_{T-1}, that is fixed equal to 6%, see [18, TP.5.21].

Using the notation given in Sects. 2.1 and 2.2 and Definition 2.1, we apply the two-steps valuation procedure on the final liability cash-flow $Z(T)$, and we obtain the following value of technical provisions $L(T-1)$ at time $T-1$,

$$\begin{aligned} L(T-1) &:= Z(T-1) + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{T-1}^T r(u)du}\, \Pi\left[Z(T)\right] |\mathcal{F}_{T-1}\right], \qquad (2.2)\\ \Pi\left[Z(T)\right] &:= \mathbb{E}^{\mathbb{P}}\left[Z(T) \mid \mathcal{F}_T^X \vee \mathcal{F}_{T-1}^Y\right] \\ &\quad + \delta_{T-1}\, VaR_\alpha^{\mathbb{P}}\left[Z(T) - \mathbb{E}^{\mathbb{P}}\left[Z(T) \mid \mathcal{F}_T^X \vee \mathcal{F}_{T-1}^Y\right] \mid \mathcal{F}_T^X \vee \mathcal{F}_{T-1}^Y\right], \end{aligned}$$

[4] In this paper, liabilities are modeled as non-negative random variables, and the VaR at the 99.5% confidence level pertains to losses. This corresponds to the SCR and the underlying VaR at the 0.5% confidence level, which refers to financial positions such as own funds, when liabilities are considered with a negative sign.

where α is the confidence level of the VaR fixed to 99.5%. Firstly, the final liability cash-flow $Z(T)$ is evaluated using $\Pi[Z(T)]$, that is the actuarial cost of capital approach under the physical measure $\mathbb{P}$, conditionally to the knowledge of market risk factors evolution at time T, $\mathcal{F}_T^X$. In the second step, the expected value of the conditional valuation $\Pi\,[Z(T)]$ is taken under a risk-neutral measure $\mathbb{Q}$.

For the proceeding of the work, it is useful to explicitly rewrite formula 2.2 decomposing the technical provision $L(T-1)$ in the BE and in the SCR, as

$$L(T-1) = Z(T-1) + BE_{T-1}[Z(T)] + \delta_{T-1} SCR_{T-1}[\Delta Z(T)], \tag{2.3}$$

where for any time $0 \leq t \leq T-1$ and any random variable $Z \in \mathcal{L}^2(\mathcal{F}_s)$ with $t \leq s \leq T$, we define the best estimate $BE_t[Z]$ as

$$BE_t[Z] := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s r(u)du}\,\mathbb{E}^{\mathbb{P}}\left[Z \mid \mathcal{F}_s^X \vee \mathcal{F}_t^Y\right] \mid \mathcal{F}_t\right]. \tag{2.4}$$

Furthermore, for any random variable $Z \in \mathcal{L}^2(\mathcal{F}_s)$ with $0 \leq t \leq s \leq T$, its variation $\Delta Z \in \mathcal{L}^2(\mathcal{F}_s)$ is defined as

$$\Delta Z = Z - \mathbb{E}^{\mathbb{P}}\left[Z|\mathcal{F}_s^X \vee \mathcal{F}_t^Y\right], \tag{2.5}$$

and the solvency capital requirement $SCR_t[\Delta Z]$ as

$$SCR_t[\Delta Z] := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s r(u)du}\,VaR_\alpha^{\mathbb{P}}[\Delta Z\,|\mathcal{F}_s^X \vee \mathcal{F}_t^Y]|\mathcal{F}_t\right]. \tag{2.6}$$

We assume that at time T the value of technical provisions and best estimate coincide with the liability cash-flow at maturity, that is $L(T) = BE_T[Z(T)] = Z(T)$.

Equations 2.4 and 2.6 are crucial for the proceeding of the work, then we report some useful remarks about them. Firstly, we note that the BE operator $BE_t : \mathcal{L}^2(\mathcal{F}_s) \to \mathcal{L}^2(\mathcal{F}_t)$ can be rewritten as an expectation under the economic valuation measure $\tilde{\mathbb{Q}}$ defined in [24], that is $BE_t[Z] = \mathbb{E}^{\tilde{\mathbb{Q}}}\left[e^{-\int_t^s r(u)du}\,Z|\mathcal{F}_t\right]$. Secondly, we note that in a one-period setting, the SCR component of liabilities in 2.3, that is $SCR_t[\Delta Z]$, coincides with the mean value at risk definition discussed in [21]. Moreover, $SCR_t[\Delta Z]$ can be interpreted as the SCR for the reference undertaking, $SCR_{RU}(t)$, required in Solvency II directive, see formula 1.1. In fact, $SCR_t[\Delta Z]$ is the result of the perfect hedging of the financial market risk part of the liability cash-flows. Finally, we remark that $SCR_t : \mathcal{L}^2(\mathcal{F}_s) \to \mathcal{L}^2(\mathcal{F}_t)$ defined in 2.6 for $0 \leq t \leq s$ is a conditional risk mapping, see Appendix A, Definition A.1. In fact, it satisfies the translation invariant property and it is normalized. This last remark is fundamental for definition of ASCR in the Sect. 3.2.

Finally, assuming that the processes $X_{0,T}$ and $Y_{0,T}$ are Markovian, the BE and SCR formulas 2.4 and 2.6 can be rewritten in a simpler way. Given any financial/actuarial claim, represented by the square integrable function $Z = f_s(X(s), Y(s))$, the BE at time t for $0 \leq t \leq s \leq T$ is given by

$$BE_t[Z] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s r(u)du}\, \mathbb{E}^{\mathbb{P}}\left[Z \mid X(s), Y(t)\right] \mid X(t), Y(t)\right],$$

and the SCR at time t is given by

$$\begin{aligned} SCR_t[\Delta Z] &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s r(u)du}\, VaR_\alpha^{\mathbb{P}}\left[\Delta Z \mid X(s), Y(t)\right] \mid X(t), Y(t)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s r(u)du}\, \left(VaR_\alpha^{\mathbb{P}}\left[Z \mid X(s), Y(t)\right] - \mathbb{E}^{\mathbb{P}}\left[Z \mid X(s), Y(t)\right]\right) \mid X(t), Y(t)\right] \end{aligned}$$

We precise that Markovianity is assumed solely to clarify the notation, and that in the following sections, Propositions 3.2, 3.4 and 3.6 remain valid even if the driving processes $X_{0,T}$ and $Y_{0,T}$ are not Markovian. Moreover, in the numerical implementation in Sect. 4.2, we adopt Markovian processes.

The application of VaR to measure the risk of capital losses is questionable. Important theoretical works have highlighted some limits of VaR as risk measure. In particular, [4, 5] postulated a set of properties that a well-behaved risk measure should satisfy and that are not fulfilled by VaR; in fact VaR for general distributions doesn't possess sub-additivity. For this reason, the adoption of risk measures different from VaR, such as convex or coherent risk measures, is widespread in the literature for capital requirements calculation, see [3, 17] and is increasing in insurance industries. However, the applications of risk measures, different from VaR to SCR calculation is out of the scope of this work. We note that all the definitions and proposition contained in this work for the MCV, RM and SCR calculations are applicable also to risk measures different from VaR, such as for instance the average value at risk as required by the Swiss Solvency Test. In particular, Definitions 3.1, 3.3, 3.5 and Propositions 3.2, 3.4 and 3.6 remain valid even if the SCR_t as defined in 2.6 is substituted with a more general conditional risk mapping as defined in Definition A.1.

3 Multi-period time and market consistent valuation (TMCV) of liabilities

In this section, we show two different procedures to achieve a time consistent calculus of the RM and then of the MCV of liabilities. The first consists in the backward iteration of the one-period valuation formula presented in Sect. 2.3. Then, we propose an alternative time consistent calculation of the RM and MCV of liabilities, through the definition of the additive-SCR: a dynamic extension of the one period SCR in formula 2.6. Then, we compare pros and cons of the two approaches. For instance [15] criticize the approach of iterated risk measures to capital requirements calculation, showing that it becomes quite expensive for long term liabilities. This is a consequence of the capital-on-capital effect produced by the backward induction, as suggested in [34].

The capital-on-capital cost is then estimated as the difference between the time and market consistent valuations obtained with the two methods.

3.1 Time consistent valuation of liabilities by backward induction

To extend the valuation formula 2.2 to a multi-period setting, a fundamental aspect to consider is time consistency. The time consistency can be achieved using backward induction, see for instance [2, 12, 29, 33] and references therein.

Definition 3.1 (*Time and market consistent valuation of liabilities* (*TMCV-1*)) We define the one period operator V_t for $0 \leq t \leq T-1$ as

$$\begin{aligned} V_t[L(t+1)] &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t+1} r(u)du}\, \mathbb{E}^{\mathbb{P}}\left[L(t+1) \mid \mathcal{F}^X_{t+1} \vee \mathcal{F}^Y_t\right] \mid \mathcal{F}_t\right] \\ &\quad + \delta_t\, \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t+1} r(u)du} \left(VaR^{\mathbb{P}}_{\alpha}\left[L(t+1) \mid \mathcal{F}^X_{t+1} \vee \mathcal{F}^Y_t\right]\right.\right. \\ &\quad \left.\left. - \mathbb{E}^{\mathbb{P}}\left[L(t+1) \mid \mathcal{F}^X_{t+1} \vee \mathcal{F}^Y_t\right]\right) \mid \mathcal{F}_t\right] \\ &= BE_t[L(t+1)] + \delta_t\, SCR_t[\Delta L(t+1)], \end{aligned} \tag{3.1}$$

where $BE_t[\cdot]$, $SCR_t[\cdot]$ and the variation $\Delta L(t+1)$ are defined in formulas 2.4, 2.6 and 2.5, respectively.

Then, by backward induction we obtain that the market and time consistent value of liabilities at any time t with $0 \leq t \leq T-1$ is equal to

$$\begin{aligned} L(t) &= Z(t) + V_t[L(t+1)] \\ &= Z(t) + V_t[Z(t+1) + V_{t+1}[Z(t+2) + \cdots + V_{T-2}[Z(T-1) \\ &\quad + V_{T-1}[Z(T)]] \cdots]], \end{aligned} \tag{3.2}$$

where $Z(s)$ for $s = t, t+1, \ldots, T$ is defined in equation 2.1.

We remark that V_t is a t-valuation as defined in [9] since it is normalized and translation invariant, or equivalently a conditional risk mapping as defined in Appendix A. Moreover, we note that the proposed backward iteration formulas for liabilities valuation is slightly different from the one proposed in [34]. In fact, we assume that VaR is conditioned on the evolution of the actuarial risk factor until time t, that is, $\mathcal{F}^Y_t$. Instead, [34] assume that the VaR is conditioned on the best estimate path of actuarial risk factor, that is, $\bar{Y}_t = BE_t[Y(t+1)]$. Our assumption is adopted also in [9] or [8].

The following proposition clarifies the structure of the technical provision $L(t)$ at time t.

Proposition 3.2 *The backward induction defined by Eqs. 3.1 and 3.2 can be equivalently rewritten in the following form*

$$L(t) = \sum_{k=t}^{T} BE_t[Z(k)] + \sum_{k=t}^{T-1} \delta_k \; BE_t \left[SCR_k[\Delta L(k+1)]\right], \tag{3.3}$$

where $BE_t[\cdot]$, $SCR_k[\cdot]$ and $\Delta L(k+1)$ are defined in 2.4, 2.6 and 2.5, respectively.
The proposition can be easily proved by induction.

From Proposition 3.2, we can define the value of the liabilities $L(t)$ as the sum of two parts: the Best Estimate (BE) and Risk Margin (RM),

$$L(t) = L^{BE}(t) + RM(t), \tag{3.4}$$

where we define the BE of liabilities at time t as the sum of the best estimate of future cash-flows

$$L^{BE}(t) := \sum_{k=t}^{T} BE_t[Z(k)], \tag{3.5}$$

and the RM at time t as the sum of expected future SCRs times the cost of capital rates δ_k for $k = t, t+1, \ldots, T-1$

$$RM(t) := \sum_{k=t}^{T-1} \delta_k \; BE_t \left[SCR_k[\Delta L(k+1)]\right]. \tag{3.6}$$

In formula 3.6, it seems that $RM(t)$ is a sum of expected annual SCRs. However, if we analyse the formula more carefully, we note that the RM contains iterated VaRs (or equivalently SCRs), that correspond to the capital-on-capital effects. We illustrate this effect, looking at the full backward iterated formula defined in 3.1 in a two period case with $t = T-2$ years. In this case the RM at time $t = T-2$ is equal to

$$\begin{aligned} RM(T-2) = \delta_{T-1} \; & BE_{T-2}\left[SCR_{T-1}[\Delta Z(T)]\right] \\ & + \delta_{T-2} SCR_{T-2} \left[\Delta Z(T-1) \right. \\ & \left. + \Delta BE_{T-1}[Z(T)] + \delta_{T-1} \Delta SCR_{T-1}[\Delta Z(T)]\right]. \end{aligned} \tag{3.7}$$

In the previous formula, the SCR at time $T-2$ is calculated not only on the variation of the best estimate of liability cash-flows but also on the variation of SCR at time $T-1$, ΔSCR_{T-1}. This is due to the capital-on-capital effect, that is, at any time the capital requirement should cover also the risk of variation of future capital buffers. This means that a quote of the regulatory capital is self-generated, growing from year to year and becoming a consistent source of cost for long term liabilities. In the next sections, we give a formal mathematical definition of the cost of capital-on-capital for a general multi-period case. Moreover, we illustrate that time consistency can be achieved even without the cost of capital-on-capital, considering in the liability valuation the expected futures SCRs on the variation of cash-flows best estimate. This means that the cost of capital-on-capital does not necessary coincide with a time consistency risk premium, as instead suggested by [34].

3.2 Definition of the additive-SCR

In this section, inspired from the result of Proposition 3.2, we propose a new time consistent dynamic extension of the one-period SCR formula, defined in 2.2, that we call Additive-SCR or ASCR.

Definition 3.3 (*Additive solvency capital requirement*) We define the additive-SCR (ASCR) and the expected-SCR (ESCR) at time t as

$$\begin{aligned} ESCR_t(k) &= BE_t\left[SCR_k[\Delta L^{BE}(k+1)]\right], \\ ASCR(t,T) &:= \sum_{k=t}^{T-1} ESCR_t(k), \end{aligned} \tag{3.8}$$

where $SCR_k[\cdot]$ is defined in 2.6, the variation $\Delta L^{BE}(k+1)$ is defined in 2.5 and the best estimate of liabilities L^{BE} is defined in 3.5.

The ASCR answers to a fundamental request of the Solvency II directive: the amount of eligible own funds equal to the SCR necessary to support the insurance and reinsurance obligations over the lifetime thereof, see [18, TP.5.3]. The ASCR indicates the best estimate of the total capital requirement for the period from time zero to the liabilities maturity. The ASCR can be decomposed in the annual expected-SCRs (ESCR), which represent the best estimates of the regulatory capital for the next year, not only at time zero but also at the futures payment dates, that is, we build a term structure of expected annual SCRs. This is important for the regulatory application of the dynamic risk measures, in fact the insurance companies should public their balance sheet at the end of each year, hence they have to be solvent each year and they can hardly postpone liabilities payments. Instead, using the iterated approach, the RM defined in Eq. 3.6 cannot be decomposed in annual components, in fact each expected SCR_t at time t depends on all futures SCR_s for $t \leq s \leq T$.

As already remarked in Sect. 2.3, $SCR_t[\cdot]$ is a conditional risk mapping, see Appendix A, Definition A.1. Then, the ASCR defined in 3.8 fits the additive structure of SEC conditional mapping, see Appendix B, Definition B.1. Then, the ASCR owns the same properties of the SEC risk mappings, including recursiveness and time consistency, as illustrated in the following proposition.

Proposition 3.4 *The ASCR defined in Eq. 3.8 is time consistent and satisfies the following backward recursive formulation for any* $0 \leq t \leq T-1$

$$\begin{aligned} ASCR(t,T) &= ESCR_t(t) + BE_t[ASCR(t+1,T)] \\ &= SCR_t\left[\Delta L^{BE}(t+1) + BE_t\left[SCR_{t+1}\left[\Delta L^{BE}(t+2)\right.\right.\right. \\ &\quad \left.\left.\left. + \cdots + BE_{T-2}\left[SCR_{T-1}\left[\Delta L^{BE}(T)\right]\right]\cdots\right]\right]\right], \end{aligned} \tag{3.9}$$

where BE_t *and* SCR_t *are defined in 2.4 and 2.6, respectively.*

The proof is based on the properties of Separable Expected Conditional (SEC) risk mappings reported in Appendix B, see also [11, 25] and references therein. In particular, Theorem A.4 in Appendix A states the equivalence between recursive and time consistent dynamic multi-period risk mappings. We remark that SEC risk mappings, as the ASCR, are recursive, but are not self-recursive. This means that ASCR is not obtained iterating the conditional risk mapping $SCR_t[\cdot]$.

3.3 An alternative time consistent valuation formula of liabilities

In this section, we build a valuation procedure for liabilities based on the definition of ASCR in Sect. 3.2.

Definition 3.5 (*Time and market consistent valuation of liabilities, TMCV-2*) We define the following alternative market and time consistent valuation formula for the liabilities

$$\widehat{L}(t) := L^{BE}(t) + \widehat{RM}(t), \tag{3.10}$$

where the Best Estimate $L^{BE}(t)$ is defined in formula 3.5 and $\widehat{RM}(t)$ is defined in analogy with formula 3.6 as

$$\begin{aligned} \widehat{RM}(t) &:= \sum_{k=t}^{T-1} \delta_k \; ESCR_t(k) \\ &= \sum_{k=t}^{T-1} \delta_k \; BE_t \left[SCR_k[\Delta L^{BE}(k+1)] \right], \end{aligned} \tag{3.11}$$

with $ESCR_t(k)$ and $SCR_k[\cdot]$ defined in formula 3.8 and 2.6, respectively.

If the cost of capital rate δ is constant, the previous formula 3.10 reduces to

$$\widehat{L}(t) = L^{BE}(t) + \delta \; ASCR(t, T), \tag{3.12}$$

where $ASCR(t, T)$ in defined in 3.8.

The following proposition proves that the liabilities valuation formula proposed in 3.10 is time consistent.

Proposition 3.6 *The valuation formula of liabilities defined in equation 3.10 is time consistent and satisfies the following backward recursive formula for* $0 \le t \le T - 1$

$$\widehat{L}(t) = Z(t) + BE_t \left[\widehat{L}(t+1) \right] + \delta_t \; ESCR_t(t), \tag{3.13}$$

where $Z(t)$, $ESCR_t(t)$ *are defined in Eqs. 2.1, 3.8, respectively.*

The proof is a simple application of the recursive formulas of SEC risk measure, reported in Appendix B for reader convenience.

3.4 The capital-on-capital cost

In the previous section, we build a time consistent valuation formula for liabilities without iterating the VaR risk measure. This means that at time t we do not consider the SCR on futures RMs, but only on the BE of liabilities. In other words, we avoid the capital-to-capital effect, induced by backward iterating the one-period valuation formula. Then, we can estimate the cost of capital on capital effect, as the difference between the two different TMCVs of liabilities, or equivalently between the two RMs.

Definition 3.7 (*Cost of the capital on capital effect*) The capital on capital cost $C(t)$ is defined as

$$C(t) = L(t) - \widehat{L}(t) = RM(t) - \widehat{RM}(t), \tag{3.14}$$

where $L(t)$ and $\widehat{L}(t)$ are defined in equations 3.4 and 3.10, respectively. Moreover, $RM(t)$ and $\widehat{RM}(t)$ are defined in 3.6 and 3.11, respectively.

We illustrate the structure of the capital-on-capital cost in a two period case with $t = T - 2$ years. In this simplified case, we calculate the cost of capital-on-capital $C(T-2)$ as

$$\begin{aligned} C(T-2) &= RM(T-2) - \widehat{RM}(T-2) \\ &= \delta_{T-2}\, SCR_{T-2}\left[\Delta Z(T-1) + \Delta BE_{T-1}[Z(T)] + \delta_{T-1}\, \Delta SCR_{T-1}[\Delta Z(T)]\right] \\ &\quad -\delta_{T-2}\, SCR_{T-2}\left[\Delta Z(T-1) + \Delta BE_{T-1}[Z(T)]\right]. \end{aligned} \tag{3.15}$$

In the iterated approach, the $RM(T-2)$ depends on the variation of the future SCR_{T-1}. This generates an extra cost, namely the capital-on-capital cost, that is not presented in our proposed RM approach, $\widehat{RM}(T-2)$, based on ASCR.

4 Application to a portfolio of equity-linked life-insurance contracts

In this section, we present an illustrative example of the proposed approach for a portfolio of equity-linked life-insurance contracts. We compare the two TMCVs of liabilities presented in Sects. 3.1 and 3.3, and we estimate the capital-on-capital cost by calculating the difference between them.

4.1 The financial and actuarial model

For the remainder of this section, we consider a financial market that consists of a risk free asset $X_0(t) = e^{rt}$ and a risky asset $X_1(t)$, $t = 0, 1, \ldots, T$, where r is the risk free rate.[5] Moreover, we assume that the insurance liability has a single cash-flow at

[5] For sake of simplicity, the risk free rate is assumed to be constant over time. However, the presented approach can be extended to a time varying and stochastic rate.

maturity T, that has the following form

$$Z(T) = \max(X_1(T), K)\, N(T), \tag{4.1}$$

where $N(T)$ counts the number of survivors at time T given by a mortality process, $X_1(T)$ is the risky asset value at maturity and K is a fixed guarantee level.

For simplicity of illustration,[6] we assume that the stock follows a geometric Brownian motion under $\mathbb{P}$:

$$dX_1(t) = X_1(t)(\mu dt + \sigma dW_1(t)). \tag{4.2}$$

Under this specification, the financial market is complete and the filtration $\mathcal{F}_T^X$ corresponds to the usual augmented filtration generated by the Brownian motion W_1. Then, under the unique equivalent martingale measure $\mathbb{Q}$, the dynamic of the risky asset is the following

$$dX_1(t) = X_1(t)(rdt + \sigma dW_1(t)).$$

The mortality process $N(t)$ counts the number of survivals among an initial population of N_0^x policyholders of age x. The mortality intensity is assumed to be stochastic and follows the dynamics under $\mathbb{P}$ given by

$$d\lambda_x(t) = c\lambda_x(t)dt + \eta dW_2(t), \tag{4.3}$$

with $c, \eta > 0$ and $W_2(t)$ a standard Brownian motion. The two Brownian motions $W_1(t)$ of the risky stock and $W_2(t)$ of the mortality intensity are correlated, that is, $W_2(t) = \rho W_1(t) + \sqrt{1-\rho^2} W_3(t)$, with $W_1(t)$ and $W_3(t)$ independent Brownian motions and correlation parameter $\rho \in [-1, 1]$. The survival function is defined as

$$S_x(t) := \mathbb{P}(T_x > t) = e^{-\int_x^{x+t} \lambda_x(s)ds}, \tag{4.4}$$

where T_x is the remaining lifetime of an individual who is aged x at time 0. Moreover, deaths of individuals are assumed to be independent events conditional on knowing population mortality, see [28] for similar assumptions. Further, if we denote $D(t+1)$ the number of deaths during year $[t, t+1]$, the dynamics of the number of active contracts can be described as a nested binomial process as follows: $N(t+1) = N(t) - D(t+1)$ with $D(t+1)|N(t), p_{x+t} \sim Bin(N(t), p_{x+t})$. Here, $p_x + t$ represents the one-year death probability

$$p_{x+t} := \mathbb{P}(T_x \leq t+1 | T_x > t) = 1 - \frac{S_x(t+1)}{S_x(t)}, \tag{4.5}$$

[6] The presented approach can be easily adapted to other stock dynamics, e.g. stochastic volatility or Lévy models.

for $t = 0, \cdot, T-1$. In particular, the number of deaths during year $[t, t+1]$ can be obtained in the following way:

$$D(t+1) = \sum_{k=1}^{N_0^x} \mathrm{I}(k < N(t))\, \mathrm{I}(U_k(t) < p_{x+t}),$$

where $I(\cdot)$ si the indicator function and $U(t) = (U_k(t))_{k=1,\ldots,N_0^x}$ is a vector of independent and identically distributed uniform random variables in $[0, 1]$. The process $U_{1,T}$ is independent from the market process $X_{0,T}$ under $\mathbb{P}$ and it represents together with the Brownian motion W_3 a pure actuarial risk factor. Then, using the notation of previous sections, the actuarial risk factor at time t is the vector composed by the components of the mortality process and the intensity, i.e. $Y(t) = (U(t), W_3(t))$.

4.2 Numerical study

In this section, we calculate the TMCVs $L(0)$ and $\widehat{L}(0)$ in formulas 3.4 and 3.10 for different maturities T; the BE of liabilities $L^{BE}(0)$ in equation 3.5 and the risk margins $RM(0)$ and $\widehat{RM}(0)$ defined in Eqs. 3.6 and 3.11. Then, we estimate the cost of capital $C(0)$ defined in equation 3.14 as the difference between the two risk margins $RM(0)$ and $\widehat{RM}(0)$. Moreover, we compare the time consistent results with the industry rule-of-thumb of SCR projections. The TP by SCR projections, that we call L^{NTC}, is obtained through the following formula

$$L^{NTC}(t) = L^{BE}(t) + RM^{NTC}(t) \tag{4.6}$$

where L^{BE} is in 3.5. The RM^{NTC} is defined as

$$RM^{NTC}(t) = \sum_{k=t}^{T-1} e^{-r(k-t)}\, \delta_k\, \mathbb{E}^{\mathbb{Q}}\left[VaR_\alpha^{\mathbb{P}}\left[L^{BE}(k+1) - L^{BE}(k) | \left(X(k+1), \bar{Y}_t^k\right)\right] \Big| X(t)\right], \tag{4.7}$$

where $\bar{Y}_t^k = BE_t[Y(k)]$ is the best estimate projection of the actuarial risk factor from time t to time k. The RM and the fair value obtained through the SCR projections are not time consistent, as discussed in [34].

The benchmark parameters for the financial market and the actuarial risk factor are the following. Following [9], the risk free rate is $r = 1.0\%$, the risky stock volatility value is $\sigma = 0.1$ and the fixed guaranteed level is equal to the initial value of the risky stock, $K = X_1(0) = 1$. The mortality parameters ($\lambda_x(0) = 0.0087$, $c = 0.0750$, $\eta = 0.000597$) follow from [27] and correspond to UK male individuals who are aged 55 at time 0. We assume that there are $N_0^x = 1000$ initial contracts at time 0. The correlation assumes four different values $\rho = [0, 0.5, 0.75, 1]$. We consider contract maturities T from 1 year to 40 years and a confidence level $\alpha = 99.5\%$. In particular, we investigate the impact of maturity T and of the correlation parameter between the risky asset and the force of mortality ρ on the risk margins and hence on the cost of capital-on-capital.

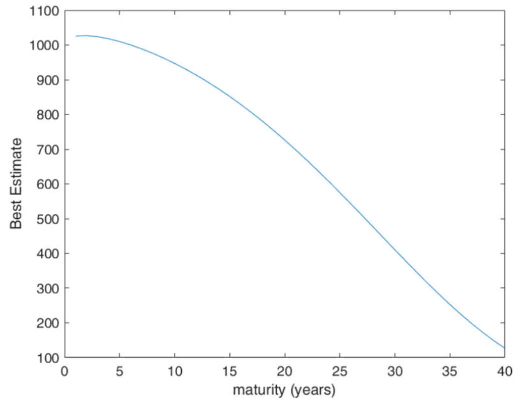

Fig. 1 The graphs show: the best estimate of liabilities L^{BE} defined in Eq. 3.5, varying the liability maturities for $\rho = 0$

To numerically estimate the quantities of interest, we use the Least Square Monte Carlo (LSMC) approach, a popular numerical technique widely applied in finance and insurance. The LSMC method was firstly proposed by [26] for the valuation of American-type options. The main concept is to perform a regression of conditional expectations on the cross-sectional information of the underlying risk drivers, such as mortality and equity risks, in order to obtain insights into the valuation of liabilities. For estimating the regressions, we use 100 outer scenarios for $X_{0,T}$ and 1000 inner simulations for $Y_{0,T}$ and polynomial of degree two, using the product of $X_1(t) \cdot N(t)$ as regressor. Moreover, to calculate the conditional VaR, we suppose that for short periods (a one-year horizon), the expected number of survivors is approximately normally distributed. Hence, the conditional α-quantile can be approximated by its mean and standard deviation.[7]

Figure 1 reports the exact analytical calculation of the best estimate L^{BE} in case of independence, that is $\rho = 0$.

Figure 2 illustrates the risk loading, that is the ratio between RM and BE of liabilities, for the two time consistent risk margins RM and $\widehat{RM}$ in formulas 3.6 and 3.11 and for RM^{NTC} in formula 4.7. The time consistent risk margins always exceed the inconsistent one. For long term liabilities, that is $T > 20$ years, the RM obtained through backward iteration is appreciably larger than $\widehat{RM}$. This means that the cost of capital-on-capital increases with respect to the maturity. All three RMs decrease, increasing the correlation value. However, for $\rho = 1$, the RMs are not exactly equals to zero. In fact, unless the stock X_1 and the force of mortality λ_x are comonotonic, the stock X_1 and the mortality process N are not perfectly correlated, due to the nested binomial process.

Finally, Fig. 3 reports the ratio between the cost of capital-on-capital and the RM, that is, $\frac{C(0)}{RM(0)} = \frac{RM(0)-\widehat{RM}(0)}{RM(0)}$ for different maturities of the liabilities and correlations. As shown in Fig. 3, the capital-on-capital cost increases with maturity and decreases with correlation. Moreover, in the uncorrelated case and for long term liabilities with

[7] A similar simplifying approximation is used also in [34], to make the problem tractable. In our numerical experiments, we verify that the probability of a negative expected number of survivors is negligible.

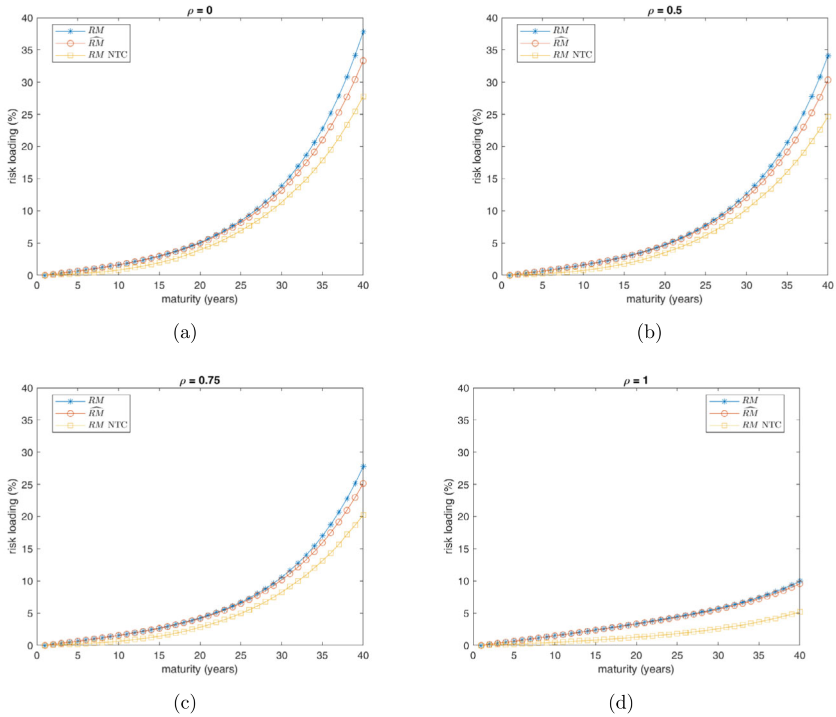

Fig. 2 The graphs show the trend of risk loading, as the ratio between RM and BE of liabilities with respect to maturity for different values of correlation ρ

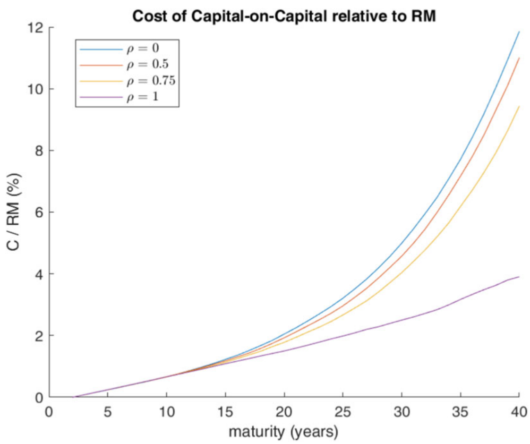

Fig. 3 The plot shows the ratio of the cost of capital-on capital to the RM with respect to maturity for different values of correlation ρ

a maturities equal to 30 and 40 years, the capital-on-capital cost can reach the 5% and 12% of the RM value obtained with backward iteration method.

The Matlab code used to obtain the numerical results reported in this section is public available at [19].

5 Conclusion

In recent times, practitioners have engaged in discussions about potential revisions to the calculation of the Solvency II risk margin. In particular, insurance companies have expressed discontent with the excessively high value of the risk margin for long-term life insurance portfolios. Additionally, the requirement to calculate the risk margin using a time consistent valuation of liabilities has led to an increase in its value, introducing a time consistent risk premium. Our work mainly focuses on demonstrating how it is possible to limit the self-generation effect of regulatory capital, called capital-on-capital effect, in the framework of a dynamic valuation that remains consistent over time. We achieve this objective by replacing the iterative approach with the development of time-consistent SEC risk mappings. This approach provides a novel and effective way to manage the impact of capital regulatory requirements within a dynamic time-consistent framework. Moreover, we formally define the concept of capital-on-capital cost as the difference between the two RMs calculated using the backward iteration method or the valuation based on SEC risk mappings, called ESCRs valutation method. Finally, we apply the two time consistent valuation formulas to a portfolio of equity-linked life-insurance contracts and we compare them also with a non-time consistent valuation formula based on SCR projections. Our numerical experiments show that the capital-on-capital cost is a non-negligible component of the RM value, especially for long term liabilities, and that it does not coincide with a time consistent risk premium. Future researches could explore the application of time consistent ESCRs valuation method in conjunction with the market-consistent hedge-based approach of [16]. In particular, it could be interesting to investigate the relation between the cost of capital-on-capital and the different hedging strategies adopted in the backward iteration and in the ESCRs valuation methods.

A Dynamic risk mappings

A conditional risk mapping at time t provides a risk estimate conditional to the information at time t of a random variable with realization in $0 \leq t \leq s$. A formal definition is the following.

Definition A.1 (*Conditional risk mapping, see* [11]) Consider a canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{L}_t := \mathcal{L}(\Omega, \mathcal{F}_t, \mathbb{P})$ be the space of $\mathcal{F}_t$-measurable random variables.[8] A conditional risk mapping is a function $\rho_t(\cdot|\mathcal{F}_t) : \mathcal{L}_s \to \mathcal{L}_t$, that is translation invariant, i.e., for $X(s) \in \mathcal{L}_s$ and $c_t \in \mathcal{L}_t$,

$$\rho_t(c_t + X(s)|\mathcal{F}_t) = c_t + \rho_t(X(s)|\mathcal{F}_t).$$

Moreover, we consider conditional risk mapping that are normalized, i.e. centred at zero $\rho_t(0|\mathcal{F}_t) = 0$.

[8] We precise that $\mathcal{L}_t^2$, that is the space of square integrable random variables, is contained in $\mathcal{L}_t$. Then, the results presented in this section and in Appendix B are valid also for square integrable random variables considered in the paper.

A multi-period risk mapping $\tilde{\rho}_t(X_{t+1,T})$ provides a risk estimate at time t of a process with realizations in $t+1, t+2, \ldots, T$. We formalize the definition as follows.

Definition A.2 (*Multi-period conditional risk mapping, see* [11]) A multi-period risk mapping is a functional $\tilde{\rho}_{t,T}(\cdot) : \mathcal{L}_{t,T} \to \mathcal{L}_t$, that is translation invariant, i.e., for any $t = 0, 1, \ldots, T$ and $X_{t,T} \in \mathcal{L}_{t,T}$,

$$\tilde{\rho}_{t,T}(X_{t,T}) = X(t) + \tilde{\rho}_{t,T}\left((0, X_{t+1,T})\right),$$

where $(0, X_{t+1,T}) \in \mathcal{L}_{t,T}$ represents a process whose value at time t is equal to zero.

Then, the sequence of multi-period risk mappings $\{\tilde{\rho}_{t,T}\}_{t=0}^{T-1}$ is a dynamic risk mapping.

Definition A.3 (*Dynamic time consistent risk mapping, see* [11]) Let $0 \leq u \leq t \leq T$ and $X_{u,T}, Y_{u,T} \in \mathcal{L}_{u,T}$ for all u. If for any u the conditions

$$\begin{aligned}&X(k) = Y(k), \text{ almost surely (a.s.) for } k = u, u+1, \ldots, t-1\\&\text{and}\\&\tilde{\rho}_{t,T}(X_{t,T}) = \tilde{\rho}_{t,T}(Y_{t,T}) \text{ a.s.,}\end{aligned}$$

imply

$$\tilde{\rho}_{u,T}(X_{u,T}) = \tilde{\rho}_{u,T}(Y_{u,T}) \text{ a.s.,}$$

then $\{\tilde{\rho}_{t,T}\}_{t=0}^{T-1}$ is a time consistent dynamic risk mapping.

The previous definition of time consistency is formulated by [6] for risk measures adding the monotonicity property and it is largely applied in the literature, see for instance [2, 11, 14, 15, 33].

Theorem A.4 [11, 33] *A dynamic risk mapping is time consistent, as in Definitions A.2 and A.3, if and only if it can be equivalently expressed in a recursive form, i.e. for* $0 \leq t \leq \theta \leq T$

$$\tilde{\rho}_{t,T}(X_{t,T}) = \tilde{\rho}_{t,\theta}\left(\left(X_{t,\theta-1}, \tilde{\rho}_{\theta,T}(X_{\theta,T})\right)\right), \tag{A.1}$$

where $\left(X_{t,\theta-1}, \tilde{\rho}_{\theta,T}(X_{\theta,T})\right) \in \mathcal{L}_{t,\theta}$ *is a process, whose final value at time* θ *is equal to* $\tilde{\rho}_{\theta,T}(X_{\theta,T})$.

B Separable expected conditional risk mapping

Consider a random vector $X_{t,T} := (X(t), X(t+1), \ldots, X(T))$ with $X(t) \in \mathcal{L}_t$ and one-period conditional risk mapping $\rho_t(\cdot|\mathcal{F}_t) : \mathcal{L}_{t+1} \to \mathcal{L}_t$, for $t = 1, 2, \ldots, T$, defined in A.1.

Definition B.1 (*Separable expected conditional* (*SEC*) *risk mapping*) A multi-period risk mapping $\tilde{\rho}_{t,T}$ is a SEC-RM, if there exists a sequence of conditional risk mappings $\{\rho_k(\cdot|\mathcal{F}_k)\}_{k=t,\dots,T-1}$ such that

$$\tilde{\rho}_{t,T}(X_{t+1,T}) := \sum_{k=t+1}^{T} \mathbb{E}[\rho_{k-1}(X(k)|\mathcal{F}_{k-1})|\mathcal{F}_t].$$

In literature, there exist three main definitions of separable expected conditional functional. All three definitions share the same additive structure reported in Definition B.1. In [25], $\tilde{\rho}_{t,T}$ is called SEC acceptability mappings and ρ_{k-1} are conditional acceptability mappings, that satisfy: normalization, translation invariance, monotonicity and concavity. Then the definition of [25] cannot be applied to VaR. Similarly [14] apply the definition of [25], using the AVaR. Finally, [11] propose a broader definition of $\tilde{\rho}_{t,T}$ as SEC risk mappings and ρ_{k-1} are conditional risk mappings that satisfy only normalization and translation invariance. Then, we decided to adopt the definition of [11] that is the less restrictive one, without loosing the time consistency property. As shown in [11, 14, 25], SEC risk mapping are time consistent. In particular, the time consistency is proved using the following recursive formulation, see for instance [25, Proposition 3.3.11.], [14, Sect. 1.3] and [11, Sect. 3]

$$\tilde{\rho}_{t,T}(X_{t+1,T}) = \rho_t(X(t+1)|\mathcal{F}_t) + \mathbb{E}[\tilde{\rho}_{t+1,T}(X_{t+2,T})|\mathcal{F}_t];$$

in fact, Theorem A.4 states the equivalence between recursive and time consistent multi-period conditional risk mapping.

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Data availability Data and code of the present work are publicly available in the GitHub repository. https://github.com/AnnaMariaGambaro/CoCcost.git.

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