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Capital Allocation Rules and Generalized Collapse to the Mean: Theory and Practice

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Abstract: In this paper, we focus on capital allocation methods based on marginal contributions. In particular, concerning the relation between linear capital allocation rules and the well-known Gradient (or Euler) allocation, we investigate an extension to the convex and non-differentiable case of the result above and its link with the "generalized collapse to the mean" problem. This preliminary result goes in the direction of applying the popular marginal contribution method, which fosters the diversification of risk, to the case of more general risk measures. In this context, we will also discuss and point out some numerical issues linked to marginal methods and some future research directions.

Keywords: risk management; capital allocation; risk measures; actuarial sciences; Euler method

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1. Introduction

In the financial and actuarial literature, risk measures have been introduced and widely studied so as to answer to the need of quantifying the riskiness of financial positions and, consequently, evaluating the adequate capital requirement (or margin) to be deposited to hedge risky exposures (see [1–5] for an axiomatic approach). In this framework, the capital allocation problem emerges whenever a financial risky position consists of different sub-positions (called business lines or sub-units). Roughly speaking, it consists in finding suitable criteria to share the margin of an aggregate position among its business sub-units in such a way that the chosen division fulfills some desirable properties. To illustrate the idea in a more precise way, given a suitable risk measure ρ and a risky position (whose profit and loss is represented by a random variable Y), the issue—better formalized afterwards—typically consists in deciding what portion $\Lambda(X, Y)$ of the margin $\rho(Y)$ to allocate to each sub-unit X of Y in a way that respects some financially sound properties. This kind of problem is typically faced by risk managers and practitioners and has given rise to an extensive literature, as allocating capital is used for performance measurements and risk–reward purposes.

The crucial elements in a capital allocation problem are the choices of the risk measure and of the capital allocation method, respectively, from which also the implementability and calculation of the capital assigned to the each business line depends. An axiomatic approach was introduced by Denault [6] and Kalkbrener [7]. In [6], also the connection with cooperative game theory was developed. For an overview on the theoretical and applied aspects of the topic, we direct the reader to [6–15], among many others. In particular, numerical methods of capital allocations of the most popular risk measures (e.g., VaR, CVaR, Entropic risk measure) are performed and discussed, e.g., in [8,9,12–14,16,17]. Implementation as well as a sensitivity analysis are of key importance in order to verify which of the most desired properties are fulfilled in practice (see [10,12]).

A popular capital allocation rule (CAR from now on) for coherent risk measures, both from a theoretical point of view and for the several financial and empirical applications, is the so-called Gradient (or Euler) allocation, which attributes to each sub-unit X its "marginal contribution" to the riskiness of the aggregate position Y, that is,

$$\Lambda^G(X,Y) \triangleq D^\rho(Y;X) \triangleq \lim_{\varepsilon \to 0} \frac{\rho(Y+\varepsilon X) - \rho(Y)}{\varepsilon}. \tag{1}$$

Moreover, it is related to the discrete marginal contribution approach which, given a risky position X and n-risky sub-positions $X_1, \ldots, X_n$, attributes the quantity $K_i = \rho(X) - \rho(X - X_i)$ to each. It has a sound meaning and is easily implementable (for a numerical application, see [10,12]). This marginal approach is furthermore somehow also in line with the classical Shapley value in cooperative game theory which assigns to each player their averaged marginal contribution with respect to all possible coalitions.

The Euler method is also very well known for its applicability to the case of coherent and differentiable risk measures where, in particular, the positive homogeneity of the risk measure implies the validity of Euler's Theorem and guarantees that all the risk capital is fully allocated among the sub-units (see again [14]). This allocation, as emphasized by [14,18], is crucial for performance measurements purposes, as it turns out to be the only return on risk adjusted capital (RORAC)-compatible capital allocation rule. Also in this simple case, there are several statistical and numerical issues linked to the estimations of the losses distributions (e.g., through Monte Carlo simulations) and to techniques to reduce the estimates' volatility (for example through non-parametric methods such as kernel smoothing). See [8,12,14,17]. For example, the Euler capital allocation rule for Value at Risk estimated through classical non-parametric methods converges more slowly than the standard rate, and in [8] some adjustments on the definition of this capital allocation rule were proposed to fix the mentioned convergence problem. In the same context, in [19], a statistical inference theory based on a consistent empirical estimator was developed. Moreover, as for the problem of reduction of volatility of the estimates, the paper of Peters et al. [20] introduced a new class of efficient Monte Carlo methods to calculate capital allocations for some specific coherent risk measures, based on a combination of Markov Chain Monte Carlo and sequential Monte Carlo. For a detailed and updated review of numerical methods for coherent capital allocation principles, we direct the interested reader to Table 1 in [13].

It is interesting to underline that, going back to the general formulation in (1) corresponding to the "continuous case", the Euler allocation "collapses to a mean" or, better, to an expected value. Under the suitable conditions Theorem 4.3 in Kalkbrener [7], indeed, it takes the form

$$\Lambda^G(X,Y) = E_{Q_Y}[-X] \text{ for any } X, \tag{2}$$

for a *unique* probability measure Q_Y, and it turns out to be the unique CAR, also satisfying at the same time the appealing properties of *linearity* and *no-undercut*.

We will refer to the specific formulation in (2), which turns out to hold, for instance, for the Expected Shortfall as a "generalized" collapse to the mean with respect to a particular probability measure. Indeed, we use the term generalized to underline the difference with the classical collapse to the mean that, in the literature, usually refers to law invariant functionals that reduce to the expectation with respect to the reference probability measure (see [21–24]). Differently from the classical collapse to the mean, whose criticalities are well described by Liebrich and Munari [24], in the case of CARs the generalized collapse to the mean appears to be more consistent with pricing theory also because of its dependence on Q_Y, which highlights a parallelism with martingale pricing measures and bid–ask spreads.

For all the reasons discussed above, conditions under which a CAR reduces to the previous form appear worth investigating. However, relying on the existence of D^ρ in order to have a CAR which exhibits a generalized collapse to the mean and the relative desirable properties is quite a strong assumption. In general, as pointed out by Grechuk [25], the directional derivative at Y (hence the uniqueness of such Q_Y) needs not exist for every direction X. Furthermore, as shown by Cherny and Orlov [26], this non-existence may occur in some important cases (e.g., if the risk measure is the Conditional Value at Risk, and Y has a discrete distribution). It is therefore natural to investigate what happens when we depart from the gradient allocation and we drop the differentiability assumption on the underlying risk measure. Indeed, in general, when dealing with a linear CAR with respect to a coherent risk measure and satisfying no-undercut, dropping differentiability at some Y results in a collapse to the mean with respect to the (non-unique) elements of the subdifferential $\partial\rho(Y)$ (if non-empty) (see the discussion in [25]).

The fundamental goal of this paper is thus to investigate what happens when we depart from Kalkbrener's result [7] to account for the case of non-differentiability but, at the same time, also to go beyond the coherent case. Indeed, the relevance of convex risk measures in the literature, as well as in the framework of capital allocation problems, is well known (see [15,27]) and deserves a deeper study. The extension of Kalkbrener's analysis to the convex and non-differentiable case comes anyway at the cost of imposing different properties on the CAR, thus losing, in general, the representation as a collapse to the mean which is substituted by linear bounds, except in particular cases. This calls for the necessity of suitable numerical techniques because of the failure of Euler's Theorem and of the unicity of the optimizing probability measure.

The paper is organized as follows: in Section 2, we recall the basic definitions and results on risk measures and capital allocation rules, and the result of Kalkbrener [7], which we extend. We also briefly discuss some numerical aspects and issues in the literature. Section 3 contains our main result on the generalized collapse to the mean. A comparison of our findings with some existing results in the literature—with a special focus on those of Kalkbrener—is also included to better clarify our contribution. In Section 4, we draw our conclusions, discuss the critical aspects of the work, and provide some future research directions which are currently under investigation.

2. Brief Review on Risk Measures and Capital Allocation

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $L^\infty(\Omega, \mathcal{F}, P)$ (shortly, L^∞) be the space of all essentially bounded random variables on $(\Omega, \mathcal{F}, P)$, endowed with the weak topology $\sigma(L^\infty, L^1)$. We assume that L^∞ represents the space of all risky positions' profits and losses. Equalities and inequalities must be understood to hold P-almost surely (a.s.).

Let $\mathcal{P}$ denote the set of all probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to P. Any probability measure $Q \in \mathcal{P}$ will be identified with its Radon–Nikodym density $\frac{dQ}{dP} \in L^1$.

2.1. Risk Measures

Risk measures were introduced in the seminal paper of Artzner et al. [1] in order to quantify the riskiness and set a capital requirement (margin) to hedge financial positions. Following their axiomatic approach, a functional $\rho : L^\infty \to \mathbb{R}$ is called *coherent* risk measure if it satisfies the following:

- (Decreasing) monotonicity: if $X \leq Y$, P-a.s., with $X, Y \in L^\infty$, then $\rho(X) \geq \rho(Y)$;
- Positive homogeneity: $\rho(bX) = b\rho(X)$ for any $X \in L^\infty, b \geq 0$;
- Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$ for any $X, Y \in L^\infty$;
- Translation invariance: $\rho(X+c) = \rho(X) - c$ for any $X \in L^\infty, c \in \mathbb{R}$.

As in the works of Föllmer and Schied [4] and Frittelli and Rosazza Gianin [23], ρ is called *convex* risk measure if it fulfills monotonicity, translation invariance, $\rho(0) = 0$, and the following:

- Convexity: $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$ for any $\alpha \in [0,1], X, Y \in L^\infty$.

Furthermore, a risk measure ρ is said to be continuous from below (respectively, continuous from above) if for any sequence $(X_n)_{n\geq 0}$ such that $X_n \nearrow_n X$, P-a.s. (resp. $X_n \searrow_n X$, P-a.s.), it holds that $\lim_{n\to+\infty} \rho(X_n) = \rho(X)$. It is well known (see [28]) that, for convex risk measures, continuity from above is equivalent to the following:

- Lower semi-continuity: $\{X \in L^\infty : \rho(X) \leq c\}$ is $\sigma(L^\infty, L^1)$-closed for any $c \in \mathbb{R}$.

We now recall the dual representation of coherent and convex risk measures.

- Any coherent risk measure $\rho : L^\infty \to \mathbb{R}$ satisfying continuity from below can be represented as follows:

$$\rho(X) = \max_{Q\in\mathcal{Q}} E_Q[-X], \quad \text{for any } X \in L^\infty, \tag{3}$$

for a suitable $\sigma(L^\infty, L^1)$-closed convex $\mathcal{Q} \subseteq \mathcal{P}$. See [3,28].

- Any convex risk measure $\rho : L^\infty \to \mathbb{R}$ satisfying continuity from below can be represented as

$$\rho(X) = \max_{Q\in\mathcal{P}} \{E_Q[-X] - c(Q)\}, \quad \text{for any } X \in L^\infty, \tag{4}$$

for some lower semi-continuous and convex penalty functional $c : L^1 \to [0; +\infty]$ with $\inf_{Q\in\mathcal{P}} c(Q) = 0$ where, with an abuse of notation, $c(Q)$ will stand for $c\left(\frac{dQ}{dP}\right)$. See [4,5,28].

It is well known (see [3,28]) that, under continuity from above ρ, the dual representations in (3) and (4) hold with a supremum instead of a maximum.

It is worth noticing that the dual representations in (3) and (4) underline the strong relationship between risk measures and option pricing theory in arbitrage-free incomplete financial markets. This connection is through the concepts of super-replication price and of dynamic certainty equivalent and when $\mathcal{P}$ consists in the set of martingale measures. See [5,29], and the references therein for a detailed discussion.

2.2. Directional Derivatives and Gateaux Differentiability

We recall now that, given a functional ρ (in our case, a risk measure), the subdifferential of ρ at $Y \in L^\infty$ is defined (with an abuse of language) as

$$\partial\rho(Y) \triangleq \{Q \in \mathcal{P} : \rho(X) - \rho(Y) \geq E_Q[-(X-Y)] \quad \text{for any } X \in L^\infty\}$$

or, equivalently, as

$$\partial\rho(Y) \triangleq \left\{\xi \in L^1 : \rho(X) - \rho(Y) \geq E_P[-\xi(X-Y)] \quad \text{for any } X \in L^\infty\right\},$$

while the directional derivative of ρ at $Y \in L^\infty$ in the direction $X \in L^\infty$ is given by

$$D^\rho(Y; X) \triangleq \lim_{\varepsilon\to 0} \frac{\rho(Y+\varepsilon X) - \rho(Y)}{\varepsilon}.$$

See [30–32] for more details. Moreover, in the following, we also consider the left and right directional derivatives, respectively defined as

$$D^{\rho}_{-}(Y;X) \triangleq \lim_{\varepsilon \uparrow 0} \frac{\rho(Y+\varepsilon X)-\rho(Y)}{\varepsilon}$$
$$D^{\rho}_{+}(Y;X) \triangleq \lim_{\varepsilon \downarrow 0} \frac{\rho(Y+\varepsilon X)-\rho(Y)}{\varepsilon}.$$

The existence of the one-sided derivative $D^{\rho}_{+}(Y;X)$ (and hence of $D^{\rho}_{-}(Y;X)$), for every $X,Y \in L^{\infty}$ is guaranteed when ρ is convex (see [30]). Instead, the directional derivative $D^{\rho}(Y;X)$ may not exist even for convex functions. On the implication of this fact, on capital allocation problems, we refer to the works of Cherny and Orlov [26] and of Grechuck [25].

We also recall (see [33,34]) that, if for every $Y \in L^{\infty}$, $D^{\rho}_{+}(Y;\cdot)$ is linear and $\sigma(L^{\infty},L^{1})$-continuous, i.e., there exists $\xi \in L^{1}$ such that $D^{\rho}_{+}(Y;X)) = E_P[-X\xi]$ holds for every $X \in L^{\infty}$, ρ is said to be Gateaux differentiable at Y with Gateaux derivative $\xi \triangleq \nabla\rho(Y)$. In this case, $\nabla\rho(Y)$ is unique and $\partial\rho(Y) = \{\nabla\rho(Y)\}$.

Furthermore, for convex risk measures $\rho : L^{\infty} \to \mathbb{R}$ as in (4) and $Y \in L^{\infty}$, it holds that the subdifferential $\partial\rho(Y)$ has the following representation (see [32], Theorem 2.4.2 and [34], Equation (3.2)):

$$\partial\rho(Y) = \text{argmax}_{Q\in\mathcal{P}}\{E_Q[-Y] - c(Q)\} \tag{5}$$

where, for coherent risk measures, the penalty $c(Q) \in \{0;+\infty\}$, with $c(Q) = 0$ on a subset $\mathcal{Q} \subseteq \mathcal{P}$. Also, under some continuity on ρ, the Gateaux differentiability of ρ at Y is equivalent to the fact that $\partial\rho(Y)$ is a singleton $\{Q^*_Y\}$ (see [32], Corollary 2.4.10), where Q^*_Y can be identified with $\nabla\rho(Y)$. Thus, the Gateaux-differentiable case yields a unique maximizer $\{Q^*_Y\}$ for every $Y \in L^{\infty}$.

2.3. Capital Allocation Rules

To answer the question about how to share the capital requirement for an aggregate risky position among its different sub-units, the first step is to recall the classical definition of capital allocation rule introduced through the axiomatic approach of Kalkbrener [7].

Definition 1. Given a risk measure $\rho : L^{\infty} \to \mathbb{R}$, a *capital allocation rule (CAR)* is a map $\Lambda : L^{\infty} \times L^{\infty} \longrightarrow \mathbb{R}$ such that $\Lambda(X,X) = \rho(X)$ for every $X \in L^{\infty}$.

We underline that a CAR Λ is defined for any pair $(X,Y) \in L^{\infty} \times L^{\infty}$, where Y is always interpreted as an aggregate position (or portfolio) and X as a sub-unit (or sub-portfolio). The condition $\Lambda(X,X) = \rho(X)$ means that the capital allocated to X when considered as a stand-alone portfolio is exactly the margin $\rho(X)$.

Kalkbrener [7] defines the following desirable properties for a capital allocation rule Λ associated to a risk measure ρ:

- Linearity: $\Lambda(aX+bY,Z) = a\Lambda(X,Z) + b\Lambda(Y,Z)$ for every $a,b \in \mathbb{R}, X,Y,Z \in L^{\infty}$.
- No-undercut: $\Lambda(X,Y) \leq \rho(X)$ for every $X,Y \in L^{\infty}$.

Linearity in the first argument is often required and satisfied by the most popular capital allocation rules at least for the coherent case.

Since $\Lambda(X,Y)$ can be interpreted as the capital allocated to the sub-portfolio X to hedge/cover the global portfolio Y, the no-undercut axiom just requires that the capital allocated to X as a sub-unit of Y does not exceed the margin required for X. In the terminology of Tsanakas [15], no-undercut corresponds to the non-split requirement of X from Y, while from a game theory perspective, it corresponds to the core property of Denault [6]. In other words, no-undercut translates the idea that no player has any convenience to leave the coalition since this would imply incurring a greater cost. For further details on the connection with game theory, see [35].

We point out that, in capital allocation problems, the capital requirement can be given exogenously (see [11,36,37]) and not necessarily be related to a pre-specified risk measure. For example, it can be derived through an optimization problem driven by economic motivations such as in [38]. In this case, the capital allocation problem consists then in deciding how to share such an exogenous risk capital according to some reasonable criteria.

We now recall the following result, which we aim at generalizing.

Theorem 1 (Kalkbrener [7], Theorems 4.1 and 4.3)**.** *Let $\rho : L^\infty \to \mathbb{R}$ be a coherent risk measure, and set*

$$\mathcal{H}_\rho \triangleq \{h : L^\infty \to \mathbb{R} : h \text{ is linear, } h(X) \leq \rho(X) \text{ for any } X \in L^\infty\}.$$

Then, it holds $\rho(X) = max_{h \in \mathcal{H}_\rho} h(X)$ for every $X \in L^\infty$.

Furthermore, for any $Y \in L^\infty$ set $\Lambda_\rho(X, Y) = h_Y(X)$ for every $X \in L^\infty$ (where $\rho(Y) = h_Y(Y)$). Then, $\Lambda_\rho(X, Y)$ is a linear capital allocation rule satisfying no-undercut, and the following conditions are equivalent:

(a) Λ_ρ satisfies continuity (C0), that is, $\lim_{\varepsilon \mapsto 0} \Lambda_\rho(X, Y + \varepsilon X) = \Lambda_\rho(X, Y)$ for every $X \in L^\infty$;
(b) There exists the directional derivative $D^\rho(Y; X)$ for any $X \in L^\infty$;
(c) There exists a unique $h_Y \in \mathcal{H}_\rho$ such that $\rho(Y) = h_Y(Y)$.

If any one among these assumptions hold, then $\Lambda_\rho(X, Y) = D^\rho(Y; X)$ for any $X, Y \in L^\infty$.

Continuity (C0) means that small changes in the composition of an aggregate portfolio only slightly affect the allocation to sub-units. In this case, Λ_ρ coincides with the *gradient allocation* (see [6,14]). Moreover, in this case, as $\Lambda_\rho(X, Y) = D^\rho(Y; X)$, the directional derivative is linear (which implies Gateaux differentiability) and, by duality, $D^\rho(Y; \cdot)$ is identified with the unique element of the subdifferential (hence $h_Y(\cdot) = E_{Q_Y^*}[\cdot]$ if the representation in (3) holds).

Therefore, under the hypotheses of the previous result, the gradient allocation exhibits a generalized collapse to the mean with respect to the unique element of the subdifferential. In this case, the gradient allocation turns out to be the unique linear CAR satisfying no-undercut. Moreover, we point out that a linear CAR (with respect to a coherent risk measure, not necessarily differentiable at every Y) and satisfying no-undercut always collapses to the mean with respect to the measures $Q_Y \in \partial\rho(Y)$ (see also [25] for a discussion on the choice among the various Q_Y).

The aim of this paper is then to generalize the above result of Kalkbrener [7] in two directions: first, by dropping coherence and differentiability of the risk measure; second, for general CARs Λ satisfying weaker assumptions. We emphasize that, as shown by Kalkbrener [7], Theorems 4.2 and 4.3, linearity, and no-undercut together are, however, possible only for CARs associated to coherent risk measures. Hence, when studying CARs beyond the coherent case, either linearity or no-undercut can be potentially dropped.

We point out that, in the case of a positive homogeneous and differentiable risk measures, numerical simulations through Monte Carlo sampling techniques can be easily implemented (despite some problems linked to the volatility of the estimates, see [8,12,14,17]). Instead, when we depart from the standard case and drop differentiability, the multiplicity of the elements in the subgradient calls for a new approach, which is currently under our investigation.

Moreover, it is crucial to point out that, by the equivalent conditions in Kalkbrener's Theorem, in the non-differentiable case, allocations can face shocks also for small changes in the composition of the business lines, which is an undesirable feature. This calls for an approach through sensitivity/stress test analysis suitable for discontinuous models. It is also well known that (see [39]), in the case of discontinuous distributions, some non-coherent tail risk measures widely used in the financial practice exhibit large changes with respect to small changes in the confidence level. Hence, suitable sensitivity analysis tools are requested. The very recent paper [40] developed an approach whose possible application to our context is currently under our investigation.

3. Capital Allocation and (Generalized) Collapse to the Mean

Driven by the popularity of convex risk measures, we now aim to generalize Theorem 1 to the convex case but not necessarily to differentiable risk measures. As a byproduct, we also deal with the problem below.

It is worth emphasizing that, although we state our results in L^∞ spaces, they hold also in L^p spaces (unless differently specified) when the risk measure ρ is assumed to be finite (and without any further continuity assumption), as both the dual representation and the characterization of the subdifferential hold as well (see [34,41], and also [42]).

Problem 1 (generalized collapse to the mean)**.** *Finding a set of mild and financially sound conditions under which, given $Y \in L^\infty$ and a CAR Λ not necessarily linear (a priori) or differentiable, there exists at least a $Q_Y \in \mathcal{P}$ such that*

$$\Lambda(X, Y) = E_{Q_Y}[-X] \quad \textit{for any } X \in L^\infty. \tag{6}$$

Note that, if $\Lambda(X, Y) = E_{Q_Y}[-X]$ for some $Q_Y \in \mathcal{P}$ (hence $\Lambda(\cdot, Y)$ is linear), then it satisfies the following continuity conditions:

$$\Lambda(X + \varepsilon Z, Y) = E_{Q_Y}[-X - \varepsilon Z] \underset{\varepsilon \to 0}{\longrightarrow} E_{Q_Y}[-X] = \Lambda(X, Y);$$
$$\Lambda(\varepsilon Z, Y) \underset{\varepsilon \to 0}{\longrightarrow} 0$$

for every $X, Y, Z \in L^\infty$. The last condition is similar to the vanishing continuity imposed by Guan et al. [37].

The study of the non-differentiable case was well justified by Cherny and Orlov [26] in the coherent case (see also Example 13 in the work of Centrone and Rosazza Gianin [27] for the convex case).

For what concerns meaningful examples which motivate our analysis, an important family of coherent/convex risk measures that is widely used in insurance and can be non-differentiable is generated by Choquet integrals and distortion–exponential Choquet integrals (see [15] for further details and discussion) with differentiable concave distortion functions but quantile functions that are not strictly increasing (see Corollary 2 in the work of Carlier and Dana [43] and also Proposition 2 in the work of Tsanakas [15]). Concerning the numerical aspects, the implementation of the elements in the subgradient is still an open problem. Indeed, as shown in [15], already in the non-coherent case for the special case of differentiable distortion-exponential risk measures, a closed-formula for the (generalized) gradient allocation is available only for continuous distributions, as it requires the quantile functions to be strictly increasing. In some specific cases, the formula holds for any distribution via the use of subgradients, and hence a formula true for general distributions and for general convex risk measures is still missing.

We now enrich the analysis of Kalkbrener [7] in the aforementioned way. As emphasized at the end of Section 2, when dealing with a CAR associated to truly convex risk measures, we have to drop at least one of the assumptions between linearity and no-undercut. Our choice is to keep no-undercut and to replace linearity with a property which is financially sound as well.

As a preliminary step, in the following lemma, we extend the validity of inequality (3.1) in Kalkbrener [7].

Lemma 1. *Let Λ be a CAR associated to a convex risk measure ρ and satisfying no-undercut.*

If Λ satisfies the marginal condition, *i.e., $\Lambda(Z + aX, Y) - \Lambda(Z, Y) \geq a\Lambda(X, Y)$ for any sufficiently small $a \in \mathbb{R}$, then*

$$\Lambda(X, Y + \varepsilon X) \leq \frac{\rho(Y + \bar{\varepsilon}X) - \rho(Y + \varepsilon X)}{\bar{\varepsilon} - \varepsilon} \leq \Lambda(X, Y + \bar{\varepsilon}X) \tag{7}$$

for any $\varepsilon, \bar{\varepsilon} \in \mathbb{R}$ with $\varepsilon < \bar{\varepsilon}$ and any $X, Y \in L^\infty$.

Note that the marginal condition is always satisfied when Λ is superlinear, i.e., homogeneous and superadditive (with superadditivity, it is meant that $\Lambda(X + Y, Z) \geq \Lambda(X, Z) + \Lambda(Y, Z)$, for any $X, Y, Z \in L^\infty$), and that both the technical conditions of marginality and superlinearity are milder than linearity. They can also be financially motivated as follows. For what concerns superlinearity, it implies $\Lambda(aX + bY, Z) \geq a\Lambda(X, Z) + b\Lambda(Y, Z)$ for any $a, b \in \mathbb{R}$, $X, Y, Z \in L^\infty$, which, in the case of CARs associated to convex risk measures, takes into account the potential extra risk created by merging two positions.

Instead, the marginal condition implies

$$\frac{\Lambda(Z + aX, Y) - \Lambda(Z, Y)}{a} \geq \Lambda(X, Y) \quad \text{for any } a > 0 \text{ sufficiently small, } Z \in L^\infty. \tag{8}$$

Thus, $\Lambda(X, Y)$ should not exceed the marginal increment (ratio) of capital due to the contribution of aX to any sub-position Z. A similar interpretation holds for sufficiently small $a < 0$.

Proof of Lemma 1. First of all, it holds that, for any $X, Y \in L^\infty$ and $\varepsilon, \bar{\varepsilon} \in \mathbb{R}$,

$$\begin{aligned}
\rho(Y + \bar{\varepsilon}X) &= \Lambda(Y + \bar{\varepsilon}X, Y + \bar{\varepsilon}X) \\
&\geq \Lambda(Y + \bar{\varepsilon}X, Y + \varepsilon X) \\
&= \Lambda(Y + \varepsilon X + (\bar{\varepsilon} - \varepsilon)X, Y + \varepsilon X)
\end{aligned}$$

where the first equality is due to the assumption that Λ is a CAR and the inequality to the no-undercut.

By the arguments above and by the marginal condition,

$$\begin{aligned}
\rho(Y + \bar{\varepsilon}X) &\geq \Lambda(Y + \varepsilon X + (\bar{\varepsilon} - \varepsilon)X, Y + \varepsilon X) \\
&\geq \Lambda(Y + \varepsilon X, Y + \varepsilon X) + (\bar{\varepsilon} - \varepsilon)\Lambda(X, Y + \varepsilon X) \\
&= \rho(Y + \varepsilon X) + (\bar{\varepsilon} - \varepsilon)\Lambda(X, Y + \varepsilon X)
\end{aligned}$$

for any $X, Y \in L^\infty$ and $\bar{\varepsilon}, \varepsilon \in \mathbb{R}$ with $\bar{\varepsilon} - \varepsilon$ sufficiently small. Hence,

$$\frac{\rho(Y + \bar{\varepsilon}X) - \rho(Y + \varepsilon X)}{\bar{\varepsilon} - \varepsilon} \geq \Lambda(X, Y + \varepsilon X)$$

holds for any $X, Y \in L^\infty$ and $\bar{\varepsilon} - \varepsilon > 0$ being sufficiently small. The remaining inequality can be obtained as before. □

We provide here below an example of CAR satisfying the no-undercut and a marginal-like condition.

Example 1. *Let ρ be a convex risk measure with $\rho(Y) = E_{Q_Y}[-Y] - c(Q_Y)$ for some probability measure Q_Y and penalty function $c \geq 0$.*

Consider now

$$\Lambda(X,Y) \triangleq E_{Q_Y}[-X] - c(Q_Y).$$

Such a Λ is a CAR (associated to a convex risk measure) satisfying the no-undercut. Indeed, CAR and no-undercut follow immediately by the dual representation of ρ and by the definition of Λ. In fact,

$$\Lambda(X,Y) = E_{Q_Y}[-X] - c(Q_Y) \leq E_{Q_X}[-X] - c(Q_X) = \Lambda(X,X) = \rho(X)$$

for any $X, Y \in L^\infty$. Moreover, Λ exhibits a sort of marginal condition, that is,

$$\begin{aligned}
\Lambda(Z + aX, Y) - \Lambda(Z, Y) &= E_{Q_Y}[-(Z + aX)] - c(Q_Y) - \big(E_{Q_Y}[-Z] - c(Q_Y)\big) \\
&= aE_{Q_Y}[-X] \\
&\geq a(E_{Q_Y}[-X] - c(Q_Y)) = a\Lambda(X,Y)
\end{aligned}$$

for any $X, Y, Z \in L^\infty$ and $a \geq 0$ sufficiently small.

In the following results, we assume the marginal condition on Λ.

Theorem 2. *Let Λ be a CAR associated to a convex risk measure ρ, satisfying no-undercut and the marginal condition (see Lemma 1). Then, it holds that*

$$(b') \Leftrightarrow (a) \begin{array}{l} \nearrow \; (c) \\ \searrow \; (b) \end{array}$$

among the following conditions:
(a) $\partial\rho(Y) \neq \emptyset$ for any $Y \in L^\infty$ (in general, not a singleton);
b) There exists $D^\rho_+(Y;X)$ (and hence $D^\rho_-(Y;X)$) for any $X, Y \in L^\infty$ and it is finite;
(b') $D^\rho_+(Y;X) \geq E_{\bar{\bar{Q}}_Y}[-X]$, for any $X, Y \in L^\infty$ and for some $\bar{\bar{Q}}_Y \in \mathcal{P}$;
(c) Continuity condition on Λ: for any fixed $Y \in L^\infty$

$$(c1) \qquad \exists\, \Lambda^-(X,Y) \triangleq \lim_{\varepsilon \uparrow 0} \Lambda(X, Y + \varepsilon X) \leq E_{\bar{Q}_Y}[-X]$$

$$(c2) \qquad \exists\, \Lambda^+(X,Y) \triangleq \lim_{\varepsilon \downarrow 0} \Lambda(X, Y + \varepsilon X) \geq E_{\bar{\bar{Q}}_Y}[-X]$$

for some $\bar{Q}_Y, \bar{\bar{Q}}_Y \in \partial\rho(Y)$ and any $X \in L^\infty$. Moreover,

$$\Lambda^-(X,Y) \leq \Lambda(X,Y) \leq D^\rho_+(Y;X) \leq \Lambda^+(X,Y) \tag{9}$$

holds for any $X, Y \in L^\infty$.

Proof. *(a) $\Rightarrow$ (b)* can be found in Lemma 7.14 in [30].
(b') $\Leftrightarrow$ (a) follows from Theorem 7.16 in [30] and from the definition of the subdifferential in our case, as ρ is convex and proper.
(a) $\Rightarrow$ (c2). By Lemma 1, we have

$$\Lambda(X, Y + \varepsilon X) \leq \frac{\rho(Y + \bar{\varepsilon}X) - \rho(Y + \varepsilon X)}{\bar{\varepsilon} - \varepsilon} \leq \Lambda(X, Y + \bar{\varepsilon}X)$$

for any $\varepsilon, \bar{\varepsilon} \in \mathbb{R}$ with $\varepsilon < \bar{\varepsilon}$ and any $X, Y \in L^\infty$. Hence, $\Lambda(X, Y + hX)$ is increasing in $h \in \mathbb{R}$ and, by taking $\varepsilon = 0$, $\bar{\varepsilon} > 0$, we have

$$\Lambda(X, Y + \bar{\varepsilon}X) \geq \frac{\rho(Y + \bar{\varepsilon}X) - \rho(Y)}{\bar{\varepsilon}}$$

for any $X, Y \in L^\infty$.

Now, from *(a)*, there exists $\bar{\bar{Q}}_Y \in \partial\rho(Y)$ such that, for any $X \in L^\infty$,

$$\rho(Y + \bar{\varepsilon}X) - \rho(Y) \geq E_{\bar{\bar{Q}}_Y}[-(Y + \bar{\varepsilon}X - Y)]$$

and so

$$\frac{\rho(Y + \bar{\varepsilon}X) - \rho(Y)}{\bar{\varepsilon}} \geq E_{\bar{\bar{Q}}_Y}[-X].$$

Therefore, for any $X \in L^\infty$,

$$\lim_{\bar{\varepsilon} \downarrow 0} \Lambda(X, Y + \bar{\varepsilon}X) \geq \lim_{\bar{\varepsilon} \downarrow 0} \frac{\rho(Y + \bar{\varepsilon}X) - \rho(Y)}{\bar{\varepsilon}} \geq E_{\bar{\bar{Q}}_Y}[-X]$$

where the former limit exists thanks to the monotonicity of $\Lambda(X, Y + \bar{\varepsilon}X)$ in $\bar{\varepsilon}$, while the latter thanks to the monotonicity of $\frac{\rho(Y+\bar{\varepsilon}X)-\rho(Y)}{\bar{\varepsilon}}$ (following from the convexity of ρ).
(a) $\Rightarrow$ *(c1)* can be derived in the same way as the previous one by taking $\varepsilon < 0$, $\bar{\varepsilon} = 0$, where $\bar{Q}_Y \in \partial\rho(Y)$.

Finally, (9) derives from increasing monotonicity of $\Lambda(X, Y + \varepsilon X)$ in $\varepsilon \in \mathbb{R}$, which in turn follows from Lemma 1. □

From Theorem 2, we obtain the following result related to the problem of the generalized collapse to the mean.

Corollary 1. *(i) (Super-collapse): if condition (c2) holds and $\Lambda(X, Y) = \Lambda^+(X, Y)$ for any $X, Y \in L^\infty$, then $\Lambda(X, Y) \geq E_{\bar{\bar{Q}}_Y}[-X] \geq \Lambda^{sub,c}(X, Y)$ for $\bar{\bar{Q}}_Y \in \partial\rho(Y)$ of condition (c2), where*

$$\Lambda^{sub,c}(X, Y) \triangleq E_{\bar{\bar{Q}}_Y}[-X] - c(\bar{\bar{Q}}_Y); \tag{10}$$

(ii) (Sub-collapse): if condition (c1) holds and $\Lambda^-(X, Y) = \Lambda(X, Y)$, for any $X, Y \in L^\infty$, then $\Lambda(X, Y) \leq E_{\bar{Q}_Y}[-X]$, for $\bar{Q}_Y \in \partial\rho(Y)$ of condition (c1);
(iii) (Subgradients bounds): if condition (c) and $\Lambda = \Lambda^+ = \Lambda^-$ hold then, for any $X, Y \in L^\infty$

$$E_{\bar{\bar{Q}}_Y}[-X] \leq \Lambda(X, Y) \leq E_{\bar{Q}_Y}[-X] \tag{11}$$

for $\bar{\bar{Q}}_Y, \bar{Q}_Y \in \partial\rho(Y)$. Moreover, $\Lambda(X, Y) = D^\rho_+(Y; X)$ for any $X, Y \in L^\infty$.
(iv) (Collapse to the mean): if condition (b') holds with equality and $\Lambda = \Lambda^+$ then, for each fixed $Y \in L^\infty$,

$$\Lambda(X, Y) = E_{\bar{Q}_Y}[-X], \text{ for any } X \in L^\infty, \tag{12}$$

for a unique $\bar{Q}_Y \in \partial\rho(Y)$. Moreover, $\Lambda(X, Y) \geq \Lambda^{sub,c}(X, Y)$ for any $X, Y \in L^\infty$.

Proof. The first three points follow immediately from the previous theorem. For the last point, notice that, from the assumption that $D^\rho_+(Y; X) = E_{\bar{Q}_Y}[-X]$, we have that, for every $Y \in L^\infty$, $D^\rho_+(\cdot)$ is linear (i.e., it is the Gateaux derivative), so $\partial\rho(Y)$ is a singleton. Hence, $\Lambda = \Lambda^+$ implies that $\Lambda(X, Y) = D^\rho_+(Y; X) = E_{\bar{Q}_Y}[-X]$. □

Note that the last statement of (iii) is also in line with [26]. Furthermore, $\Lambda^{sub,c}$ in items (i) and (iv) can be interpreted as a penalized subdifferential CAR. We recall indeed that, for a not necessarily Gateaux differentiable ρ,

$$\Lambda^{sub}(X,Y) \triangleq E_{\bar{\bar{Q}}_Y}[-X], \quad \text{with } \bar{\bar{Q}}_Y \in \partial\rho(Y),\ X,Y \in L^\infty,$$

generalizes the gradient allocation for a coherent risk measure (see [27]). $\Lambda^{sub,c}$, as above, is then the corresponding version for convex risk measures.

Corollary 1 underlines that working with a general CAR Λ entails only subgradient bounds (under condition c) plus $\Lambda = \Lambda^+ = \Lambda^-$) but guarantees neither the collapse to the mean nor the uniqueness of the optimal scenario. It is also worth emphasizing that, in the case of subgradient bounds, the lower and upper bounds depend on the aggregate position Y, once X is fixed. Loosely speaking, the generalized collapse to the mean and sub/super collapse to the mean try to make a bridge (through the expected value) between the generality of the model in terms of the properties of risk measures and the more easily implementable cases.

The subgradient bounds also make evident, once Y is fixed, the parallelism with the pricing of financial derivatives in incomplete markets (see [44–46], among others). Indeed, when $\mathcal{P}$ is the set of martingale measures in an arbitrage-free incomplete financial market, the lower and upper bounds can be interpreted as two different no-arbitrage prices evaluated with respect to two different martingale measures. Loosely speaking, then, inequality (11) can be seen as a kind of no-arbitrage interval formulated for capital allocation principles.

Comparison with Existing Results

We start by comparing our results with those of Kalkbrener [7] on the collapse to the mean.

Firstly, notice that in our investigation, we weaken the following assumptions of Kalkbrener's results:

- Coherence and Gateaux differentiability of ρ;
- Linearity of Λ.

To be more precise, we consider convex risk measures and Λ satisfying either superlinearity or the marginal condition.

We now investigate our results under Kalkbrener's assumptions. Suppose that ρ is a coherent risk measure with $\partial\rho(Y) \neq \emptyset$, for every $Y \in L^\infty$. Hence, by the representation of the subdifferential, for every $Y \in L^\infty$, there exists a—not necessarily unique—probability measure $\bar{Q}_Y$ such that $\rho(Y) = E_{\bar{Q}_Y}[-Y]$. In this case, the functionals h_Y in Theorem 1 are exactly the probability measures $\bar{Q}_Y$, and so the CAR Λ_ρ becomes $\Lambda_\rho(X,Y) = E_{\bar{Q}_Y}[-X]$ for some $\bar{Q}_Y \in \partial\rho(Y)$ and for each $X \in L^\infty$. In this context, condition (c) of Theorem 1 (hence, Kalkbrener's condition) can be rewritten as follows: for every $Y \in L^\infty$, there exists a unique $\bar{Q}_Y$ such that $\rho(Y) = E_{\bar{Q}_Y}[-Y]$, that is $\partial\rho(Y)$ is a singleton. Moreover, notice that, from the representation of the subdifferential, we always obtain, for every $X, Y \in L^\infty$

$$D^\rho_-(Y;X) \leq E_{\bar{Q}_Y}[-X] \leq D^\rho_+(Y;X), \tag{13}$$

and hence condition (b') in Theorem 2 is weaker than asking for the existence of $D^\rho(Y;X)$. Indeed, the previous inequalities also show that the existence of the directional derivative $D^\rho(Y;X)$ yields a singleton subdifferential. Furthermore, as we have already noticed, $\Lambda^-_\rho \leq \Lambda_\rho \leq \Lambda^+_\rho$, so by $\Lambda_\rho(X,Y) = E_{\bar{Q}_Y}[-X]$ for some $\bar{Q}_Y \in \partial\rho(Y)$, also assumption (c) in our Theorem 2 turns out to be fulfilled by Λ_ρ. In conclusion, when we deal with $\Lambda = \Lambda_\rho$, our assumptions are milder than Kalkbrener's ones and that is why, also in this case, we do not obtain an equivalence among all the conditions stated in Theorem 2. Our conditions enable us to deal with CARs Λ more general than Λ_ρ, so we can speak of a generalization of Kalkbrener's result in this sense.

For the reader's convenience, Table 1 summarizes the main assumptions and results in Kalkbrener's paper and in ours.

Table 1. Comparison with Kalkbrener's paper and assumptions.

		Kalkbrener's Paper	**Our Paper**
hypothesis on Λ	linearity	x	- (no, only marginal)
	CAR	x	x
	no-undercut	x	x
	marginal	x (by linearity of Λ)	x
	continuity (C0)	x	-
hypothesis on ρ	coherence	x	- (no, generalized to convex)
	differentiability	x	- (no, only subdifferentiability)
results		collapse to the mean	collapse to the mean under milder conditions subgradient bounds and generalized collapse to the mean under further conditions

We also point out that, in a recent paper, Guan et al. [37] obtained a result of the collapse to the mean of a CAR with respect to the reference probability P, thus showing that some assumptions are sound per se but too stringent when taken together as noticed by the authors themselves. Indeed, in general, this stronger collapse to the mean is too restrictive in many financial contexts (see also the observation in [24]). In particular, when a risk measure is involved, this representation fails to fully capture the risk and is not useful in applications. In the case of CARs, the dependence on the aggregated capital of the allocation to the various sub-units is completely missing.

4. Conclusions and Future Research Directions

In this work, we have faced the problem of the representation of capital allocation rules as expectations with respect to probability measures which depend on an aggregate risky position (generalized collapse to the mean) and that do not necessarily agree with the reference probability on the space, without making any assumption of the Gateaux differentiability on the underlying risk measure.

We underline that a generalized collapse to the mean is of key importance for implementing capital allocations in real practice. Indeed, the classical Euler method is so popular because of its ease of usein empirical studies with standard statistical methods. As soon as Gateaux differentiability fails—which anyway occurs in relevant contexts (see [26])—the multiplicity of optimizing measures calls for the development of alternative and more sophisticated numerical approach which are currently under our study.

To summarize our contribution, we have provided a version, for capital allocation rules with respect to convex and non-differentiable risk measures, of the well-known result of Kalkbrener (Theorem 4.3 in [7]). His result guarantees that, in the coherent and differentiable case, any linear CAR satisfying no-undercut reduces to the Gateaux derivative (and hence, for every unit Y, the gradient allocation collapses to a mean with respect to a unique probability measure Q_Y) if a continuity assumption on the CAR—equivalent to the existence of the directional derivative at every unit Y in the direction of any sub-unit X—is fulfilled. Instead, our findings are obtained under weaker assumptions also on the CARs and, in particular cases, reduce to a generalized collapse to the mean. Nevertheless, the generalization to a wider class of risk measures (beyond the coherent case) implies the impossibility of obtaining a "full" collapse to the mean but only lower and upper bounds, paying the price of losing the expected value representation.

Somehow related to the present study, different ongoing and future research directions are currently under study or planned for the next future. Both for risk measures and for capital allocations in the non-differentiable case, a numerical approach that could provide the optimal generalized scenario among the multiplicity of the elements in the subgradient is currently under investigation.

Deep/reinforcement learning methodologies can be also a possible first step in the direction above as already discussed in [47,48]. See also the work of Agram et al. [49] for a recent study in the BSDE setting.

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