Moment-matching approximations for stochastic sums in non-Gaussian Ornstein–Uhlenbeck models

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Abstract

In this paper, we recall actuarial and financial applications of sums of dependent random variables that follow a non-Gaussian mean-reverting process and contemplate distribution approximations. Our work complements previous related studies restricted to lognormal random variables; we revisit previous approximations and suggest new ones. We then derive moment-based distribution approximations for random sums attuned to Asian option pricing and computation of risk measures of random annuities. Various numerical experiments highlight the speed-accuracy benefits of the proposed methods.

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JEL classification: G13, C63, C15, G22

1. Introduction

We devise accurate distribution approximations for discrete and continuous sums of random variables that follow an exponential Lévy Ornstein–Uhlenbeck (OU) process.

Contingent claims on sums of random variables have various applications in actuarial practice. For example, the valuation and risk management of annuities are important topics in actuarial science. The literature is rich with cases of annuities with fixed payouts or variable annuity guarantees (e.g., see Vanduffel *et al.*, 2008, Pirjol and Zhu, 2016, Feng and Volkmer, 2012 and references therein). These will be revisited later in the paper in greater detail. In addition, as emphasized in Plat and Pelsser (2009), a common embedded option in insurance products in Europe is a profit-sharing rule based on a (moving average) fixed income rate, combined with a minimum guarantee. In practice, these options are often valued using an (average) forward swap rate as an approximation for the profit-sharing rate, which turns out to be an equivalent problem to that of valuing Asian options. Also, due to periodic investment over a long period, the embedded option in equity-linked pension schemes with a guarantee is of the average type, therefore the pricing of the pension scheme is inherently linked to the pricing of

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the Asian option (see Nielsen *et al.*, 2011). Guaranteed equity-linked life insurance contracts are closely related (e.g., see Schrager and Pelsser, 2004, Hürlimann, 2010), hence similar principles are shared; refer to Nielsen *et al.* (2011) for a more detailed literature review.

Average-based derivatives are also prevalent in commodity markets where risk management is everlasting. For example, in Europe, contracts on the CME Cumulative Average Temperature (CAT) Indexes allow businesses to hedge against monthly volatility. In the freight market, charterers typically face freight rate exposure during a voyage, therefore most freight derivatives are settled against average spot freight rates. But also, crude oil consumers use average-based options to hedge against spikes in oil prices during the supply period. The path-dependent character of these options inevitably adds to the complexity of the pricing and hedging using these contracts.

The key points of our approach are its modelling capability of mean reversion and jumps in the dynamics of the underlying variable, but also its generality in the sense that its sole requirement is the availability of a closed-form expression for the cumulant generating function of the background driving process of interest. Mean reversion in different variables' dynamics is spotlighted in researches such as Poterba and Summers (1988), Wong and Lo (2009) and Liang *et al.* (2011). In addition, the deficiencies of a more basic model, such as the lognormal model, with absent skewness and excess kurtosis of the log-returns, are clearly acknowledged, for example, in the American Academy of Actuaries (2005) report, which highlights the importance of distributions with varying skewness and fat tails that better reflect the market realism. Jang (2007) explains the usefulness of jump diffusions and, in general, Lévy processes for modelling in insurance, such as aggregate claim amounts, and in finance, e.g., in zero-coupon bond pricing or in pricing credit default swaps. As Bakshi and Madan (2002) argue, a model in catastrophe insurance, with "low-frequency, high-severity" risks, that omits the jump loss feature, or alternatively relies only on diffusion loss dynamics, is likely misspecified. But also, when studying the price dynamics of commodity products mentioned earlier, jump and mean reversion are important stylized properties that need to be taken into account (e.g., see Kyriakou *et al.*, 2016, Kyriakou *et al.*, 2017, Kyriakou *et al.*, 2018).

In this paper, we consider a general exponential Lévy OU process for the underlying asset (e.g., equity) price, index, or variable, in general, or loss. The modelling philosophy is based on the fact that large fluctuations of the underlying lead to non-normal deviations from the long-term mean towards which it reverts. Boyle and Potapchik (2008) uniquely survey extensively different methods for pricing average-based derivatives, including seminal contributions such as those of Geman and Yor (1993) and Milevsky and Posner (1998), which are limited, though, to the basic Black–Scholes model framework. Since then, we have seen various contributions in the literature under more sophisticated models; we do not reiterate them here, but rather refer to Fusai and Kyriakou (2016) for more details. The literature so far comes to a standstill when start thinking of nonlinear functions of linear combinations of dependent random variables in this model. The purpose of this paper is to fill this void using moment-based approximations for this unknown distribution law. We reconsider approximating laws from the literature, such as the 3-parameter shifted versions of the lognormal, gamma and reciprocal gamma distributions (see Lo *et al.*, 2014) as well as the series expansion of Willems (2019). These have been applied originally in the geometric Brownian motion model setting, therefore their efficiency in different model settings is debatable. More importantly, aiming to account for non-Gaussian driving dynamics, we also come up with new suggestions, including the 3-parameter modified lognormal power-law for a better control of the tail behaviour, and the 4-parameter Pearson system and Johnson family of transformations. In addition, we derive a lower bound for the Asian option price along the lines of Fusai and Kyriakou (2016), adapted to the proposed model framework. We end up with a full battery of option pricing expressions with grounded cornerstone the moments of the discrete or continuous arithmetic average, which we derive under general underlying model assumptions and show how to compute fast and accurately. We find that the Pearson system offers the flexibility required to capture the skewness and excess kurtosis of the distribution of the arithmetic average and, in the largest part of our numerical experimentation, it is found to be the most precise.

So, what are the sought merits of a formula? First, it is easy and can be better understood than numerical algorithms, is fast to implement, and expectantly accurate. Second, the inversion of a pricing formula smooths the way for the inference of the underlying model parameter values from option market quotes based on different maturities and strikes traded on the market. Third, in practical actuarial applications (e.g., see Laeven *et al.*, 2005), one may be interested in computing popular risk measures for annuities, which can be already challenging enough under simpler driving dynamics. Our method provides access to the cumulative distribution function, which is extremely useful as it offers itself to direct computation of the risk measures. A part of the paper is devoted to the case of random, continuous or discrete, annuities under general model assumptions. Finally, replicating the payoff of an option leads to a perfect hedge for the risk associated with the sale of this option. Traders and risk managers favour uncomplicated formulae for option prices as they also yield formulae for the price sensitivities with respect to the changes of various model parameters that constitute the components of the replicating portfolio and facilitate also the analysis of these changes.

The remainder of the article is structured as follows. In Section 2, we present the underlying model assumptions accompanied by an empirical validation and the basic results about the moments of the arithmetic average required for the mathematical treatment that follows. Section 3 introduces the moment-matching approach, whereas Sections 4–5 introduce the various 3 and 4-parameter distribution approximations for the arithmetic average. In Section 6, we additionally propose a lower bound for the Asian option price. In Section 7, we assess the accuracy of the different pricing expressions on numerical simulations. In Section 8, we extend to annuities and the efficient computation of popular risk measures. Section 9 concludes the article.

2. The model

Consider the stochastic process

$$dX(t) = \alpha(\beta - X(t))dt + dL(t), \tag{1}$$

where $\alpha > 0$ is the speed of mean reversion, β is the long-run mean, and L is a general background driving Lévy process (BDLP). The solution of the stochastic differential equation

(1) is

$$X(t) = X(0)e^{-\alpha t} + \beta(1 - e^{-\alpha t}) + \int_0^t e^{-\alpha(t-s)} dL(s). \tag{2}$$

By means of a preliminary empirical analysis, we aim to investigate how this model fares in reproducing the stylized properties of the dynamics of underlying variables in index-linked products[1]. To this end, we consider a Gaussian OU and an example of non-Gaussian OU process with double exponential jump diffusion BDLP as well as their special counterparts with parameter $\alpha = 0$. We calibrate to market prices of options written on assets from different classes, such as the S&P500 index, VIX, Emerging Markets Ishares MSCI ETF (EEM ETF) and Apple stock; we report the relevant results in Table 1. In Table 2, we study the impact of varying model assumptions on the ability to reproduce the market prices of options based on minimization of the mean squared error (MSE). First, we find that permitting non-Gaussian dynamics can significantly reduce the MSE: 17% reduction for the Apple stock options up to 98% for the VIX options (bottom panel, second column); these figures are generally boosted in analogous models with mean reversion allowed (last column). Second, accounting for both jumps and mean reversion yields further reductions ranging from 6% for EEM ETF up to 69% for S&P500 (fourth column). Third, moving from the basic Black–Scholes model setting in the Asian options literature, that is, Gaussian dynamics (log-scale) without mean reversion to a non-Gaussian mean-reverting model results in a percentage MSE reduction from a minimum of 30% for EEM ETF up to a maximum of 99% for VIX (third column). Looking at the estimated parameters of the DEJD-driven mean-reverting model in Table 1, we see that α ranges from 0.3 (Apple) to 9.97 (VIX) and the jump arrival intensity λ is between 0.93 (S&P500) and 2.99 (VIX). Obviously, the presence of mean reversion and jumps and their effect on the option pricing error is undoubtedly significant.

In view of the above empirical results, we adopt the general model (1), which can also be flexibly reduced to a simpler without mean reversion depending on the needs of the user. Then, as per the scope of this paper, we consider an option with payoff at some terminal time $T > 0$ contingent on the average Y

$$\begin{cases} \text{(continuous)} & Y(T) = \frac{1}{T}\int_0^T e^{X(t)} dt \\ \text{(discrete)} & Y_n(T) = \frac{1}{n}\sum_{j=1}^n e^{X(t_j)}, \text{ where } t_1 < \cdots < t_n = T \end{cases} . \tag{3}$$

For example, an Asian call option with fixed strike price K has payoff at maturity time T

$$(Y - K)^+,$$

where $x^+ := \max(x, 0)$, and time-0 price

$$e^{-rT} p_0$$

where

$$p_0 = E\left[(Y - K)^+\right]. \tag{4}$$

[1]For example, in `https://equitable.com/retirement/products/variable-annuities`, there are several indices, exchange-traded funds (ETFs) and funds linked to annuities reported which are actively traded.

The pricing expectation is traditionally evaluated under the risk neutral measure with underlying asset price dynamics

$$S(t) = S(0)\frac{e^{rt+X(t)}}{E(e^{X(t)})}, \tag{5}$$

which no longer depends on β and satisfies the standard martingale condition. However, it may be the case, for example, of an underlying (non-economic) loss process that is uncorrelated with economy-wide risk factors. In such and other similar cases, the expected payoffs can be discounted under the physical measure using the risk free rate. For the purposes of the option pricing application and the computation of the risk measures of annuities we consider in the later sections, we adhere, respectively, to the risk neutral and the physical measure.

Computing (4) by means of an exact closed-form solution is an unsolvable problem. For this, we present, next, important results relating to the moments of Y that will be the building block of our proposed solution. In particular, we aim to provide suitable approximations of the distribution of the arithmetic average exploiting knowledge of its moments.

2.1. The moments of the arithmetic average

In this section, we show how to obtain the moments of the arithmetic average within the general model framework defined in the previous section. First, we derive the multivariate characteristic function of the intertemporal joint distribution of $(X(t_1), X(t_2), \ldots, X(t_n))$ for any partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ of the interval $[0, T]$ using the following useful lemma.

Lemma 1 (Eberlein and Raible, 1999). *Let L be a Lévy process. If $h : \mathbb{R}_+ \to \mathbb{C}$ is a complex-valued, left-continuous function with limits from the right, such that $|Re(h)| \leq M$, then*

$$E\left[\exp\left(\int_0^t h(s)dL(s)\right)\right] = \exp\left(\int_0^t \psi(h(s))ds\right),$$

where $\psi(u) := \ln E[\exp(uL(1))]$.

Lemma 2. *For parameter vector $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ where $\gamma_j \in \mathbb{C}\ \forall j$, we have that*

$$E\left[\exp\left(\sum_{j=1}^n \gamma_j X(t_j)\right)\right] = \exp\left\{\sum_{j=1}^n \left(X(0)\gamma_j e^{-\alpha t_j} + \beta\gamma_j(1-e^{-\alpha t_j}) + \int_{t_{j-1}}^{t_j} \psi\left(\sum_{i=j}^n \gamma_i e^{-\alpha(t_i - s)}\right) ds\right)\right\}. \tag{6}$$

Proof. See Appendix A. ■

Having access to the characteristic function of $(X(t_1), X(t_2), \ldots, X(t_n))$, we can now state the results for the moments of the continuous and discrete arithmetic average.

Proposition 3 (Discrete average). *The mth moment of the discrete arithmetic average price is given by*

$$\begin{aligned} E[Y_n^m(T)] \;&=\; \frac{1}{n^m} \sum_{\gamma_1+\cdots+\gamma_n=m} \binom{m}{\gamma_1, \ldots, \gamma_n} \\ &\exp\left\{\sum_{j=1}^n \left(X(0)\gamma_j e^{-t_j} + \beta\gamma_j\left(1-e^{-t_j}\right) + \int_{t_{j-1}}^{t_j} \psi\left(\sum_{i=j}^n \gamma_i e^{-\alpha(t_i - s)}\right) ds\right)\right\}, \end{aligned} \tag{7}$$

where $\begin{pmatrix} m \\ \gamma_1, \ldots, \gamma_n \end{pmatrix} = \frac{m!}{\gamma_1!\gamma_2!\cdots\gamma_n!}$ *is a multinomial coefficient and the sum* $\sum_{\gamma_1+\cdots+\gamma_n=m}$ *is taken over all combinations of non-negative integer indices* γ_1 *through* γ_n *such that their sum is* m*.*

Proof. (7) follows from the multinomial theorem, which yields

$$(\exp(X(t_1)) + \cdots + \exp(X(t_n)))^m = \sum_{\gamma_1+\cdots+\gamma_n=m} \begin{pmatrix} m \\ \gamma_1, \ldots, \gamma_n \end{pmatrix} \prod_{j=1}^{n} \exp(\gamma_j X(t_j)),$$

and (6). ■

Proposition 4 (Continuous average)**.** *The nth moment of the continuous arithmetic average price is given by*

$$\begin{aligned} E[Y^n(T)] &= \frac{n!}{T}\int_{t_0}^{T} dt_1 \int_{t_1}^{T} dt_2 \cdots \int_{t_{n-1}}^{T} \exp\left\{\sum_{j=1}^{n} \left(X(t_0)e^{-\alpha t_j} + \beta\left(1-e^{-\alpha t_j}\right)\right.\right. \\ &\quad \left.\left. + \int_{t_{j-1}}^{t_j} \psi\left(\sum_{i=j}^{n} e^{-\alpha(t_i-s)}\right) ds\right)\right\} dt_n. \end{aligned} \tag{8}$$

Proof. From Bharucha-Reid (1960, p. 344–345),

$$\begin{aligned} E[Y^n(T)] &= \frac{1}{T}\int_{t_0}^{T} \cdots \int_{t_0}^{T} E[\exp(X(t_1) + \cdots + X(t_n))] dt_1 \cdots dt_n \\ &= \frac{n!}{T}\int_{t_0}^{T} dt_1 \int_{t_1}^{T} dt_2 \cdots \int_{t_{n-1}}^{T} E[\exp(X(t_1) + \cdots + X(t_n))] dt_n. \end{aligned}$$

Then, (8) follows from (6). ■

Our general results presented in Propositions 3–4 are based on the key quantity

$$\int_{t_{j-1}}^{t_j} \psi\left(\sum_{i=j}^{n} \gamma_i e^{-\alpha(t_i-s)}\right) ds. \tag{9}$$

This may admit an explicit expression depending on the choice of the background driving Lévy process L. Table 3 exhibits various examples of Lévy processes that may or may not lead to an explicit solution for (9). To avoid illegible long mathematical expressions, we present, instead, in Appendix B snippets of Mathematica® codes that generate those expressions for (9) corresponding to models including the Gaussian, normal inverse Gaussian (Barndorff-Nielsen, 1998) and double exponential jump diffusion (Kou, 2002). Solutions for other popular model choices, such as the variance gamma, Carr–Geman–Madan–Yor, and hyperexponential jump diffusion (Cai and Kou, 2011), can be obtained similarly. In the absence of an analytic solution, (9) can still be computed numerically fast and accurately, e.g., consider the cases of Merton jump diffusion and the Meixner (Schoutens and Teugels, 1998) and generalized hyperbolic (Eberlein and Prause, 2002) models.

[Insert Table 3]

The unknown distribution law of the arithmetic average precludes the existence of true solutions for expected values of nonlinear functions of the average. Therefore, in the following few sections, we study different candidates for the unknown probability distribution of the arithmetic average (3).

3. Moment-matching approximations

We focus here on the Asian option pricing problem, although any conclusions drawn about the validity of a proposed approximation are transferable to other problems requiring access to the distribution of the average or, in general, sums of random variables. We denote by $\tilde{p}_0$ the approximate expression for the true pricing expectation (4) given by

$$\tilde{p}_0 = \int_0^\infty (x-K)^+ \tilde{f}(x;\underline{\theta})dx = \int_K^\infty x\tilde{f}(x;\underline{\theta})dx - K(1-\tilde{F}(K;\underline{\theta})), \tag{10}$$

where $\tilde{f}(x;\underline{\theta})$ is a supposed probability density function, $\tilde{F}(x;\underline{\theta})$ the associated cumulative distribution function, and $\underline{\theta}$ the vector of its parameters.

Two natural questions arising in this context are the following. What probability distributions could be potentially assigned to the arithmetic average? What is the resulting error from such an approximation? The answer to the first question varies; we postpone showcasing a collection of likely distribution proxies until the next sections. Before that, we look into the first questions. According to the literature (see Solomon and Stephens, 1978), the intrinsic accuracy of the approximation is difficult to assess in a mathematical way and has so far been determined by examples. Here, we improve on this by adapting Akhiezer's 1965 error bound for cumulative distribution functions and turning it into an error bound for expected value representations. We present an upper bound to the error from replacing the unknown distribution F of the arithmetic average with an approximating $\tilde{F}$ that shares the first $2n$ moments.

From Akhiezer (1965, p. 66),

$$\left|F(x) - \tilde{F}(x)\right| \leq \rho_n(x), \tag{11}$$

where

$$\frac{1}{\rho_n(x)} = \begin{pmatrix} 1 & x & \dots & x^n \end{pmatrix} \begin{pmatrix} 1 & \mu_1 & \dots & \mu_n \\ \mu_1 & \dots & \dots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ x \\ \vdots \\ x^n \end{pmatrix}$$

and μ_n is the nth moment of the unknown distribution. The error bound (11) is then applicable to expected values of functions of the underlying random quantity. For example, the price of the Asian put option with fixed strike K is given by

$$e^{-rT}E\left[(K-Y)^+\right] = e^{-rT}\int_0^K F(x)dx \approx e^{-rT}\int_0^K \tilde{F}(x)dx.$$

Then, from (11),

$$e^{-rT}\int_0^K \left|F(x) - \tilde{F}(x)\right| dx \leq e^{-rT}\int_0^K \rho_n(x)dx. \tag{12}$$

When $n = 1$, for example, we have that

$$\int_0^K \rho_1(x)dx = \sqrt{\mu_2 - \mu_1^2}\left[\arctan\left(\frac{K-\mu_1}{\sqrt{\mu_2-\mu_1^2}}\right) + \arctan\left(\frac{\mu_1}{\sqrt{\mu_2-\mu_1^2}}\right)\right],$$

but in general (12) can be computed easily and accurately numerically for any n.

Figure 1 shows a simple implementation of (11)–(12) based on a Gaussian OU process. Two comments are in order. The error upper bound becomes tighter the more moments are taken into account. This improves particularly with decreasing option moneyness, consistently with Lindsay and Basak (2000) who show that this bound is accurate only in the tails of the distributions.

[Insert Figure 1]

As it is not possible to exactly characterize the distribution of the arithmetic average, we will use Propositions 3–4 and moment-matching to define approximating distributions for the arithmetic average. In Section 4, we present 3-parameter approximations and resulting closed-form price solutions for Asian options; in Section 5, we study 4-parameter approximations which lead to solutions that require a simple and fast numerical integration. Later on in Section 7, we will assess the performance of the different candidate solutions based on accuracy–computational burden tradeoffs.

4. Approximations based on the first three moments

For brevity, in what follows we omit the details of the associated probability density and cumulative distribution functions, which can be found in the cited original works, and rather merely present the resulting pricing expressions following from (10).

4.1. Shifted lognormal (SLN)

Posner and Milevsky (1998) and Lo *et al.* (2014) suggest approximating the distribution of the arithmetic average Y by a lognormal distribution with parameters $m \in \mathbb{R}$, $s > 0$ and addition of a shift parameter h to the originally proposed approximation of Turnbull and Wakeman (1991). Matching the mean $\mu_Y = \mu_1$, variance $v_Y = \mu_2 - \mu_1^2$ and skewness $\gamma_Y = (\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3)/v_Y^{3/2}$, where μ_1, μ_2, μ_3 are the first three raw moments of the arithmetic average Y in (3), yields the following expressions for the parameters:

$$h = \mu_Y - \frac{\sqrt{v_Y}}{\gamma_Y}\left(1 + B^{\frac{1}{3}} + B^{-\frac{1}{3}}\right), \quad s^2 = \ln\left(1 + \frac{\gamma_Y}{(\mu_Y - h)^2}\right) \tag{13}$$

and

$$m = \ln(\mu_Y - h) - \frac{s^2}{2}, \tag{14}$$

where $B := (\gamma_Y + 2 - \sqrt{\gamma_Y^4 + 4\gamma_Y})/2 \in (0, 1]$. For Y approximated by a shifted lognormal law with parameter vector $\underline{\theta} = (h, s, m)$ given by (13)–(14), the pricing formula (10) leads to

$$\tilde{p}_0^{\text{SLN}} := (\mu_Y - h)\,\Phi(d_1) - (K - h)\Phi(d_2),$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function and

$$d_{1,2} := \frac{1}{s} \ln \left| \frac{\mu_Y - h}{K - h} \right| \pm \frac{s}{2}.$$

4.2. Shifted gamma (SG)

Following Chang and Tsao (2011) and Lo *et al.* (2014), another possibility is an approximation of the unknown distribution using a shifted gamma distribution with shape and scale parameters $a > 0$ and $b > 0$, and shift parameter h. In this case, moment-matching yields the following solution for the unknown parameters:

$$a = \frac{4}{\gamma_Y^2}, \quad b = \sqrt{\frac{v_Y}{a}}, \quad h = \mu_Y - ab.$$

Applying the parameter vector $\underline{\theta} = (a, b, h)$ in formula (10) under the assumption of a shifted gamma law for Y gives us

$$\tilde{p}_0^{\mathrm{SG}} := (\mu_Y - h) \left(1 - \frac{\Gamma_\omega(a+1)}{\Gamma(a+1)} \right) - (K - h) \left(1 - \frac{\Gamma_\omega(a)}{\Gamma(a)} \right),$$

where $\omega := t(K-h)/b$ and

$$\Gamma_\omega(z) = \int_\omega^\infty t^{z-1} e^{-t} dt$$

is the upper incomplete gamma function; Γ corresponds to $\omega \equiv 0$. We refer to Lo *et al.* (2014) for the details of this derivation.

4.3. Shifted reciprocal gamma (SRG)

In addition, Lo *et al.* (2014) consider an approximation based on a shifted reciprocal gamma distribution with parameters a, b and h. Here, the relevant parameters are given by

$$h = \mu_Y - \frac{\sqrt{v_Y}}{\gamma_Y} \left(2 + \sqrt{4 + \gamma_Y} \right), \quad a = 2 + \frac{(\mu_Y - h)^2}{v_Y}$$

and

$$b = (\mu_Y - h)(a - 1).$$

Substituting the resulting parameter vector $\underline{\theta} = (h, a, b)$ in (10) yields under the assumption of a shifted reciprocal gamma law for Y the option price approximation

$$\tilde{p}_0^{\mathrm{SRG}} := (\mu_Y - h) \frac{\Gamma_\omega(a-1)}{\Gamma(a-1)} - (K - h) \frac{\Gamma_\omega(a)}{\Gamma(a)}.$$

4.4. Modified lognormal power-law (MLP)

Finally, we consider for the first time in this context the case of the modified lognormal power-law distribution with initial lognormal distribution parameters m, s and parameter a controlling the tail behaviour, which we calculate by matching raw moments given by

$$\mu_k = \frac{a}{a-k} \exp\left(\frac{s^2 k^2}{2} + mk \right), \quad a > k$$

(see Basu and Jones, 2004). Parameter a can be calculated from

$$\frac{(a-2)^3 a\mu_2^3}{(a-3)(a-1)^3\mu_Y^3} = \mu_3$$

numerically. Given a, the remaining parameters are given by

$$s^2 = \ln\frac{(a-2)\mu_2}{a(a-1)^2\mu_Y^2}, \quad m = -\frac{s^2}{2} + \ln\frac{(a-1)\mu_Y}{a}.$$

The relevant pricing formula in this case is

$$\begin{aligned}\tilde{p}_0^{\mathrm{MLP}} \quad &:= \quad \int_K^\infty \frac{ae^{am+\frac{a^2s^2}{2}}x}{2x^{1+a}}\mathrm{erfc}\left\{\frac{as^2-\ln x+m}{s\sqrt{2}}\right\}dx \\ &\quad -K\left(1-\frac{1}{2}\mathrm{erfc}\left\{\frac{-\ln K+m}{s\sqrt{2}}\right\}-\frac{e^{am+\frac{a^2s^2}{2}}}{2K^a}\mathrm{erfc}\left\{\frac{as^2-\ln K+m}{s\sqrt{2}}\right\}\right),\end{aligned}$$

where $\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt$ is the complementary error function.

5. Approximations based on the first four moments

5.1. Pearson system

The Pearson system is a family of solutions $\tilde{f}(x)$ to the differential equation

$$\frac{1}{\tilde{f}(x)}\frac{d\tilde{f}(x)}{dx} = -\frac{a_0+x}{a_1+a_2x+a_3x^2}$$

whereby well-defined density functions can be derived. The shape of the distribution depends on the Pearson parameters (a_0, a_1, a_2, a_3) and these parameters can be expressed in terms of the first four moments of the distribution, here, of the arithmetic average Y. Therefore, if we know the first four moments, as in the case of Y, we can construct a density function that is consistent with these moments. The different types of probability distributions are classified based on the skewness γ_Y and now also the kurtosis $\varepsilon_Y = (\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4)/v_Y^2$, where μ_4 is the fourth raw moment of Y in (3). Then, we get that

$$a_0 = a_2 = \frac{\sqrt{\gamma_Y}\,(\varepsilon_Y+3)}{10\varepsilon_Y - 12\gamma_Y - 18}\sqrt{\mu_2}, \quad a_1 = \frac{4\varepsilon_Y - 3\gamma_Y}{10\varepsilon_Y - 12\gamma_Y - 18}\mu_2, \quad a_3 = \frac{2\varepsilon_Y - 3\gamma_Y - 6}{10\varepsilon_Y - 12\gamma_Y - 18}$$

(e.g., see Johnson *et al.*, 1994, p. 22). The classification consists of several types according to the η-criterion (Elderton and Johnson, 1969)

$$\eta = \frac{\gamma_Y^2\,(\varepsilon_Y+3)^2}{4\left(4\varepsilon_Y - 3\gamma_Y^2\right)\left(2\varepsilon_Y - 3\gamma_Y^2 - 6\right)}.$$

In particular, we have the *main* types I, IV and VI with $\eta < 0$, $0 < \eta < 1$ and $\eta > 1$, respectively; and the *transition* types, i.e., the normal $\eta = 0$ ($\varepsilon_Y = 3$), II $\eta = 0$ ($\varepsilon_Y < 3$), III $\eta = \pm\infty$, V $\eta = 1$ and VII $\eta = 0$ ($\varepsilon_Y > 3$).

The (undiscounted) option price is given from (10) computed using numerical integration.

5.2. *Johnson transformations*

The Pearson family provides a unique distribution for every possible $(\gamma_Y, \varepsilon_Y)$ combination. The full 4-parameter Johnson (1949) family offers the same flexibility using transformations of the standard normal distribution. In particular, if $Z \sim \mathcal{N}(0,1)$ and Y is a transform of Z, the Johnson family is given by

$$
\begin{array}{lll}
S_L \text{ (lognormal)} & Y(Z) = c + d\exp\left(\frac{Z-a}{b}\right) \Leftrightarrow Z = a + b\ln\left(\frac{Y-c}{d}\right), \\
S_U \text{ (unbounded)} & Y(Z) = c + d\sinh\left(\frac{Z-a}{b}\right) \Leftrightarrow Z = a + b\sinh^{-1}\left(\frac{Y-c}{d}\right), \\
S_B \text{ (bounded)} & Y(Z) = c + d\frac{1}{1+\exp\left(-\frac{Z-a}{b}\right)} \Leftrightarrow Z = a + b\ln\left(\frac{Y-c}{d-Y+c}\right),
\end{array}
$$

where the last two are the *main* types separated by a *transition* type, that is, the lognormal, $a \in \mathbb{R}$ and $b > 0$ are shape parameters, $c \in \mathbb{R}$ a location factor and $d > 0$ a scale factor. By design, a unique family is chosen for any mathematically feasible pair $(\gamma_Y, \varepsilon_Y)$. On the upside, density and distribution functions are given in closed forms, nevertheless, whereas the Pearson system can be fitted spontaneously, the Johnson system is more involved in this respect.

The (undiscounted) option price is given from (10),

$$
\int_{-\infty}^{\infty} (Y(z) - K)^+ \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz,
$$

which is computed using numerical integration.

5.3. *Polynomial expansions*

Polynomial expansions provide possible alternatives for probability density function estimation based on moments. Willems (2019) and Dufresne (2000) present approaches based on orthogonal polynomial expansions and show that their methods converge in the basic Black–Scholes setting. The main downside of the approach of Dufresne (2000) is that it relies on the reciprocal average to ensure convergence and its moments need to be calculated by numerical integration, which raises the computational cost and potentially introduces numerical errors. We exclude the Edgeworth expansion (e.g., see Turnbull and Wakeman, 1991 and Ritchken *et al.*, 1993) as increasing the number of matched moments does not ensure improvement of the approximation, but also the Gram–Charlier expansion based on the choice of an auxiliary normal density as it diverges in most cases of interest[2]. The efficacy of such density approximations lies in the appropriate choice of the auxiliary density function and the corresponding orthonormal polynomials (see Filipović *et al.*, 2013); this becomes even more questionable when the underlying is not modelled as a geometric Brownian motion.

In light of the above discussion, we choose the method which ensures a fair runtime–accuracy balance. Consistently with the logic of Filipović *et al.* (2013) of a density expansion being as close as possible to the unknown density function, we revisit the expansion approach of Willems (2019) based on polynomials that are orthogonal with respect to the lognormal, rather than the normal, distribution to approximate the distribution of the arithmetic average. We recall,

[2]For this, Dufresne and Li (2014) apply the Gram–Charlier expansion to the log-average, but again the moments of this are not available in closed form raising serious computational concerns.

though, that convergence may not be guaranteed beyond the Black–Scholes setting. More specifically, the weight function is then given by the lognormal density

$$w(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{ -\frac{(\ln x - \mu)^2}{2\sigma^2} \right\},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. The approximating density function using the first N integer moments is given by

$$\tilde{f}_N(x) = w(x) \sum_{n=0}^{N} l_n b_n(x),$$

where

$$\begin{pmatrix} l_0 & l_1 & \cdots & l_N \end{pmatrix}^T = L^{-1} \begin{pmatrix} 1 & \mu_1 & \cdots & \mu_N \end{pmatrix}^T$$

and the orthonormal polynomial basis satisfies

$$\begin{pmatrix} b_0(x) & b_1(x) & \cdots & b_N(x) \end{pmatrix}^T = L^{-1} \begin{pmatrix} 1 & x & \cdots & x^N \end{pmatrix}^T$$

with L the Cholesky factor of the Hankel moment matrix M with entries

$$M_{ij} = \int_0^\infty x^{i+j} w(x) dx = e^{(i+j)\mu + (i+j)^2\sigma^2/2}, \quad i, j = 0, \ldots, N,$$

and μ and σ^2 matched with the first two moments of the unknown distribution.

6. Lower bound price approximation

Lastly, in this section we consider a price approximation for Asian options given by a lower bound (LB) which we derive based on the principles delineated in Fusai and Kyriakou (2016). Here, we adapt to the case of the stochastic process (1) and present an analytical expression for the bound in the Fourier domain. For the Asian call option, this is given by

$$\tilde{p}_0^{\text{LB}} := E\left[(Y - K)\, \mathbf{1}_{\{\tilde{Y} > \gamma\}} \right] \leq E\left[(Y - K)^+ \right], \tag{15}$$

where Y is given in (3) and $\tilde{Y}$ is for each of the continuous and discrete averages given by

$$\begin{cases} \text{(continuous)} & \tilde{Y}(T) = \frac{1}{T} \int_0^T X(t) dt \\ \text{(discrete)} & \tilde{Y}_n(T) = \frac{1}{n} \sum_{j=1}^n X(t_j), \text{ where } t_1 < \cdots < t_n = T \end{cases}.$$

This lower bound has the following inverse Fourier transform representation:

$$\tilde{p}_0^{\text{LB}} = \frac{e^{-\delta\gamma}}{2\pi} \int_{\mathbb{R}} e^{-iu\gamma} \frac{\Phi(u;\delta)}{iu + \delta} du, \tag{16}$$

where the constant $\delta > 0$ ensures integrability and $\Phi(u;\delta)$ is given by

$$\begin{cases} \text{(continuous)} & \frac{1}{T} \int_0^T E\left[e^{X(t) + i(u - i\delta)\tilde{Y}(T)} \right] dt - K E\left[e^{i(u - i\delta)\tilde{Y}(T)} \right] \\ \text{(discrete)} & \frac{1}{n} \sum_{k=1}^n E\left[e^{X(t_k) + i(u - i\delta)\tilde{Y}_n(T)} \right] - K E\left[e^{i(u - i\delta)\tilde{Y}_n(T)} \right] \end{cases}. \tag{17}$$

The optimal bound is given for

$$\gamma^* = \underset{\gamma}{\operatorname{argmax}}\ \tilde{p}_0^{\mathrm{LB}}.$$

Computing (17) apparently requires knowledge of the characteristic functions of $\left(X(t), \tilde{Y}(T)\right)$ and $\left(X(t_k), \tilde{Y}_n(T)\right)$. This paper contributes towards that direction for the model dynamics in question. The relevant results are derived in the following two propositions. These involve time integrals of the cumulant generating function, which, as explained in Section 2.1, depending on the BDLP may admit analytic solutions or, alternatively, may need to be computed fast numerically.

Proposition 5 (Continuous average)**.** *The characteristic function of* $\left(X(t), \tilde{Y}(T)\right)$, $0 \le t \le T$, *is given by*

$$\begin{aligned} & E\left[\exp\left(i\xi X(t) + \frac{i\zeta}{T}\int_0^T X(s)ds\right)\right] \\ = \; & \exp\left\{ i\xi X(0)e^{-\alpha t} + i\xi\beta(1-e^{-\alpha t}) + i\zeta\beta + \frac{i\zeta(X(0)-\beta)\left(1-e^{-\alpha T}\right)}{\alpha T} \right. \\ & \left. + \int_0^t \psi\left(i\xi e^{-\alpha(t-s)} + \frac{i\zeta\left(1-e^{-\alpha(T-s)}\right)}{\alpha T}\right)ds + \int_t^T \psi\left(\frac{i\zeta\left(1-e^{-\alpha(T-s)}\right)}{\alpha T}\right)ds \right\}. \end{aligned} \tag{18}$$

Proof. See Appendix A. ■

Proposition 6 (Discrete average)**.** *The characteristic function of* $\left(X(t_k), \tilde{Y}_n(T)\right)$, $0 \le k \le n$, *is given by*

$$\begin{aligned} & E\left[\exp\left(i\xi X(t_k) + \frac{i\zeta}{n}\sum_{j=1}^n X(t_j)\right)\right] \\ = \; & \exp\left\{ \frac{i\zeta\beta}{n}\sum_{j=k+1}^n \left(1-e^{-\alpha(t_j-t_k)}\right) + \sum_{j=k+1}^n \int_{t_{j-1}}^{t_j} \psi\left(\frac{i\zeta}{n}\sum_{i=j}^n e^{-\alpha(t_i-s)}\right)ds \right. \\ & + \left(\frac{i\zeta}{n}\sum_{j=1}^k e^{-\alpha t_j} + i\tilde{\xi}e^{-\alpha t_k}\right)X(0) + \left(\frac{i\zeta}{n}\sum_{j=1}^k \left(1-e^{-\alpha t_j}\right) + i\tilde{\xi}\left(1-e^{-\alpha t_k}\right)\right)\beta \\ & \left. + \sum_{j=1}^k \int_{t_{j-1}}^{t_j} \psi\left(i\tilde{\xi}e^{-\alpha(t_k-s)} + \frac{i\zeta}{n}\sum_{i=j}^k e^{-\alpha(t_i-s)}\right)ds \right\}, \end{aligned} \tag{19}$$

where

$$\tilde{\xi} := \frac{\zeta}{n}\sum_{j=k+1}^n e^{-\alpha(t_j-t_k)} + \xi.$$

Proof. See Appendix A. ■

7. Numerical experiments

In this section, we perform runtime–accuracy comparisons of the various moment-based approximations for the arithmetic average option price presented in Sections 4–5. Computations are done in Matlab R2017b running in Microsoft Windows 10 on an Intel(R) Core(TM) i7-6700HQ machine with CPU @2.60GHz and 16.0GB of RAM. We use as benchmark the results from a very accurate Monte Carlo simulation using the lower bound (15), which is computed exactly in (16), as control variate (henceforth referred to as CV-LB). To this end, we employ standard CV Monte Carlo setup with the CV coefficient estimated in a pilot run, e.g., see Glasserman (2004) and Cont and Tankov (2004). Our choice of the Monte Carlo benchmark is justified by its high accuracy and flexible adaptability to different dynamics, with a, nevertheless, notable computational burden (see later reports in the tables).

In this analysis, we consider three examples of background driving Lévy processes for (1) and different parameter sets: a Brownian motion, DEJD, and NIG process with cumulant generating functions shown in Table 3, i.e., those models that lead to explicit solutions for (9). Nevertheless, with some additional computational effort, one can apply to the other models in Table 3; such results can be made available upon request.

7.1. Computation of moments: the case of continuous average

Apparently, implementing our approximations for the continuous average requires fast and accurate computation of the iterated integrals for the moments given in (8). To this end, we considered different approaches. We found the Gauss–Legendre quadrature (see Press *et al.*, 1992 and Abramowitz and Stegun, 1968, Ch. 25.4) with 24 nodes to be achieving an ideal speed–accuracy balance, requiring for the Gaussian model about a hundredth (half) of a second for the first three (four) integer moments of the continuous average for an, at least, 4-decimal place precision. We confirmed this against Matlab's built-in global adaptive quadrature that took 14 seconds for the first three moments as well as computationally heavy Monte Carlo moment estimates. Changing to NIG or DEJD models slightly uplifts the computing time, due to the more involved computation of (9), to around 1 or 4 seconds (for four moments), respectively. Calculating the moments occupies most of the total time (indeed, Tables 4–5 report total times) required for a continuous Asian option price which is then given from a simple closed-form expression or by (fast) numerical integration for the 4-parameter approximations as shown in Sections 4 and 5. In light of these preliminary results, in what follows we adhere to the use of Gauss–Legendre quadrature.

7.2. Discussion of results

The base parameter values are from Fusai and Kyriakou (2016) and Černý and Kyriakou (2011), which we then vary aiming to assess the accuracy of our moment-based approaches under stressed conditions of volatility and speed of mean reversion. Our comparative analysis encompasses the 3-parameter approximations SLN, SG, SRG, MLP and the orthogonal polynomial expansion based on 3 moments (OP-3), as well as the 4-parameter approximations Pearson (P), Johnson (J) and the OP-4.

In Tables 4–5 we report results for a continuous Asian call option from an implementation of Gauss–Legendre quadrature using 6 and 24 nodes for comparison purposes. Increasing the nodes inevitably increases the computing time, which may vary depending on the underlying model assumption. However, in absolute terms this does not exceed 5 seconds for a 4-moment-based approximation such as J, P or OP-4 for the NIG model. This, nevertheless, reduces also the error.

[Insert Tables 4–5]

Consistently with our expectations, the 4-parameter approximations represent the group of front runners in terms of accuracy. Pearson is the clear winner across the different background driving models, parameter values and monitoring frequency of the averaging (continuous or discrete). This is sometimes also more accurate than the LB. Generally low in the precision rankings we find SG and OP-3 to be. We calculate absolute relative errors against the CV-LB benchmark. More specifically, under market conditions of either low or high volatility for a given speed of mean reversion α, we report in Table 4, respectively, an error (for 24 nodes) of 0.011% and 0.012% for the Gaussian case, 0.042% and 0.001% for the DEJD, and 0.056% and 0.049% for the NIG: the discrepancies for different models and sets of parameter values are generally small. There is an overall homogeneity in the precision achieved of $\pm 10^{-3}$ that these errors map to. In fact, in the high volatility case, Pearson surmounts the lower bound both in terms of computing time and precision. Switching to a higher α in Table 5, still shows Pearson as the best with approximation errors of 0.006% and 0.014% (Gaussian), 0.114% and 0.027% (DEJD), and 0.016% and 0.002% (NIG) which translate to a similar precision as for low α.

In Tables 6–7 we focus on a discretely sampled arithmetic Asian call option and present results for the same model parameterizations. Pearson is again the winning competitor. The impact of changing α is minor for a given low volatility and the error turns out to be for $\alpha = 0.1$ and $\alpha = 0.5$, respectively, 0.001% and 0.003% (Gaussian), 0.067% and 0.067% (DEJD), and 0.072% and 0.040% (NIG); this translates to a precision of up to $\pm 10^{-3}$. We also study the impact of changing α when volatility is high and obtain errors of 0.002% and 0.058% (Gaussian), 0.034% and 0.021% (DEJD), and 0.164% and 0.124% (NIG). Quite similarly to the continuous average case, SG and SLN appear low in the precision rankings and OP-3 around the middle of the rankings but beaten by MLP. The computing time is no more than a second, depending on the model choice, for 12 dates, and this is generally faster than the LB which is sometimes also surpassed in terms of accuracy by Pearson for non-Gaussian dynamics.

[Insert Tables 6–7]

To shed some more light on our discussion of the different approximations, we provide also a few visualizations. Figure 2 presents the density of the log-return process $\ln S$ (from equation 5, where for convenience we assume $S(0) = 1$ and one-year time horizon) and, on the same scale for comparison purposes, also the densities of the log-arithmetic average $\ln Y$ based on a Pearson fit and a shifted lognormal fit to the average Y and a true probability density estimate of $\ln Y$ from exact simulation of Y. We consider the Gaussian as well as the DEJD and NIG background driving processes. From these plots, it first becomes obvious from comparing with the log-return distribution that averaging tapers the thickness of the tails. Second, it is clear that the Pearson fit is the closest to the true distribution of the average (noting that this is only indicative given the simulation error), while the basic shifted lognormal fit diverges from the true distribution of Y especially when we depart from Gaussian driving dynamics. We confirm our observation based on the outcome of a two-sample Kolmogorov–Smirnov test of the difference between the distribution of each of the Pearson and SLN fits from the true distribution. In the case of the Gaussian driving dynamics, the p-value is 1 and 0.9921 for the Pearson fit and 1 and 0.8938 for the SLN fit, for each of the low and high σ, which is overall good news for the standard choice of the SLN approximation although Pearson is performing better. Nevertheless, in favour of Pearson, the discrepancy from the SLN fit increases dramatically when we switch to non-Gaussian driving dynamics with SLN's p-values dropping to 0.003 and 0.01 (for the two DEJD parameter sets) and 0.093 and 0.069 (for the two NIG parameter sets). On the other hand, the p-values remain high for the Pearson fit, in particular, 0.961 and 1 (DEJD parameter sets) and 0.8839 and 0.8938 (NIG parameter sets). Clearly, this confirms the superiority of our 4-parameter approximation against traditional 3-parameter alternatives.

[Insert Figure 2]

8. Risk measures for random annuities

In finance and insurance, typical problems, such as setting of provisions and optimal portfolio selection, come down to computing risk measures related to random sums of the type

$$Y_n(T) := \sum_{j=1}^{n} Z_j e^{X(t_j)}, \tag{20}$$

where $\{Z_j\}$ are non-negative representing payments and $(X(t_1), X(t_2), \ldots, X(t_n))$ is a random vector of returns or discount factors (Vanduffel *et al.*, 2008). The continuous-time analogue of (20) is

$$Y(T) := \int_0^T Z(t) e^{X(t)} dt, \tag{21}$$

where $\{Z(t) : t \geq 0\}$ here is a continuous stream of payments. The discrete case (sum of random variables) or the continuous counterpart (integrated product of stochastic processes) is interpreted as the stochastically discounted value of all future obligations Z. In Norberg's (1999) terminology, Z and $\exp(X)$ are, respectively, the liability/insurance risk related to the insurance portfolio and the asset/financial risk related to the investment portfolio (see also Tang and Tsitsiashvili, 2003).

In this section, based on general driving dynamics for X with possible mean reversion and jumps under the physical probability measure, we extend our theorerical framework to the more general sum-product structure (20)–(21) which traditionally appears in the study of the interplay of insurance and financial risks[3]. We also treat Z generally as a Lévy process. The basic special case of constant Z and Gaussian process X is encompassed in our formulation. We present exact formulae for the moments of (20) and (21) and, based on these, derive an accurate approximation for the cumulative distribution function of Y, thereby allowing an also fast evaluation of the risk measures. This way we tackle the important problem of efficient computation of risk measures of random, continuous or discrete, finite annuities under universal underlying model assumptions.

Proposition 7 (Continuous sum-product)**.** *Let $Y(T)$ be as in (21), where X is given by (1)–(2) and $Z(t) := \exp\left(\left(\mu - \tilde{\psi}(1)\right) t + \tilde{L}(t)\right)$ for $\mu \in \mathbb{R}$ and $\tilde{L}$ an independent Lévy process with*

[3] We thank the editor for pointing towards this direction.

cumulant generating function $\tilde{\psi}$. The nth moment of $Y(T)$ is then given by

$$\begin{aligned} E[Y^n(T)] &= \int_{t_0}^{T} dt_1 \int_{t_0}^{T} dt_2 \cdots \int_{t_0}^{T} \exp\left\{ \sum_{j=1}^{n} \left(X(t_0) e^{-\alpha t_j} + \beta \left(1 - e^{-\alpha t_j}\right) \right.\right. \\ &\quad \left.\left. + \int_{t_0}^{t_j} \psi \left(\sum_{i=j}^{n} e^{-\alpha(t_i - s)} \right) ds + \left(\mu - \tilde{\psi}(1)\right) t_j + \tilde{\psi}(n-j+1)(t_j - t_{j-1}) \right) \right\} dt_n. \end{aligned} \tag{22}$$

Proof. See Appendix A. ■

Proposition 8 (Discrete sum-product). *Let $Y_n(T)$ be as in (20). Based on the same assumptions as in Proposition 7 for the driving dynamics, the mth moment of $Y_n(T)$ is given by*

$$E[Y_n^m(T)] = \sum_{\gamma_1 + \cdots + \gamma_n = m} \begin{pmatrix} m \\ \gamma_1, \ldots, \gamma_n \end{pmatrix} E\left[e^{\sum_{j=1}^{n} \gamma_j \ln Z_j} \right] E\left[e^{\sum_{j=1}^{n} \gamma_j X(t_j)} \right],$$

where

$$\begin{aligned} E\left[e^{\sum_{j=1}^{n} \gamma_j X(t_j)}\right] &= \exp\left\{ \sum_{j=1}^{n} \left(X(0)\gamma_j e^{-t_j} + \beta \gamma_j \left(1 - e^{-t_j}\right) + \int_{t_{j-1}}^{t_j} \psi \left(\sum_{i=j}^{n} \gamma_i e^{-\alpha(t_i - s)} \right) ds \right) \right\}, \\ E\left[e^{\sum_{j=1}^{n} \gamma_j \ln Z_j}\right] &= \exp\left\{ \sum_{j=1}^{n} \left(\left(\mu - \tilde{\psi}(1)\right) t_j \gamma_j + \tilde{\psi} \left(\sum_{i=j}^{n} \gamma_i \right) (t_j - t_{j-1}) \right) \right\}, \end{aligned} \tag{23}$$

and $\begin{pmatrix} m \\ \gamma_1, \ldots, \gamma_n \end{pmatrix}$ *and the sum* $\sum_{\gamma_1 + \cdots + \gamma_n = m}$ *are as defined in Proposition 3.*

Proof. The proof follows that of Proposition 3. In addition, for (23) refer to the last part of the proof of Proposition 7. ■

Although in Propositions 7–8 we assumed independent Lévy processes, the more general case of correlated BDLPs for Z and X can also be accommodated based on the multivariate Lévy model of Ballotta *et al.* (2019) and relevant results can be derived. Finally, adapting to the case of correlated Z and X is straightforward in a Gaussian-driven model setting. To this end, define

$$Z(t) := \exp\left(\left(\mu - \tilde{\sigma}^2/2\right) t + \rho \tilde{\sigma} W_2(t) + \tilde{\sigma} \sqrt{1 - \rho^2} W_1(t) \right) \tag{24}$$

and, without loss of generality upon setting $X(0) = 0$,

$$X(t) := \sigma \int_0^t e^{-\alpha(t-s)} dW_2(s), \tag{25}$$

where $\rho \in [-1, 1]$ controls the correlation between Z and X, and $W_1(t)$ and $W_2(t)$ are independent standard Brownian motions. As is obvious from Propositions 7–8, key quantity in the derivation of the moments' results is

$$E\left[e^{\sum_{j=1}^{n}\gamma_j \ln Z(t_j)+\gamma_j X(t_j)}\right] = \exp\left\{\sum_{j=1}^{n}\left(\left(\mu-\frac{\tilde{\sigma}^2}{2}\right)\gamma_j t_j+\psi\left(\sum_{i=j}^{n}\gamma_i\tilde{\sigma}\sqrt{1-\rho^2}\right)(t_j-t_{j-1})\right.\right.$$
$$\left.\left.+\int_{t_{j-1}}^{t_j}\psi\left(\sum_{i=j}^{n}\gamma_i\left(\tilde{\sigma}\rho+\sigma e^{-\alpha(t_i-s)}\right)\right)ds\right)\right\},$$

where $\psi(u) := u^2/2$ and the integrated cumulant generating function can be computed as explained in Section 2.1.

Adhering to the notation introduced in Section 3 and following Vanduffel *et al.* (2008), we define the p-quantile risk measure

$$Q_p[Y] = \inf\left\{x \in \mathbb{R} \mid \tilde{F}(x) \geq p\right\}, \tag{26}$$

where $p \in (0,1)$ and $\tilde{F}$ is the (approximate) cumulative distribution function of Y in (20) or (21); investing an initial amount equal to $Q_p[Y]$ will enable one to meet all future payments with probability p. We also consider the Conditional Tail Expectation defined by

$$\mathrm{CTE}_p[Y] = E[Y \mid Y \geq Q_p[Y]] = \frac{\int_p^1 Q_q[Y]\,dq}{1-p} \tag{27}$$

for a continuous and strictly increasing function $\tilde{F}$. Finally, the stop-loss premium with retention $K > 0$ of Y, that is, $E\left[(Y-K)^+\right]$ follows from (10) based on the earlier analysis on Asian options in the paper.

In Table 8 we compute the risk measures (26) and (27) under the model assumptions specified in Proposition 7 based on a Pearson curve fit (see Section 5.1) to the first four moments of $Y(T)$ in (21) given by (22). The results from our method are obtained in less than 1 second and are very close to the estimates from Monte Carlo simulation, consistently with our numerical experiments in the previous section. Nevertheless, Monte Carlo converges very slowly taking more than 1 hour for a comparable level of accuracy.

[Insert Table 8]

Given the convenience offered by our fast and accurate computational approach, we perform a sensitivity analysis of Q_p and CTE_p with respect to changes in the parameters of the models underlying the insurance and financial risks. In particular, we want to see how mean reversion and the interplay between the liabilities and the asset returns affect the risk measures. In the spirit of Tang and Tsitsiashvili (2003), we want to look into the impact of the heaviness of the two risks on the risk measures[4]. Therefore, for given $p = 0.95$, in Figure 3 we vary the volatility $\tilde{\sigma}$ of the process Z in (24) and the long-term mean level β and mean reversion speed α of the process X, whereas in Figure 4 we study the effect of changing volatility σ of X and of the instantaneous correlation ρ between the two processes. Qualitatively similar results for

[4]We thank our third reviewer for bringing the particular research to our attention.

$p = 0.75$, for example, are not currently presented in the interest of space, but can be made available upon request.

[Insert Figure 3]

As seen in Figure 3, mostly significant is the boost-up effect of the volatility $\tilde{\sigma}$ of the liabilities on the risk measures. For Q_p, we obtain up to 36% increase when α is high and $\tilde{\sigma}$ grows from 0.1 to 0.5 and a further increase by up to 42% when $\tilde{\sigma}$ is 1. The corresponding increases of CTE_p are slightly higher in the range of 40–50%. This is due to the dominating kurtosis[5] effect of Z over e^X which becomes stronger for large $\tilde{\sigma}$. This is in analogy with Tang and Tsitsiashvili (2003) in the ruin probability context where it is concluded that the main determinant is the heaviest-tailed risk. The risk measures' discrepancies for different β values, and likewise the kurtosis discrepancies between Z and e^X, become more obvious with increasing α. Increasing α has a clear decreasing (increasing) effect on the risk measures when β and $\tilde{\sigma}$ are low (high). These match, respectively, the cases of diametrically smallest kurtosis of Z and e^X (with the former's being smaller) and largest kurtosis of Z and e^X (with the former's being substantially larger).

[Insert Figure 4]

In Figure 4, the effect of the volatility σ of the asset risk on the risk measures Q_p and CTE_p varies with the correlation ρ: the stronger the correlation, the larger the volatility boost-up effect which is now due to the dominating kurtosis of e^X. The risk measures' discrepancies for varying ρ become more obvious for low α and increasing σ. Increasing α clearly reduces the risk measures when σ is moderately high to very high and ρ is positive enough and, again, this is attributed to the more evident decline in the kurtosis of both risk factors in this case; when the correlation is negative there is more uncertainty as to the direction of the volatility effect due to mixing kurtosis levels of Z and e^X.

Our analysis leads us to the conclusion that the parameters that prevail in the kurtosis of the insurance risk or financial risk (or both) are those that ultimately drive the risk measures.

9. Conclusions

We propose and survey different distribution approximations for discrete and continuous sums of random variables which play a pivotal role in various financial and actuarial applications. Our approach relies on moment-based approximations that are flexible with conventional empirical regularities, such as mean reversion and discontinuous movements. To this end, we present a general expression for the raw moments of weighted stochastic sums which we implement within our proposed model framework.

In light of the discussion in the previous sections, we can draw some interesting conclusions. Despite the attention that series expansions have received in the literature of derivatives' pricing, they are shown not to be the best method. Having tested different background driving models under mild or stressed market conditions, we find that a 4-parameter Pearson approximation

[5] We do not explicitly report the kurtosis in the paper in the interest of space.

of the distribution of the continuous or discrete sum yields results which are accurate up to the third decimal place usually. Sometimes this is more accurate than the optimized lower bound for Asian options which we also derive, and perceptibly faster especially under non-Gaussian dynamics where the computations generally become more involved. Other approximations that are popular in Gaussian-driven models, such as the shifted lognormal, gamma and reciprocal gamma, cease performing well once we switch to non-Gaussian models. All in all, we conclude that the Pearson approximation achieves a favourable runtime–accuracy balance which remains robust to changing dynamics, skewness, excess kurtosis, parameter combinations, but also computation of quantities of interest such as option prices or risk measures.

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Appendix A. Proofs

Proof of Lemma 2. From (2),

$$\begin{aligned}\sum_{j=1}^{n}\gamma_j X(t_j) &= \sum_{j=1}^{n}\left(X(0)\gamma_j e^{-\alpha t_j}+\beta\gamma_j(1-e^{-\alpha t_j})+\int_0^{t_j}\gamma_j e^{-\alpha(t_j-s)}dL(s)\right)\\ &= \sum_{j=1}^{n}\left(X(0)\gamma_j e^{-\alpha t_j}+\beta\gamma_j(1-e^{-\alpha t_j})+\int_{t_{j-1}}^{t_j}\sum_{i=j}^{n}\gamma_i e^{-\alpha(t_i-s)}dL(s)\right).\end{aligned}$$

Then, for $h(s) := \sum_{i=j}^{n}\gamma_i e^{-\alpha(t_i-s)}$ and by applying Lemma 1, we get, by independence and stationarity of the increments of L,

$$E\left[\exp\left(\int_{t_{j-1}}^{t_j}\sum_{i=j}^{n}\gamma_i e^{-\alpha(t_i-s)}dL(s)\right)\right]=\exp\left(\int_{t_{j-1}}^{t_j}\psi\left(\sum_{i=j}^{n}\gamma_i e^{-\alpha(t_i-s)}\right)ds\right),$$

from which (6) follows. ■

Proof of Proposition 5. From (2) and conditional expectations,

$$\begin{aligned}& E\left[\exp\left(i\xi X(t)+\frac{i\zeta}{T}\int_0^T X(s)ds\right)\right]\\ =\;& E\left[\exp\left(i\xi X(t)+\frac{i\zeta}{T}\int_0^t X(s)ds\right.\right.\\ & \left.\left.+\frac{i\zeta}{T}\int_t^T\left(X(t)e^{-\alpha(s-t)}+\beta(1-e^{-\alpha(s-t)})+\int_t^s e^{-\alpha(s-w)}dL(w)\right)ds\right)\right]\\ =\;& E\left[\exp\left(\frac{i\zeta(X(t)-\beta)(1-e^{-\alpha(T-t)})}{\alpha T}+\frac{i\zeta\beta(T-t)}{T}+\frac{i\zeta}{T}\int_0^t X(s)ds+i\xi X(t)\right)\right.\\ & \left.E_t\left[\exp\left(\frac{i\zeta}{T}\int_t^T\int_w^T e^{-\alpha(s-w)}ds\,dL(w)\right)\right]\right]\\ =\;& \exp\left(\frac{i\zeta\beta(T-t)}{T}-\frac{i\zeta\beta(1-e^{-\alpha(T-t)})}{\alpha T}+\int_t^T\psi\left(\frac{i\zeta\left(1-e^{-\alpha(T-s)}\right)}{\alpha T}\right)ds\right)\\ & E\left[\exp\left(i\left(\frac{\zeta(1-e^{-\alpha(T-t)})}{\alpha T}+\xi\right)X(t)+\frac{i\zeta}{T}\int_0^t X(s)ds\right)\right]\end{aligned}$$

where the last equality follows from Lemma 1. Replacing where necessary above by

$$X(s)=X(0)e^{-\alpha s}+\beta(1-e^{-\alpha s})+\int_0^s e^{-\alpha(s-w)}dL(w),\quad s\le t,$$

we reach (18) by a further application of the same steps. ■

Proof of Proposition 6. Before proceeding with the proof of the required result, we state the following two equations which are straightforward implications of (2):

$$\sum_{j=k+1}^{n}X(t_j)=X(t_k)\sum_{j=k+1}^{n}e^{-\alpha(t_j-t_k)}+\beta\sum_{j=k+1}^{n}\left(1-e^{-\alpha(t_j-t_k)}\right)+\sum_{j=k+1}^{n}\int_{t_{j-1}}^{t_j}\sum_{i=j}^{n}e^{-\alpha(t_i-s)}dL(s)$$

for $0 \leq k < n$, and

$$X(t_k) = X(0)e^{-\alpha t_k} + \beta\left(1 - e^{-\alpha t_k}\right) + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} e^{-\alpha(t_k - s)} dL(s).$$

Using conditional expectations, the above two results and by applying Lemma 1, we get

$$\begin{aligned}
& E\left[\exp\left(i\xi X(t_k) + \frac{i\zeta}{n}\sum_{j=1}^{k} X(t_j) + \frac{i\zeta}{n}\sum_{j=k+1}^{n} X(t_j)\right)\right] \\
= \; & E\left[\exp\left(i\xi X(t_k) + \frac{i\zeta}{n}\sum_{j=1}^{k} X(t_j) + \frac{i\zeta}{n}\sum_{j=k+1}^{n} \left(X(t_k)e^{-\alpha(t_j - t_k)} + \beta\left(1 - e^{-\alpha(t_j - t_k)}\right)\right)\right)\right. \\
& \left. E_{t_k}\left[\exp\left(\frac{i\zeta}{n}\sum_{j=k+1}^{n}\int_{t_{j-1}}^{t_j}\sum_{i=j}^{n} e^{-\alpha(t_i - s)} dL(s)\right)\right]\right] \\
= \; & \exp\left(\frac{i\zeta\beta}{n}\sum_{j=k+1}^{n}\left(1 - e^{-\alpha(t_j - t_k)}\right) + \sum_{j=k+1}^{n}\int_{t_{j-1}}^{t_j}\psi\left(\frac{i\zeta}{n}\sum_{i=j}^{n} e^{-\alpha(t_i - s)}\right) ds\right) \\
& E\left[\exp\left(i\tilde{\xi} X(t_k) + \frac{i\zeta}{n}\sum_{j=1}^{k} X(t_j)\right)\right] \\
= \; & \exp\left(\frac{i\zeta\beta}{n}\sum_{j=k+1}^{n}\left(1 - e^{-\alpha(t_j - t_k)}\right) + \sum_{j=k+1}^{n}\int_{t_{j-1}}^{t_j}\psi\left(\frac{i\zeta}{n}\sum_{i=j}^{n} e^{-\alpha(t_i - s)}\right) ds\right. \\
& \left. + \left(\frac{i\zeta}{n}\sum_{j=1}^{k} e^{-\alpha t_j} + i\tilde{\xi} e^{-\alpha t_k}\right) X(0) + \left(\frac{i\zeta}{n}\sum_{j=1}^{k}\left(1 - e^{-\alpha t_j}\right) + i\tilde{\xi}\left(1 - e^{-\alpha t_k}\right)\right)\beta\right) \\
& E\left[\exp\left(\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\left(i\tilde{\xi} e^{-\alpha(t_k - s)} + \frac{i\zeta}{n}\sum_{i=j}^{k} e^{-\alpha(t_i - s)}\right) dL(s)\right)\right].
\end{aligned}$$

A further application of Lemma 1 then leads to (19). ■

Proof of Proposition 7. (22) follows the proof of Proposition 4. We have that

$$E\left[Y^n(T)\right] = E\left[\left(\int_0^T Z(s)e^{X(s)} ds\right)^n\right] = \int_{t_0}^{T}\cdots\int_{t_0}^{T} E\left[\prod_{j=1}^{n} Z(t_j)e^{X(t_j)}\right] dt_1 \cdots dt_n. \qquad (28)$$

By independence of Z and X, (28) is equal to

$$\int_{t_0}^{T}\cdots\int_{t_0}^{T} E\left[e^{\sum_{j=1}^{n}\ln Z(t_j)}\right] E\left[e^{\sum_{j=1}^{n} X(t_j)}\right] dt_1 \cdots dt_n,$$

where $E\left[e^{\sum_{j=1}^{n} X(t_j)}\right]$ follows from (6). In addition,

$$\begin{aligned}
E\left[e^{\sum_{j=1}^{n}\ln Z(t_j)}\right] &= e^{\sum_{j=1}^{n}\left(\mu - \tilde{\psi}(1)\right)t_j} E\left[e^{\sum_{j=1}^{n}\tilde{L}(t_j)}\right] = e^{\sum_{j=1}^{n}\left(\mu - \tilde{\psi}(1)\right)t_j} E\left[e^{\sum_{j=1}^{n}(n-j+1)\int_{t_{j-1}}^{t_j} d\tilde{L}(s)}\right] \\
&= e^{\sum_{j=1}^{n}\left(\left(\mu - \tilde{\psi}(1)\right)t_j + \tilde{\psi}(n-j+1)(t_j - t_{j-1})\right)}.
\end{aligned}$$

This completes the proof. ∎

Appendix B. Towards computing the moments of the arithmetic average price

We present here Mathematica® codes that produce the integral (9). In particular, we provide the codes for each of the Gaussian OU, OU with DEJD BDLP and NIG BDLP.

Gaussian OU

```
PSI_G[u_] := u^2*\[Sigma]^2/2
INT_PSI_G[n_] := Simplify[Integrate[
     PSI_G[Sum[Exp[-\[Alpha]*(t[i] - s)], {i, j, n}]], {s, t[j - 1],
      t[j]}]]
```

OU with NIG BDLP

```
PSI_NIG[u_] := (1 -
    Sqrt[1 - 2*\[Theta]*\[Kappa]*u - \[Kappa]*\[Sigma]^2*u^2])/\[Kappa]
INT_PSI_NIG[n_] := Simplify[Integrate[
     PSI_NIG[Sum[Exp[-\[Alpha]*(t[i] - s)], {i, j, n}]], {s, t[j - 1],
      t[j]}]]
```

OU with DEJD BDLP

```
PSI_DEJD[u_] :=
 u^2*\[Sigma]^2/
    2 + \[Lambda]*(p*eta1/(eta1 - u) + (1 - p)*eta2/(eta2 + u) - 1)
INT_PSI_DEJD[n_] := Simplify[Integrate[
     PSI_DEJD[Sum[Exp[-\[Alpha]*(t[i] - s)], {i, j, n}]], {s, t[j - 1],
      t[j]}]]
```

Index	Parameters							No. of contracts
	α	σ	λ	p	η_1	η_2	MSE	
SP500	0.810	0.168	0.931	0.10%	48.551	2.872	163	173
VIX	9.972	1.602	2.997	1.65%	1.013	0.928	0.769	218
ETF	5.220	0.200	2.332	2.38%	1.102	2.576	0.149	62
AAPL	0.308	0.257	1.654	0.05%	1.056	5.729	1.189	537

Table 1: DEJD-driven OU model parameters based on calibration to options based on different asset classes (as on 16/6/2020). MSE: minimized mean squared difference between market prices of plain vanilla put and call options (based on reported number of contracts) with different strikes and maturities and corresponding model prices based on indicated parameter values.

Model	Gaussian		DEJD	
Mean reversion (MR)	×	✓	×	✓
SP500	1306.85	1506.71	521.20	163.01
VIX	75.46	15.18	1.64	0.77
ETF	0.21	0.21	0.16	0.15
AAPL	3.93	2.54	3.28	1.19
	% MSE reduction			
	DEJD vs. Gaussian	MR-DEJD vs. Gaussian	MR-DEJD vs. DEJD	MR-DEJD vs. MR-Gaussian
SP500	60%	88%	69%	89%
VIX	98%	99%	53%	95%
ETF	26%	30%	6%	30%
AAPL	17%	70%	64%	53%

Table 2: MSE of different calibrated models and % reductions based on indicated transitions from models in the ninth to the eighth row of the table.

BDLP	$\psi(u)$	Solution of (9)
Gaussian	$\frac{\sigma^2 u^2}{2}$	Explicit
DEJD	$\lambda\left(\frac{\eta_1 p}{\eta_1 - u} + \frac{\eta_2(1-p)}{\eta_2+u} - 1\right) + \frac{\sigma^2 u^2}{2}$	Explicit
HEJD	$\lambda\left(\sum_{i=1}^{m}\frac{p_i \eta_i}{\eta_i - u} + \sum_{j=1}^{n}\frac{(1-p_j)\theta_j}{\theta_j+u} - 1\right) + \frac{\sigma^2 u^2}{2}$	Explicit
MJD	$\lambda\left(e^{\frac{\delta^2 u^2}{2}+\mu u} - 1\right) + \frac{\sigma^2 u^2}{2}$	Non-analytic
NIG	$\frac{1-\sqrt{-k\sigma^2 u^2 - 2k\theta u + 1}}{k}$	Explicit
VG	$-\frac{\ln\left(-\frac{1}{2}k\sigma^2 u^2 - k\theta u + 1\right)}{k}$	Special function
CGMY	$C\Gamma(-y)\left((G+u)^y - G^y - M^y + (M-u)^y\right)$	Special function
Meixner	$\log\left(\left(\cos\left(\frac{b}{2}\right)\operatorname{sech}\left(\frac{1}{2}(-iau - ib)\right)\right)^{2\delta}\right)$	Non-analytic
GH	$\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{\frac{\lambda}{2}} \frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+u)^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}$	Non-analytic

Table 3: Cumulant generating functions of different background driving Lévy processes (BDLP) and corresponding solutions of (9): Gaussian, double exponential jump diffusion (DEJD), hyperexponential jump diffusion (HEJD), Merton jump diffusion (MJD), normal inverse Gaussian (NIG), variance gamma (VG), Carr–Geman–Madan–Yor (CGMY), Meixner, generalized hyperbolic (GH). Note: $K_\lambda(\cdot)$ is the modified Bessel function of the second kind.

		Nodes = 6			Nodes = 24			Nodes = 24	
Method	Price	Abs. error	Time (sec.)	Price	Abs. err.	Time (sec.)	Price	Abs. err.	Time (sec.)
			Gaussian: $\sigma = 0.1$					Gaussian: $\sigma = 0.5$	
CV-LB	3.1768	–	8.39	3.1768	–	8.39	11.6954	–	7.44
(std. error)	7.8E-08			7.8E-08			5.2E-06		
LB	3.1766	0.006%	0.605	3.1766	0.006%	0.61	11.6809	0.124%	0.35
SLN	3.1875	0.338%	0.002	3.1776	0.024%	0.02	11.7039	0.072%	0.02
SG	3.1883	**0.362%**	0.002	3.1784	**0.049%**	0.02	11.7930	**0.835%**	0.02
SRG	3.1873	0.329%	0.001	3.1773	0.016%	0.02	11.6785	0.144%	0.02
MLP	3.1874	0.334%	0.008	3.1764	0.013%	0.03	11.6774	0.154%	0.02
OP-3	3.1867	0.313%	0.002	3.1775	0.021%	0.02	11.7144	0.163%	0.02
J	3.1875	0.338%	0.016	3.1776	0.024%	0.58	11.6854	0.086%	0.57
P	3.1863	**0.300%**	0.020	3.1771	**0.011%**	0.58	11.6968	**0.012%**	0.57
OP-4	3.1873	0.329%	0.024	3.1773	0.016%	0.56	11.6935	0.017%	0.56
			DEJD: $(\sigma, \lambda) = (0.1, 3)$					DEJD: $(\sigma, \lambda) = (0.5, 5)$	
CV-LB	3.9255	–	19.40	3.9255	–	19.40	12.0331	–	19.51
(std. error)	2.3E-07			2.3E-07			6.0E-06		
LB	3.9253	0.005%	2.893	3.9253	0.005%	2.89	12.0165	0.138%	3.01
SLN	4.0039	1.997%	0.003	3.9898	1.639%	0.04	12.0433	0.085%	0.04
SG	4.0109	**2.176%**	0.004	3.9968	**1.815%**	0.04	12.1400	**0.888%**	0.04
SRG	4.0016	1.939%	0.003	3.9876	1.582%	0.04	12.0164	0.138%	0.04
MLP	3.9871	1.568%	0.005	3.9732	1.214%	0.04	12.0166	0.137%	0.04
OP-3	3.9945	1.759%	0.005	3.9806	1.404%	0.04	12.0542	0.176%	0.04
J	3.9430	0.445%	0.021	3.9307	0.132%	1.42	12.0178	0.127%	1.41
P	3.9176	**0.201%**	0.027	3.9238	**0.042%**	1.44	12.0330	**0.001%**	1.48
OP-4	3.9109	0.373%	0.020	3.9297	0.106%	1.40	12.0276	0.046%	1.46
			NIG: $(\theta, \sigma) = (-0.4091, 0.2637)$					NIG: $(\theta, \sigma) = (-0.6819, 0.4395)$	
CV-LB	7.0756	–	51.97	7.0756	–	51.97	10.8344	–	52.25
(std. error)	1.8E-06			1.8E-06			5.7E-06		
LB	7.0726	0.042%	21.228	7.0726	0.042%	21.23	10.8200	0.133%	22.16
SLN	7.1847	1.542%	0.007	7.1543	1.112%	0.11	10.9743	1.291%	0.09
SG	7.1854	1.552%	0.007	7.1550	1.122%	0.12	11.0056	1.580%	0.10
SRG	7.1845	1.539%	0.009	7.1540	1.108%	0.11	10.9640	1.196%	0.10
MLP	7.1847	1.542%	0.010	7.1543	1.112%	0.11	10.9743	1.291%	0.09
OP-3	7.2828	**2.928%**	0.009	7.2507	**2.475%**	0.11	11.0828	**2.293%**	0.10
J	7.1007	0.355%	0.050	7.0688	0.096%	4.61	10.8202	0.131%	4.09
P	7.0978	**0.313%**	0.042	7.0717	**0.056%**	4.58	10.8398	**0.049%**	4.44
OP-4	7.1968	1.713%	0.040	7.1654	1.269%	4.47	11.0515	2.004%	4.46

Table 4: Continuous arithmetic Asian option prices: case of mean reversion speed $\alpha = 0.1$. Gauss–Legendre quadratures nodes $= \{6, 24\}$. Other parameters: (DEJD) $p = 0.6$, $\eta_1 = \eta_2 = 25$; (NIG) $k = 0.1222$; and $S(0) = K = 100$, $r = 0.0367$, $T = 1$. LB: lower bound; CV-LB (std. error): control variate lower bound (standard error); SLN: shifted lognormal; SG: shifted gamma; SRG: shifted reciprocal gamma; MLP: modified lognormal power law; OP-m = m-moment orthogonal polynomial expansion; J: Johnson; P: Pearson. Boldface entries indicate cases of smallest and largest absolute error (abs. err.) excluding the LB. Computing times are in seconds (sec.).

Method	Price	Abs. err.	Time (sec.)	Price	Abs. err.	Time (sec.)	Price	Abs. err.	Time (sec.)
		Nodes = 6			Nodes = 24			Nodes = 24	
			Gaussian: $\sigma = 0.1$					Gaussian: $\sigma = 0.5$	
CV-LB	2.9075	–	7.48	2.9075	–	7.48	10.2898	–	8.02
(std. error)	2.9E-07			2.9E-07			1.6E-05		
LB	2.9074	0.003%	0.32	2.9074	0.003%	0.32	10.2748	0.146%	0.41
SLN	2.9196	0.417%	0.02	2.9084	0.030%	0.01	10.2938	0.039%	0.02
SG	2.9201	**0.433%**	0.01	2.9088	**0.046%**	0.02	10.3550	**0.633%**	0.01
SRG	2.9195	0.411%	0.01	2.9082	0.025%	0.02	10.2753	0.141%	0.01
MLP	2.9195	0.414%	0.01	2.9083	0.028%	0.02	10.2711	0.182%	0.02
OP-3	2.9192	0.402%	0.01	2.9079	0.015%	0.01	10.3205	0.298%	0.02
J	2.9197	0.421%	0.04	2.9084	0.030%	0.60	10.2834	0.062%	0.56
P	2.9189	**0.393%**	0.03	2.9077	**0.006%**	0.57	10.2884	**0.014%**	0.56
OP-4	2.9195	0.412%	0.03	2.9082	0.025%	0.56	10.3172	0.266%	0.54
			DEJD: $(\sigma, \lambda) = (0.1, 3)$					DEJD: $(\sigma, \lambda) = (0.5, 5)$	
CV-LB	3.5504	–	18.07	3.5504	–	18.07	10.5839	–	18.15
(std. error)	5.4E-07			5.4E-07			1.7E-05		
LB	3.5502	0.006%	2.53	3.5502	0.006%	2.53	10.5669	0.161%	2.58
SLN	3.6193	1.941%	0.003	3.6034	1.491%	0.03	10.5897	0.055%	0.04
SG	3.6239	**2.071%**	0.002	3.6079	**1.619%**	0.03	10.6568	**0.688%**	0.04
SRG	3.6178	1.899%	0.005	3.6019	1.450%	0.03	10.5696	0.135%	0.03
MLP	3.6070	1.593%	0.003	3.5912	1.149%	0.03	10.5653	0.176%	0.04
OP-3	3.6134	1.774%	0.005	3.5976	1.328%	0.03	10.6151	0.295%	0.04
J	3.5604	0.282%	0.018	3.5457	0.133%	1.26	10.5742	0.091%	1.24
P	3.5409	**0.268%**	0.019	3.5545	**0.114%**	1.25	10.5810	**0.027%**	1.23
OP-4	3.5694	0.536%	0.010	3.5553	0.139%	1.27	10.6092	0.239%	1.22
			NIG: $(\theta, \sigma) = (-0.4091, 0.2637)$					NIG: $(\theta, \sigma) = (-0.6819, 0.4395)$	
CV-LB	6.3024	–	43.34	6.3024	–	43.34	9.6307	–	44.54
(std. error)	5.0E-06			5.0E-06			1.7E-05		
LB	6.2996	0.044%	17.18	6.2996	0.044%	17.18	9.6164	0.148%	17.42
SLN	6.4031	1.598%	0.009	6.3685	1.049%	0.10	9.7183	0.910%	0.10
SG	6.4032	1.599%	0.007	6.3685	1.049%	0.09	9.7332	1.065%	0.09
SRG	6.4031	1.598%	0.007	6.3685	1.049%	0.10	9.7133	0.857%	0.09
MLP	6.4031	1.598%	0.007	6.3685	1.049%	0.09	9.7183	0.910%	0.09
OP-3	6.4930	**3.024%**	0.007	6.4564	**2.444%**	0.10	9.8131	**1.894%**	0.09
J	6.3343	0.506%	0.038	6.2997	0.042%	3.97	9.6528	0.229%	3.95
P	6.3326	**0.479%**	0.034	6.3014	**0.016%**	3.96	9.6309	**0.002%**	3.97
OP-4	6.3982	1.519%	0.038	6.3631	0.962%	3.97	9.8010	1.768%	3.95

Table 5: Continuous arithmetic Asian option prices: case of mean reversion speed $\alpha = 0.5$. Other notes: see Table 4.

Method	Price	Abs. err.	Time (sec.)	Price	Abs. err.	Time (sec.)
			Gaussian: $\sigma = 0.1$			
		$\alpha = 0.1$			$\alpha = 0.5$	
CV-LB	3.3970	–	5.77	3.1023	–	5.81
(std. error)	1.1E-07			1.0E-07		
LB	3.3968	0.004%	0.05	3.1021	0.004%	0.08
SLN	3.3960	**0.029%**	0.15	3.1016	**0.020%**	0.16
SG	3.3978	0.026%	0.13	3.1027	0.013%	0.12
SRG	3.3967	0.008%	0.16	3.1020	0.008%	0.12
MLP	3.3967	0.007%	0.15	3.1020	0.007%	0.12
OP-3	3.3963	0.020%	0.14	3.1018	0.015%	0.11
J	3.3970	**0.001%**	0.23	3.1022	**0.003%**	0.18
P	3.3970	0.001%	0.25	3.1022	0.003%	0.20
OP-4	3.3969	0.003%	0.22	3.1021	0.005%	0.18
			DEJD: $\sigma = 0.1$, $\lambda = 3$, $p = 0.6$, $\eta_1 = \eta_2 = 25$			
		$\alpha = 0.1$			$\alpha = 0.5$	
CV-LB	4.1933	–	27.89	3.7816	–	27.99
(std. error)	5.9E-05			5.1E-05		
LB	4.1929	0.009%	2.85	3.7812	0.010%	4.34
SLN	4.2574	1.529%	0.14	3.8334	1.371%	0.11
SG	4.2647	**1.703%**	0.13	3.8381	**1.495%**	0.14
SRG	4.2551	1.473%	0.13	3.8319	1.331%	0.13
MLP	4.2409	1.136%	0.13	3.8216	1.058%	0.14
OP-3	4.2472	1.285%	0.14	3.8271	1.203%	0.13
J	4.1838	0.227%	0.27	3.7753	0.168%	0.25
P	4.1961	**0.067%**	0.24	3.7841	**0.067%**	0.24
OP-4	4.1812	0.289%	0.24	3.7734	0.217%	0.23
			NIG: $\theta = -0.4091$, $k = 0.1222$, $\sigma = 0.2637$			
		$\alpha = 0.1$			$\alpha = 0.5$	
CV-LB	7.5363	–	9.91	6.6864	–	9.99
(std. error)	4.0E-04			3.4E-04		
LB	7.5348	0.020%	2.52	6.6848	0.023%	2.74
SLN	7.4320	1.385%	0.46	6.5901	1.440%	0.35
SG	7.7192	**2.427%**	0.45	6.8462	**2.389%**	0.38
SRG	7.6419	1.400%	0.41	6.7569	1.054%	0.33
MLP	7.6145	1.038%	0.42	6.7512	0.969%	0.31
OP-3	7.6156	1.052%	0.42	6.7513	0.970%	0.35
J	7.5279	0.112%	1.28	6.6820	0.066%	0.97
P	7.5309	**0.072%**	1.27	6.6837	**0.040%**	0.98
OP-4	7.6141	1.032%	1.27	6.7512	0.969%	0.95

Table 6: Discrete arithmetic Asian option prices: case of low volatility. 12 monitoring dates. Other notes: see Table 4.

Method	Price	Abs. err.	Time (sec.)	Price	Abs. err.	Time (sec.)
			Gaussian: $\sigma = 0.5$			
		$\alpha = 0.1$			$\alpha = 0.5$	
CV-LB	12.4356	–	5.91	10.8942	–	5.68
(std. error)	1.9E-03			1.9E-03		
LB	12.4154	0.162%	0.13	10.8723	0.201%	0.05
SLN	12.3781	0.462%	0.13	10.8546	0.364%	0.11
SG	12.5325	**0.779%**	0.12	10.9537	**0.546%**	0.11
SRG	12.4087	0.216%	0.12	10.8682	0.239%	0.12
MLP	12.4142	0.172%	0.12	10.8696	0.226%	0.12
OP-3	12.4031	0.262%	0.12	10.8604	0.311%	0.13
J	12.4289	0.054%	0.21	10.8825	0.108%	0.19
P	12.4358	**0.002%**	0.21	10.8880	**0.058%**	0.19
OP-4	12.4160	0.158%	0.20	10.8771	0.158%	0.19
			DEJD: $\sigma = 0.5$, $\lambda = 5$, $p = 0.6$, $\eta_1 = \eta_2 = 25$			
		$\alpha = 0.1$			$\alpha = 0.5$	
CV-LB	12.7912	–	27.19	11.2023	–	24.61
(std. error)	2.0E-03			2.1E-03		
LB	12.7715	0.154%	3.14	11.1807	0.192%	1.74
SLN	12.7319	0.464%	0.17	11.1614	0.365%	0.15
SG	12.9001	**0.851%**	0.17	11.2720	**0.623%**	0.14
SRG	12.7670	0.190%	0.16	11.1786	0.212%	0.15
MLP	12.7684	0.179%	0.17	11.1800	0.199%	0.14
OP-3	12.7627	0.223%	0.17	11.1698	0.290%	0.14
J	12.7856	0.044%	0.27	11.1914	0.097%	0.26
P	12.7956	**0.034%**	0.28	11.1999	**0.021%**	0.25
OP-4	12.7740	0.135%	0.26	11.1840	0.164%	0.25
			NIG: $\theta = -0.6819$, $k = 0.1222$, $\sigma = 0.4395$			
		$\alpha = 0.1$			$\alpha = 0.5$	
CV-LB	11.5623	–	10.34	10.1895	–	9.93
(std. error)	1.3E-03			1.2E-03		
LB	11.5551	0.062%	3.46	10.1890	0.005%	2.63
SLN	11.7024	1.211%	0.38	10.3035	1.119%	0.42
SG	11.7484	1.609%	0.33	10.3891	1.959%	0.40
SRG	11.6663	0.899%	0.31	10.2863	0.950%	0.44
MLP	11.6663	0.899%	0.31	10.2863	0.950%	0.44
OP-3	11.8058	**2.106%**	0.32	10.3956	**2.023%**	0.43
J	11.5311	0.270%	0.95	10.1660	0.231%	1.03
P	11.5813	**0.164%**	0.97	10.1769	**0.124%**	1.01
OP-4	11.5097	0.455%	0.96	10.1521	0.367%	1.02

Table 7: Discrete arithmetic Asian option prices: case of high volatility. 12 monitoring dates. Other notes: see Table 4.

	p-quantile risk measure			Conditional Tail Expectation	
p	Monte Carlo	4-moment Pearson fit	p	Monte Carlo	4-moment Pearson fit
0.25	0.9387	0.9385	0.25	0.7934	0.7912
0.5	1.0058	1.0057	0.5	0.5577	0.5476
0.75	1.0862	1.0786	0.75	0.2683	0.2875
0.95	1.1935	1.1947	0.95	0.0579	0.0626
0.99	1.2739	1.2853	0.99	0.0124	0.0134
0.995	1.3007	1.3204	0.995	0.0069	0.0069

Table 8: p-quantile risk measure $Q_p[Y(T)]$ and Conditional Tail Expectation $\mathrm{CTE}_p[Y(T)]$ for varying p and $Y(T)$ as in (21). Moments of (21) follow from formula (22) by solving iterated integrals using Gauss–Legendre quadrature (30 nodes). Model parameters: $X(0) = 0$, $\alpha = 0.1$, $\beta = 0.2$, $\sigma = 0.1$, $\mu = 0$, $\tilde{\sigma} = 0.15$, $\rho = 0$. Monte Carlo estimates correspond to 5×10^7 simulation trials and 2000 time steps over the interval $[0, T]$ with $T = 1$.

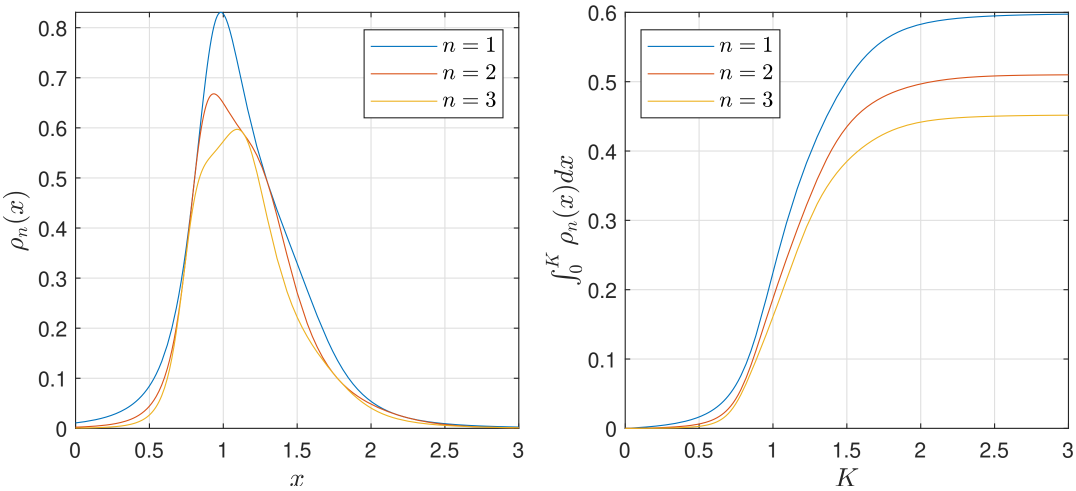

Figure 1: Error upper bounds for a continuum of strike prices K. ρ_n (left plot): the case of an approximating distribution function that shares $2n$ moments with the original distribution function (see 11). $\int_0^K \rho_n(x)dx$ (right plot): the case of the fixed-strike Asian put option price with the unknown distribution function of the arithmetic average approximated by a distribution function that has the first $2n$ moments matched (see 12).

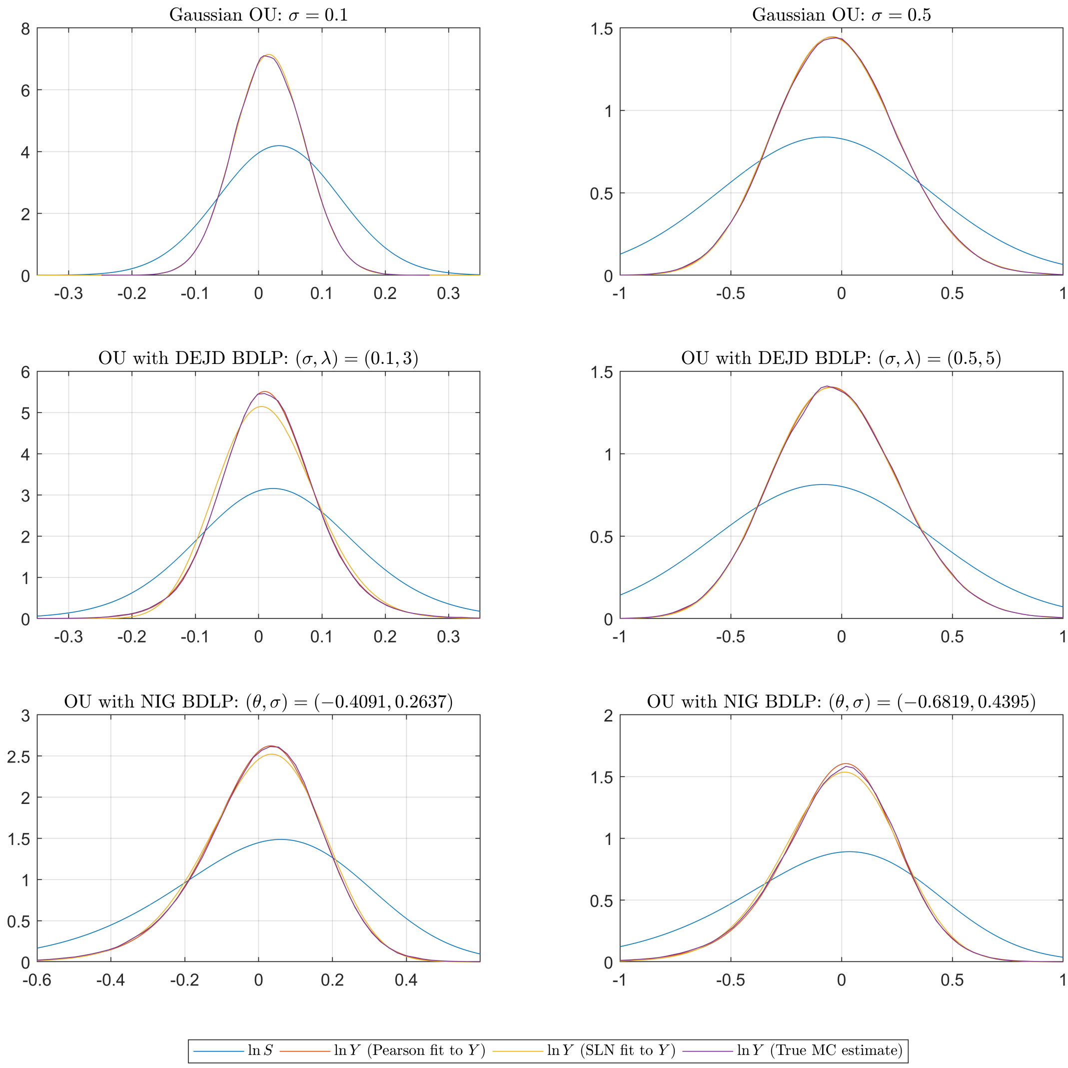

Figure 2: Probability density functions of the log-return process $\ln S$ (from equation 5, where for convenience we assume $S(0) = 1$ and one-year time horizon); log-arithmetic average $\ln Y$: based on fitted Pearson distribution to Y; based on fitted shifted lognormal distribution to Y; true probability density estimate obtained using Matlab's *ksdensity* based on exact Monte Carlo samples of $\ln Y$.

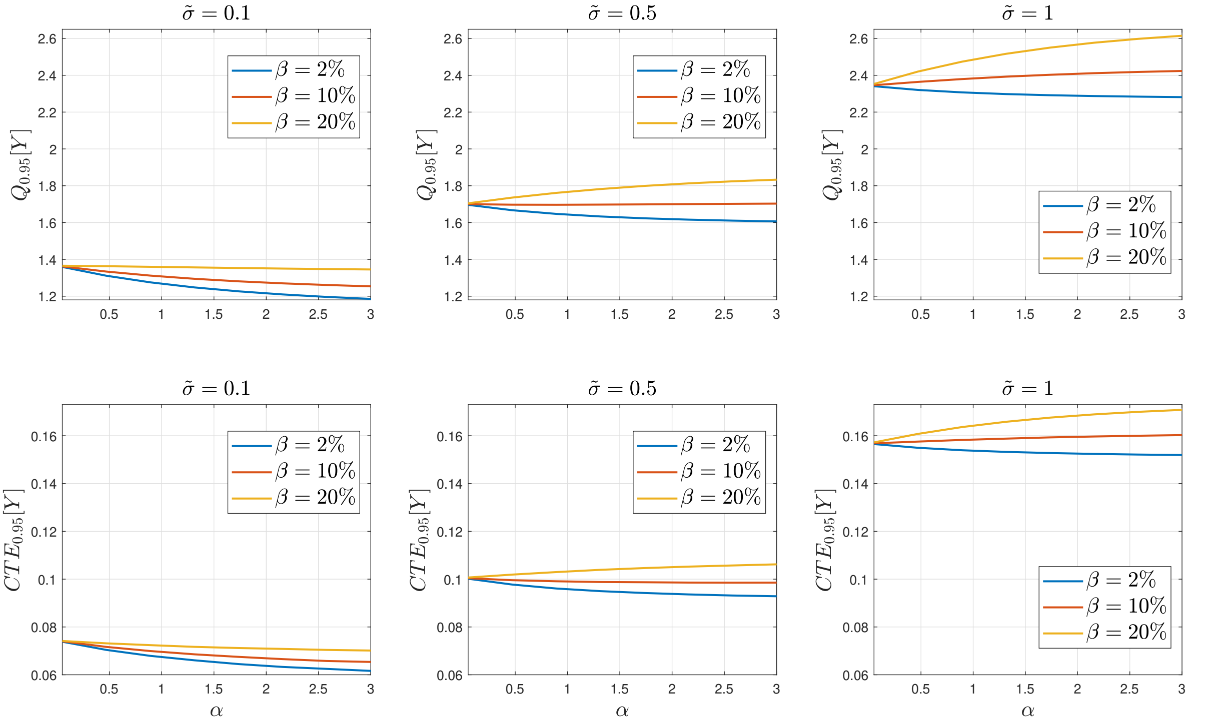

Figure 3: p-quantile (Q_p) and Conditional Tail Expectation (CPE$_p$) for varying parameters α, β and $\tilde{\sigma}$. Other parameters: $p = 0.95, X(0) = 0, \mu = 0, \sigma = 0.3, \rho = 0$ and $T = 1$.

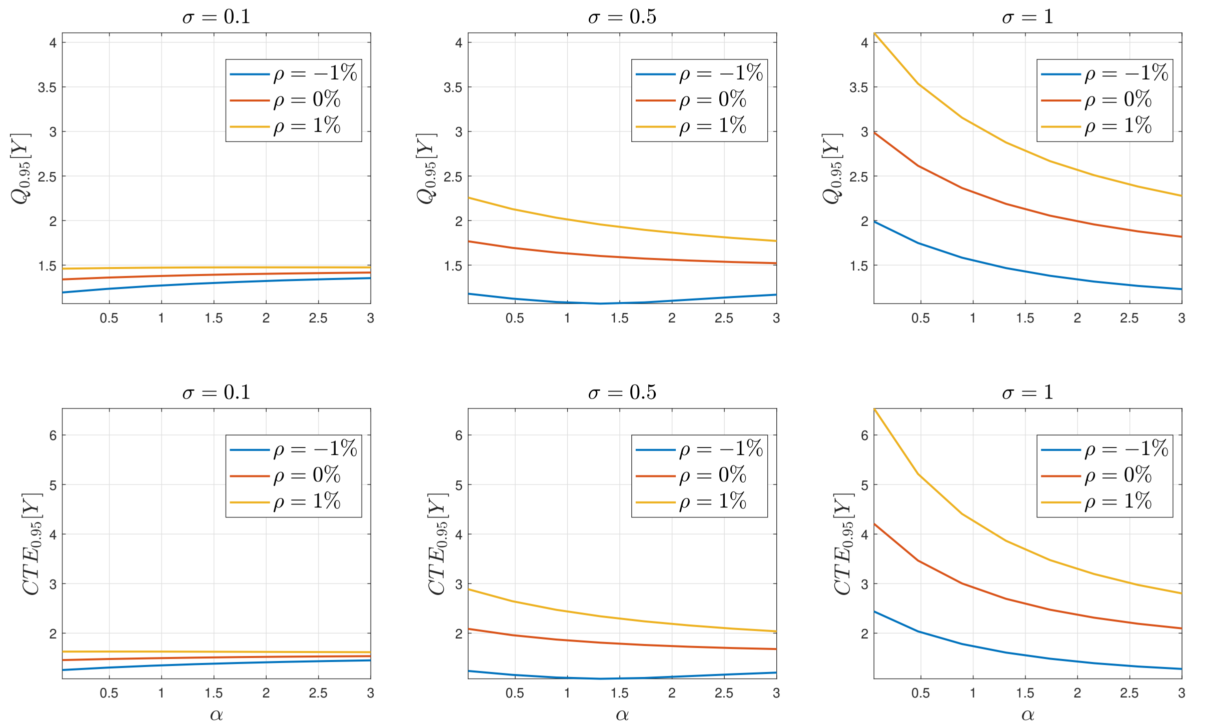

Figure 4: p-quantile (Q_p) and Conditional Tail Expectation (CPE$_p$) for varying parameters α, ρ, σ. Other parameters: $p = 0.95, X(0) = 0, \mu = 0, \tilde{\sigma} = 0.3, \beta = 0.1$ and $T = 1$.