QMLSV: a gretl function for quasi maximum likelihood estimation of stochastic volatility models

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Abstract

The paper presents a gretl package, QMLSV, for quasi-maximum likelihood (QML) estimation of univariate stochastic volatility (SV) models. Unlike univariate GARCH models, which only include a single random component (return innovation), stochastic volatility models also include a volatility innovation. In addition to the standard QLM method (Ruiz, 1994; Harvey, Ruiz, Shephard, 1994), suitable only for models where volatility innovation is not correlated with return innovation (symmetric volatility), the package also includes two methods for asymmetric volatility models, i.e. models in which volatility innovation is correlated with return innovation. The first of these is the modified QML method proposed by Harvey and Shephard in 1996, which provides a volatility predictor mainly similar to the Threshold Garch model; the second is the iterative QML method recently proposed by Chirico (2024), which provides a volatility predictor similar to the EGARCH predictor. All three methods can also be applied in case of random-walk volatility as well as in case of Student-t-distributed return innovation. Some examples illustrate the practical use of the package.

Keywords: stochastic volatility, quasi maximum likelihood, leverage effect, gretl

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1 Introduction

Volatility measures the uncertainty about future price movements, generally log-returns, of a good or financial asset. Econometric models for volatility are typically classified into two categories: (i) models for conditional heteroskedasticity; (ii) stochastic volatility models. In the first category, often identified with the family of GARCH models (Bollerslev, 1986), volatility is a function of the previous information about the return series, and its square, σ_t^2, corresponds to the conditional variance of the returns. On the other hand, stochastic volatility (SV) models assume that σ_t^2 is a latent stochastic process, whose logarithm is usually shaped as an autoregressive process (Melino and Turnbull, 1990; Taylor, 1994).

Most statistical software includes modules/packages for estimating GARCH models, but few deal with stochastic volatility with dedicated packages. One of the reasons for this lack is certainly that ML estimation of SV models is a tough challenge since the likelihood of the models is difficult to manage, especially if the models try to deal with asymmetric volatility like in case of leverage effect[1]. There are several proposals in the literature (Sandmann and Koopman, 1998), but at present there is no unanimously recognized best estimation method.

This paper presents a package for gretl, *"QMLSV"*, that estimate leveraged and unleveraged SV models using the standard QML method (Ruiz, 1994; Harvey et al, 1994) and two variants known in literature (Harvey and Shephard, 1996; Chirico, 2024); Section 2 illustrates some aspects of the QML method applied to SV models; Section 3 illustrates the package functions and some examples which explain the package use; final comments summarize the main features of the package, highlighting the specificities of the three estimation methods.

2 QML estimation of stochastic volatility models

A fairly simple and quite general modelling to represent the stochastic volatility of financial returns, r_t, is the following one:

$$\begin{aligned} r_t &= \mu + \sigma_t^2 \varepsilon_t \\ h_{t+1} &= \omega + \beta h_t + \eta_t \end{aligned} \qquad \begin{bmatrix} \varepsilon_t \\ \eta_t \end{bmatrix} \sim NID\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \gamma \\ \gamma & \sigma_\eta^2 \end{bmatrix} \right) \tag{1}$$

where: r_t is the log-return on an asset, simply "return" in the following; $h_t = \ln \sigma_t^2$, i.e. the log-square volatility, is an order-1 autoregressive latent process; ε_t and η_t are zero-mean Gaussian innovations, serially independent, with $E(\eta_t \varepsilon_{t-k}) = \gamma$ if $k = 0$, and zero otherwise.

Model (1) admits that the volatility innovation may be correlated with the return innovation, i.e. $Corr(\eta_t, \varepsilon_t) = \gamma / \sigma_\eta$. As result, the expected volatility at the next time will depend not only on the current volatility, but also on the magnitude and the sign

[1] In financial series increases of volatility are often observed after negative returns, and generally the greater the extent of the loss the greater the increase. This evidence is called "leverage effect" as price drop in a stock decreases the value of the firm equity, and increases the leverage-ratio.The increased leverage-ratio will involve higher risk on the equity which will be more volatile during next period (Black, 1976).

of the shock on returns:

$$E(h_{t+1} \mid h_t, \varepsilon_t) = \omega + \beta h_t + \gamma \varepsilon_t \tag{2}$$

If the correlation is negative, negative shocks on returns will increase expected volatility, revealing the presence of *leverage*.

The standard QML

In the absence of leverage effect (i.e. $\gamma = 0$), Model (1) can be easily estimated with the Quasi Maximum Likelihood (QML) method (Ruiz, 1994; Harvey et al, 1994), which consists in the estimation, via Kalman Filter, of the auxiliary state space model:

$$\begin{array}{l} y_t = \alpha + h_t + \xi_t \\ h_{t+1} = \omega + \beta h_t + \eta_t \end{array} \qquad \begin{bmatrix} \xi_t \\ \eta_t \end{bmatrix} \sim ID\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_\xi^2 & 0 \\ 0 & \sigma_\eta^2 \end{bmatrix} \right) \tag{3}$$

where $y_t = \ln(r_t - \mu)^2$, and μ is the returns mean which, if not known, has to be preliminarily estimated so as to have the y_t series determined. The parameters α and σ_ξ^2 represent the mean and the variance of $\ln \varepsilon_t^2$ respectively; when ε_t is Gaussian, α and σ_ξ^2 correspond to -1.2704 and 4.9348 respectively (Zelen and Severo, 1972). Finally, $\xi_t = \ln \varepsilon_t^2 - \alpha$ represents a zero-mean disturbance whose distribution in clearly not Gaussian. In the remainder of the paper, this approach will be called "standard QML".

The standard QML is a good compromise between simplicity and efficiency: it maximizes a function that is not the actual exact likelihood function, but it provides an optimal linear estimator (and predictor) of h_t, consistent and asymptotically Gaussian, according with the results of Dunsmuir (1979). However, if there is a leverage effect in the returns, the standard QML is not useful to directly estimate the parameter γ: the disturbances ξ_t and η_t are found to be uncorrelated when the distribution of ε_t and η_t is symmetric, as Gaussian or Student's t, so that the original covariance (correlation) between ε_t and η_t is definitely lost in model (3) (Harvey et al, 1994).

The Harvey-Shephard modified QML

In order to overcome the limitations of the standard QML, Harvey and Shephard (1996) proposed a modified QML procedure, in the following named HS-QML, by which all parameters of (1) can be estimated applying the QML method to the time-varying state space model:

$$\begin{array}{rl} y_t &= \alpha + h_t + \xi_t \\ h_{t+1} &= \omega + \theta_1 \gamma s_t + \beta h_t + \eta_t^* \end{array} \tag{4}$$

$$\begin{bmatrix} \xi_t \\ \eta_t^* \end{bmatrix} \mid s_t \sim ID\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_\xi^2 & \theta_2 \gamma s_t \\ \theta_2 \gamma s_t & \sigma_\eta^2 - \theta_1^2 \gamma^2 \end{bmatrix} \right)$$

where s_t is the sign of ε_t (i.e. the sign of the observable $r_t - \mu$); $\theta_1 = E|\varepsilon_t|$ and

$\theta_2 = E[|\varepsilon_t| \ln \varepsilon_t^2] - \theta_1 E[\ln \varepsilon_t^2]$. Under the assumption that (ε_t, η_t) is bivariate Gaussian, θ_1 and θ_2 correspond to 0.7979 and 1.061 respectively (Harvey and Shephard, 1996). Volatility innovation η_t^* can be decomposed as $\eta_t^* = (\theta_2 \gamma s_t / \sigma_\xi^2)\xi_t + \eta_t^+$, where η_t^+ represents the exogenous volatility innovation[2].

The one step ahead predictor of h_t provided by the Kalman Filter presents similarities with the Threshold-GARCH predictor (Glosten et al, 1993; Zakoian, 1994)[3]:

$$\begin{aligned} \hat{h}_{t+1|t} &= \omega + \beta \hat{h}_{t|t-1} + \theta_1 \gamma s_t + \kappa_t (y_t - \hat{y}_{t|t-1}) \\ \kappa_t &= \frac{\beta P_{t|t-1} + \theta_2 \gamma s_t}{P_{t|t-1} + \sigma_\xi^2} \end{aligned} \tag{5}$$

where κ_t is the gain of the Kalman filter (Harvey, 1990); $\hat{y}_{t|t-1}$ is the one step ahead prediction of y_t; $P_{t|t-1}$ is the MSE of $\hat{h}_{t|t-1}$.

The iterative QML

Recently, Chirico (2024) has proposed an alternative QML procedure to estimate Model (1) which consists of iterating the QML estimation on the state space model:

$$\begin{aligned} y_t &= \alpha + h_t + \xi_t \\ h_{t+1} &= \omega + \gamma \varepsilon_t + \beta h_t + \eta_t^+ \end{aligned} \qquad \begin{bmatrix} \xi_t \\ \eta_t^+ \end{bmatrix} \sim ID\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_\xi^2 & 0 \\ 0 & \sigma_\eta^2 - \gamma^2 \end{bmatrix} \right) \tag{6}$$

where η_t^+ represents the exogenous volatility innovation.

As Model (6) is not linear[4], the QML estimation cannot be directly applied to the model. To overcome this impasse ε_t is repeatedly replaced by a proxy in the following iterative procedure (named here IQML):

1. ε_t is initially replaced by $(r_t - \mu)/s_r$ where s_r is the sample standard deviation of the returns series;
2. Model (6) is estimated by standard QML;
3. ε_t is updated by $(r_t - \mu)/\exp(\tilde{h}_t/2)$, where $\tilde{h}_t$ is the smoothed h_t filtered with the Kalman smoother;
4. steps 2 and 3 are repeated successively according with a pre-set stopping rule[5].

Empirical evidence shows that the parameter estimates converge to realistic values after few steps. The steady-state predictor of h_t can be formalized as:

$$\hat{h}_{t+1|t} = \omega + \beta \hat{h}_{t|t-1} + \gamma \tilde{\varepsilon}_t + \alpha \xi(|\tilde{\varepsilon}_t|) \tag{7}$$

[2] if $\gamma = 0$, it follows that $\eta_t = \eta_t^+$, i.e. i.e. the volatility innovation is entirely exogenous.
[3] Actually, predictor (5) also presents similarities with EGARCH one (Nelson, 1991), as pointed out by Harvey and Shephard (1996).
[4] Indeed $\varepsilon_t = (r_t - \mu)/\exp(h_t/2)$).
[5] In the *QMLSV* package the iterative procedure is stopped when the increment of the quasi log-likelihood becomes less than 0.001. With this stopping rule, the relative difference in the estimates between the last two iterations becomes generally less than 0.1%

where the function $\xi(|\tilde{\varepsilon}_t|) = 2\ln|\tilde{\varepsilon}_t| + \alpha$ is a monotonically increasing, mean-corrected, function of the (approximated) magnitude of ε_t. Predictor (7) shows a clear similarity with the EGARCH predictor (Nelson, 1991) in which $\xi(|\varepsilon_t|) = |\varepsilon_t| - 0.7979$ is the mean-corrected magnitude of ε_t when $\varepsilon_t \sim N(0,1)$.

2.1 Student's t return innovations

Each of the three methods described above can be generalized to the case where ε_t has a scaled Student's t-distribution with v degrees of freedom, i.e. $\varepsilon_t \sim t_v\sqrt{(v-2)/v}$. In this case the parameters $\alpha, \sigma_\xi^2, \theta_1, \theta_2$ assume values that depend on the degrees of freedom (see Appendix A):

$$\alpha = = -1.2701 + \ln(v/2-1) - \psi_0(v/2) \tag{8}$$
$$\sigma_\xi^2 = 4.9348 + \psi_1(v/2) \tag{9}$$
$$\theta_1 = 0.7979 \cdot (v/2-1)^{1/2} \cdot \Gamma(v/2-1/2)/\Gamma(v/2) \tag{10}$$
$$\theta_2 = \theta_1 \cdot (1.3863 + \psi_0(v/2) - \psi_0(v/2-1/2)) \tag{11}$$

where ψ_0 and ψ_1 are the *digamma* and *trigamma* function, respectively (Davis, 1972).

These formulas can be used in two distinct approaches: (i) determining the parameters above for a given value of v and then estimating the remaining parameters of models (3), (4), or (6); (ii) not fixing v and estimating the parameters of the model, including v. The first approach significantly simplifies parameter estimation, but requires assuming the distribution of ε_t (i.e., v) is completely known. The second approach, theoretically preferable if v is unknown, involves a more complex estimation, the numerical solution of which often does not converge to a solution. For this reason, the QMLSV package adopts the first approach at the moment.

3 The *QMLSV* package

The package consists in a main (public) function, `qml_sv`, and the following auxiliary (private) functions: `TV1` and `TV2`, which update the time-varying parts of models (4) and (6) respectively; `tS_param`, which calculates the values of $\alpha, \sigma_\xi^2, \theta_1, \theta_2$ given the degrees of freedom, v, of the ε_t distribution; `degrees_v` which provides the ML estimate of v using the standardised residuals derived from the last model estimation. This last function is optional to have feedback on the assumed value of v.

The QML estimation of the models illustrated in Section 2 is provided by `qml_sv` with the following syntax:

```
bundle Mod = qml_sv(returns, method, integ, constant, smooth,
degrees, horizon, vest)
```

where:

- `returns (series)` is the series of return;
- `method (int)` sets the estimation method: 1 = standard QML, 2 = HS-QML, 3 = IQML;

- `integ (bool)` fixes the type of volatility: 0 = stationary volatility 1 = integrated volatility (see below);
- `constant (bool)` set the model with the constant μ or without it: 0 = no constant, 1 = constant;
- `smooth (bool)` set the output in the bundle: if =1, the smoothed state $\tilde{h}_t$ is given, otherwise (=0) the one-step-ahead expected state $\hat{h}_{t|t-1}$ is given[6];
- `degrees (scalar)` are the degrees of freedom set for the Student-t distribution of ε_t. The maximum accepted value is 100, which is also the default value. By convention the default values implies Gaussian ε_t;
- `horizon (int)` fixes the forecast horizon, i.e. up to which time beyond T, volatility forecasts are calculated;
- `vest (bool)` if =1, provides the ex-post ML estimate of the Student-t degrees of ε_t distribution using the standardised residuals derived from the last model estimation.

Function `qml_sv` allows to estimate models in Section 2 even in the case of integrated volatility. In this case the parameters ω and β are fixed to 0 and 1 respectively, and the gretl's Kalman filter option `diffuse=2` is set. This option adopts the sequential procedure suggested by Durbin and Koopman (2012), from an idea of Anderson and Moore (1979), to determine the initial value of the variance of h_t[7].

Bundle objects

The *qml_sv* function returns a bundle containing several estimation and filtering results, in particular:

- `vol (series)` is the series of the filtered (or smoothed if `smooth=1`) volatilities;
- `stduhat (series)` is the series of the filtered (smoothed) return innovations;
- `coeff (matrix)` is the vector of the model parameter estimates;
- `SE (matrix)` is the vector of the corresponding standard errors.

Graphs

The graphing functions of the gretl gig package (Lucchetti and Baglietti, 2018) can be used to quickly plot the volatility time series and/or the residual distribution. More specifically the functions:

- `gig_plot` performs the time-series plot of mean-corrected returns, y_t, and predicted/smoothed volatilities[8];
- `gig_dplot` compares the kernel-estimated density distribution of $\tilde{\varepsilon}_t$ with the theoretical distribution assumed for ε_t.

In the following examples we will also see the use of these functions.

[6] In both cases, step 3. of the procedure uses the smoothed state $\tilde{h}_t$.
[7] See Chapter 36 of the Gretl User's Guide (Cottrell and Lucchetti, 2025).
[8] But in this case the volatilities do not correspond to conditional standard deviations as in Garch models.

3.1 Examples

The following examples are carried out using the Bollerslav-Ghysels dataset which reports 1974 nominal returns on Mark/Pound exchange rate in the period from January 3, 1984, to December 31, 1991. The dataset is available in the gretl's library with the command:

```
open b-g.gdt
```

Standard QML estimation (3) can be performed by QMLSV with the following commands:

```
include QMLSV.gfn
bundle Mod1 = qml_sv(Y,1,0,1,1,100,0,1)
```

where Y is the series of the nominal return on Mark/Pound exchange rate. The command above also includes the following settings: stationary volatility; constant; smoothed volatility; Gaussian return innovation (i.e. `degrees=100`); no forecast; ex-post Student-t degrees estimation. The command produces the output:

```
STOCHASTIC VOLATILITY MODEL BY QML ESTIMATION
Dependent variable: Y
Sample: 1 -- 1974 (T = 1974)

Constant estimation:

           coefficient   std. error     z      p-value
-----------------------------------------------------
const      -0.0164268    0.0105840    -1.552   0.1207


Volatility model:

Type of volatility: stationary volatility
Estimation method: Standard QML
Gaussian return innovations


            coefficient   std. error      z       p-value
-------------------------------------------------------------
s_eta        0.249054     0.0345244      7.214   5.44e-013 ***
omega       -0.0670505    0.0193613     -3.463   0.0005    ***
beta         0.967820     0.00921848   105.0     0.0000    ***


Llik: -4533.5058       AIC:  9075.0115
BIC:   9097.3628       HQC:  9083.2239

Ex-post estimation of Student-t degrees:
```

```
            coefficient   std. error     z      p-value
-------------------------------------------------------
v            9.20530      1.04801     8.784   1.58e-018 ***
```

The ex-post estimate of v indicates that the standardized residuals have heavy tails and that it would seem more appropriate setting a value of v close to the estimated value (9.20). Repeating the previous command with `degrees=10`, we get[9]:

```
Volatility model:

Type of volatility: stationary volatility
Estimation method: Standard QML
Student-t return innovations, v = 10.0000

            coefficient   std. error      z       p-value
---------------------------------------------------------
s_eta        0.227149     0.0339927      6.682   2.35e-011 ***
omega       -0.0549912    0.0172638     -3.185   0.0014    ***
beta         0.971996     0.00868813   111.9     0.0000    ***


Llik: -4531.1526       AIC:  9070.3053
BIC:   9092.6565       HQC:  9078.5176

Ex-post estimation of Student-t degrees:

            coefficient   std. error     z      p-value
-------------------------------------------------------
v            11.4814      1.46295     7.848   4.22e-015 ***
```

The new model represents a step forward compared to the previous one as it presents a higher log-likelihood, as well as lower criteria. The set value of v (i.e. 10) seems acceptable since it is about the value of the SE (1.46) away from the estimated value (11.48).

However, the standard QML method is unable to estimate the covariance between volatility innovation and return innovation. It therefore seems reasonable to re-estimate the model (1) with the HS-QML and IQML methods:

```
bundle Mod2 = qml_sv(Y,2,0,1,1,10,0,1)  # HS-QML method
bundle Mod3 = qml_sv(Y,3,0,1,1,10,0,1)  # IQML method
```

The commands produce the output:

```
Volatility model:

Type of volatility: stationary volatility
Estimation method: Harvey-Shephard modified QML
```

[9] The output regarding the constant μ is no longer reported, since it does not change.

```
Student-t return innovations, v = 10.0000

             coefficient   std. error      z       p-value
-----------------------------------------------------------
s_etaplus     0.220412     0.0337083      6.539   6.20e-011 ***
omega        -0.0570103    0.0170756     -3.339   0.0008    ***
beta          0.969648     0.00867671   111.8     0.0000    ***
gamma        -0.0650312    0.0218390     -2.978   0.0029    ***

rho = -0.285

Llik: -4527.1181       AIC:  9062.2363
BIC:   9084.5875       HQC:  9070.4486

Ex-post estimation of Student-t degrees:

             coefficient   std. error     z      p-value
-------------------------------------------------------
v             10.8030      1.29264     8.357   6.42e-017 ***


Volatility model:

Type of volatility: stationary volatility
Estimation method: Iterative QML
Student-t return innovations, v = 10.0000

             coefficient   std. error      z       p-value
-----------------------------------------------------------
s_etaplus     0.216656     0.0332237      6.521   6.98e-011 ***
omega        -0.0569699    0.0167735     -3.396   0.0007    ***
beta          0.970653     0.00843470   115.1     0.0000    ***
gamma        -0.0633877    0.0200411     -3.163   0.0016    ***

rho = -0.281

Llik: -4526.6352       AIC:  9061.2705
BIC:   9083.6217       HQC:  9069.4828

Ex-post estimation of Student-t degrees:

             coefficient   std. error     z      p-value
-------------------------------------------------------
v             10.4694      1.20231     8.708   3.10e-018 ***
```

The results of the two methods are quite similar (as usually happens if the series is long enough), and both provide a better model than the Gaussian one. The covariance γ is significant in both models, implying a correlation (rho) of 0.281-0.285. However, even if the parameter estimates are quite similar between the two methods, it is worth

remembering that forecasts $\hat{h}_{t+1|t}$ are obtained with two different formulas, as proved by (5) and (7).

Being based on an iterative procedure, the IQML method is obviously slower than the HS-QML method, which is in turn quite slower than the standard QML method (the latter uses a state space invariant model, which is faster than a similar time-varying one). While with the standard QML method the parameter estimation is practically instantaneous, with the other two methods it takes a few seconds to obtain the parameter estimates.

Finally, once a model is estimated, the time series plot of volatilities and the density distribution plot of residuals can be obtained with the gig functions (Fig. 1):

```
include gig.gfn
gig_plot(&Mod3)
gig_dplot(&Mod3)
```

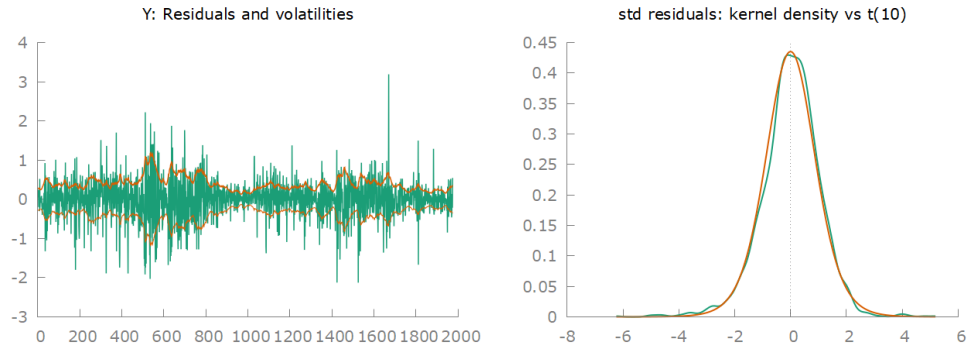

Fig. 1 volatility time series and kernel density of residuals

4 Final comments

The gretl package QMLSV allows the estimation of univariate stochastic volatility models using three distinct QML methods: (i) the standard QML method (Ruiz, 1994; Harvey et al, 1994); (ii) the modified QML method proposed by Harvey and Shephard (1996); (iii) the Iterative QML method (Chirico, 2024). The first method, fast and well-established, is only suitable for no leveraged financial returns, i.e. when volatility and return innovations are not correlated. The second method, slower than the first, takes into account leverage effect and provides a volatility predictor mainly similar to that of the Threshold GARCH model. Finally, the third method, the slowest of the three, also takes leverage into account and provides a volatility predictor similar to that of the EGARCH model.

All three methods can be applied under two alternative assumptions about the distribution of the return innovation: Gaussian distribution or Student-t distribution

with given degrees of freedom. Given the estimated model parameters, the package also provides, as an option, the estimation of the Student-t degrees of freedom. QMLSV does not currently provide simultaneous estimation of all parameters, including the degrees, because quasi-likelihood optimization algorithms do not always converge to a solution. However, ex post estimation of the degrees allows the model to be refined by updating the degrees in a subsequent estimation.

The package allows, under appropriate restrictions, the application of the three methods also to the case of random walk volatility.

In conclusion, QMLSV enriches the repertoire of univariate time series models in Gretl, filling the previous gap in stochastic volatility models.

Appendix A Proofs of Formulas (8) - (11)

A scaled Student's t-distribution, with v degrees of freedom, is the distribution of a random variable (r.v.), t_v^*, defined as:

$$t_v^* = t_v\sqrt{(v-2)/v} = \frac{Z}{\sqrt{X_v}}\sqrt{(v-2)} \tag{A1}$$

where t_v is a Student-t r.v. with v degrees of freedom, Z is a standard normal r.v., and X_v is a chi-square r.v. with v degrees of freedom, independent of Z.

If the return innovations ε_t have a t_v^* distribution, it is proved (see Appendix B in Chirico (2024)) that the parameters α and σ_ξ^2 are linked to the degrees v as follows:

$$\alpha = E[\ln t_v^{*2}] = -1.2701 - \psi_0(v/2) + \ln(v/2-1) \tag{A2}$$
$$\sigma_\xi^2 = Var[\ln t_v^{*2}] = 4.9348 + \psi_1(v/2) \tag{A3}$$

where ψ_0 and ψ_1 are the *digamma* and *trigamma* function, respectively (Davis, 1972).

In order to derive the dependence of θ_1 and θ_2 on v, we use the following known formulas[10]:

$$E[X_v^a] = 2^a\Gamma\Big(\frac{v}{2}+a\Big)/\Gamma\Big(\frac{v}{2}\Big) \tag{A4}$$
$$E[X_v^{\pm 1/2}\ln X_v] = E[X_v^{1/2}]\cdot E[\ln X_{v\pm 1}] \tag{A5}$$
$$E|t_v| = v^{1/2}E|Z|\cdot E[X_v^{-1/2}] \tag{A6}$$
$$E\big[\log X_v\big] = \psi_0(v/2) + \log(2) \tag{A7}$$
$$E|Z| = (2/\pi)^{1/2} = 0.7979 \tag{A8}$$

According to (A4)-(A8)[11], if $\varepsilon_t \sim t_v^*$, it follows that:

$$\theta_1 = E|t_v^{*2}| = E|t_v|\Big(\frac{v-2}{v}\Big)^{1/2}$$

[10] Formulas (A4)-(A6) can be got from Harvey and Shephard (1996), whereas formula (A7) from Zelen and Severo (1972).

[11] From these formulas, it also results that $E\big[|Z|\log Z^2\big] = E[X_1^{1/2}\ln X_1] = 0,0925$.

$$
\begin{aligned}
&= E|Z| v^{1/2} \Big(\frac{v-2}{v}\Big)^{1/2} (1/2)^{1/2} \Gamma\big(v/2-1/2\big)/\Gamma\big(v/2\big) \\
&= 0.7979 \cdot \big(v/2-1\big)^{1/2} \Gamma\big(v/2-1/2\big)/\Gamma\big(v/2\big) \qquad \text{(A9)}
\end{aligned}
$$

$$
\begin{aligned}
\theta_2 &= E[|t_v^*| \ln t_v^{*2}] - \theta_1 E[\ln t_v^{*2}] \\
&= E\big[(v-2)^{1/2} X_v^{-1/2} |Z| \cdot (\log Z^2 - \log X_v + \log(v-2))\big] - \theta_1 \alpha \\
&= (v-2)^{\frac{1}{2}} E\big[X_v^{-\frac{1}{2}}\big] E\big[|Z| \log Z^2\big] - (v-2)^{\frac{1}{2}} E|Z| E\big[X_v^{-\frac{1}{2}} \log X_v\big] \\
&\quad + E\big[t_v^*\big] \log(v-2) - \theta_1 \alpha \\
&= (v-2)^{\frac{1}{2}} E\big[X_v^{-\frac{1}{2}}\big] E|Z| \cdot \Big(\frac{0.0925}{0.7979} - \log X_{v-1} + \log(v-2)\Big) - \theta_1 \alpha_1
\end{aligned}
$$

From (A1) we know that $\theta_1 = E|t_v^{*2}| = (v-2)^{\frac{1}{2}} E\big[X_v^{-\frac{1}{2}}\big] E|Z|$, then:

$$
\begin{aligned}
\theta_2 &= \theta_1 \big(0.1159 - \psi(v/2-1/2) + \log(v/2-1) - \alpha\big) \\
&= \theta_1 \big(1.3863 + \psi_0(v/2) - \psi_0(v/2-1/2)\big) \qquad \text{(A10)}
\end{aligned}
$$

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