




Review

# Capital Allocation Rules and the No-Undercut Property

Gabriele Canna <sup>1,†</sup> , Francesca Centrone <sup>2,\*</sup>  and Emanuela Rosazza Gianin <sup>1,†</sup> 

<sup>1</sup> Department of Statistics and Quantitative Methods, Università di Milano Bicocca, 20126 Milano, Italy; gabriele.canna@unimib.it (G.C.); emanuela.rosazza1@unimib.it (E.R.G.)

<sup>2</sup> Department of Studies for Economics and Business, Università del Piemonte Orientale, 28100 Novara, Italy

\* Correspondence: francesca.centrone@uniupo.it

† These authors contributed equally to this work.

**Abstract:** This paper makes the point on a well known property of capital allocation rules, namely the one called *no-undercut*. Its desirability in capital allocation stems from some stability game theoretical features that are related to the notion of core, both for finite and infinite games. We review these aspects, by relating them to the properties of the risk measures that are involved in capital allocation problems. We also discuss some problems and possible extensions that arise when we deal with non-coherent risk measures.

**Keywords:** capital allocation; risk measures; cooperative games; Choquet integral

## 1. Introduction

Banks and financial institutions face different sources of riskiness (market risk, credit risk, ...): for this reason and in order to guarantee the ability of financial institutions to hedge such riskiness, regulators impose a capital requirement (or margin) on them. In the literature, there are many possible ways to measure risk, e.g., by means of the well known Value at Risk or of the Conditional Value at Risk (see [1–4] for a detailed treatment, while [5–7] for Value at Risk's models). Conditional Value at Risk belongs to the class of coherent risk measures (see [1,2,8,9]), which is, risk measures that are defined by four axioms: monotonicity, cash-additivity, subadditivity, and positive homogeneity. In order to enhance the role that is played by the idea of risk diversification (pooling of risky portfolios should not increase risk), the notion of coherent risk measure has been later extended to include the class of convex risk measures (see [10,11]) and the one of quasi-convex risk measures (see [12]), where the former is motivated by liquidity arguments, while the latter takes the right formulation of diversification when cash-additivity is dropped into account.

Anyway, whatever risk measure is chosen, one of the most relevant problems that is connected to the use of risk measures for firms and insurances is the one of *capital allocation*. Once a suitable risk measure  $\rho$  is fixed and the corresponding risk capital  $\rho(X)$  that is associated to a risky position  $X$  is determined, the capital allocation problem consists in finding a division of this aggregate capital among the constituents of the activity, such as business units or various insurance lines (later called sub-portfolios). For instance, this problem is particularly meaningful in the context of risk management, or when comparing the return of various business units in order to remunerate managers.

As it can be easily understood, there are many possible ways to allocate the aggregate capital of a company to its sub-units, according to the features that one wants to capture and to the properties that one wishes to verify. In this respect, a huge literature has grown over the years, and several methods have been proposed (see, for example, [2,13–19]), where the different approaches have motivations that can be either axiomatic or financial. One among the several approaches to the problem is linked to cooperative game theory and, in particular, to transferable utility games. Indeed, in cooperative game theory, the focus is on methods satisfying some desirable properties and allowing to share or allocate the gains (or costs) that are derived from the cooperation of the players. In this respect,



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solution concepts, such as the core and the Shapley value, are among the most popular ones (see [20]). In this stream, starting with the work of Denault [14], firms are initially seen as indivisible players of a cooperative transferable utility (TU) cost game derived by a risk measure, and an allocation is defined as *coherent*, if, among some desirable properties, it also fulfills the one of belonging to the core of the cost game itself. Denault [14]'s work also then concentrates on extending the cost game approach to games with fractional players (fuzzy games, see also [21]) and defines the desirable allocations through the concept of Shapley value (where the idea is to assign to each player its marginal contribution to the overall risk). This approach is only suitable for coherent and differentiable risk measures, but some extensions have been proposed in the literature also inspired by the game theoretical concept of Shapley value. For example, the capital allocation method that was proposed in [19] and in [13] makes use of the so-called *Aumann–Shapley* capital allocation rule, which is also suitable for (quasi-)convex and non-differentiable risk measures.

In an axiomatic framework, instead, Kalkbrener [16] defines a *capital allocation rule* as a map whose values depend on the profit and loss (or on the return) of both a portfolio and its sub-portfolios, and that is required to satisfy some suitable properties w. r. t. the chosen risk measure. More precisely, a capital allocation rule (CAR) for a monetary risk measure  $\rho: L^\infty \rightarrow \mathbb{R}$  is a map  $\Lambda: L^\infty \times L^\infty \rightarrow \mathbb{R}$  such that  $\Lambda(X, X) = \rho(X)$  for every  $X \in L^\infty$ , where  $\Lambda(X, Y)$  represents the capital that is allocated to  $X$  when seen as a sub-unit of  $Y$  and it is interpreted as the risk contribution of a sub-position  $X$  to the risk of the aggregated position  $Y$ . For any coherent risk measure, Kalkbrener [16] shows that there exists a capital allocation rule satisfying reasonable properties. When the coherent risk measure  $\rho$  is also Gateaux differentiable, the Kalkbrener [16]'s approach yields the so-called *gradient* or *Euler* allocation, which is largely applied in practice and well known in the literature (see also [17,18]).

Dhaene et al. [15] highlight some of the financial aspects of capital allocation: indeed, its key purposes for a firm consist in distributing the cost of capital among the various business units and being able to make a comparison of their performances through the return of allocated capital. The approach of Centrone and Rosazza Gianin [13] and Tsanakas [19] refer both to the axiomatic approach and the game theoretic stream that was proposed by Denault [14] (also see Csóka et al. [22]).

It is customary to require some properties on the capital allocation rule that is specified by the method. One of the most relevant is called full allocation (see [14,16]), and it requires that the whole risk capital has to be allocated among the sub-units. Instead, another one requires that the capital allocated to a coalition of sub-portfolios does not exceed the risk capital of the coalition itself, and it is stated in the literature by Denault [14] under the name of *no-undercut*, while it is formulated in a more general way under the name of *diversifying* in Kalkbrener [16]'s work. Although meaningful, full allocation is not always needed (see [23] and the references therein for a proper discussion), while no-undercut seems to be the main property. Indeed, the appeal of no-undercut consists in its relation with the existence of core allocations (for games with a finite number of players, see Denault [14]), which, in turn, allows for interpreting it as a stability requirement where no coalition of sub-units is incentivized to split from the main one, since, whenever split, it would incur a greater cost.

It is worth recalling that, more recently, the problem of capital allocation has been defined also in a set-valued context (see [24,25]): indeed, in the latest years, the theory of risk measures has been extended to the more general setting, where they can be set-valued (see [26,27]), motivated by financial considerations. For example, in the case of portfolios of financial positions in different currencies that can not be aggregated due to liquidity constraints and/or transaction costs (see [26,27]), it is reasonable to consider risk measures that associate, to any portfolio in different currencies, a set of hedging deterministic positions. In this framework, Centrone and Rosazza Gianin [24] defined a set-valued capital allocation rule (with respect to a set-valued risk measure  $R$ ) as a set-valued map  $\Lambda$  that associates to every  $X, Y \in L_d^\infty$  (here,  $X$  and  $Y$  are  $d$ -dimensional vectors

of random variables, with each component in  $L^\infty$ , a set  $\Lambda(X, Y)$  of deterministic positions that can be allocated to  $X$  in order to compensate for its risk as a sub-portfolio of  $Y$ , and such that  $\Lambda(X, X) = R(X)$ . Hence, also in this context, no-undercut has been defined and studied: its interpretation is that, given a portfolio  $Y$  and a sub-portfolio  $X$ , all of the deterministic positions that hedge the risk of  $X$  when it is seen as a sub-portfolio of itself, also covers the risk of  $X$  as a sub-portfolio of  $Y$ . Similarly to the scalar case, a set-valued capital allocation rule satisfying no-undercut and based on directional derivatives of set-valued risk measures has been obtained for the case of coherent risk measures (see [24]).

In the present work, we recover the game theoretic stream that was proposed by Denault [14] (also see Csóka et al. [22]) and reframe some results in the context of general TU games, especially concentrating on the no-undercut property. We make use of the version of this property given by Kalkbrenner [16] in order to investigate the relation between the no-undercut property in the context of risk measures that are derived from Choquet integrals, and the core of TU games that are defined by submodular capacities. More precisely, we consider a monotone game  $v: \mathcal{F} \rightarrow \mathbb{R}$  and define a risk measure  $\rho_v$  in terms of a Choquet integral with respect to  $v$ . We then define a linear CAR by means of a Choquet integral with respect to an additive game  $\mu$ . Finally, we show that no-undercut is equivalent to  $\mu$  belonging to the core of  $v$ . We also investigate how the no-undercut property is linked to the existence of core allocations that are derived from capital allocation rules in the case of distorted probabilities.

The paper is organized, as follows. In Section 2 we recall some standard notions and results about risk measures, capital allocation rules, and games; in Section 3, we briefly review the work of [14] and provide some extensions to the general case, and Section 4 is devoted to the conclusions with some possible future research lines.

## 2. Preliminaries

In this section, we recall some notions and results regarding games, risk measures, and capital allocations that will be useful throughout the work.

### 2.1. Transferable Utility Games and Choquet Integrals

For a proper discussion on this topic, we refer to [28–30].

Let  $(\Omega, \mathcal{F})$  be a couple, where  $\Omega$  is a set and  $\mathcal{F}$  is an algebra of sets on  $\Omega$ . A *transferable utility (TU) game* is a set function  $v: \mathcal{F} \rightarrow \mathbb{R}$  satisfying  $v(\emptyset) = 0$ . For every  $S \in \mathcal{F}$ , the real number  $v(S)$  is interpreted as the maximal worth that the members of the coalition  $S$  can obtain when they cooperate. A game  $v$  is called finite when  $\Omega$  is a finite set and  $\mathcal{F}$  is the set of all subsets of  $\Omega$ , and it is normalized if  $v(\Omega) = 1$ . Furthermore, a game is called a:

**Capacity** if it is monotone:  $v(A) \leq v(B)$  whenever  $A \subseteq B$ , with  $A, B \in \mathcal{F}$ .

**Charge** if it is finitely additive:  $v(A \cup B) = v(A) + v(B)$  for all pairwise disjoint sets  $A, B \in \mathcal{F}$ .

**Measure** if it is countably additive:  $v(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} v(A_i)$  for all pairwise disjoint sequence of sets  $(A_i)_{i \in \mathbb{N}} \subseteq \mathcal{F}$ . (Notice that some authors define a measure as a positive countably additive game.)

**Submodular** if  $v(A \cup B) + v(A \cap B) \leq v(A) + v(B)$  for all  $A, B \in \mathcal{F}$ .

Given a game  $v$ , its variation norm is defined as

$$\|v\| := \sup \left\{ \sum_{i=1}^n |v(A_i) - v(A_{i-1})| \right\},$$

where the supremum is taken over all finite chains  $\emptyset = A_0 \subseteq A_1 \subseteq \dots \subseteq A_n = \Omega$ . We denote by  $\text{ba}(\Omega, \mathcal{F})$  the space of all charges with finite variation norm.

The *core* of a game  $v$  is classically defined (see [29]) as:

$$\mathcal{C}_v := \{ \mu \in \text{ba}(\Omega, \mathcal{F}) \mid \mu(A) \geq v(A) \text{ for all } A \in \mathcal{F}, \text{ and } \mu(\Omega) = v(\Omega) \}. \tag{1}$$

When the game has a finite set of players  $N = \{1, \dots, n\}$ , the core can be then expressed as:

$$C_\nu := \left\{ K \in \mathbb{R}^n \mid \sum_{i \in S} K_i \geq \nu(S) \text{ for all } S \subseteq N \text{ and } \sum_{i \in N} K_i = \nu(N) \right\}, \tag{2}$$

where  $K_i$  denotes the  $i$ -th component of  $K \in \mathbb{R}^n$ .

Thus, if an allocation is in the core, then it is undominated in the sense that no coalition has incentive to split off from the grand coalition  $N$ , because the total amount which is allocated to  $S$  is not smaller than the amount  $\nu(S)$  that the players can achieve by forming the sub-coalition  $S$ .

The nonemptiness of the core of a TU game  $\nu$  is well known to be equivalent to the so called *balancedness* property of the game itself (see, for example, [29,31]), which is, for all  $\lambda_1, \dots, \lambda_n \geq 0$  and all  $A_1, \dots, A_n \in \mathcal{F}$ , such that  $\sum_{i=1}^n \lambda_i \mathbb{1}_{A_i} = 1$ , holds:

$$\sum_{i=1}^n \lambda_i \nu(A_i) \leq \nu(\Omega). \tag{3}$$

Bondareva [32] and Shapley [33] have proved the equivalence between nonemptiness of the core and balancedness for finite games, and Schmeidler [34] for infinite games.

Let  $B(\mathcal{F})$  be the set of all bounded  $\mathcal{F}$ -measurable functions. The Choquet integral of  $X \in B(\mathcal{F})$  with respect to a capacity  $\nu$  is defined as

$$E_\nu[X] := \int X d\nu := \int_{-\infty}^0 (\nu(X \geq t) - \nu(\Omega)) dt + \int_0^{+\infty} \nu(X \geq t) dt.$$

It is well known that the Choquet integral is monotone, positively homogeneous, and translation invariant. Furthermore, if  $\nu$  is submodular, then the Choquet integral is also subadditive.

A very popular class of Choquet integrals is the one generated by (*concave*) *distorted probabilities*  $\nu := f \circ \mathbb{P}$ , where the function  $f: [0, 1] \rightarrow [0, 1]$  is (concave), increasing, and satisfies  $f(0) = 0$  and  $f(1) = 1$ .

We refer to the work of Carlier and Dana for game theoretical applications of distorted probabilities and their relations with the notion of core [35].

It is well known that the Choquet integral is finitely additive when  $\nu$  is a charge and it coincides with the standard Lebesgue integral when  $\nu$  is a positive measure. Furthermore, it is monotone with respect to games, which is, if  $\nu$  and  $\mu$  are two games such that  $\nu(A) \leq \mu(A)$  for all  $A \in \mathcal{F}$ , then  $E_\nu[X] \leq E_\mu[X]$  for all  $X \in B(\mathcal{F})$ .

### 2.2. Risk Measures and Capital Allocation Rules

In the following, given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a time horizon  $T$ ,  $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$  denotes the space of all  $\mathbb{P}$ -essentially bounded random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$ . Equalities and inequalities have to be understood to hold  $\mathbb{P}$ -almost surely. It is well known that the norm dual space of  $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$  can be identified with the space  $\text{ba}(\Omega, \mathcal{F}, \mathbb{P})$  of all charges in  $\text{ba}(\Omega, \mathcal{F})$  absolutely continuous w. r. t.  $\mathbb{P}$  (see [36]).

For the following definitions, we mainly refer to Föllmer and Schied [37].

The first risk measure that has been introduced in the literature is Value at Risk (VaR) (see [3]). To be more precise, the Value at Risk of a random variable  $X$  at the level  $\alpha \in (0, 1)$  is defined as

$$\text{VaR}_\alpha(X) := -q_\alpha^+(X) := -\inf\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) > \alpha\},$$

where  $q_\alpha^+(X)$  denotes the upper  $\alpha$ -quantile of  $X$ . In particular,  $\text{VaR}_\alpha(X)$  represents the maximal loss that one can have with a probability greater than  $\alpha$ . Value at Risk has been quite popular as a risk measure, mainly because the Basel Accord initially prescribed its use to financial institutions in order to evaluate risk margins. Different models of VaR

computations have been considered in the literature, e.g., parametric or numerical models (see [5–7] and the references therein for an exhaustive treatment). Because, for normal distributions with zero mean,  $\text{VaR}_\alpha(X)$  is proportional to the standard deviation of  $X$ , a key point for some numerical models of VaR consists in the volatility estimation by means, e.g., of GARCH models (see, among many others, [7,38,39], and [40,41] for the relative backtesting).

Despite its wide use, Value at Risk presents different drawbacks; the most relevant is that, in general, it does not encourage diversification of risk (see [1] for a detailed discussion). Therefore, the family of coherent risk measures has been introduced in order to go beyond the drawbacks of VaR and define risk measures axiomatically. A relevant example of coherent risk measure that is related to Value at Risk is Conditional Value at Risk is defined as

$$\text{CVaR}_\alpha(X) := \frac{\mathbb{E}[(q_\alpha(X) - X)^+]}{\alpha} - q_\alpha(X),$$

where  $q_\alpha(X)$  is any  $\alpha$ -quantile of  $X$ , or, equivalently, as

$$\text{CVaR}_\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\gamma(X) d\gamma.$$

Note that  $\text{CVaR}_\alpha(X)$  can be interpreted as the average of VaRs at levels that are smaller than or equal to  $\alpha$ ; hence, as the expected loss on the tail of the distribution corresponding to losses exceeding  $\text{VaR}_\alpha(X)$  (see [4,37] for further details).

In general, a map  $\rho: L^\infty \rightarrow \mathbb{R}$  is called a *coherent* risk measure if it satisfies the following conditions:

**Monotonicity:** if  $X \leq Y$  ( $X, Y \in L^\infty$ ), then  $\rho(X) \geq \rho(Y)$ .

**Cash-additivity:**  $\rho(X + m) = \rho(X) - m$  for any  $m \in \mathbb{R}$  and  $X \in L^\infty$ .

**Subadditivity:**  $\rho(X + Y) \leq \rho(X) + \rho(Y)$  for any  $X, Y \in L^\infty$ .

**Positive homogeneity:**  $\rho(\lambda X) = \lambda\rho(X)$  for any  $\lambda \geq 0$  and  $X \in L^\infty$ .

See Artzner et al. [1] and Delbaen [9] for further details.

Roughly speaking, for any  $Y \in L^\infty$  (representing the profit and loss at time  $T$  of a financial position),  $\rho(Y)$  quantifies the riskiness of  $Y$  in terms of the margin (or capital requirement) needed to hedge the riskiness of such a position. In particular, when positive,  $\rho(Y)$  denotes the minimal cash amount (margin) that is needed to make the new position  $Y + \rho(Y)$  acceptable.

More, in general, a map  $\rho: L^\infty \rightarrow \mathbb{R}$  only satisfying monotonicity and cash-additivity is called a *monetary* risk measure, while *normalized* if  $\rho(0) = 0$ .

By liquidity arguments, coherent risk measures have been generalized to *convex* risk measures (see Föllmer and Schied [10], Frittelli and Rosazza Gianin [11]), which is,  $\rho$  satisfying monotonicity, cash-additivity and

**Convexity:**  $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$  for any  $\lambda \in [0, 1]$ ,  $X, Y \in L^\infty$ .

Positive homogeneity and convexity, together, are equivalent to positive homogeneity and subadditivity.

For a normalized capacity  $\nu$  that is absolutely continuous with respect to  $\mathbb{P}$  (see [28]),

$$\rho_\nu(X) := \mathbb{E}_\nu[-X], \quad X \in L^\infty,$$

defines a coherent risk measure (see [9,37] for further details). A very popular class of such risk measures is the one that is generated by (*concave*) *distorted probabilities*  $\nu := f \circ \mathbb{P}$ , where the function  $f: [0, 1] \rightarrow [0, 1]$  is (concave), increasing, and satisfies  $f(0) = 0$  and  $f(1) = 1$  (see [2,9]).

It is worth mentioning the class of *distortion exponential* convex risk measures that were introduced by Tsanakas [19] as a generalization of the exponential premium principle of Gerber [42].

The class of games given by distorted probabilities is also popular in actuarial mathematics. Indeed, Wang et al. [43] showed that, under suitable assumptions, the premium of an insurance contract has a representation as a Choquet integral w. r. t. a distorted probability.

Moreover, any coherent risk measure that satisfies a continuity property and comonotone additivity is induced by a normalized and submodular capacity  $\nu$  by means of

$$\rho(X) = \sup_{\mu \in \mathcal{C}_\nu} E_\mu[-X], \quad X \in L^\infty;$$

see [2,9,44] for the precise statement and further details.

Capital allocation is a relevant problem for risk measures that arise from their use in the context of firms and insurances or of portfolios formed by different positions in risky assets, as mentioned in the introduction. The main idea of the capital allocation problem can be summarized, as follows (see [14,16]). Given a risk measure  $\rho$  and an aggregate position  $X$  (for instance,  $X$  can be the Profit and Loss of a portfolio that formed by positions in different stocks or the return of a firm with different business lines or sectors), the capital allocation problem consists in finding how to share the margin  $\rho(X)$  among the different sub-units, in a suitable way. Because the key point of capital allocation is to evaluate what the impact of any sub-unit on the overall position is, such a problem is somehow related to systemic risk measures (see [45] and the references therein).

We now recall the standard definition of a capital allocation rule.

**Definition 1** (see [16]). *Given a risk measure  $\rho$ , a capital allocation rule (CAR) with respect to  $\rho$  is a map  $\Lambda_\rho: L^\infty \times L^\infty \rightarrow \mathbb{R}$  such that*

$$\Lambda_\rho(X, X) = \rho(X) \quad \text{for any } X \in L^\infty.$$

We refer the reader to Denault [14], Kalkbrener [16], and Centrone and Rosazza Gianin [13] for more details.

Given a pair  $(X, Y) \in L^\infty \times L^\infty$ ,  $\Lambda(X, Y)$  can be interpreted as the capital to be allocated to  $X$  as a sub-portfolio of the whole position  $Y$  and, in general, it represents a cost being linked to  $\rho(Y)$ . To be more precise, given a generic set of random variables  $\mathcal{X}$ , we say that  $X \in \mathcal{X}$  is a sub-portfolio (or sub-unit) of  $Y \in \mathcal{X}$  if there exists  $Z \in \mathcal{X}$  such that  $Y = X + Z$ . Because, in the present paper, we work with  $\mathcal{X} = L^\infty$  (hence, with a vector space), every pair of random variables can be seen as a pair of, respectively, a sub-portfolio and a portfolio.

It is customary in the literature to assume some properties on capital allocation rules. We remind the most significant (see [14,16,19]):

**Full allocation:**  $\Lambda(Y, Y) = \sum_{i=1}^n \Lambda(Y_i, Y)$  for any  $Y_1, \dots, Y_n, Y \in L^\infty$  with  $\sum_{i=1}^n Y_i = Y$ .

**Linearity:**  $\Lambda(aX + bZ, Y) = a\Lambda(X, Y) + b\Lambda(Z, Y)$  for any  $a, b \in \mathbb{R}$  and  $X, Y, Z \in L^\infty$ .

**Diversifying:**  $\Lambda(X, Y) \leq \Lambda(X, X)$  for any  $X, Y \in L^\infty$ .

Note that, for the capital allocation rules where  $\Lambda(X, X) = \rho(X)$  holds by definition (see also [23] for a discussion on the weakening of this condition), diversifying can be rewritten as

$$\Lambda(X, Y) \leq \rho(X) \quad \text{for any } X, Y \in L^\infty. \tag{4}$$

Full allocation requires that the sum of the capital that is allocated to any sub-unit  $Y_i$  is equal to the capital that is allocated to the whole portfolio  $Y = \sum_{i=1}^n Y_i$ . Notice that linearity is stronger than full allocation. Diversifying property means that the capital that is allocated to  $X$  considered as a sub-portfolio of  $Y$  does not exceed the capital that is allocated to  $X$  considered to be a stand-alone portfolio, which is, when a risk measure is involved, the risk contribution of  $X$  does not exceed its risk capital. This is a well known property in standard capital allocation problems, which, together with other ones, is required for a

fair (also called coherent) allocation of risk capital (see Centrone and Rosazza Gianin [13], Denault [14], and Kalkbrener [16]). The following section will provide a further discussion on the diversifying property (and on its link with the no-undercut property).

To end this section, we just briefly review the definition and interpretation of set-valued capital allocation rule, given the recent increasing popularity of set-valued risk measures. For the sake of brevity, we will omit the definition of the set-valued risk measure as well as some technical details, which are not essential for the understanding of the subject. We just recall that, for a set-valued risk measure  $R$ ,  $R(X)$  is interpreted as the set of all the deterministic positions hedging the risk of  $X$ . The reader can refer to [24,26,27] for further details.

Let  $L_d^\infty$  be the usual linear space of  $\mathbb{P}$ -equivalence classes of  $\mathcal{F}$ -measurable functions  $X: \Omega \rightarrow \mathbb{R}^d$ , such that  $\text{ess sup}_{\omega \in \Omega} |X(\omega)| < +\infty$ . The value of each component  $X_i$ ,  $i = 1, \dots, d$ , of the random vector  $X$  is interpreted as the profit and the loss of asset  $i$  in some market, at maturity  $T$ . The classical notion of capital allocation rule of Kalkbrener [16] can be extended to this framework, as follows (see also [24]). Given a set-valued risk measure  $R$ , a *set-valued capital allocation rule* (associated to  $R$ ) is a set-valued map

$$\Lambda: L_d^\infty \times L_d^\infty \rightarrow \mathbb{F}_M$$

satisfying  $\Lambda(X, X) = R(X)$  for any  $X$ , where  $\mathbb{F}_M$  denotes a suitable family of subsets of a linear space  $M$  of  $\mathbb{R}^d$  with dimension  $1 \leq m \leq d$ . Additionally, in this setting, the property of no-undercut can be stated as:

**No-undercut:**  $\Lambda(X, Y) \supseteq R(X)$  for any  $X, Y \in L_d^\infty$ .

We interpret  $\Lambda(X, Y)$  as the set of all deterministic positions that can be allocated to  $X$  in order to compensate for its risk as a sub-portfolio of  $Y$ . Hence, no-undercut means that all of the deterministic positions that hedge the risk of  $X$  as a sub-portfolio of itself, also cover the risk of  $X$  as a sub-portfolio of  $Y$ . Therefore, as in the scalar case, under no-undercut there is no incentive to split the sub-portfolio  $X$  from the whole  $Y$ , since the riskiness  $R(X)$  of  $X$  as a stand-alone portfolio is contained in  $\Lambda(X, Y)$ .

Under suitable assumptions, some capital allocation rules satisfying no-undercut, which are based on the representation theorems for coherent/convex set-valued risk measures and on directional derivatives, have been proposed in ([24], Propositions 3, 8), thus extending some ideas of the scalar case, in particular, the one that is linked to the gradient allocation, where, to each sub-portfolio, is allocated its marginal contribution to the risk of the whole portfolio.

### 3. On the No-Undercut Property

In this section, we discuss various aspects of the no-undercut property (as defined by Denault [14]) in connection with the properties of both the capital allocation rules and the risk measures involved. We start by examining the link with the game theoretical features.

The game theoretical interpretation of the no-undercut property allows for us to interpret it as a stability requirement, since it implies that a sub-portfolio  $X$  has no incentive to split from the whole portfolio  $Y$ , as staying alone would be more costly.

In the framework of TU cooperative games, assuming that each financial institution or portfolio is composed by a fixed finite number  $n$  of sub-units, a capital allocation problem can be rephrased in terms of a so-called cost game, where each sub-unit can be seen as a player. Indeed, some results in this perspective can be found in Denault [14] and they are summarized here below. Other results regarding capital allocation and cooperative games can be found in Csóka et al. [22].

Now, set  $N := \{1, \dots, n\}$  for  $n \in \mathbb{N}$ , consider  $n$  portfolios  $X_i \in L^\infty, i \in N$ , and let  $X = \sum_{i \in N} X_i$ . Additionally, assume that we are provided with a risk measure  $\rho$  and define the cost game with  $n$  players by:

$$c(S) := \rho\left(\sum_{i \in S} X_i\right) \tag{5}$$

for all coalitions  $S \subseteq N$ . Subsequently, in the terminology of Denault [14], an allocation principle is defined as a function that maps every  $(X_1, \dots, X_n)$  into a vector:

$$\Gamma := \begin{bmatrix} \Lambda(X_1, X) \\ \Lambda(X_2, X) \\ \vdots \\ \Lambda(X_n, X) \end{bmatrix}, \tag{6}$$

with  $X = \sum_{i \in N} X_i$  and such that  $\sum_{i=1}^n \Lambda(X_i, X) = \rho(X)$ . It is easily noticed that this amounts to considering a linear capital allocation rule  $\Lambda$ , for a subdivision into  $n$  sub-portfolios of each portfolio  $X$ .

We recall that, for a cost game  $c$ , the core is expressed by

$$\mathcal{C}_c := \left\{ K \in \mathbb{R}^n \mid \sum_{i \in S} K_i \leq c(S) \text{ for all } S \subseteq N \text{ and } \sum_{i \in N} K_i = c(N) \right\}. \tag{7}$$

See [14] for more details.

For every coalition,  $S$ ,  $c(S)$  represents a cost to be shared among its players, the stability feature—expressed by the core condition—requires that the sum of the costs allocated to  $S$  is smaller than the cost that the coalition would face if it splits from  $N$ .

Additionally, in general, in terms of charges, the core of a cost game  $v$  is also defined (see [2,29,37]) as

$$\mathcal{C}_v := \{ \mu \in \text{ba}(\Omega, \mathcal{F}) \mid \mu(A) \leq v(A), \text{ for all } A \in \mathcal{F}, \mu(\Omega) = v(\Omega) \}. \tag{8}$$

Note that the link between this notion of core and classical one recalled in (1) is given by the notion of dual game. Indeed, the dual of a game  $v$  is defined as the game  $\bar{v}(A) = v(\Omega) - v(A^c)$ , for every  $A \in \mathcal{F}$ . Therefore, it is immediate to see that  $\mu \in \mathcal{C}_v$  (where core is defined in the standard way, see (1) and [29]) if and only if  $\mu \in \mathcal{C}_{\bar{v}}$  (as defined above, see also [2,37]).

In the setting above and for a linear capital allocation rule  $\Lambda$ , the no-undercut property can be formulated as

$$\Lambda\left(\sum_{i \in S} X_i, X\right) \leq \rho\left(\sum_{i \in S} X_i\right) \tag{9}$$

for all  $S \subseteq N$ , see [14].

Therefore, it can be easily seen that the diversifying property of Kalkbrener [16] reduces to the no-undercut of Denault [14] for linear CARs  $\Lambda$ . However, in Kalkbrener [16]’s work, the number of sub-units constituting a firm is not a priori fixed, but the notions of sub-unit and sub-portfolio are fully general. From now on, we will then call no-undercut the diversifying property. The only assumption that is required in Kalkbrener [16]’s work is that the capital assigned to a sub-unit  $X$  of a firm  $Y$  only depends on  $X$  and  $Y$ , and not on how  $Y$  is decomposed.

We recall the following result, which was proved by Denault [14] for finite games.

**Proposition 1** (see [14]). *Let  $\Lambda$  be a linear CAR with respect to  $\rho$  and let  $c$  be the cost game associated to  $\rho$ . Subsequently,  $\Lambda$  satisfies no-undercut if and only if  $\Gamma$  in (6) belongs to the core of  $c$ .*



Hence, the stability feature of the allocations ensured by the no-undercut property is evident: if an allocation rule  $\Lambda$  satisfies no-undercut, then no player is incentivized to refuse the subdivision of costs resulting from  $\Lambda$ . Indeed, forming a subcoalition  $S$  of  $N$  would yield to  $S$  the cost  $c(S)$ , which is higher than the aggregate cost that results from the sum of the allocated quantities.

It is well known that the core of a game can be empty. Hence, one natural question to ask, in order to have the no-undercut property for a CAR, is under which conditions on the risk measure  $\rho$  defining the cost game the core is nonempty. Denault [14] shows that subadditivity and positive homogeneity of the risk measure are sufficient in guaranteeing that the core is nonempty: indeed, such properties imply the balancedness of the cost game, hence the nonemptiness of its core. In fact, the equivalence between nonemptiness of the core and the property of balancedness is well known (see [32,33]).

Note that subadditivity (at least on sums of  $X_i$ 's) is also necessary for the core to be nonempty (see also [14]). Assume, indeed, that the core of  $c$  is non-empty or, equivalently,  $c$  is balanced. Subsequently, by taking  $\lambda_i = 1$  and  $A_i = \{i\}$  for any  $i \in N$  in the formulation of (3) for cost games,

$$\rho\left(\sum_{i \in N} X_i\right) = c(N) \leq \sum_{i \in N} \lambda_i c(A_i) = \sum_{i \in N} \rho\left(\sum_{j \in A_i} X_j\right) = \sum_{i \in N} \rho(X_i).$$

Therefore, it follows that the subadditivity of  $\rho$  at least on the sum  $\sum_{i \in N} X_i$  is necessary for the nonemptiness of the core of  $c$ .

Now, motivated by the literature on infinite games as well as by the more general definition of Kalkbrenner [16], which does not assume just a fixed number of sub-portfolios for each portfolio, we investigate whether the general form of Kalkbrenner [16]'s no-undercut can still be seen as a core property. Hence, consider the Choquet integral w. r. t. a capacity  $\nu$ : if  $\nu$  is also submodular and normalized, then  $\rho_\nu$  is coherent and the following representation holds:

$$\rho_\nu(X) = \max_{\mu \in \mathcal{C}_\nu} E_\mu[-X],$$

with  $\mathcal{C}_\nu$  as in (8) (see [9]). We henceforth assume that  $\nu$  is submodular and thus that the previous representation holds. Then, for every fixed  $Y \in L^\infty$ , we can choose a positive and normalized  $\mu_Y \in \text{ba}(\Omega, \mathcal{F})$  and define

$$\Lambda(X, Y) := E_{\mu_Y}[-X]$$

for all  $X \in L^\infty$ . Therefore, the following is straightforward.

**Proposition 2.**  $\Lambda$  satisfies the no-undercut (as formulated in (4)) if and only if  $\mu_Y \in \mathcal{C}_\nu$ , for all  $Y \in L^\infty$ . Moreover,  $\Lambda$  is a CAR with respect to  $\rho_\nu$  if and only if  $\mu_Y \in \arg \max_{\mu \in \mathcal{C}_\nu} E_\mu[-Y]$ , for all  $Y \in L^\infty$ .

**Proof.** Fix  $Y \in L^\infty$ . If  $\Lambda$  satisfies no-undercut, then

$$\mu_Y(A) = \Lambda(-\mathbb{1}_A, Y) \leq \rho_\nu(-\mathbb{1}_A) = \nu(A)$$

for every  $A \in \mathcal{F}$ . Since  $\mu_Y \in \text{ba}(\Omega, \mathcal{F})$  is positive and normalized by construction, the only if of the first part of the Proposition follows. Conversely, if  $\mu_Y \in \mathcal{C}_\nu$ , then

$$\Lambda(X, Y) = E_{\mu_Y}[-X] \leq \max_{\mu \in \mathcal{C}_\nu} E_\mu[-X] = \rho_\nu(X)$$

holds for all  $X, Y \in L^\infty$ ; that is, no-undercut holds.

If  $\Lambda$  is a CAR with respect to  $\rho_\nu$ , then

$$E_{\mu_Y}[-Y] = \Lambda(Y, Y) = \rho_\nu(Y) = \max_{\mu \in \mathcal{C}_\nu} E_\mu[-Y],$$

thus,  $\mu_Y$  is a maximizer. Conversely, if  $\mu_Y \in \arg \max_{\mu \in \mathcal{C}_v} E_\mu[-Y]$  then

$$\Lambda(Y, Y) = E_{\mu_Y}[-Y] = \max_{\mu \in \mathcal{C}_v} E_\mu[-Y] = \rho_v(Y).$$

Because  $Y \in L^\infty$  is arbitrary, the Proposition follows.  $\square$

The previous proposition shows how the no-undercut property is equivalent to  $\mu_Y$  belonging to the core of  $v$ , thus it extends the result of [14] to the case where the number of sub-portfolios is not necessarily finite.

We point out that using the monotonicity of the Choquet integral with respect to the game is an alternative way to prove the previous result (see [28]).

The following result, also giving a link between the no-undercut property and the core of a game in the particular case of a distorted probability, can be proved by duality arguments and by Proposition 2. Note that an alternative and direct proof can be driven fully by duality, without using Proposition 2.

**Proposition 3.** *Let  $v = f \circ \mathbb{P}$  be a concave distorted probability and  $\Lambda$  be a  $\|\cdot\|_\infty$ -continuous (w. r. t. the first argument) linear capital allocation rule of the coherent risk measure  $\rho(X) = E_v[-X]$ . If no-undercut holds, then  $\Lambda$  gives rise to an element of  $\mathcal{C}_v$ .*

**Proof.** For every fixed  $Y \in L^\infty$ , the map  $\Lambda(\cdot, Y): L^\infty \rightarrow \mathbb{R}$  is a  $\|\cdot\|_\infty$ -continuous linear functional on  $L^\infty$  and, thus, it belongs to the norm dual  $(L^\infty)'$ . The same holds for the map  $-\Lambda(\cdot, Y)$ . By Theorem IV.8.16 of Dunford and Schwartz [36], there exists a  $\mu_Y$  (depending on the fixed  $Y \in L^\infty$ ) that belongs to  $\text{ba}(\Omega, \mathcal{F}, \mathbb{P})$ , such that

$$-\Lambda(X, Y) = E_{\mu_Y}[X], \quad \text{for all } X \in L^\infty;$$

thus (by the linearity of the integral)

$$\Lambda(X, Y) = E_{\mu_Y}[-X], \quad \text{for all } X \in L^\infty.$$

Furthermore,  $\mu_Y$  is positive and normalized (also see Remark A.51 of Föllmer and Schied [37]). Since  $\Lambda$  has the same form of the map that is involved in Proposition 2,  $\mu_Y \in \mathcal{C}_v$  by no-undercut of  $\Lambda$ . Moreover, as  $\Lambda$  is assumed to be a CAR with respect to  $\rho$ , by Proposition 2, it follows that  $\mu_Y \in \arg \max_{\mu \in \mathcal{C}_v} E_\mu[-X]$ . The latter clearly implies  $\Lambda(-\mathbb{1}, Y) = \mu_Y \in \mathcal{C}_v$ .  $\square$

The main interest of the previous propositions lies in the fact that they show that the link between the no-undercut property and the existence of stable allocations, in a game theoretical sense, is also maintained in a general setting. Indeed, in principle, there is no rationale for a set of portfolios in being decomposed into an a priori fixed set of sub-portfolios, as in Denault [14] and Csóka et al. [22]; hence, this assumption can be too restrictive. For instance, the very well known Gradient or Euler allocation is derived in a general setting in the work of Kalkbrener [16] without requiring a fixed number of sub-portfolios; the same is true for the extensions that are provided by Centrone and Rosazza Gianin [13] and Tsanakas [19]. Moreover, our Proposition 2 can be seen as an extension of Proposition 1 to the framework of Kalkbrener [16]’s work. Indeed, thanks to the representation of  $\rho_v$  that is derived in [2], following Kalkbrener [16], we obtain the following CAR for  $\rho_v$ :

$$\Lambda^K(X, Y) := E_{\mu_Y}[-X]$$

where  $X, Y \in L^\infty$  and  $\mu_Y \in \arg \max_{\mu \in \mathcal{C}_v} E_\mu[-Y]$ . Notice that the latter is in line with the framework of Proposition 2. In particular, our proposition points out that  $\Lambda^K$  satisfies the no-undercut property thanks to the representation of  $\rho_v$  and the properties of the core. Thus, it provides a further interpretation of Kalkbrener [16]’s approach for the special case of risk measures that are defined by means of a Choquet integral with respect to a sub-modular

capacity. A future goal would be to go beyond the case of games that are submodular capacities, despite their interest in risk measure theory (see, among many, [19,43]) and game theory (see, for example, [20,35,46]).

#### 4. Conclusions

The problem of *capital allocation* is crucial to financial entities, as it represents the process by which the total capital requirement is assigned to the different sub-portfolios or business lines. In the literature, there is a consistent number of capital allocation methods that correspond to different objectives; indeed, one popular approach is based on a set of reasonable axioms (see [16]), and it is also related to cooperative game theory, as, for example, in the works of Denault [14] and Csóka et al. [22], while others are mainly based on the evaluation of performance of portfolios/activities (see [17]), optimization principles (see [15]), or pricing issues (see [47]).

In this paper, we have focused on the first streamline initiated by the works of Denault [14] and Kalkbrenner [16], by concentrating, in particular, on the study of the property known in the literature as *no-undercut* (or, sometimes, as *diversifying*). This property entails game theoretical features in that, when satisfied, it guarantees that the allocation to the various sub-portfolios of a portfolio is stable (no sub-coalition of sub-portfolios can benefit from splitting).

It is easily seen that, in the context of games with a fixed and finite number  $n$  of players (where, in the framework of Denault [14], the players are financial institutions or portfolios), a capital allocation rule satisfying no-undercut is equivalent to the corresponding allocation vector being in the core of a particular cost game. Starting from this result, we have examined the general case of a measure space of players, by obtaining an analogous result, which is a first attempt to link the no-undercut property with the notion of core in the case of games with a not-necessarily finite and a priori fixed set of players, although in the particular case of monotone games. Although not fully general, the result is given anyway for a popular class of games: indeed, distorted probabilities (which are monotone games) are widely considered in the game theoretical literature, from the seminal work of Aumann and Shapley [20]. Future possible investigation can take several directions: (a) trying to extend the results beyond the class of capacities through methods involving derivatives of games à la Aumann–Shapley (see [20]); (b) clarifying the link between the capital allocation rules that are defined in Propositions 1 and 2, the core and the gradient allocation under Gateaux differentiability of the Choquet integral, possibly making use of results on the core of supermodular games (see [46]); and, (c) extending the framework and characterization of Csoka et al. [22] to the case of infinite games in order to apply them in the general capital allocation framework.

For what concerns, instead, the range of validity of the no-undercut property, it is well known that, if the risk measure is subadditive and positive homogeneous, then no-undercut holds for the Aumann–Shapley allocation rule (see [19,21]). Indeed, recall that this allocation rule requires the Gateaux differentiability of the risk measure involved. Instead, the fulfilling of this property can become problematic when leaving the world of coherent risk measures. Indeed, for convex risk measures that are not-coherent, some “subadditivity-like” conditions for the exponential-distortion risk measures are given in Tsanakas [19]. Moreover, Theorem 4.2 (a) of Kalkbrenner [16] implies that, when considering capital allocation problems with respect to a convex but not coherent risk measure, it is impossible to have a capital allocation rule satisfying linearity and no-undercut at the same time. The trade-off between these two properties must be then taken into consideration when dealing with capital allocation (see [23] for a discussion on this topic). Instead, the family of “Aumann–Shapley-like” capital allocation rules  $\Lambda^{\text{AS}}$  of [13] for convex (but not coherent) risk measures, satisfies a *generalized* no-undercut property, i.e.,

$$\Lambda^{\text{AS}}(X, Y) \leq \rho(X) + A_F(Y), \quad \text{for all } X, Y \in L^\infty;$$

where  $A_F(Y)$  represents a sort of penalty term, due to the liquidity issues that arise when dealing with convex risk measures.

Turning to the quasi-convex case, the maps  $\Lambda^{\text{AS}}$ , in general, do not satisfy full allocation, hence no-undercut can be fulfilled in some cases. For this case, it holds:

$$\Lambda^{\text{AS}}(X, Y) \leq \rho(X) + M_K(X, Y), \quad \text{for all } X, Y \in L^\infty;$$

where  $M_K$  is a term depending on the function  $K$  that appears in the dual representation of quasi-convex risk measures (see [12]). When  $M \leq 0$ , the no-undercut property is satisfied.

For the sake of completeness on the subject, we have also made a short focus on the problem of capital allocation in the set-valued context (see [24,25]) whose definition has been motivated by the recent introduction (due to financial reasons) of the theory of set-valued risk measures (see [26,27]). Additionally, in this context, no-undercut has been defined and studied, and some of the results in the streamline of the gradient allocation have been derived (see [24]).

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