

Group manifold approach to supergravity

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Abstract

We present a short review of the group-geometric approach to supergravity theories, from the point of view of recent developments. The central idea is the unification of usual diffeomorphisms, gauge symmetries and supersymmetries into superdiffeomorphisms in a supergroup manifold. The example of $N = 1$ supergravity in $d = 4$ is discussed in detail, and used to illustrate all the steps in the construction of a group manifold action. In the Appendices we summarize basic notions of group manifold geometry, and of integration on supermanifolds.

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1 Introduction

Fundamental interactions are described by field theories with local invariances: the actions that govern their dynamics are invariant under field transformations involving parameters that are (arbitrary) functions of spacetime. This holds true both for gravity and gauge theories, where the local symmetries are general coordinate, and gauge transformations, respectively.

The essential difference between these two types of local transformations, in their infinitesimal versions, is that variations under diffeomorphisms always contain a derivative of the field, which is absent in gauge transformations. As well known, this is due to the fact that general coordinate transformations relate fields at different spacetime points, whereas gauge transformations relate fields at the same spacetime point.

Nonetheless, it is possible to give a unified description of diffeomorphisms and gauge transformations. This we achieve in a group geometrical framework.

The main idea is to consider as basic fields of the theory the *components of the vielbein* one-form $\sigma^A = \sigma(z)^\Lambda dz^\Lambda$ on the manifold of a Lie (super)group G , A being an index in the G Lie (super)algebra, and z^Λ the coordinates of the group manifold. This vielbein satisfies the Maurer-Cartan (MC) equations

$$d\sigma^A + \frac{1}{2}C_{BC}^A \sigma^B \wedge \sigma^C = 0 \quad (1.1)$$

where C_{BC}^A are the structure constants of the G Lie algebra. A brief account of group manifold geometry is given in Appendix A.

The G vielbein $\sigma^A(z)$ has a fixed dependence on the coordinates z , and therefore cannot be a dynamical object. We must consider a “soft” group manifold, diffeomorphic to G and denoted by \tilde{G} , with a vielbein μ^A not satisfying anymore the MC equations. The amount of deformation from the original “rigid” group manifold is measured by the *curvature* two-form:

$$R^A \equiv d\mu^A + \frac{1}{2}C_{BC}^A \mu^B \wedge \mu^C \quad (1.2)$$

Tangent vectors on \tilde{G} , dual to the vielbein μ^A , are denoted by t_B , so that $\mu^A(t_B) = \delta_B^A$.

Diffeomorphisms along tangent vectors $\varepsilon = \varepsilon^A t_A$ on \tilde{G} are generated by the Lie derivative ℓ_ε . When applied to the \tilde{G} vielbein, the variation under diffeomorphisms takes the form:

$$\ell_\varepsilon \mu^A = d\varepsilon^A + C_{BC}^A \mu^B \varepsilon^C + \iota_\varepsilon R^A \quad (1.3)$$

where ι_ε is the contraction operator, see Appendix A. On the right-hand side one recognizes the G -covariant derivative of the infinitesimal parameter ε^A plus a curvature term. When the curvature term vanishes, i.e. when $\iota_\varepsilon R^A = 0$, the diffeomorphism takes the form of a *gauge transformation*, and the curvature is said to be *horizontal* along the t_A 's entering the sum in $\varepsilon = \varepsilon^A t_A$. Thus in group manifold geometry *gauge transformations* can be interpreted as *particular diffeomorphisms*, along the directions on which the curvatures are horizontal.

This group geometric setting is particularly suited to supergravity theories, where local supersymmetry variations can be interpreted as diffeomorphisms in the super Poincaré group manifold, along the fermionic directions. It is then clear how to proceed to find theories invariant under local supersymmetry transformations: we must devise a procedure that yields actions, invariant under superdiffeomorphisms. This is very similar in spirit to the superspace approach [1, 2], where supergravity multiplets of dynamical (and auxiliary) fields are contained into a single superfield depending on superspace coordinates, or to the gauge supersymmetry framework, based on the super-Poincaré group, initiated in [3]. However the group manifold approach has important differences, as we explain in the coming Sections.

The action is obtained with an algorithmic procedure, as the integral of a d -form, “living” on the whole supergroup (soft) manifold \tilde{G} , but integrated on a

d -dimensional bosonic *submanifold* of \tilde{G} . This leads to an ordinary spacetime action containing the dynamical fields (and possibly also the auxiliary fields) of a d -dimensional supergravity theory. This algorithm will be discussed in detail and applied to obtain the action of $N = 1$, $d = 4$ supergravity.

The original references, where this approach was first proposed, are given in [4]-[7]. Reviews can be found in [8]-[13], and [14] is a standard reference for the use of differential forms in gravity and gauge theories.

The paper is organized as follows. In Section 2 we recall the algebraic basis of $d = 4$, $N = 1$ supergravity as a theory on the (soft) superPoincaré manifold, and the passage to a spacetime action. Section 3 deals with the symmetries of the spacetime action, as inherited from the diffeomorphism invariances of the group manifold action. In Section 4 the variational principle is formulated for the group manifold action, and equations of motion are derived. The building rules for (super)group manifold actions are discussed in Section 5, and applied to arrive unambiguously at the group manifold Lagrangian for $N = 1$, $d = 4$ supergravity. Some conclusions, and a selected list of applications and advantages of the group-geometric approach are discussed in Section 6. Finally, the Appendices contain brief accounts of group manifold geometry, integration on supermanifolds and gamma matrix properties.

2 Supergravity from superPoincaré geometry

2.1 Soft superPoincaré manifold

Supergravity in first order vierbein formalism can be recast in a supergroup geometric setting as follows. Consider $G = \text{superPoincaré group}$, and denote the vielbein on the \tilde{G} manifold as $\mu^A = (V^a, \omega^{ab}, \psi^\alpha)$. The index $A = (a, ab, \alpha)$ runs over the translations P_a , Lorentz rotations M_{ab} and supersymmetry charges \bar{Q}_α of the superPoincaré Lie algebra:

$$[P_a, P_b] = 0 \tag{2.1}$$

$$[M_{ab}, M_{cd}] = -\frac{1}{2}(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}) \tag{2.2}$$

$$[M_{ab}, P_c] = -\frac{1}{2}(\eta_{bc}P_a - \eta_{ac}P_b) \tag{2.3}$$

$$[P_a, \bar{Q}_\alpha] = 0 \tag{2.4}$$

$$[M_{ab}, \bar{Q}_\beta] = -\frac{1}{4}\bar{Q}_\alpha(\gamma_{ab})^\alpha_\beta \tag{2.5}$$

$$\{\bar{Q}_\alpha, \bar{Q}_\beta\} = -i(C\gamma^a)_{\alpha\beta}P_a, \tag{2.6}$$

η being the flat Minkowski metric, and $C_{\alpha\beta}$ the charge conjugation matrix. The spinorial generator $\bar{Q}_\alpha \equiv Q^\beta C_{\beta\alpha}$ is a Majorana spinor, i.e. $Q^\beta C_{\beta\alpha} = Q^\dagger_\beta(\gamma_0)^\beta_\alpha$. Thus the super-Poincaré manifold has 10 bosonic directions with coordinates x^a , y^{ab} , parametrizing translations and Lorentz rotations, and 4 fermionic directions with Grassmann coordinates θ^α , corresponding to the 4 supercharges \bar{Q}_α , $\alpha = 1, \dots, 4$.

The components of the supervielbein of the \tilde{G} =(soft) superPoincaré manifold are the vierbein V^a , the spin connection ω^{ab} and the gravitino ψ^α , corresponding respectively to the generators P_a , M_{ab} and \bar{Q}_α ,

Using the structure constants of the Lie superalgebra, the curvature (1.2) becomes :

$$R^a = dV^a - \omega^a_c V^c - \frac{i}{2} \bar{\psi} \gamma^a \psi \equiv DV^a - \frac{i}{2} \bar{\psi} \gamma^a \psi \quad (2.7)$$

$$R^{ab} = d\omega^{ab} - \omega^a_c \omega^{cb} \quad (2.8)$$

$$\rho = d\psi - \frac{1}{4} \omega^{ab} \gamma_{ab} \psi \equiv D\psi \quad (2.9)$$

defining respectively the supertorsion, the Lorentz curvature and the gravitino field strength. D is the Lorentz covariant exterior derivative. Wedge products between forms are understood when omitted.

Taking the exterior derivative of these definitions yields the Bianchi identities:

$$dR^a - \omega^a_b R^b + R^a_b V^b - i \bar{\psi} \gamma^a \rho \equiv DR^a + R^a_b V^b - i \bar{\psi} \gamma^a \rho = 0 \quad (2.10)$$

$$dR^{ab} - \omega^a_c R^{cb} + \omega^b_c R^{ca} \equiv DR^{ab} = 0 \quad (2.11)$$

$$d\rho - \frac{1}{4} \omega^{ab} \gamma_{ab} \rho + \frac{1}{4} R^{ab} \gamma_{ab} \psi \equiv D\rho + \frac{1}{4} R^{ab} \gamma_{ab} \psi = 0 \quad (2.12)$$

At this stage all the fields depend on all \tilde{G} manifold coordinates, corresponding to the generators of the Lie superalgebra: thus $V^a = V^a(x, y, \theta)$, $\omega^{ab} = \omega^{ab}(x, y, \theta)$, $\psi^\alpha(x, y, \theta)$, where the coordinates x^a , corresponding to the translations P_a , describe usual spacetime. Moreover the one-forms V^a , ω^{ab} , ψ live on the whole \tilde{G} , and therefore can be expanded as:

$$V^a = V^a_\mu(x, y, \theta) dx^\mu + V^a_{\mu\nu}(x, y, \theta) dy^{\mu\nu} + V^a_\alpha(x, y, \theta) d\theta^\alpha \quad (2.13)$$

$$\omega^{ab} = \omega^{ab}_\mu(x, y, \theta) dx^\mu + \omega^{ab}_{\mu\nu}(x, y, \theta) dy^{\mu\nu} + \omega^{ab}_\alpha(x, y, \theta) d\theta^\alpha \quad (2.14)$$

$$\psi^\alpha = \psi^\alpha_\mu(x, y, \theta) dx^\mu + \psi^\alpha_{\mu\nu}(x, y, \theta) dy^{\mu\nu} + \psi^\alpha_\beta(x, y, \theta) d\theta^\beta \quad (2.15)$$

2.2 Group manifold action

The overabundance of field components, and their dependence on y and θ coordinates can be tamed by defining an appropriate action principle. To end up with a geometrical theory in four spacetime dimensions, we first construct a 4-form Lagrangian L made out of the \tilde{G} vielbein μ^A and its curvature R^A , according to a few building rules to be discussed in Section 5. The resulting Lagrangian for super-Poincaré supergravity is given by:

$$L = R^{ab} V^c V^d \epsilon_{abcd} + 4 \bar{\psi} \gamma_5 \gamma_a \rho V^a \quad (2.16)$$

We then define an action by integrating this Lagrangian on a 4-dimensional submanifold M^4 of the \tilde{G} manifold, spanned by the x coordinates.

Integration on submanifolds M^d of a d -form L that lives on a g -dimensional bigger space \tilde{G} is carried out as follows: we multiply L by the *Poincaré dual* of M^d , a (singular) closed $(g-d)$ -form η_{M^d} that localizes the Lagrangian on the submanifold M^d , and integrate the resulting g -form on the whole \tilde{G} . Thus the group manifold action has the general expression

$$S = \int_{\tilde{G}} L \wedge \eta_{M^d} \quad (2.17)$$

The fields of the theory are those contained in L , i.e. the \tilde{G} vielbein components, and the embedding functions that define the M^d submanifold of \tilde{G} , present in η_{M^d} . We will see later that the embedding functions do not enter the field equations obtained from the variation of (2.17). This program makes use of standard integration theory when \tilde{G} is a bosonic space, but requires some new ingredients when \tilde{G} is a supermanifold, discussed in Appendix B.

In our $d = 4$ supergravity example the group manifold action is the integral on the 14-dimensional $\tilde{G} = \text{soft superPoincaré}$ manifold:

$$S = \int_{\tilde{G}} (R^{ab}V^cV^d\epsilon_{abcd} + 4\bar{\psi}\gamma_5\gamma_a\rho V^a) \eta_{M^4} \quad (2.18)$$

2.3 Spacetime action

A spacetime action, i.e. an action that is the integral on M^4 of a Lagrangian containing fields depending only on x , is obtained from (2.18) with a *particular choice* of η_{M^4} . Integration on y and θ coordinates produces then the spacetime action. This particular Poincaré dual is the product of two pieces: $\eta_{M^4} = \eta_y \wedge \eta_\theta$, where η_y is a (singular) 6-form that localizes the Lagrangian on the $y = 0$ hypersurface:

$$\eta_y = \delta(y^{01})\delta(y^{02}) \cdots \delta(y^{23}) dy^{01} \wedge dy^{02} \wedge \cdots \wedge dy^{23} \quad (2.19)$$

Integration on the y coordinates reduces (2.18) to an integral on the superspace $M^{4|4}$ spanned by x and θ . The y dependence of all fields in L disappears because of the delta functions in η_y , and the “legs” of L along dy differentials are killed by the product of all independent $dy^{\mu\nu}$ in η . The other piece of η_{M^4} (see Appendix B), after integration on θ coordinates, produces an integral on M^4 of a Lagrangian 4-form, not depending any more on the θ and on the $d\theta$ differentials. Thus the action

$$\begin{aligned} S_{spacetime} &= \int_{\tilde{G}} L \wedge \eta_{M^4} = \int_{M^4} L_{y=0, dy=0, \theta=0, d\theta=0} \\ &= \int_{M^4} R^{ab}V^cV^d\epsilon_{abcd} + 4\bar{\psi}\gamma_5\gamma_a\rho V^a \end{aligned} \quad (2.20)$$

contains only the usual fields $V_\mu^a(x)$ and $\omega_\mu^{ab}(x)$ and $\psi_\mu^\alpha(x)$ of $N = 1$ supergravity, and reproduces the first order supergravity action.

Note 1: η_y is closed (because it contains “functions” depending on y multiplied by all the dy differentials) and not exact (because of the Dirac deltas $\delta(y)$), and thus belongs to a nontrivial de Rham cohomology class. In general deformations of the M^4 surface generated by diffeomorphisms leave the Poincaré dual η in the same cohomology class, since the Lie derivative commutes with the exterior derivative.

Note 2: We will always assume that integration on the Lorentz coordinates has been carried out, so that all fields depend only on x and θ coordinates. Moreover all curvatures are taken to be horizontal in the Lorentz directions. As a consequence the theory lives in a superspace $M^{4|4}$ spanned by four bosonic coordinates x^a and four fermionic coordinates θ^α .

Note 3: the spacetime action (2.20) and its invariance under the supersymmetry transformations (3.22)-(3.24) were first found in ref. [15] in second order formalism and in [16] in first order formalism, see also the standard references [17, 18] on supergravity.

3 Symmetries

The action (2.18) is the integral on \tilde{G} of a top form: it is clearly invariant under diffeomorphisms on \tilde{G} . But what we are really interested in are the symmetries of the spacetime action as given in (2.20), where the variations are carried out only in the x -dependent fields in $L|_{y=dy=0, \theta=d\theta=0}$. The only symmetries guaranteed a priori are the 4-dimensional spacetime diffeomorphisms, the spacetime action being an integral of a 4-form on M^4 .

Here resides most of the power of the group manifold formalism: if one considers the “mother” action (2.17) on \tilde{G} , the guaranteed symmetries are *all* the diff.s on \tilde{G} , generated by the Lie derivative ℓ_ε along the tangent vectors $\varepsilon = \varepsilon^A t_A$ of \tilde{G} . But how do these symmetries transfer to the spacetime action ?

The variation of the group manifold action under diff.s generated by ℓ_ε is¹

$$\delta S = \int_{\tilde{G}} \ell_\varepsilon(L \wedge \eta) = \int_{\tilde{G}} (\ell_\varepsilon L) \wedge \eta + L \wedge \ell_\varepsilon \eta = 0 \quad (3.1)$$

modulo boundary terms. One has to vary the fields² in L as well as the submanifold embedded in \tilde{G} : the sum of these two variations gives zero³ on the group manifold action S . But what we need in order to have a *spacetime* interpretation of all the symmetries of S , is really

$$\delta S = \int_{\tilde{G}} (\ell_\varepsilon L) \wedge \eta = 0 \quad (3.2)$$

¹Recall $\ell_\varepsilon = \iota_\varepsilon d + d\iota_\varepsilon$ so that $\ell_\varepsilon(\text{top form}) = d(\iota_\varepsilon \text{ top form})$

²Since ℓ_ε satisfies the Leibniz rule, $\ell_\varepsilon L$ can be computed by varying in turn all fields inside L .

³In the following the vanishing of action variations will always be understood modulo boundary terms.

If this holds, varying the fields ϕ inside L with the Lie derivative ℓ_ϵ as in (1.3), and then projecting on spacetime, yields spacetime variations

$$\delta\phi(x) = \ell_\epsilon\phi(x, y, \theta)|_x \quad (3.3)$$

that leave the spacetime action (2.20) invariant. We have denoted by $|_x$ the projection on spacetime due to the integration on y and θ coordinates in (3.2). We call these variations *spacetime invariances*, since they leave invariant the spacetime action. They originate from the diff. invariance of the group manifold action, and give rise to symmetries of the spacetime action (2.20) only when (3.2) holds. This happens if one of the following conditions is satisfied:

- the Lie derivative on η vanishes:

$$\ell_\epsilon\eta = 0 \quad (3.4)$$

- the spacetime projection of the Lie derivative of L is exact:

$$(\ell_\epsilon L)|_x = d\alpha \quad (3.5)$$

In this case the variation (3.2)

$$\delta S = \int_{\tilde{G}} (\ell_\epsilon L) \wedge \eta = \int_{M^4} (\ell_\epsilon L)|_x \quad (3.6)$$

vanishes after integration by parts. The requirement (3.5) is equivalent to

$$(\iota_\epsilon dL)|_x = d\alpha' \quad (3.7)$$

since $\ell_\epsilon = \iota_\epsilon d + d\iota_\epsilon$, and $d\eta = 0$.

The Lagrangian L depends on the \tilde{G} -vielbein μ^A and its curvature R^A , so that also dL , after use of Bianchi identities, is expressed in terms of μ^A and R^A . Then condition (3.7) translates into a *condition on the contractions $\iota_\epsilon R^A$* , i.e. a condition on the curvature components.

Let us see how this works for superPoincaré supergravity.

3.1 Symmetries of $d = 4$ supergravity

The symmetries of the spacetime action (spacetime invariances) are those generated by a Lie derivative ℓ_ϵ such that $\iota_\epsilon dL|_x = d\alpha'$, cf. (3.7). We need to compute dL . Using the Bianchi identities (2.11) and (2.12), and the definition of the torsion R^a in (2.7) we find:

$$\begin{aligned} dL = & 2R^{ab}R^cV^d\varepsilon_{abcd} + iR^{ab}\bar{\psi}\gamma^c\psi V^d\varepsilon_{abcd} + 4\bar{\rho}\gamma_5\gamma_a\rho V^a + \\ & + \bar{\psi}\gamma_5\gamma_c\gamma_{ab}\psi R^{ab}V^c - 4\bar{\psi}\gamma_5\gamma_a\rho R^a - 2i\bar{\psi}\gamma_5\gamma_a\rho\bar{\psi}\gamma^a\psi \end{aligned} \quad (3.8)$$

The gamma matrix identity

$$\gamma_c \gamma_{ab} = \eta_{ac} \gamma_b - \eta_{bc} \gamma_a + i \varepsilon_{abcd} \gamma_5 \gamma^d \quad (3.9)$$

implies $\bar{\psi} \gamma_5 \gamma_c \gamma_{ab} \psi = i \varepsilon_{abcd} \bar{\psi} \gamma^d \psi$, so that the second and the fourth term cancel in (3.8). Moreover from the Fierz identity in Appendix C one deduces

$$\gamma_a \psi \bar{\psi} \gamma^a \psi = 0 \quad (3.10)$$

and since $\bar{\psi} \gamma_5 \gamma_a \rho = \bar{\rho} \gamma_5 \gamma_a \psi$ also the last term in (3.8) vanishes due to (3.10). Therefore

$$dL = 2R^{ab} R^c V^d \varepsilon_{abcd} + 4\bar{\rho} \gamma_5 \gamma_a \rho V^a - 4\bar{\psi} \gamma_5 \gamma_a \rho R^a \quad (3.11)$$

Lorentz gauge transformations

It is immediate to see that if all curvatures are horizontal in the Lorentz directions (no ‘‘legs’’ along ω) then indeed $\iota_{\varepsilon^{ab} t_{ab}} dL = 0$, and Lorentz transformations are a spacetime invariance of the supergravity action. This is essentially due to the absence of bare ω^{ab} in L . The general diffeomorphism formula (1.3) yields then the usual Lorentz transformations

$$\ell_{\varepsilon^{cd} t_{cd}} V^a = \varepsilon^a_b V^b \quad (3.12)$$

$$\ell_{\varepsilon^{cd} t_{cd}} \omega^{ab} = d\varepsilon^{ab} - \omega^a_c \varepsilon^{cb} + \omega^b_c \varepsilon^{ca} = D\varepsilon^{ab} \quad (3.13)$$

$$\ell_{\varepsilon^{cd} t_{cd}} \psi = \frac{1}{4} \varepsilon^{ab} \gamma_{ab} \psi \quad (3.14)$$

We can check directly the invariance of the action under these variations: all curvatures and vierbeins appearing in (2.20) transform homogeneously, and Lorentz indices are contracted with Lorentz invariant tensors.

Spacetime diffeomorphisms

Ordinary diff.s along tangent vectors ∂_μ dual to dx^μ are invariances of the spacetime action, since (2.20) is an integral on a 4-dimensional manifold of a 4-form.

Supersymmetry transformations

Diff.s along tangent vectors t_α dual to ψ^α are spacetime invariances provided $\iota_\epsilon dL|_x = \text{total derivative}$ with $\epsilon = \epsilon^\alpha t_\alpha$, that is to say

$$\begin{aligned} \iota_\epsilon dL &= 2(\iota_\epsilon R^{ab}) R^c V^d \varepsilon_{abcd} + 2R^{ab} (\iota_\epsilon R^c) V^d \varepsilon_{abcd} + 8\bar{\rho} \gamma_5 \gamma_a (\iota_\epsilon \rho) V^a \\ &\quad - 4\bar{\varepsilon} \gamma_5 \gamma_a \rho R^a - 4\bar{\psi} \gamma_5 \gamma_a (\iota_\epsilon \rho) R^a - 4\bar{\psi} \gamma_5 \gamma_a \rho (\iota_\epsilon R^a) = \text{tot. der.} \end{aligned} \quad (3.15)$$

once projected on spacetime. This is a condition for the contractions on the curvatures, and it is satisfied by:

$$\iota_\epsilon R^a = 0 \quad (3.16)$$

$$\iota_\epsilon R^{ab} = -\varepsilon^{abef} \bar{\rho}_{ef} \gamma_5 \gamma_g \epsilon V^g - \varepsilon^{efg[a} \bar{\rho}_{ef} \gamma_5 \gamma_g \epsilon V^{b]} \equiv \bar{\theta}_c^{ab} \epsilon V^c \quad (3.17)$$

$$\iota_\epsilon \rho = 0 \quad (3.18)$$

Thus we have supersymmetry invariance of the spacetime action if the curvatures have the following parametrization on a basis of 2-forms:

$$R^a = R^a{}_{bc} V^b V^c \quad (3.19)$$

$$R^{ab} = R^{ab}{}_{cd} V^c V^d + \bar{\theta}_c^{ab} \psi V^c \quad (3.20)$$

$$\rho = \rho_{ab} V^a V^b \quad (3.21)$$

where we have taken into account also horizontality in the Lorentz directions. The conditions (3.16)-(3.18) are called ‘‘rheonomic conditions’’, and similarly (3.19)-(3.21) are called ‘‘rheonomic parametrizations’’ of the curvatures.

The diff.s along $\epsilon = \epsilon^\alpha t_\alpha$ (supersymmetry transformations) act on the fields according to the general formula (1.3), where the contractions on the curvatures are given in (3.16)-(3.18):

$$\ell_\epsilon V^a = i\bar{\epsilon}\gamma^a \psi \quad (3.22)$$

$$\ell_\epsilon \omega^{ab} = \bar{\theta}_c^{ab} \epsilon V^c \quad (3.23)$$

$$\ell_\epsilon \psi = D\epsilon \equiv d\epsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon \quad (3.24)$$

with $\bar{\theta}_c^{ab}$ defined in (3.17).

4 Variational principle and field equations

The group manifold action (2.17) is a functional of L and of the embedded submanifold M , and therefore varying the action means varying both L and M . Varying M corresponds to varying η_M . Then the variational principle reads:

$$\delta S[L, M] = \int_{\tilde{G}} (\delta L \wedge \eta_M + L \wedge \delta \eta_M) = 0. \quad (4.1)$$

Any (continuous) variation of M can be obtained by acting on η_M with a diffeomorphism generated by a Lie derivative ℓ_ξ . An arbitrary variation is generated by an arbitrary ξ vector, and the variational principle becomes

$$\delta S[L, M] = \int_{\tilde{G}} (\delta L \wedge \eta_M + L \wedge \ell_\xi \eta_M) = 0. \quad (4.2)$$

Since field variations in L and variation of M are independent, the two terms in (4.2) must vanish separately. From the vanishing of the first we deduce

$$\int_{\tilde{G}} (\delta\phi \wedge \frac{\partial L}{\partial\phi} + d\delta\phi \wedge \frac{\partial L}{\partial(d\phi)}) \wedge \eta_M = 0 \quad (4.3)$$

where $L = L(\phi, d\phi)$ is considered a function of the 1-form fields ϕ and their ‘‘velocities’’ $d\phi$. A summation on all fields is understood. Integrating by parts and recalling $d\eta_M = 0$ yields

$$\int_{\tilde{G}} \delta\phi \wedge (\frac{\partial L}{\partial\phi} + d\frac{\partial L}{\partial(d\phi)}) \wedge \eta_M = 0 \quad (4.4)$$

and since the $\delta\phi$ are arbitrary we find

$$\left(\frac{\partial L}{\partial\phi} + d\frac{\partial L}{\partial(d\phi)}\right) \wedge \eta_M = 0 \quad (4.5)$$

This must hold for any η_M (i.e. for generic embedding functions): we arrive therefore at equations that hold on the whole \tilde{G} , and are the form version of the Euler-Lagrange equations:

$$\frac{\partial L}{\partial\phi} + d\frac{\partial L}{\partial(d\phi)} = 0 \quad (4.6)$$

If L is a d -form, these equations are $(d-1)$ -forms. Their content can be examined by expanding them along a complete basis of $(d-1)$ -forms in \tilde{G} .

Requiring the vanishing of the second term in the variation (4.2) does not imply further equations besides the Euler-Lagrange field equations (4.6): indeed this term vanishes on the shell of solutions of Euler-Lagrange equations. To prove it, notice that

$$\int_{\tilde{G}} L \wedge \ell_\xi \eta_M = - \int_{\tilde{G}} \ell_\xi L \wedge \eta_M = 0 \quad (\text{on shell}) \quad (4.7)$$

because $\ell_\xi L$ is just a particular variation of L , under which the action remains stationary on-shell.

Thus the group manifold variational principle leads to the field equations (4.6), holding as $(d-1)$ -form equations on the whole \tilde{G} .

Note 1: The variational principle *does not determine* the embedding of M into \tilde{G} .

Note 2: the field equations (4.6) are form equations, and therefore invariant under the action of a Lie derivative. More precisely, if ϕ is a solution of (4.6), so is $\phi + \ell_\varepsilon \phi$: Lie derivatives generate symmetries of the field equations.

Finally, we have the following

Theorem: $dL = 0$ (on shell)

i.e. the Lagrangian, as a d -form on \tilde{G} , is closed on shell. To prove it recall that η_M is closed, so that on shell we find, cf. (4.7):

$$0 = \int_{\tilde{G}} L \wedge \ell_\xi \eta_M = \int_{\tilde{G}} L \wedge d\iota_\xi \eta_M = -(-)^d \int_{\tilde{G}} dL \wedge \iota_\xi \eta_M \quad (4.8)$$

ξ being arbitrary, this implies $dL = 0$ (on shell)⁴ \square

Let us apply the preceding discussion to the superPoincaré supergravity example.

⁴In fact, this is just Stokes theorem applied to a $d+1$ dimensional region of \tilde{G} bounded by two different hypersurfaces M and M' , with $\eta_{M'} = \eta_M + \ell_\xi \eta_M$.

4.1 Supergravity field equations

The variational equations (4.6) for the group manifold action (2.18) read:

$$2R^c V^d \epsilon_{abcd} = 0 \quad (4.9)$$

$$2R^{ab} V^c \epsilon_{abcd} + 4\bar{\psi} \gamma_5 \gamma_d \rho = 0 \quad (4.10)$$

$$8\gamma_5 \gamma_a \rho V^a - 4\gamma_5 \gamma_a \psi R^a = 0 \quad (4.11)$$

obtained varying ω^{ab} , V^d and ψ respectively. The analysis proceeds as follows: we first expand the curvatures on a basis of 2-forms⁵, with a set of coefficient functions that are a priori unknown:

$$R^a = R^a{}_{bc} V^b V^c + \bar{\theta}^a{}_c \psi V^c + \bar{\psi} K^a \psi \quad (4.12)$$

$$R^{ab} = R^{ab}{}_{cd} V^c V^d + \bar{\theta}^{ab}{}_c \psi V^c + \bar{\psi} K^{ab} \psi \quad (4.13)$$

$$\rho = \rho_{ab} V^a V^b + H_c \psi V^c + \Omega_{\alpha\beta} \psi^\alpha \psi^\beta \quad (4.14)$$

and then insert them into the field equations (4.9)-(4.11). These, being 3-form equations, can be expanded on the basis $\psi\psi\psi$, $\psi\psi V$, ψVV , VVV . Their content is given below (the three lines correspond to the three eq.s of motion):

$\psi\psi\psi$ sector:

$$\Omega_{\alpha\beta} = 0 \quad (4.15)$$

$$0 = 0 \quad (4.16)$$

$$K^a = 0 \quad (4.17)$$

$\psi\psi V$ sector:

$$2\bar{\psi} K^{ab} \psi V^c \epsilon_{abcd} + 4\bar{\psi} \gamma_5 \gamma_d H_c \psi V^c = 0 \quad (4.18)$$

$$0 = 0 \quad (4.19)$$

$$\bar{\theta}^a{}_c = 0 \quad (4.20)$$

ψVV sector:

$$2\bar{\theta}^{ab}{}_c \psi V^e V^c \epsilon_{abcd} + 4\bar{\psi} \gamma_5 \gamma_d \rho_{ab} V^a V^b = 0 \quad (4.21)$$

$$0 = 0 \quad (4.22)$$

$$\gamma_5 \gamma_a H_b \psi V^b V^a - 4\gamma_5 \gamma_c \psi R^c{}_{ab} V^a V^b = 0 \quad (4.23)$$

VVV sector:

$$R^a{}_{bc} = 0 \quad (4.24)$$

$$R^a{}_{bc} - \frac{1}{2} \delta_b^a R^c{}_{cd} = 0 \quad (4.25)$$

$$\gamma^a \rho_{ab} = 0 \quad (4.26)$$

⁵assuming horizontality in the Lorentz directions. This amounts to consider configurations satisfying the Lorentz horizontality constraints on the curvatures.

Inserting $R^a{}_{bc} = 0$ into (4.23) yields $H_c = 0$, which used in (4.18) gives $K^{ab} = 0$. Thus the only nontrivial relation in the “outer” projections is (4.21), that determines $\theta^{ab}{}_c$ to be

$$\theta^{ab}{}_c = -\varepsilon^{abef} \bar{\rho}_{ef} \gamma_5 \gamma_c - \delta_c^{[a} \varepsilon^{b]efg} \bar{\rho}_{ef} \gamma_5 \gamma_g \quad (4.27)$$

in agreement with the $\theta^{ab}{}_c$ obtained from the condition (3.17). In this way we arrive at the same curvature parametrizations (3.19)-(3.21) obtained in Sect. 3.1 by requiring spacetime supersymmetry invariance.

Finally, the VVV sector reproduces the (super)torsion equation, and the propagation equations for the vierbein and the gravitino.

Note: from the torsion equation

$$2R^a{}_{\mu\nu} \equiv \partial_\mu V_\nu^a - \partial_\nu V_\mu^a - \omega^a{}_{b,\mu} V_\nu^b + \omega^a{}_{b,\nu} V_\mu^b - i\bar{\psi}_\mu \gamma^a \psi_\nu = 0 \quad (4.28)$$

we can express the spin connection in terms of V and ψ , recovering second order formalism:

$$\begin{aligned} \omega_{ab,\mu} = & \frac{1}{2} V_a^\nu V_b^\rho \eta_{cd} (\partial_{[\mu} V_{\nu]}^c V_\rho^d - \partial_{[\mu} V_{\rho]}^c V_\nu^d + \partial_{[\nu} V_{\rho]}^c V_\mu^d) + \\ & + \frac{i}{4} V_a^\nu V_b^\rho (\bar{\psi}_\mu \gamma_\nu \psi_\rho + \bar{\psi}_\nu \gamma_\rho \psi_\mu - \bar{\psi}_\rho \gamma_\mu \psi_\nu - (\nu \leftrightarrow \rho)) \end{aligned} \quad (4.29)$$

5 Building rules

The group geometric approach provides a systematic set of building rules [8] for constructing Lagrangians of supersymmetric theories:

1) Choose a Lie (super)algebra G , containing generators P_a that can be associated to d spacetime directions, and a Lorentz-like subalgebra H . Examples are the superPoincaré algebras in d dimensions or their uncontracted versions (orthosymplectic superalgebras $OSp(N|2^{[d/2]})$). The fields of the theory are the vielbein components of the soft group manifold \tilde{G} .

2) Construct the most general d -form on \tilde{G} , by multiplying (with exterior products) 1-form vielbein components σ^A and 2-form curvatures R^A , without bare Lorentz connection and contracting indices with H -invariant tensors, so that the resulting Lagrangian is a Lorentz scalar.

3) Require that the variational equations admit the “vacuum solution” $R^A = 0$, described by the vielbein of the rigid group manifold G .

4) The construction is greatly helped by scaling properties of the fields, dictated by the structure of the Lie (super)algebra G , or equivalently by the Maurer-Cartan equations for the G vielbein. Consider for example the superPoincaré algebra: it is invariant under the rescalings $P_a \rightarrow \lambda P_a, M_{ab} \rightarrow M_{ab}, \bar{Q}_\alpha \rightarrow \lambda^{\frac{1}{2}} \bar{Q}_\alpha$. Then the

curvature definitions (2.7)-(2.9) are invariant under

$$V^a \rightarrow \lambda V^a, \quad \omega^{ab} \rightarrow \omega^{ab}, \quad \psi^\alpha \rightarrow \lambda^{\frac{1}{2}} \psi^\alpha \quad (5.1)$$

The field equations must be invariant under these rescalings, and therefore the action must scale homogeneously under (5.1). Since the Einstein-Hilbert term scales as λ^2 , all terms must scale in the same way, and this restricts the candidate terms in the Lagrangian.

5) Finally, requiring that all terms have the same parity as the Einstein-Hilbert term further narrows the list of candidates.

5.1 The Lagrangian for $d = 4$ supergravity

Following the above rules, one arrives at the $d = 4$ supergravity action (2.18). We recall here the steps of the procedure [8]. The most general lagrangian 4-form satisfying Rule 1 can at most contain two curvatures, and is therefore of the type:

$$L = R^A R^B \nu_{AB} + R^A \nu_A + \Lambda \quad (5.2)$$

with⁶

$$R^A R^B \nu_{AB}^{(2)} = c_1 R^{ab} R^{cd} \varepsilon_{abcd} + c_2 R^{ab} R^{ab} + c_3 R^a R^a + c_4 \bar{\rho} \rho + c_5 \bar{\rho} \gamma_5 \rho \quad (5.3)$$

The first two are total derivatives, and are related to the Euler characteristic and to the Pontriagyn number of M^4 . The last three can be reduced to linear terms in the curvatures plus total derivatives. Actually scaling invariance eliminates all the terms in (5.3) except $R^a R^a$, since the Einstein term scales as λ^2 . The torsion-squared term can be reduced to a linear term since

$$R^a R^a = (DV^a - \frac{i}{2} \bar{\psi} \gamma^a \psi) R^a = d(V^a R^a) + V^a (-R^{ab} V^b + i \bar{\psi} \gamma^a \rho) - \frac{i}{2} \bar{\psi} \gamma^a \psi R^a \quad (5.4)$$

in virtue of the Bianchi identity (2.10). This leaves us with a lagrangian of the form:

$$L = \Lambda + \nu_{ab} R^{ab} + \nu_a R^a + \bar{\nu} \rho \quad (5.5)$$

where

$$\Lambda = \alpha_1 \varepsilon_{abcd} V^a V^b V^c V^d + i \alpha_2 \varepsilon_{abcd} \bar{\psi} \gamma^{ab} \psi V^c V^d + i \alpha_3 \bar{\psi} \gamma^{ab} \psi V^a V^b \quad (5.6)$$

$$\nu_{ab} = \beta_1 \varepsilon_{abcd} V^c V^d + \beta_2 V^a V^b + i \beta_3 \bar{\psi} \gamma_{ab} \psi + i \beta_4 \varepsilon_{abcd} \bar{\psi} \gamma^{cd} \psi \quad (5.7)$$

$$\nu_a = i \eta_1 \bar{\psi} \gamma_a \psi \quad (5.8)$$

$$\nu = \delta_1 \gamma_5 \gamma_a \psi V^a + i \delta_2 \gamma_a \psi V^a \quad (5.9)$$

⁶repeated indices are contracted with the Minkowski flat metric.

are the most general Lorentz covariant terms. Notice that the only nonvanishing $\psi\psi$ currents are $\bar{\psi}\gamma^a\psi$ and $\bar{\psi}\gamma^{ab}\psi$. Correct λ^2 scaling of L drastically reduces the possible terms: $\alpha_1 = \alpha_2 = \alpha_3 = \beta_3 = \beta_4 = 0$. Moreover parity implies $\beta_2 = \eta_1 = \delta_2 = 0$ (all terms must have the same parity as the Einstein term $R^{ab}V^cV^d\varepsilon_{abcd}$, i.e. must be pseudoscalars). Thus we finally have:

$$L = \beta_1\varepsilon_{abcd}R^{ab}V^cV^d + \delta_1\bar{\psi}\gamma_5\gamma_a\rho V^a \quad (5.10)$$

The requirement 3) that the vacuum be a solution of the field equations fixes the last parameter $a = \delta_1/\beta_1$. Indeed the field equations obtained from the Lagrangian (5.10) by varying V^a, ω^{ab} and ψ are respectively:

$$2R^{ab}V^c\varepsilon_{abcd} + a\bar{\psi}\gamma_5\gamma_d\rho = 0 \quad (5.11)$$

$$2DV^cV^d\varepsilon_{abcd} + \frac{1}{4}a\bar{\psi}\gamma_5\gamma_d\gamma_{ab}\psi V^d = 0 \quad (5.12)$$

$$2a\gamma_5\gamma_a\rho V^a - a\gamma_5\gamma_a\psi R^a = 0 \quad (5.13)$$

To find the first is immediate; for the second we only have to recall that varying ω^{ab} in R^{ab} yields $\delta R^{ab} = D(\delta\omega^{ab})$, and that, integrating by parts, the Lorentz covariant derivative D can be transferred on V^a . Finally for the gravitino variation we have

$$\frac{1}{a}\delta L = (\delta\bar{\psi})\gamma_5\gamma_a D\psi V^a + \bar{\psi}\gamma_5\gamma_a D(\delta\psi)V^a = \quad (5.14)$$

$$= (\delta\bar{\psi})\gamma_5\gamma_a D\psi V^a + \bar{\psi}\gamma_5\gamma_a\delta\psi DV^a + \delta\bar{\psi}\gamma_5\gamma_a D\psi V^a = \quad (5.15)$$

$$= 2(\delta\bar{\psi})\gamma_5\gamma_a D\psi V^a - \delta\bar{\psi}\gamma_5\gamma_a\psi(R^a + \frac{i}{2}\bar{\psi}\gamma^a\psi) = \quad (5.16)$$

$$= (\delta\bar{\psi})(2\gamma_5\gamma_a D\psi V^a - \gamma_5\gamma_a\psi R^a) \quad (5.17)$$

$$(5.18)$$

in virtue of $\bar{\psi}\gamma_5\gamma_a(\delta\psi) = -(\delta\bar{\psi})\gamma_5\gamma_a\psi$ and the Fierz identity

$$\gamma_a\psi\bar{\psi}\gamma^a\psi = 0 \quad (5.19)$$

Note that using the gamma-algebra identity:

$$\gamma_5\gamma_d\gamma_{ab} = 2\gamma_5\delta_{d[a}\gamma_{b]} - i\varepsilon_{abcd}\gamma^c \quad (5.20)$$

the variational equation (5.12) can be recast in the form:

$$2R^cV^d\varepsilon_{abcd} + \frac{i}{4}(4-a)\bar{\psi}\gamma_5\gamma_d\gamma_{ab}\psi V^d = 0 \quad (5.21)$$

so that the vacuum, defined by vanishing curvatures, is a solution of the field equations (5.21) only if $a = 4$.

In conclusion: applying the building rules with $G = \text{superPoincaré}$ yields the $N = 1, d = 4$ supergravity action (2.18).

6 Conclusions

In this review we have focused mostly on the logic of the group manifold approach, applied to a single example, i.e. $N = 1$, $d = 4$ supergravity. Comprehensive discussions on the applications of the method for the construction of supergravity theories in diverse dimensions can be found in the recent reviews [12, 13].

We list here some of the advantages/motivations:

- all fields have a group-geometric origin, even if they are not all gauge fields.
- all symmetries have a common origin as diffeomorphisms on \tilde{G} .
- there is a systematic procedure based on group geometry to construct actions, invariant under diffeomorphisms, and under gauge symmetries closing on a subgroup of G .
- supersymmetry is formulated in a very natural way as a diffeomorphism in Grassmann directions of a supermanifold.
- closer contact is maintained with the usual component actions, whereas in the superfield formalism the actions looks quite different. In fact the group manifold action interpolates between the component and the superfield actions of the same supergravity theory, see [19]-[21].
- in the group manifold formulation of $d = 6$ supergravity [22] and $d = 10$ supergravity [23] the selfdual conditions for the 3-form (in $d = 6$) and 5-form (in $d = 10$) curvatures are a yield of the field equations in the respective superspaces, and do not need to be imposed as external constraints.

Finally, we recall some conceptual advances due to the group-geometric treatment of supergravity:

- the generalization to p -form potentials, necessary to treat supergravity theories with p -form fields, in the framework of Free Differential Algebras (FDA) [24, 25, 8, 26, 27], and their dual formulation [28] - [31].
- the bridge between superspace and group manifold methods provided by superintegration, developed in ref.s [19]-[21].
- a covariant hamiltonian formalism, initially proposed in [32]-[34], based on the definition of field momenta as derivatives of the Lagrangian with respect to the exterior derivative of the fields, not involving a preferred direction (time). Recent developments [35, 36] include the construction of all canonical symmetry generators for $N = 1$, $d = 4$ supergravity [36]. This covariant hamiltonian formalism can also be generalized to a noncommutative (twisted) setting [37], describing noncommutative twisted (super)gravity [38, 39].

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A Group manifold geometry

This brief resumé is taken from Sec. 2 of [11]. We start from a Lie algebra $\text{Lie}(G)$, with generators T_A satisfying the commutation relations

$$[T_A, T_B] = C^C{}_{AB} T_C \quad (\text{A.1})$$

For simplicity we consider only usual Lie algebras. The extension to superalgebras is straightforward and only necessitates extra signs (for ex. anticommutators for fermionic generators) due to gradings.

A generic group element $g \in G$ connected with the identity ⁷ can be expressed as

$$g = \exp(y^A T_A) \equiv y \quad (\text{A.2})$$

where y^A are the (exponential) coordinates of the group manifold. Each element of G is labelled by the coordinates y^A , and for notational economy we denote it simply by y . Similarly yx stands for $\exp(y^A T_A) \exp(x^B T_B)$, the product of two group elements, and by $(yx)^M$ we denote the corresponding coordinates.

Consider now $(yx)^M$ as a function⁸ of x^A :

$$(yx)^M = y^M + e_A{}^M(y) x^A + e_{AB}{}^M(y) x^A x^B + \dots \quad (\text{A.3})$$

For infinitesimal x :

$$(yx)^M = y^M + (x^A t_A) y^M = (1 + x^A t_A) y^M, \quad t_A \equiv e_A{}^N(y) \frac{\partial}{\partial y^N} \quad (\text{A.4})$$

so that the t_A are a differential representation of the abstract generators T_A , and satisfy therefore the same algebra:

$$[t_A, t_B] = C^C{}_{AB} t_C \quad (\text{A.5})$$

The geometrical meaning of the components $e_A{}^N(y)$ in eq. (A.3) is clear: consider the infinitesimal displacement $\delta_A y^M$ due to the (right) action of $1 + \varepsilon T_A$ ($\varepsilon =$ infinitesimal parameter). Then

$$\delta_A y^M = \varepsilon e_A{}^M(y) \quad (\text{A.6})$$

⁷Hereafter G indicates the part of the group connected with the identity.

⁸Since G is a Lie group, this function is smooth.

and the $\dim G$ vectors $e_A^M(y)$, $A=1,\dots,\dim G$ are simply the tangent vectors at y in the direction of the displacements $\delta_A y^M$. It is customary to call tangent vector along the T_A direction the whole differential operator $t_A \equiv e_A^N(y) \frac{\partial}{\partial y^N}$. Note that e_A^M is an invertible matrix, since the map $y \rightarrow yx$ is a diffeomorphism.

The $t_A(y)$ span the tangent space of G at y : they form a contravariant basis. The ‘‘coordinate’’ basis, given by the vectors $\frac{\partial}{\partial y^N}$, is related to the t_A (the intrinsic basis) via the nondegenerate matrix e_A^N . The indices A,B,\dots are tangent space indices (‘‘flat’’ indices) and are inert under y coordinate transformations. The indices M,N,\dots are coordinate indices (‘‘world’’ indices) and do transform under coordinate transformations in the usual way (see later). Next we define the one-forms $\sigma^A(y)$ as the duals of the t_A :

$$\sigma^A(t_B) = \delta_A^B \quad (\text{A.7})$$

The σ^A are a covariant basis (the intrinsic vielbein basis) for the dual of the tangent space, called cotangent space (the space of 1-forms). The ‘‘coordinate’’ cotangent basis dual to the $\frac{\partial}{\partial y^N}$ vectors is given by the differentials dy^M ($dy^M(\frac{\partial}{\partial y^N}) = \delta_N^M$). The components of $\sigma^A(y)$ on the coordinate basis are denoted $e_M^A(y)$:

$$\sigma^A(y) = e_M^A(y) dy^M \quad (\text{A.8})$$

From the duality of the tangent and cotangent bases we find:

$$e_M^A e_B^M = \delta_B^A \quad (\text{A.9})$$

$$e_A^M e_N^A = \delta_N^M \quad (\text{A.10})$$

Note 1: Substituting t_A by $e_A^N(y) \frac{\partial}{\partial y^N}$ into the commutator (A.5) leads to the differential condition on $e_A^M(y)$:

$$-2e_{[A}^N e_{B]}^M \partial_N e_M^C = C^C{}_{AB} \quad (\text{A.11})$$

Note 2: computing the exterior derivative of σ^A , using eq.s (A.8) and (A.11) leads to the equations

$$d\sigma^A + \frac{1}{2} C^A{}_{BC} \sigma^B \wedge \sigma^C = 0 \quad (\text{A.12})$$

These are called *Maurer-Cartan (MC) equations*, and provide a dual formulation of Lie algebras in terms of the one-forms σ^A . It is immediate to verify that the closure of the exterior derivative ($d^2 = 0$) is equivalent to the Jacobi identities for the structure constants:

$$C^A{}_{B[C} C^B{}_{DE]} = 0 \quad (\text{A.13})$$

(apply d to eq. (A.12)).

Note 3:

Defining $\sigma(y) \equiv \sigma^A(y) T_A$ the MC equations (A.12) take the form

$$d\sigma + \sigma \wedge \sigma = 0 \quad (\text{A.14})$$

The Lie-valued one-form $\sigma(y)$ can also be constructed directly from the group element y :

$$\sigma(y) = y^{-1}dy \quad (\text{A.15})$$

It is easy to verify that (A.15) satisfies the MC equation (A.14) (use $dy^{-1} = -y^{-1}dy y^{-1}$). Moreover, it takes the same value as $e_M^A dy^M T_A$ at the origin $y = 0$. Indeed from the definition of e_A^M in eq. (A.3) one sees that $e_A^M(y = 0) = \delta_A^M$, and therefore $e_M^A(0)dy^M T_A = dy^A T_A$. This value coincides with $y^{-1}dy|_{y=0}$ since $y^{-1}|_{y=0} = [\text{group unit}]$, and $dy|_{y=0} = dy^A T_A$ (from (A.2)). This observation suffices to conclude that $y^{-1}dy$ is equal to $e_M^A(y)dy^M T_A$.

Soft group manifold

Consider a smooth deformation \tilde{G} of the group manifold G . Its vielbein field is given by the intrinsic cotangent basis, defined for any differentiable manifold:

$$\mu^A(y) = \mu_M^A(y)dy^M \quad (\text{A.16})$$

where we use the symbol μ for the soft vielbein. In general μ^A does not satisfy the MC equations any more, so that

$$d\mu^A + \frac{1}{2}C^A_{BC}\mu^B \wedge \mu^C \equiv R^A \neq 0 \quad (\text{A.17})$$

The extent of the deformation $G \rightarrow \tilde{G}$ is measured by the curvature two-form R^A . $R^A = 0$ implies $\mu^A = \sigma^A$ and viceversa.

Applying the external derivative d to the definition (A.17), using $d^2 = 0$ and the Jacobi identities on C^A_{BC} , yields the Bianchi identities

$$(\nabla R)^A \equiv dR^A - C^A_{BC}R^B \wedge \mu^C = 0 \quad (\text{A.18})$$

Diffeomorphisms and Lie derivative

First we discuss the variation under diffeomorphisms of the vielbein field $\mu^A(y)$:

$$\begin{aligned} \mu^A(y + \delta y) - \mu^A(y) &= \delta[\mu_M^A(y)dy^M] = \\ &= (\partial_N \mu_M^A) \delta y^N dy^M + \mu_M^A (\partial_N \delta y^M) dy^N = \\ &= dy^N [\partial_N \delta y^A + \delta y^M (\partial_M \mu_N^A - \partial_N \mu_M^A)] = \\ &= d\delta y^A - 2\mu^B \delta y^C (d\mu^A)_{BC} = d(\iota_{\delta y} \mu^A) + \iota_{\delta y} d\mu^A \end{aligned} \quad (\text{A.19})$$

where

$$\delta y^A \equiv \delta y^M \mu_M^A, \quad \delta y \equiv \delta y^M \partial_M, \quad d\mu^A \equiv (d\mu^A)_{BC} \mu^B \wedge \mu^C, \quad (\text{A.20})$$

and the contraction ι_t along a tangent vector t is defined on p -forms

$$\omega_{(p)} = \omega_{B_1 \dots B_p} \mu^{B_1} \wedge \dots \wedge \mu^{B_p} \quad (\text{A.21})$$

as

$$\iota_t \omega_{(p)} = p t^A \omega_{AB_2 \dots B_p} \mu^{B_2} \wedge \dots \wedge \mu^{B_p} \quad (\text{A.22})$$

Note that ι_t maps p -forms into $(p-1)$ -forms. The operator

$$\ell_t \equiv d \iota_t + \iota_t d \quad (\text{A.23})$$

is called the *Lie derivative* along the tangent vector t and maps p -forms into p -forms. As shown in eq. (A.19), the Lie derivative of the one-form μ^A along δy gives its variation under the diffeomorphism $y \rightarrow y + \delta y$. This holds true for any p -form.

We now rewrite the variation $\delta \mu^A$ of eq. (A.19) in a suggestive way, by adding and subtracting $C^A{}_{BC} \mu^B \delta y^C$:

$$\delta \mu^A = d \delta y^A + C^A{}_{BC} \mu^B \delta y^C - 2 \mu^B \delta y^C (d \mu^A)_{BC} - C^A{}_{BC} \mu^B \delta y^C \quad (\text{A.24})$$

$$= (\nabla \delta y)^A + \iota_{\delta y} R^A \quad (\text{A.25})$$

$$(\text{A.26})$$

where we have used the definition (A.17) for the curvature, and the G -covariant derivative ∇ acts on δy^A as

$$(\nabla \delta y)^A \equiv d \mu^A + C^A{}_{BC} \mu^B \delta y^C \quad (\text{A.27})$$

The algebra of Lie derivatives

The algebra of diffeomorphisms is given by the commutators of Lie derivatives:

$$\left[\ell_{\varepsilon_1^A t_A}, \ell_{\varepsilon_2^B t_B} \right] = \ell_{\varepsilon_3^C t_C} \quad (\text{A.28})$$

with

$$\varepsilon_3^C = \varepsilon_1^A \partial_A \varepsilon_2^C - \varepsilon_2^A \partial_A \varepsilon_1^C - 2 \varepsilon_1^A \varepsilon_2^B \mathcal{R}_{AB}^C \quad (\text{A.29})$$

and

$$\mathcal{R}_{AB}^C \equiv R_{AB}^C - \frac{1}{2} C_{AB}^C \quad (\text{A.30})$$

The components R_{BC}^A are defined by $R^A = R_{BC}^A \mu^B \wedge \mu^C$. The closure of the algebra requires the Bianchi identities (A.18), that we can rewrite in the form

$$\partial_{[B} \mathcal{R}_{CD]}^A + 2 \mathcal{R}_{E[B}^A \mathcal{R}_{CD]}^E = 0 \quad (\text{A.31})$$

To prove (A.28) just apply both sides of the equation to the soft vielbein μ^A .

B Integration on supermanifolds: integral forms

In this Appendix, taken from Section 4 of [12], we recall basic results in supermanifold integration (see for ex. [40] for a recent review, or [41] for a textbook), and new developments concerning integral forms, discussed in ref.s [19]-[21].

We have defined the supergravity action (2.18) as an integral of a top form on the superPoincaré group manifold. We have given explicitly only the 4-form Lagrangian, postponing the precise expression of η_{M^4} to the present Appendix. In fact in the supergravity case we have tacitly assumed typical properties of bosonic integration, as for ex. the existence of a top form and Stokes' theorem. Here we want to justify these assumptions, and give a short account of superintegration theory.

The construction of actions invariant under diffeomorphisms is solved *ab initio* in ordinary integration theory by *form* integration. The integral of a d -form

$$\omega^{(d)} = \omega_{[\mu_1 \dots \mu_d]}(x) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_d} \quad (\text{B.1})$$

on a d -dimensional manifold M^d is defined by

$$I = \int_{M^d} \omega^{(d)} \equiv \int_{M^d} \frac{1}{d!} \omega_{[\mu_1 \dots \mu_d]}(x) \epsilon^{\mu_1 \dots \mu_d} d^d x \quad (\text{B.2})$$

i.e. by usual (Riemann-Lebesgue) integration on M^d of the function $\frac{1}{d!} \omega_{[\mu_1 \dots \mu_d]}(x) \epsilon^{\mu_1 \dots \mu_d}$, where $\epsilon^{\mu_1 \dots \mu_d}$ is the Levi-Civita antisymmetric symbol in the coordinate basis, a tensor density of weight -1 . Therefore

$$\epsilon^{\mu_1 \dots \mu_d} d^d x = \epsilon^{\mu_1 \dots \mu_d} dx^1 \wedge \dots \wedge dx^d \quad (\text{B.3})$$

is a tensor, and the integrand of (B.2) is a scalar.

As in the previous Sections, we can consider infinitesimal diffeomorphisms as active transformations, generated by the Lie derivative $\ell_\epsilon = \iota_\epsilon d + d\iota_\epsilon$. Then the form integral (B.2) transforms as

$$\delta I = \int_{M^d} \ell_\epsilon \omega^{(d)} = \int_{M^d} (\iota_\epsilon d + d\iota_\epsilon) \omega^{(d)} = 0 \quad (\text{B.4})$$

since $d\omega^{(d)} = 0$ ($\omega^{(d)}$ is a top form) and $\int_{M^d} d(\iota_\epsilon \omega) = 0$ for appropriate boundary conditions. Thus we have checked invariance of the form integral under infinitesimal diff.s generated by the Lie derivative. Note that the existence of a top form, namely the fact that a d -form is closed on M^d , is crucial to ensure action invariance under diff.s.

Can we generalize form integration to *supermanifolds*, and use it to construct actions automatically invariant under superdiffeomorphisms? The answer to both questions is affirmative.

In analogy with the bosonic case, integration on forms living on supermanifolds is defined via integration of functions in superspace. Consider a function $\Phi(x, \theta)$,

defined on a supermanifold $M^{d|m}$ with d bosonic coordinates x and m fermionic (anticommuting) coordinates θ^α . It is called a *superfield*, and can be expanded in the θ^α coordinates:

$$\Phi(x, \theta) = \phi(x) + \phi_{\alpha_1}(x)\theta^{\alpha_1} + \phi_{\alpha_1\alpha_2}(x)\theta^{\alpha_1}\theta^{\alpha_2} + \cdots + \phi_{\alpha_1\cdots\alpha_m}(x)\theta^{\alpha_1}\cdots\theta^{\alpha_m} \quad (\text{B.5})$$

The functions $\phi_{\alpha_1\cdots\alpha_p}(x)$ are called *superfield components*, and have antisymmetrized indices due to the anticommuting θ 's in the expansion (B.5). The integral of the superfield on $M^{d|m}$ is defined by Berezin integration:

$$\int_{M^{d|m}} \Phi(x, \theta) d^d x d^m \theta \equiv \int_{M^d} \frac{1}{m!} \phi_{\alpha_1\cdots\alpha_m}(x) \varepsilon^{\alpha_1\cdots\alpha_m} d^d x \quad (\text{B.6})$$

Only the highest component of Φ (corresponding to the maximal number of θ 's) enters the integral on M^d .

Note the striking similarity between the two integrals (B.2) and (B.6). In fact we can define form integration in terms of Berezin integration. Consider the differentials dx in the d -form (B.1) as *anticommuting coordinates* $\xi^\mu = dx^\mu$, so that $\omega^{(d)}$ becomes a *function* of x and ξ :

$$\omega^{(d)}(x, \xi) = \omega_{[\mu_1\cdots\mu_d]}(x) \xi^{\mu_1} \cdots \xi^{\mu_d} \quad (\text{B.7})$$

Then its Berezin integral on $M^{d|d}$ exactly yields the form integral (B.2). This observation is the key for a definition of superform integration on supermanifolds.

A natural generalization of a bosonic top form (B.1) is a $(d+m)$ -superform:

$$\omega^{(d+m)}(x, \theta) = \omega_{[\mu_1\cdots\mu_d]\{\alpha_1\cdots\alpha_m\}}(x, \theta) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_d} \wedge d\theta^{\alpha_1} \wedge \cdots \wedge d\theta^{\alpha_m} \quad (\text{B.8})$$

Note that the $d\theta$ differentials *commute* (since the θ 's are anticommuting), so that the indices α_i are symmetrized. For this reason $\omega^{(d+m)}$ cannot be a top form: a superform can have an arbitrary number of $d\theta$ differentials, and its exterior derivative does not vanish. Let's ignore for the moment this difficulty, and try to define a superform integral. Inspired by the observation in the preceding paragraph, we consider the superform $\omega^{(d+m)}(x, \theta)$ as a function of $x, \theta, dx, d\theta$, i.e. a function of the commuting variables $x, d\theta$ and anticommuting variables θ, dx . Its integral can be defined by Berezin integration on θ, dx , and usual Riemann-Lebesgue integration on $x, d\theta$. Here a second difficulty arises: the ordinary integration on the $u = d\theta$ coordinates produces integrals of the type

$$\int u^m d^m u \quad (\text{B.9})$$

and there is no algorithmic way to assign a C -number to it. For the integral on the even variables $u = d\theta$ to make sense, the integrand must have compact support as a function of u . For this reason we consider functions of the $d\theta$'s which are *distributions* in $d\theta$ with support at the origin:

$$\omega(x, \theta, dx, d\theta) = \omega_{[\mu_1\cdots\mu_d]}(x, \theta) dx^{\mu_1} \cdots dx^{\mu_d} \delta(d\theta^1) \cdots \delta(d\theta^m) \quad (\text{B.10})$$

These “functions” can be integrated on the supermanifold $M^{d+m|d+m}$ spanned by the $d + m$ bosonic variables $x, d\theta$ and $d + m$ fermionic variables dx, θ . The integral

$$\int_{M^{d+m|d+m}} \omega(x, \theta, dx, d\theta) d^d x d^m \theta d^d(dx) d^m(d\theta) \quad (\text{B.11})$$

is defined by Berezin integration on the odd variables dx, θ and usual Riemann-Lebesgue integration on the even variables $x, d\theta$. Carrying out integration on the variables dx and $d\theta$ the integral becomes

$$\int_{M^{d|m}} \omega_{[\mu_1 \dots \mu_d]}(x, \theta) \epsilon^{\mu_1 \dots \mu_d} d^d x d^m \theta \quad (\text{B.12})$$

This integral can also be seen as an integral of the *form*:

$$\omega^{d|m} = \omega_{[\mu_1 \dots \mu_d]}(x, \theta) \delta(u^1) \dots \delta(u^m) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_d} \wedge du^1 \wedge \dots \wedge du^m \quad (\text{B.13})$$

where the even variables u are the differentials $d\theta$. Indeed, let us integrate this form with the recipe of considering it a function of x, θ, u and of the differentials dx, du , and then using Berezin and Riemann integration according to the odd or even grading of the variables. The result coincides with (B.12).

Thus the form $\omega^{d|m}$ can be integrated, even if it contains $d\theta$ differentials. We achieve this by confining the $d\theta$'s inside delta functions, and in this way overcome the first difficulty encountered with the superforms (B.8). But can $\omega^{d|m}$ overcome also the second difficulty, and be a *top form*? The answer is yes: the dx and du differentials are all anticommuting, so that their number in $\omega^{d|m}$ is already maximal, and multiplying it by $d\theta$ differentials gives zero because of the presence of the deltas. Therefore $d\omega^{d|m} = 0$, and $\omega^{d|m}$ is a *bona fide* top form. Since it can be integrated and it is a top form, $\omega^{d|m}$ is called an *integral top form*.

Finally, using the notation

$$\delta(u^1) \dots \delta(u^m) du^1 \wedge \dots \wedge du^m \equiv \delta(u^1) \wedge \dots \wedge \delta(u^m) \quad (\text{B.14})$$

the integral top form can be rewritten (using $u = d\theta$):

$$\omega^{d|m} = \omega_{[\mu_1 \dots \mu_d]}(x, \theta) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_d} \wedge \delta(d\theta^1) \wedge \dots \wedge \delta(d\theta^m) \quad (\text{B.15})$$

or also

$$\omega^{d|m} = \omega_{[\mu_1 \dots \mu_d][\alpha_1 \dots \alpha_m]}(x, \theta) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_d} \wedge \delta(d\theta^{\alpha_1}) \wedge \dots \wedge \delta(d\theta^{\alpha_m}) \quad (\text{B.16})$$

where indices α are antisymmetrized since the $\delta(d\theta^\alpha)$ anticommute, and

$$m! \omega_{[\mu_1 \dots \mu_d]} \equiv \omega_{[\mu_1 \dots \mu_d][1 \dots m]} \quad (\text{B.17})$$

In this notation μ and α indices play a similar role, and are both antisymmetrized. The numbers d, m are respectively called the *form number* and the *picture number*,

and for integral top forms they coincide with the numbers of bosonic and fermionic dimensions of the supermanifold $M^{d|m}$.

We call “superforms” the forms of the kind (B.8), with dx and $d\theta$ differentials, without $\delta(d\theta)$'s. Thus superforms have a form number that counts the $dx, d\theta$ differentials, and zero picture number. For example the Lagrangian in (2.18) is a superform $L^{4|0}$.

Integration on submanifolds of supermanifolds

Supergravity actions on supergroup manifolds \tilde{G} are given by integrals of a d -form Lagrangian L on a d -dimensional (bosonic) submanifold M^d of \tilde{G} . They can be written as integrals on the whole \tilde{G} of the Lagrangian multiplied by an appropriate Poincaré dual η_{M^d} of M^d , such that $L \wedge \eta_{M^d}$ becomes an integral top form. Let us see how this works for $N = 1, d = 4$ supergravity.

The supergravity Lagrangian in (2.18) is a $(4|0)$ superform. For simplicity we now assume that fields satisfy the Lorentz horizontality constraints on all the curvatures, and thus effectively depend only on the superspace coordinates x^μ, θ^α , with $\mu = 1, \dots, 4, \alpha = 1, \dots, 4$. Then \tilde{G} is $M^{4|4}$ superspace, and only integral top forms of type $(4|4)$ can be integrated on $M^{4|4}$. We therefore need a Poincaré dual of type $(0|4)$, so that

$$L^{4|0} \wedge \eta_{M^4}^{0|4} \quad (\text{B.18})$$

is an integral top form, i.e. of type $(4|4)$. For this purpose we choose:

$$\eta_{M^4}^{0|4} = \varepsilon_{\alpha\beta\gamma\delta} \theta^\alpha \theta^\beta \theta^\gamma \theta^\delta \varepsilon_{\alpha'\beta'\gamma'\delta'} \delta(d\theta^{\alpha'}) \wedge \delta(d\theta^{\beta'}) \wedge \delta(d\theta^{\gamma'}) \wedge \delta(d\theta^{\delta'}) \quad (\text{B.19})$$

so that

$$\int_{M^{4|4}} L^{4|0} \wedge \eta_{M^4}^{0|4} = \int_{M^4} L^{4|0}(\theta = 0, d\theta = 0) \quad (\text{B.20})$$

and we obtain a spacetime action, where all fields depend only on x -coordinates (the terms containing θ 's are annihilated by the presence of the 4 θ 's in η) and have no “legs” $d\theta$ because of the $\delta(d\theta)$ in η . Note that $\eta_{M^4}^{0|4}$ is closed, and the explicit θ 's prevent it to be exact.

Since multiplying by the Poincaré dual changes the picture number of the resulting form, η is also called Picture Changing Operator (PCO), a name borrowed from string theory and string field theory.

The Poincaré dual is by no means unique: we can orient the M^4 surface inside \tilde{G} in many different ways. For example consider the PCO obtained by acting on η with an infinitesimal diffeomorphism in the θ directions:

$$\eta' = \eta + \ell_\epsilon \eta = \eta + d(\iota_\epsilon \eta) \quad (\text{B.21})$$

This is still a PCO, being closed and not exact⁹, and dual to a submanifold diffeomorphic to the original M^4 . Note also that the change in η is exact.

⁹because η is closed and not exact, and d commutes with ℓ_ϵ .

C Gamma matrices in $d = 3 + 1$

$$\eta_{ab} = (1, -1, -1, -1), \quad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \quad [\gamma_a, \gamma_b] = 2\gamma_{ab}, \quad (\text{C.1})$$

$$\gamma_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_5\gamma_5 = 1, \quad \varepsilon_{0123} = -\varepsilon^{0123} = 1, \quad (\text{C.2})$$

$$\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0, \quad \gamma_5^\dagger = \gamma_5 \quad (\text{C.3})$$

$$\gamma_a^T = -C\gamma_aC^{-1}, \quad \gamma_5^T = C\gamma_5C^{-1}, \quad C^2 = -1, \quad C^T = -C \quad (\text{C.4})$$

C.1 Useful identities

$$\gamma_a\gamma_b = \gamma_{ab} + \eta_{ab} \quad (\text{C.5})$$

$$\gamma_{ab}\gamma_5 = -\frac{i}{2}\varepsilon_{abcd}\gamma^{cd} \quad (\text{C.6})$$

$$\gamma_{ab}\gamma_c = \eta_{bc}\gamma_a - \eta_{ac}\gamma_b + i\varepsilon_{abcd}\gamma_5\gamma^d \quad (\text{C.7})$$

$$\gamma_c\gamma_{ab} = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a + i\varepsilon_{abcd}\gamma_5\gamma^d \quad (\text{C.8})$$

$$\gamma_a\gamma_b\gamma_c = \eta_{ab}\gamma_c + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b + i\varepsilon_{abcd}\gamma_5\gamma^d \quad (\text{C.9})$$

$$\gamma^{ab}\gamma_{cd} = i\varepsilon_{cd}^{ab}\gamma_5 - 4\delta_{[c}^{[a}\gamma_{d]}^{b]} - 2\delta_{cd}^{ab} \quad (\text{C.10})$$

C.2 Charge conjugation and Majorana condition

$$\text{Dirac conjugate } \bar{\psi} \equiv \psi^\dagger\gamma_0 \quad (\text{C.11})$$

$$\text{Charge conjugate spinor } \psi^c = C(\bar{\psi})^T \quad (\text{C.12})$$

$$\text{Majorana spinor } \psi^c = \psi \Rightarrow \bar{\psi} = \psi^T C \quad (\text{C.13})$$

C.3 Fierz identity for two spinor one-forms

$$\psi\bar{\chi} = \frac{1}{4}[(\bar{\chi}\psi)1 + (\bar{\chi}\gamma_5\psi)\gamma_5 + (\bar{\chi}\gamma^a\psi)\gamma_a + (\bar{\chi}\gamma^a\gamma_5\psi)\gamma_a\gamma_5 - \frac{1}{2}(\bar{\chi}\gamma^{ab}\psi)\gamma_{ab}] \quad (\text{C.14})$$

C.4 Fierz identity for two Majorana spinor one-forms

$$\psi\bar{\psi} = \frac{1}{4}[(\bar{\psi}\gamma^a\psi)\gamma_a - \frac{1}{2}(\bar{\psi}\gamma^{ab}\psi)\gamma_{ab}] \quad (\text{C.15})$$

As a consequence

$$\gamma_a\psi\bar{\psi}\gamma^a\psi = 0, \quad \psi\bar{\psi}\gamma^a\psi - \gamma_b\psi\bar{\psi}\gamma^{ab}\psi = 0 \quad (\text{C.16})$$

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