Capital Allocation Rules and Acceptance Sets

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Abstract

This paper introduces a new approach to face capital allocation problems from the perspective of acceptance sets, by defining the family of sub-acceptance sets. We study the relations between the notions of sub-acceptability and acceptability of a risky position as well as their impact on capital allocation rules; in this context, indeed, capital allocation rules are interpretable as tools for assessing the contribution of a sub-portfolio to a given portfolio in terms of acceptability instead of necessarily involving a risk measure. Furthermore, we investigate under which conditions on a capital allocation rule a representation of an acceptance set holds in terms of the capital allocation rule itself, thus extending to this setting the interpretation, classical in risk measures theory, of minimal amount required to hedge a risky position.

Keywords: Capital allocation, acceptance sets, convex risk measures, quasi-convex risk measures.

1 Introduction

In the literature, capital allocation problems are classically studied and associated to risk measures. Indeed, given a monetary risk measure $\rho : L^\infty \to \mathbb{R}$, a capital allocation rule (CAR) is a map $\Lambda_\rho : L^\infty \times L^\infty \to \mathbb{R}$ such that $\Lambda_\rho(Y;Y) = \rho(Y)$ for every $Y \in L^\infty$ (Kalkbrener [16]). Many popular methods (e.g. Euler's method, Aumann-Shapley allocation (Centrone and Rosazza Gianin [2], Kalkbrener [16], Tasche [18]), beyond requiring linearity of Λ_ρ in the first variable, are also such that $\Lambda_\rho(X;Y)$ is interpretable as the risk contribution of a sub-unit X to a position Y.

It is also well known that monetary risk measures are the natural counterpart of acceptance sets (Artzner et al.[1], Föllmer and Schied [13]) hence, in the classical sense, any capital allocation rule also takes into account the acceptability of a stand-alone risky position X allocating no positive capital to acceptable positions. What is instead missing is the consideration of what happens in terms of acceptability when X is "merged into another position" Y and how this possibly affects the allocation of capital. Indeed, consider a situation where we are provided with a monetary risk measure ρ that qualifies a position X as non-acceptable. If X is anyway considered as a sub-portfolio of another position Y, and we look at the marginal contribution $\rho_Y(X) := \rho(Y) - \rho(Y - X)$, the risk of X can potentially change, and it can become acceptable w.r.t. the monetary risk measure $\rho_Y(\cdot)$, not contributing to the risk of Y. We wish thus to rephrase the problem of capital allocation in a way that takes into account this eventuality, instead of simply sharing $\rho(Y)$ among its sub-units. In other words, if we consider a portfolio Y, we want to define CAR as maps assigning to each sub-portfolio X of Y a capital that reflects their acceptability as sub-units of Y, and does not necessarily assign a share $\Lambda(X;Y)$ of $\rho(Y)$.

The capital allocation problem is thus disentangled from the use of risk measures, and revisited in terms of a different definition and a newly introduced concept, that is, the one of a *sub-acceptance* family of sets. Under suitable assumptions, we derive capital allocation rules reflecting the idea above starting from acceptance and sub-acceptance sets and, conversely, we show that capital allocation rules having some natural properties give rise to acceptance and sub-acceptance sets in terms of which they can be represented, thus extending to these capital allocation rules the classical interpretation of capital requirement typical of risk measures. The situation becomes even more interesting when we consider quasi-convex risk measures, where every quasi-convex risk measure is associated to a family of acceptance sets and one speaks of *acceptability at different levels*. In analogy with what happens with monetary risk measures, in this case we will need *families* of sub-acceptance sets.

To sum up, the main contributions of the paper are the following: firstly, we introduce a new approach to capital allocation problems by means of the concepts of sub-acceptance and acceptance sets. Secondly, under suitable assumptions, we derive a representation theorem for capital allocation rules in terms of families of sub-acceptance sets in quite a general (convex, quasi-convex, S-additive) framework. Finally, we investigate the correspondence between properties at the level of capital allocation rules and those at the level of sub-acceptance families.

The paper is organized as follows. In Section 2 we recall some standard notions and results about risk measures, acceptance sets and capital allocation rules. In Section 3 we introduce the notion of sub-acceptance sets, while in Section 4 we define the notion of risk measure-free capital allocation rule and prove the main results of the paper. Finally, Section 5 contains extensions to the S-additive and quasi-convex cases.

2 Preliminaries

In this section we recall the definitions and the results which we are going to use in the following. Throughout the work, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a time horizon T, $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ denotes the space of all $\mathbb{P}$-essentially bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Equalities and inequalities have to be understood to hold $\mathbb{P}$-almost surely. A random variable $Y \in L^\infty$ will represent the profit and loss at time T of a financial position.

2.1 Risk measures

We firstly recall from Föllmer and Schied [13] the standard definitions of monetary, convex and coherent risk measures, as well as some key results about them.

A map $\rho\colon L^\infty \to \mathbb{R}$ is called a *monetary* risk measure if it satisfies the following conditions:

- monotonicity: if $X \leq Y$ $(X, Y \in L^\infty)$, then $\rho(X) \geq \rho(Y)$
- cash-additivity: $\rho(X + m) = \rho(X) - m$ for any $m \in \mathbb{R}$ and $X \in L^\infty$.

A monetary risk measure ρ is called *normalized* if $\rho(0) = 0$, while *convex* if it satisfies:

- convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for any $\lambda \in [0, 1]$, $X, Y \in L^\infty$.

A convex risk measure ρ is called *coherent* if it satisfies:

- positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for any $\lambda \geq 0$ and $X \in L^\infty$.

Positive homogeneity and convexity, together, are equivalent to subadditivity: $\rho(X+Y) \leq \rho(X)+\rho(Y)$ for any $X, Y \in L^\infty$.

Different classes of risk measures generalizing cash-additivity have been introduced in the literature. In particular, we recall (see Farkas et al. [10] and Munari [17]) that a risk measure $\rho\colon L^\infty \to \mathbb{R}$ is called *S-additive* if it satisfies monotonicity and

- S-additivity: $\rho(X + mS_T) = \rho(X) - mS_0$ for any $m \in \mathbb{R}$, S eligible and $X \in L^\infty$,

where an eligible asset is a couple $S = (S_0, S_T)$ with initial value $S_0 > 0$ and terminal (random) payoff S_T satisfying $\mathbb{P}(S_T \geq 0) = 1$.

Instead, a risk measure ρ is called *cash-subadditive* (see El Karoui and Ravanelli [9]) if it satisfies

- cash-subadditivity: $\rho(X+m) \geq \rho(X) - m$ for any $m \geq 0$ and $X \in L^\infty$.

Notice that both convexity and subadditivity express the idea of diversification of risk (see Artzner et al. [1], Delbaen [4] and [5], Föllmer and Schied [12], Frittelli and Rosazza Gianin [15]). As pointed out by Cerreia-Vioglio et al. [3] (see also Drapeau and Kupper [8], Frittelli and Maggis [14]), once cash-additivity is dropped the right formulation of diversification is given by the weaker property of quasi-convexity. This gives rise to the class of *quasi-convex* risk measures, that is satisfying monotonicity and

- quasi-convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \max\{\rho(X), \rho(Y)\}$ for any $\lambda \in [0,1]$ and $X, Y \in L^\infty$.

2.2 Risk measures and acceptance sets

We recall (see Farkas et al. [10]) that a subset $\mathcal{A} \subseteq L^\infty$ is called an *acceptance set* if it is

- non-trivial: $\emptyset \neq \mathcal{A} \neq L^\infty$
- monotone: $X \in \mathcal{A}$ and $Y \geq X$ imply $Y \in \mathcal{A}$.

Any position $X \in \mathcal{A}$ is then called acceptable.

It is well known that there exists the following one-to-one correspondence between monetary risk measures and acceptance sets (see Föllmer and Schied [13] (Propositions 4.6 and 4.7)). A monetary risk measure ρ induces the *acceptance set* of ρ, that is the set

$$\mathcal{A}_\rho := \{X \in L^\infty \mid \rho(X) \leq 0\} \tag{1}$$

of positions which are acceptable in the sense that they do not require additional capital. Moreover, ρ can be recovered from $\mathcal{A}_\rho$ as

$$\rho(X) = \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_\rho\}\,. \tag{2}$$

A similar result holds also for S-additive risk measures (see [17]). Indeed, if ρ is a S-additive risk measure then $\mathcal{A}_\rho$ is non-trivial and monotone. Moreover, ρ can be recovered from $\mathcal{A}_\rho$ as

$$\rho(X) = \inf \left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0} S_T + X \in \mathcal{A}_\rho \right\}.$$

Conversely, given a class $\mathcal{A} \subseteq L^\infty$ of acceptable positions and an eligible asset S one can define

$$\rho_{\mathcal{A},S}(X) := \inf \left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0} S_T + X \in \mathcal{A} \right\}.$$

If $\mathcal{A} \subseteq L^\infty$ is non-trivial and monotone then $\rho_{\mathcal{A},S}$ is a S-additive risk measure.

The correspondence between acceptance sets and risk measures become more complex in the quasi-convex case, as we will illustrate in section 5.2.

2.3 Capital allocation rules

We now recall the standard definition of a capital allocation rule.

Definition 1 *Given a risk measure ρ, a capital allocation rule (CAR) with respect to ρ is a map $\Lambda_\rho \colon L^\infty \times L^\infty \to \mathbb{R}$ such that*

$$\Lambda_\rho(X; X) = \rho(X) \quad \forall X \in L^\infty.$$

We refer the reader to Denault [6], Kalkbrener [16], Centrone and Rosazza Gianin [2] for more details.

Given a generic set of random variables $\mathcal{X}$, we say that $X \in \mathcal{X}$ is a sub-portfolio (or sub-unit) of $Y \in \mathcal{X}$ if there exists $Z \in \mathcal{X}$ such that $Y = X + Z$. Notice that every random variable is a sub-portfolio of any other, whenever $\mathcal{X}$ is a vector space. Since we fix $\mathcal{X} = L^\infty$ throughout the work, we simply consider every pair of random variables as a pair of respectively a sub-portfolio and a portfolio.

It is customary in the literature to assume some properties on capital allocation rules. We remind some of the most significant (see [16]):

- monotonicity: if $Z \leq X$ ($Z, X \in L^\infty$), then $\Lambda(Z; Y) \geq \Lambda(X; Y)$ for any $Y \in L^\infty$
- no-undercut: $\Lambda(X; Y) \leq \Lambda(X; X)$ for any $X, Y \in L^\infty$
- riskless allocation: $\Lambda(a; Y) = -a$ for any $a \in \mathbb{R}$ and $Y \in L^\infty$.

Monotonicity means that the capital allocated to a position with a higher profit and loss has to be less or equal than the capital allocated to another position with a lower profit and loss. No-undercut property means that the capital allocated to X considered as a sub-portfolio of Y does not exceed the capital allocated to X considered as a stand-alone portfolio, that is, when a risk measure is involved, the risk contribution of X does not exceed its risk capital. This is a well-known property in standard capital allocation problems which, together with other ones, is required for a fair (also called coherent) allocation of risk capital (see Centrone and Rosazza Gianin [2], Denault [6], Kalkbrener [16]).

3 Acceptance and sub-acceptance sets

In the classical approach to capital allocation, given a position Y and a sub-portfolio X, $\Lambda(X;Y)$ reflects $\rho(Y)$. However, the capital allocation problem can be seen from another standpoint, as the following example shows.

Example 2 *Suppose we are provided with a monetary risk measure ρ to quantify the riskiness of financial positions, together with its acceptance set $\mathcal{A}$. Given a portfolio $Y \in \mathcal{A}$ we can look for those positions which do not increment the risk of Y, that is belonging to the set*

$$\mathcal{A}_Y = \{X \in L^\infty \mid \rho(Y) - \rho(Y - X) \leq 0\}.$$

Roughly speaking, $\mathcal{A}_Y$ is formed by positions such that the risk of the portfolio containing the position is at most equal to the risk of the portfolio without the position.

Note that $\rho_Y(\cdot) := \rho(Y) - \rho(Y - \cdot)$ is still a monetary risk measure and evaluates the riskiness of X as a sub-portfolio of Y. In this case, $\mathcal{A}_Y$ can be viewed as the set of all acceptable positions with respect to ρ_Y, i.e. the acceptance set of ρ_Y.

Notice that it is possible to find a position Z which is not acceptable but belongs to $\mathcal{A}_Y$. A simple example is the following. For the probability space $(\Omega = [-1,1], \mathcal{F} = \mathcal{B}(\Omega), \mathbb{P} = \frac{\lambda}{2})$, where λ is the Lebesgue measure on $[-1,1]$, we consider $Y = \frac{1}{2}$ and $Z = \mathbf{1}_{[0,1]} - \mathbf{1}_{[-1,0)}$. Then, for $\rho(X) = \operatorname{ess\,sup}(-X)$ we have that $\rho(Z) = 1$. Hence $Z \notin \mathcal{A}$ while

$$\rho(Y) - \rho(Y - Z) = -\frac{1}{2} - \frac{1}{2} = -1 < 0,$$

so $Z \in \mathcal{A}_Y$.

The previous example shows that there may exist some positions which do not contribute to the risk of the portfolio, even if they require extra capital when considered as stand-alone portfolios. Hence, in that case, ρ provides a non-correct measure of the risk of the sub-portfolios of Y. It would be more suitable, instead, to measure the risk of sub-portfolios by using ρ_Y and not to allocate any part of the risk capital to those sub-portfolios belonging to $\mathcal{A}_Y$. The relevance of this fact and the lack of literature about it lead us to formalize the idea with the following definition.

Definition 3 *Let $\mathcal{A}$ be an acceptance set. A family of sets $(\mathcal{A}_Y)_{Y\in L^\infty}$ is called a sub-acceptance family of $\mathcal{A}$ if both the following properties hold:*

1. *$\mathcal{A}_Y$ is an acceptance set for every $Y \in L^\infty$*
2. *$\mathcal{A} = \{Y \in L^\infty \mid Y \in \mathcal{A}_Y\}$*

Any $\mathcal{A}_Y$ is called sub-acceptance set of Y and any position $X \in \mathcal{A}_Y$ is called sub-acceptable with respect to Y.

Condition 1. of the previous definition means that the positions belonging to $\mathcal{A}_Y$ are acceptable with respect to a fixed position Y, that is when they are considered as sub-portfolios of Y. This implies also that the sub-acceptance criterion, i.e. the one which leads us to detect $\mathcal{A}_Y$, involves features of both the position itself and of Y. Condition 2. requires that Y is acceptable if and only if it belongs to $\mathcal{A}_Y$, that is it is sub-acceptable with respect to itself.

In the following we provide an example of a sub-acceptance family, pointing out that the criterion defining the above family depends also on the fixed acceptance set $\mathcal{A}$.

Example 4 *Consider the acceptance set*

$$\mathcal{A} = \{Y \in L^\infty \mid \mathbb{E}[Y] \geq 0\}$$

and the following sub-acceptance set

$$\mathcal{A}_Y = \{X \in L^\infty \mid \mathbb{E}[X+Y] \geq 0\},$$

for a fixed $Y \in L^\infty$.

It is easy to check that $(\mathcal{A}_Y)_{Y\in L^\infty}$ is a sub-acceptance family of $\mathcal{A}$. Suppose now we slightly modify $\mathcal{A}$ and consider the acceptance set

$$\mathcal{A}' = \{Y \in L^\infty \mid \mathbb{E}[Y] \geq \lambda\} \text{ for some } \lambda > 0.$$

In this case, $(\mathcal{A}_Y)_{Y\in L^\infty}$ is no more a sub-acceptance family of $\mathcal{A}'$, since Condition 2. of Definition 3 fails. Indeed,

$$\{Y \in L^\infty \mid Y \in \mathcal{A}_Y\} = \{Y \in L^\infty \mid \mathbb{E}[Y] \geq 0\} \neq \mathcal{A}'.$$

4 Risk measure-free capital allocation rules

Acceptance and sub-acceptance sets are tools to detect whether a position needs to be covered by extra capital or not, both when considered as a stand-alone portfolio and when considered as a sub-portfolio of another position. We now provide a tool suitable for assessing the contribution of a sub-portfolio to a given portfolio in terms of acceptability. As shown in the previous section, we need to go beyond the standard approach by linking directly capital allocation rules and (sub-)acceptance sets. To this aim, we define a map Λ, where $\Lambda(X;Y)$ is interpreted as the capital allocated to X as a sub-portfolio of Y.

Definition 5 *A function $\Lambda\colon L^\infty \times L^\infty \to \mathbb{R}$ is called a risk measure-free capital allocation rule if it satisfies*

- *1-cash-additivity: $\Lambda(X+c;Y) = \Lambda(X;Y) - c$ for any $c \in \mathbb{R}$ and $X, Y \in L^\infty$*
- *normalization: $\Lambda(0;Y) = 0$ for any $Y \in L^\infty$.*

When no ambiguity arises we will simply refer to such a map as a capital allocation rule, while we will speak of ρ-capital allocation rule when we refer to the classical definition (see definition 1).

1-cash-additivity means that if we add a cash amount c to the sub-portfolio X, the capital allocated to it decreases exactly of c. Notice that some known capital allocation rules in the literature satisfy 1-cash additivity, as for example those based on directional derivatives and extensions (see Centrone and Rosazza Gianin [2], Denault [6], Kalkbrener [16]).
Normalization property is quite clear: there is no reason to allocate any capital to a position which yields an almost surely null profit and loss.

Let us now consider the following examples of capital allocation rules based on two capital allocation methods that are well-known in the classical approach, that is the marginal method and the proportional method (see Dhaene et al.[7] and Tasche [18]).

Example 6 *Given a monetary normalized risk measure ρ, the marginal method is given by*

$$\Lambda_\rho^M(X;Y) = \rho(Y) - \rho(Y-X), \quad X, Y \in L^\infty,$$

while the proportional method by

$$\Lambda_\rho^P(X;Y) = \frac{\rho(X)}{\rho(X) + \rho(Y-X)}\rho(Y), \quad X, Y \in L^\infty.$$

It is easy to check that Λ_ρ^M is a ρ-CAR satisfying normalization and 1-cash-additivity, since ρ is cash-additive, while Λ_ρ^P is normalized but it is not

1-cash-additive. Hence Λ_ρ^P is not a risk measure-free capital allocation rule, despite it is a ρ-capital allocation rule because $\Lambda_\rho^P(X;X) = \rho(X)$.

The following additional properties will be sometimes required:

- enlargement: if $\widetilde{Y} \geq Y$ $(Y, \widetilde{Y} \in L^\infty)$, then $\Lambda(X;\widetilde{Y}) \leq \Lambda(X;Y)$ for any $X \in L^\infty$
- cash-additivity: $\Lambda(Y+c;Y+c) = \Lambda(Y;Y) - c$ for any $c \in \mathbb{R}$ and $Y \in L^\infty$

Enlargement requires that the risk contribution of X considered as a sub-portfolio of a portfolio Y is higher than the risk contribution of X when it is considered as a sub-portfolio of a "dominating" portfolio $\widetilde{Y}$. In other words, the "greater" is the portfolio, the lower are the risk contribution of the sub-portfolios. While the properties of monotonicity and no-undercut are well established in the literature, to the best of our knowledge we are the first to introduce the enlargement property. Note, moreover, that the cash-additivity property is automatically satisfied in the standard case when a monetary risk measure is involved, while this does not necessarily hold for 1-cash-additivity.

Although, as we will see in the following, no-undercut and enlargement can be both used to prove some key results, they are not equivalent as next example shows.

Example 7 *Consider the capital allocation rule given by the marginal method:*

$$\Lambda_\rho(X;Y) = \rho(Y) - \rho(Y-X), \quad X, Y \in L^\infty,$$

for a given monetary risk measure ρ. Then, whenever ρ is also coherent, Λ_ρ satisfies the no-undercut property, because

$$\Lambda_\rho(X;Y) = \rho(Y) - \rho(Y-X) \leq \rho(X).$$

However, it fails to satisfy enlargement. Take indeed the sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with $\mathbb{P}(\omega_i) > 0$, for any $i = 1,2,3$ and take the positions

$$Y = 0, \quad \widetilde{Y} = \begin{cases} 0, & \omega_1 \\ 2, & \omega_2 \\ 1, & \omega_3 \end{cases}, \quad Z = \begin{cases} -2, & \omega_1 \\ 0, & \omega_2 \\ -1, & \omega_3 \end{cases}$$

and the risk measure $\rho(X) = \operatorname{ess\,sup}(-X)$. Then $\widetilde{Y} > Y$ while

$$\Lambda_\rho(Z;Y) = -\rho(-Z) = 0 < \Lambda_\rho(Z;\widetilde{Y}) = 2.$$

that is enlargement fails.

4.1 From acceptance sets to capital allocation rules

We now investigate the connections between capital allocation rules and acceptance sets. To this aim, take an acceptance set $\mathcal{A}$, a sub-acceptance family $(\mathcal{A}_Y)_{Y\in L^\infty}$ and define

$$\Lambda_{\mathcal{A}}(X;Y) := \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\} \qquad \text{for any } X, Y \in L^\infty, \tag{3}$$

where the subscript $\mathcal{A}$ will be omitted when no misunderstandings can arise. Here, $\Lambda(X;Y)$ can be seen as the capital allocated to X, considered as a sub-portfolio of Y, in terms of the minimum amount of capital which should be added to X to make it sub-acceptable. Notice that, in general, $\Lambda(Y;Y)$ does not define the minimum amount of capital which should be added to Y to make it acceptable but only the minimum amount of capital m to make $m+Y$ sub-acceptable with respect to $\mathcal{A}_Y$. However, under additional conditions on the sub-acceptance family, the previous property is fulfilled.

We introduce now the following definition.

Definition 8 *A sub-acceptance family $(\mathcal{A}_Y)_{Y\in L^\infty}$ is said to be translation invariant if it satisfies:*

$$\mathcal{A}_Y = \mathcal{A}_{Y+m} \qquad \text{for any } m \in \mathbb{R} \text{ and } Y \in L^\infty.$$

Translation invariance can be interpreted as follows: no matter if we add or remove a fixed amount of capital m to the portfolio Y, the sub-acceptable positions keep being so. This property can be too restrictive, as we are going to show in the following examples.

Example 9 *Consider the set*

$$\mathcal{A}_Y = \{X \in L^\infty \mid \rho(Y) - \rho(Y - X) \le 0\}, \quad Y \in L^\infty,$$

for a given (normalized) monetary risk measure ρ. By cash-additivity of ρ, it follows that $(\mathcal{A}_Y)_{Y\in L^\infty}$ is translation invariant. Indeed, for any $m \in \mathbb{R}$ and $Y \in L^\infty$ it holds that

$$\begin{aligned}\mathcal{A}_{Y+m} &= \{X \in L^\infty \mid \rho(Y+m) - \rho(Y+m-X) \le 0\} \\ &= \{X \in L^\infty \mid \rho(Y) - \rho(Y-X) \le 0\} = \mathcal{A}_Y.\end{aligned}$$

Example 10 *Consider instead*

$$\mathcal{A}_Y = \{X \in L^\infty \mid \mathbb{E}[X+Y] \ge 0\}, \quad Y \in L^\infty.$$

It is easy to check that $(\mathcal{A}_Y)_{Y\in L^\infty}$ is not translation invariant. However, the following inclusions hold for any $Y \in L^\infty$:

$$\begin{aligned}\mathcal{A}_{Y+m} &\subseteq \mathcal{A}_Y \quad \text{if } m < 0 \\ \mathcal{A}_{Y+m} &\supseteq \mathcal{A}_Y \quad \text{if } m > 0.\end{aligned}$$

To continue our study, we need to define the following property which an acceptance set $\mathcal{A}$ can fulfill or not:

- no certain losses: $\inf \{m \in \mathbb{R} \,|\, m \in \mathcal{A}\} = 0$

No certain losses property means that the smallest constant random variable which is acceptable is 0, i.e. no positions with a (certain) negative profit and loss can be acceptable. We will show in the following that no certain losses is strictly related to the normalization property of a capital allocation rule.

We are now ready to state a result generalizing the one true for risk measures; see Section 2 or, for more details, Föllmer and Schied [13, Prop. 4.7].

Proposition 11 *If $\mathcal{A}$ is an acceptance set, $(\mathcal{A}_Y)_{Y \in L^\infty}$ is a sub-acceptance family and they both satisfy no certain losses property, then Λ defined in (3) is a monotone capital allocation rule.*

Moreover, if the sub-acceptance family is also translation invariant then

$$\Lambda(Y;Y) = \inf \{m \in \mathbb{R} \,|\, m + Y \in \mathcal{A}\}, \quad \textit{for any}\, Y \in L^\infty.$$

Proof. *Finiteness of $\Lambda(X;Y)$:* by the essential boundedness of X and the monotonicity of $\mathcal{A}_Y$ it holds that

$$\{m \in \mathbb{R} \,|\, m + X \in \mathcal{A}_Y\} \supseteq \{m \in \mathbb{R} \,|\, m + \operatorname{ess\,inf} X \in \mathcal{A}_Y\} \neq \emptyset.$$

No certain losses implies that $\Lambda(X;Y) < +\infty$.
Moreover, by similar arguments,

$$\Lambda(X;Y) \geq \inf \{m \in \mathbb{R} \,|\, m + \operatorname{ess\,sup} X \in \mathcal{A}_Y\} = -\operatorname{ess\,sup} X > -\infty$$

by essential boundedness of X, monotonicity of $\mathcal{A}_Y$ and no certain losses of $\mathcal{A}_Y$.

1-cash-additivity: for any $X, Y \in L^\infty$ and $c \in \mathbb{R}$ it holds that

$$\begin{aligned} \Lambda(X + c; Y) &= \inf \{m \in \mathbb{R} \,|\, m + X + c \in \mathcal{A}_Y\} \\ &= \inf \{k \in \mathbb{R} \,|\, k + X \in \mathcal{A}_Y\} - c \\ &= \Lambda(X;Y) - c \end{aligned}$$

by taking $k = m + c$.

Normalization: no certain losses of $\mathcal{A}_Y$ implies that

$$\Lambda(0;Y) = \inf \{m \in \mathbb{R} \,|\, m \in \mathcal{A}_Y\} = 0 \quad \text{for any } Y \in L^\infty.$$

Monotonicity: fix $Y \in L^\infty$ and consider $Z \geq X$ (with $Z, X \in L^\infty$). By monotonicity of $\mathcal{A}_Y$,

$$\{m \in \mathbb{R} \,|\, m + Z \in \mathcal{A}_Y\} \supseteq \{m \in \mathbb{R} \,|\, m + X \in \mathcal{A}_Y\},$$

hence $\Lambda(Z;Y) \leq \Lambda(X;Y)$.

It remains to prove the last statement. For any $Y \in L^\infty$ it holds that

$$\begin{aligned}\Lambda(Y;Y) &= \inf\{m \in \mathbb{R} \,|\, m + Y \in \mathcal{A}_Y\} \\ &= \inf\{m \in \mathbb{R} \,|\, m + Y \in \mathcal{A}_{Y+m}\} \\ &= \inf\{m \in \mathbb{R} \,|\, m + Y \in \mathcal{A}\},\end{aligned}$$

where the second equality holds by translation invariance and the last one by definition of sub-acceptance family. ■

Remark 12 *Notice that, when the sub-acceptance family is translation invariant, $\Lambda(Y;Y)$ defines exactly the minimum amount of capital which should be added to Y to make it acceptable, even if the acceptance set $\mathcal{A}$ is not involved in the definition of Λ.*

We will give now some examples of capital allocation rules associated to the acceptance and sub-acceptance sets presented in the previous section.

Example 13 *Consider the acceptance set and the sub-acceptance family given by*

$$\begin{aligned}\mathcal{A} &= \{X \in L^\infty \,|\, \rho(X) \leq 0\} \\ \mathcal{A}_Y &= \{X \in L^\infty \,|\, \rho(Y) - \rho(Y - X) \leq 0\}\end{aligned}$$

for a given monetary risk measure ρ and $Y \in L^\infty$. By cash-additivity of ρ, $\Lambda_{\mathcal{A}}$ defined in (3) becomes

$$\begin{aligned}\Lambda_{\mathcal{A}}(X;Y) &= \inf\{m \in \mathbb{R} \,|\, m + X \in \mathcal{A}_Y\} \\ &= \inf\{m \in \mathbb{R} \,|\, \rho(Y) - \rho(Y - (X+m)) \leq 0\} \\ &= \inf\{m \in \mathbb{R} \,|\, \rho(Y) - \rho(Y - X) \leq m\} \\ &= \rho(Y) - \rho(Y - X),\end{aligned}$$

hence corresponding to the so-called marginal method; see, among others, Dhaene et al. [7] and Tasche [18].

Moreover, $\Lambda_{\mathcal{A}}$ is a capital allocation rule. Indeed, 1-cash-additivity is immediate and normalization follows by no certain losses property of $\mathcal{A}_Y$. Furthermore, $\Lambda_{\mathcal{A}}(Y;Y) = \rho(Y)$ because of translation invariance of $\mathcal{A}_Y$.

Example 14 *Consider now the acceptance set and the sub-acceptance family given by*

$$\begin{aligned}\mathcal{A} &= \{Y \in L^\infty \,|\, \mathbb{P}(Y \leq 0) \leq \alpha\} \\ \mathcal{A}_Y &= \{X \in L^\infty \,|\, \mathbb{P}(X + Y \leq 0) \leq \alpha\}\end{aligned}$$

for some $\alpha \in (0,1)$ *and for any* $Y \in L^\infty$. *Then*

$$\begin{aligned}\Lambda_{\mathcal{A}}(X;Y) &= \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\} \\ &= \inf\{m \in \mathbb{R} \mid \mathbb{P}(X + m + Y \leq 0) \leq \alpha\} \\ &= -\sup\{k \in \mathbb{R} \mid \mathbb{P}(X + Y \leq k) \leq \alpha\} \\ &= -q_\alpha^+(X+Y) = VaR_\alpha(X+Y),\end{aligned}$$

where $q_\alpha^+(Z) := \inf\{m \in \mathbb{R} \mid \mathbb{P}(Z \leq m) > \alpha\}$ *and* $VaR_\alpha(Z)$ *stands for the Value at Risk at level* α *of* Z. *Hence,* $\Lambda_{\mathcal{A}}$ *is 1-cash-additive but not normalized, so* $\Lambda_{\mathcal{A}}$ *is not a capital allocation rule. Moreover,*

$$\Lambda_{\mathcal{A}}(Y;Y) = -2q_\alpha^+(Y) \neq \inf\{m \in \mathbb{R} \mid m + Y \in \mathcal{A}\} = -q_\alpha^+(Y)$$

Indeed, the sub-acceptance family is not translation invariant.

To end this section, in the following we define some properties on acceptance sets corresponding to those already introduced on capital allocation rules.

First of all, it may be reasonable to require that any acceptable positions is also sub-acceptable for every portfolio. That is,

- $\mathcal{A}$-no-undercut: $\mathcal{A} \subseteq \mathcal{A}_Y \quad \forall Y \in L^\infty$

As shown in the following result, $\mathcal{A}$-no-undercut corresponds to no-undercut of the associated Λ.

Proposition 15 *Let* $\mathcal{A}$ *be an acceptance set and let* $(\mathcal{A}_Y)_{Y \in L^\infty}$ *be a translation invariant sub-acceptance family. If* $(\mathcal{A}_Y)_{Y \in L^\infty}$ *satisfies* $\mathcal{A}$*-no-undercut, then* $\Lambda_{\mathcal{A}}$ *defined in* (3) *satisfies no-undercut.*

Proof. Given arbitrary $X, Y \in L^\infty$, $\mathcal{A}$-no-undercut implies

$$\{m \in \mathbb{R} \mid m + X \in \mathcal{A}\} \subseteq \{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\}.$$

Hence,

$$\Lambda_{\mathcal{A}}(X;Y) \leq \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}\} = \Lambda_{\mathcal{A}}(X;X)$$

where the last equality holds by translation invariance of the sub-acceptance family. ■

We investigate now which conditions on a sub-acceptance family imply the enlargement property of the induced capital allocation rule. To this goal, we introduce the following property:

- $\mathcal{A}$-enlargement: if $\widetilde{Y} \geq Y$ $(Y, \widetilde{Y} \in L^\infty)$, then $\mathcal{A}_Y \subseteq \mathcal{A}_{\widetilde{Y}}$.

In other words, $\mathcal{A}$-enlargement requires that any sub-acceptable position with respect to a portfolio Y is also sub-acceptable for any portfolio $\widetilde{Y} \geq Y$, that is dominating Y. The following result shows that $\mathcal{A}$-enlargement guarantees enlargement of the corresponding capital allocation rule.

Proposition 16 *Let $(\mathcal{A}_Y)_{Y\in L^\infty}$ be a sub-acceptance family. If $(\mathcal{A}_Y)_{Y\in L^\infty}$ satisfies $\mathcal{A}$-enlargement, then $\Lambda_{\mathcal{A}}$ defined as in (3) satisfies enlargement.*

Proof. Take any $\widetilde{Y}, Y \in L^\infty$ such that $\widetilde{Y} \geq Y$. $\mathcal{A}$-enlargement implies then

$$\{m \in \mathbb{R} \,|\, m + X \in \mathcal{A}_Y\} \subseteq \left\{m \in \mathbb{R} \,\middle|\, m + X \in \mathcal{A}_{\widetilde{Y}}\right\},$$

therefore

$$\Lambda_{\mathcal{A}}(X;\widetilde{Y}) = \inf \left\{m \in \mathbb{R} \,\middle|\, m + X \in \mathcal{A}_{\widetilde{Y}}\right\} \leq \inf \{m \in \mathbb{R} \,|\, m + X \in \mathcal{A}_Y\} = \Lambda_{\mathcal{A}}(X;Y).$$

■

4.2 From capital allocation rules to acceptance sets

So far, we have defined a capital allocation rule starting from an acceptance set and a sub-acceptance family and studied some properties of that capital allocation rule corresponding to those required for the sets. We now investigate the converse.

Let us start with the case when Λ is a CAR which also satisfies the standard definition, that is induced by a monetary risk measure ρ such that $\Lambda(X;X) = \rho(X)$ for any $X \in L^\infty$.

Consider the acceptance set $\mathcal{A}$ of ρ (that is $\mathcal{A} = \{X \in L^\infty \,|\Lambda(X;X) \leq 0\}$) and set, for any $Y \in L^\infty$,

$$\mathcal{A}_Y := \{X \in L^\infty | \; \Lambda(X;Y) \leq 0\}.$$

Every position X in $\mathcal{A}_Y$ is sub-acceptable in the sense that it does not need any capital injection when seen as a sub-portfolio of Y. Notice that $\mathcal{A} = \{Y \in L^\infty \,|Y \in \mathcal{A}_Y\}$.

The following representation result is then straightforward.

Proposition 17 *If Λ is a capital allocation rule induced by a monetary risk measure ρ, then*

$$\Lambda(X;Y) = \inf\{m \in \mathbb{R} \,|m + X \in \mathcal{A}_Y\}, \quad \Lambda(Y;Y) = \inf\{m \in \mathbb{R} \,|m + Y \in \mathcal{A}\}$$

for any $X, Y \in L^\infty$.

If, moreover, Λ is monotone, then $\mathcal{A}_Y$ is an acceptance set for any $Y \in L^\infty$, and $\Lambda(\cdot;Y) = \rho_Y(\cdot)$ is a monetary risk measure satisfying $\rho_Y(Y) = \rho(Y)$.

Proof. If Λ is a capital allocation rule then, by 1-cash-additivity of Λ,

$$\inf\{m \in \mathbb{R}\,|m + X \in \mathcal{A}_Y\} = \inf\{m \in \mathbb{R}\,|\Lambda(m + X; Y) \leq 0\} = \Lambda(X; Y)$$

for any $X, Y \in L^\infty$.

Moreover, $\Lambda(Y; Y) = \rho(Y) = \inf\{m \in \mathbb{R}\,|m + X \in \mathcal{A}\}$, where the former equality holds since Λ is induced by ρ, the latter from the relation between monetary risk measures and acceptance sets. If Λ is monotone, monotonicity of each $\mathcal{A}_Y$ follows from monotonicity of Λ. ■

The modified monetary risk measure ρ_Y reflects the "true" risk of X as a sub-portfolio of Y. It is true that this capital allocation rule is not linear in general, but this is justified by the fact that we are not trying to share the risk $\rho(Y)$ among the various sub-units of Y but to reward each sub-unit exactly with its risk contribution as a sub-unit of Y.

We now investigate if the previous representation result still holds true for a general Λ not necessarily induced by a monetary risk measure. Unfortunately, this is not the case without imposing some additional properties on the capital allocation rule. The main problems are related to the lack of cash-additivity and to monotonicity which are instead automatically fulfilled in the standard framework, whenever a monotone risk measure is involved. There are several ways to fill those lacks: in the following, we will discuss and investigate the different properties to be required to obtain results similar to Proposition 17.

Given a risk measure-free capital allocation rule Λ (capital allocation rule in the following), we define the following sets:

$$\mathcal{A}_\Lambda := \{Y \in L^\infty|\ \Lambda(Y; Y) \leq 0\} \tag{4}$$

$$\mathcal{A}_{Y,\Lambda} := \{X \in L^\infty|\ \Lambda(X; Y) \leq 0\}, \quad Y \in L^\infty, \tag{5}$$

where the subscript Λ will be omitted when it is clear which capital allocation is involved.

Proposition 18 *If Λ is a capital allocation rule satisfying monotonicity, cash-additivity and no-undercut, then $\mathcal{A}$ defined in (4) is an acceptance set and $(\mathcal{A}_Y)_{Y \in L^\infty}$ given by (5) is a sub-acceptance family with respect to $\mathcal{A}$. Moreover, Λ can be written as:*

$$\Lambda(X; Y) = \begin{cases} \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\}, & \text{if } X \neq Y \\ \inf\{m \in \mathbb{R} \mid m + Y \in \mathcal{A}\}, & \text{if } X = Y \end{cases} \tag{6}$$

Proof. *Non triviality of $\mathcal{A}$:* first of all, $\mathcal{A} \neq \emptyset$ since $0 \in \mathcal{A}$ by normalization. In order to check $\mathcal{A} \neq L^\infty$ let us consider $c < \Lambda(Y; Y)$. Then, by cash-additivity of Λ, $\Lambda(Y + c; Y + c) = \Lambda(Y; Y) - c > 0$ so that $Y + c \notin \mathcal{A}$. A similar argument clearly holds for each $\mathcal{A}_Y$.

Monotonicity of each $\mathcal{A}_Y$: consider $X \in \mathcal{A}_Y$ and $Z \geq X$ with $Z, X \in L^\infty$. Then, by monotonicity of Λ and (5), it follows that

$$\Lambda(Z; Y) \leq \Lambda(X; Y) \leq 0,$$

hence $Z \in \mathcal{A}_Y$.

Monotonicity of $\mathcal{A}$: take $X \in \mathcal{A}$ and $Y \geq X$. Then

$$\Lambda(Y; Y) \leq \Lambda(X; Y) \leq \Lambda(X; X) \leq 0 \tag{7}$$

where the first inequality holds by monotonicity of Λ, the second one by no-undercut and the last one because $X \in \mathcal{A}$. Therefore, $Y \in \mathcal{A}$ and $\mathcal{A}$ is an acceptance set. Since

$$\mathcal{A} = \{Y \in L^\infty \mid Y \in \mathcal{A}_Y\} = \{Y \in L^\infty \mid \Lambda(Y; Y) \leq 0\},$$

then $(\mathcal{A}_Y)_{Y \in L^\infty}$ is a sub-acceptance family with respect to $\mathcal{A}$.

It remains to show that Λ can be represented as in (6). Consider, firstly, the case where $X \neq Y$. Then

$$\begin{aligned}
\Lambda(X; Y) &= \inf\{m \in \mathbb{R} \mid \Lambda(X; Y) \leq m\} \\
&= \inf\{m \in \mathbb{R} \mid \Lambda(X + m; Y) \leq 0\} \\
&= \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}_Y\}
\end{aligned}$$

where the second equality holds by 1-cash-additivity of Λ and the last one by definition of $\mathcal{A}_Y$. Finally, by cash-additivity of Λ and by definition of $\mathcal{A}$, it follows that

$$\begin{aligned}
\Lambda(Y; Y) &= \inf\{m \in \mathbb{R} \mid \Lambda(Y; Y) \leq m\} \\
&= \inf\{m \in \mathbb{R} \mid \Lambda(Y + m; Y + m) \leq 0\} \\
&= \inf\{m \in \mathbb{R} \mid Y + m \in \mathcal{A}\}
\end{aligned}$$

holds for any $Y \in L^\infty$. This concludes the proof. ■

As already mentioned, other (sets of) properties on a capital allocation rule could guarantee the same thesis of the previous result. Monotonicity is clearly needed to prove that each $\mathcal{A}_Y$ is monotone and there are no significant alternatives, while no-undercut can be replaced by enlargement. Replacing no-undercut with enlargement impacts just on proving that $\mathcal{A}$ is monotone, in particular inequalities of (7) still hold but thanks to the enlargement. Notice that either no-undercut or enlargement are required to fill the lack of the following property:

- full monotonicity: if $Y \geq X$ ($Y, X \in L^\infty$), then $\Lambda(Y; Y) \leq \Lambda(X; X)$.

The previous property is automatically satisfied in the standard framework when Λ is induced by a monotone risk measure. However, full monotonicity follows from either monotonicity and no-undercut or monotonicity and enlargement, as we can see from inequalities in (7), even if no-undercut and enlargement are not equivalent conditions.

Notice, moreover, that Λ satisfying no-undercut does not imply the same property on acceptance and sub-acceptance sets.

5 Some extensions

So far, we focused on the cash-additive case, that is related to a 1-cash-additive CAR or to a translation invariance sub-acceptance family. In the following, we generalize the approach above to the case where translation invariance of the acceptance family either holds with respect to a reference asset (not necessarily a risk-free asset) or is dropped. More precisely, we will focus both on the S-additive case and on the quasi-convex case.

5.1 S-additivity

As pointed out by Farkas et al. [10] and Munari [17], the idea of the milestone work of Artzner et al. [1] is to measure the risk of a position by describing how close or how far from acceptance a position is, given a "reference instrument" that does not necessarily correspond to a cash account. In our framework, capital allocation rules assess the capital to be allocated to a sub-portfolio by means of the distance to a sub-acceptance set, which is, in some cases, related to the risk of the sub-portfolio. Therefore, in general, it is too restrictive to impose the cash-additivity assumption to capital allocation rules. Following the approach of Farkas et al. [10] and Munari [17] who introduced the so-called S-additive risk measures, we would like to admit the possibility to make a portfolio acceptable or sub-acceptable by adding not necessarily cash but also shares of a "suitable" asset.

Fix now a time horizon T and an asset S given by $S = (S_0, S_T)$, where $S_0 \in \mathbb{R}$ is the initial value and $S_T \in L^\infty$ is the value of S at time T. We assume the existence of a financial market where assets are traded. We recall the following definition to clarify which are the "suitable" assets we wish to add to sub-portfolios in order to reach acceptability.

Definition 19 *(see Farkas and Smirnow [11]) Given a time horizon $T \geq 0$ and an acceptance set $\mathcal{A}$, an asset $S = (S_0, S_T)$ is called eligible if $S_T \in \mathcal{A}$ and its initial value S_0 is strictly positive.*

In the following, S will denote, with an abuse of notation, both the asset and its terminal value S_T, while $\mathcal{E}$ will denote the set of all eligible assets. The previous definition slightly differs from the one of Farkas et al. [10]

where they require the same condition on S_0 but a different one on S_T, i.e. $\mathbb{P}(S_T > 0) = 1$.

Our aim is now to investigate whether the results of the previous section can be generalized to the present case where we introduce the following definition of S-capital allocation rules.

Definition 20 *A function $\Lambda\colon L^\infty \times L^\infty \to \mathbb{R}$ is called S-capital allocation rule if it satisfies*

- *1-S-additivity:*

  $$\Lambda(X + mS; Y) = \Lambda(X; Y) - mS_0 \quad \textit{for any } m \in \mathbb{R}, S \in \mathcal{E}, X, Y \in L^\infty$$

- *normalization: $\Lambda(0; Y) = 0$ for any $Y \in L^\infty$*

Compared to capital allocation rules of Definition 5, in S-capital allocation rules the assumption of 1-cash-additivity has been replaced by 1-S-additivity.

Similarly to the previous section, given an acceptance set $\mathcal{A}$ and a sub-acceptance family $(\mathcal{A}_Y)_{Y \in L^\infty}$ we define

$$\Lambda_{\mathcal{A}}(X; Y) := \inf\left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0} S + X \in \mathcal{A}_Y \right\} \quad for any X, Y \in L^\infty, \tag{8}$$

where the subscript $\mathcal{A}$ will be omitted when no misunderstandings can arise. Before going further, we need to define the following notions for a sub-acceptance family.

Definition 21 *A sub-acceptance family $(\mathcal{A}_Y)_{Y \in L^\infty}$ is said to satisfy:*

- *S-translation invariance if $\mathcal{A}_Y = \mathcal{A}_{Y+mS}$ for any $m \in \mathbb{R}$ and $S \in \mathcal{E}$;*
- *S-no certain losses if* $\inf\left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0} S \in \mathcal{A} \right\} = 0$ *for any $S \in \mathcal{E}$.*

S-translation invariance property means that those positions which are sub-acceptable with respect to a given portfolio Y are also sub-acceptable with respect to any sum $Y + mS$ where S is eligible and m is any cash amount. In other words, no matter if we add or remove any quantity (even negative) of eligible asset S to the portfolio Y, the sub-acceptable positions keep being so. S-no certain losses property, instead, requires that the smallest share of eligible asset which is acceptable is 0, i.e. no short positions on S can be acceptable.

Proposition 22 *If $\mathcal{A}$ is an acceptance set, $(\mathcal{A}_Y)_{Y \in L^\infty}$ is a sub-acceptance family and they both satisfy no certain losses property, then $\Lambda_{\mathcal{A}}$ defined in (8) is a monotone S-capital allocation rule.*

Moreover, if the sub-acceptance family is also S-translation invariant then

$$\Lambda_{\mathcal{A}}(Y; Y) = \inf\left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0} S + Y \in \mathcal{A} \right\} \quad \textit{for any } Y \in L^\infty.$$

Proof. *Finiteness of* $\Lambda_{\mathcal{A}}(X;Y)$*:* since $\mathcal{A}_Y$ is an acceptance set (hence it is monotone and $\mathcal{A}_Y \neq \emptyset, L^\infty$) and $X \in L^\infty$,

$$\left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0}S + X \in \mathcal{A}_Y \right\} \neq \emptyset, \mathbb{R}.$$

Hence $\Lambda_{\mathcal{A}}(X;Y) \in \mathbb{R}$.

1-S*-additivity:* for any $X, Y \in L^\infty$, S eligible and $k \in \mathbb{R}$ we consider

$$\begin{aligned}
\Lambda_{\mathcal{A}}(X + kS; Y) &= \inf \left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0}S + X + kS \in \mathcal{A}_Y \right\} \\
&= \inf \left\{ (c-k)S_0 \in \mathbb{R} \,|\, cS + X \in \mathcal{A}_Y \right\} \\
&= \inf \left\{ cS_0 \in \mathbb{R} \,|\, cS + X \in \mathcal{A}_Y \right\} - kS_0 \\
&= \inf \left\{ \beta \in \mathbb{R} \,\middle|\, \frac{\beta}{S_0}S + X \in \mathcal{A}_Y \right\} - kS_0 \\
&= \Lambda_{\mathcal{A}}(X;Y) - kS_0.
\end{aligned}$$

Normalization: S-no certain losses implies, for every Y, that

$$\Lambda_{\mathcal{A}}(0;Y) = \inf \left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0}S \in \mathcal{A}_Y \right\} = 0.$$

Monotonicity: fix any $X, Y \in L^\infty$ and consider $Z \geq X$. By monotonicity of $\mathcal{A}_Y$,

$$\left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0}S + Z \in \mathcal{A}_Y \right\} \supseteq \left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0}S + X \in \mathcal{A}_Y \right\}$$

hence $\Lambda_{\mathcal{A}}(Z;Y) \leq \Lambda_{\mathcal{A}}(X;Y)$.

Finally, by S-translation invariance of $\mathcal{A}_Y$ and by definition of sub-acceptance family it follows that for any $Y \in L^\infty$

$$\begin{aligned}
\Lambda_{\mathcal{A}}(Y;Y) &= \inf \left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0}S + Y \in \mathcal{A}_Y \right\} \\
&= \inf \left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0}S + Y \in \mathcal{A}_{Y + \frac{m}{S_0}S} \right\} \\
&= \inf \left\{ m \in \mathbb{R} \,\middle|\, \frac{m}{S_0}S + Y \in \mathcal{A} \right\}.
\end{aligned}$$

■

Assume now that an S-capital allocation rule Λ is given. We can wonder which properties are fulfilled by the acceptance sets induced by Λ. To this aim, we introduce the following property:

- S-additivity:

$$\Lambda(Y + mS; Y + mS) = \Lambda(Y;Y) - mS_0 \quad \text{for any } m \in \mathbb{R}, S \in \mathcal{E}, Y \in L^\infty$$

generalizing cash-additivity of Λ.

Proposition 23 *If Λ is a S-capital allocation rule satisfying monotonicity, S-additivity and no-undercut, then the corresponding $\mathcal{A}$ and $(\mathcal{A}_Y)_{Y\in L^\infty}$ are, respectively, an acceptance set and a sub-acceptance family with respect to $\mathcal{A}$. Moreover, Λ is given by*

$$\Lambda(X;Y)=\begin{cases}\inf\left\{m\in\mathbb{R}\;\middle|\;\dfrac{m}{S_0}S+X\in\mathcal{A}_Y\right\}, & \text{if } X\neq Y\\ \inf\left\{m\in\mathbb{R}\;\middle|\;\dfrac{m}{S_0}S+Y\in\mathcal{A}\right\}, & \text{if } X=Y\end{cases}\tag{9}$$

Proof. $\mathcal{A}\neq\emptyset, L^\infty$: $0\in\mathcal{A}$ by normalization of Λ. Given an arbitrary $S\in\mathcal{E}$ there exists $m\in\mathbb{R}$ such that $mS_0<\Lambda(Y;Y)$. S-additivity implies then

$$\Lambda(Y+mS;Y+mS)=\Lambda(Y;Y)-mS_0>0,$$

hence $Y+mS\notin\mathcal{A}$. Non triviality of any $\mathcal{A}_Y$ can be checked similarly.

Monotonicity of each $\mathcal{A}_Y$: consider $X\in\mathcal{A}_Y$ and $Z\geq X$. Then, by monotonicity of Λ,

$$\Lambda(Z;Y)\leq\Lambda(X;Y)\leq 0.$$

Hence $Z\in\mathcal{A}_Y$.

Monotonicity of $\mathcal{A}$: take $X\in\mathcal{A}$ and $Y\geq X$ then, by monotonicity and no-undercut of Λ,

$$\Lambda(Y;Y)\leq\Lambda(X;Y)\leq\Lambda(X;X)\leq 0$$

where the last inequality is due to $X\in\mathcal{A}$.

Therefore $Y\in\mathcal{A}$ and $\mathcal{A}$ is an acceptance set. Since

$$\mathcal{A}=\{Y\in L^\infty\mid Y\in\mathcal{A}_Y\}=\{Y\in L^\infty\mid\Lambda(Y;Y)\leq 0\},$$

$(\mathcal{A}_Y)_{Y\in L^\infty}$ is a sub-acceptance family with respect to $\mathcal{A}$.

It remains to show that Λ can be represented as in (9). For any $X,Y\in L^\infty$ with $X\neq Y$ it holds that

$$\begin{aligned}\Lambda(X;Y)&=\inf\{m\in\mathbb{R}\mid\Lambda(X;Y)\leq m\}\\&=\inf\left\{m\in\mathbb{R}\;\middle|\;\Lambda\left(X+\frac{m}{S_0}S;Y\right)\leq 0\right\}\\&=\inf\left\{m\in\mathbb{R}\;\middle|\;X+\frac{m}{S_0}S\in\mathcal{A}_Y\right\},\end{aligned}$$

where the second equality holds by 1-S-additivity and the last one by definition of $\mathcal{A}_Y$. Finally, by S-additivity, it follows that, for any $Y \in L^\infty$,

$$\begin{aligned}\Lambda(Y;Y) &= \inf\{m \in \mathbb{R} \,|\, \Lambda(Y;Y) \leq m\} \\ &= \inf\left\{m \in \mathbb{R} \,\middle|\, \Lambda\left(Y + \frac{m}{S_0}S; Y + \frac{m}{S_0}S\right) \leq 0\right\} \\ &= \inf\left\{m \in \mathbb{R} \,\middle|\, Y + \frac{m}{S_0}S \in \mathcal{A}\right\}.\end{aligned}$$

This concludes the proof. ■

Notice that when the only eligible asset is the risk-free asset with $S_T = S_0$, the previous results reduce to Proposition 11 and to Proposition 18, respectively.

5.2 Quasi-convex case

Consider now the case of families of sub-acceptance sets in quite a general framework. This is in line with the approach of quasi-convex risk measures where no cash-additivity is assumed on the risk measure and, consequently, neither on the family of acceptance sets. See Cerreia-Vioglio et al. [3], Drapeau and Kupper [8] and Frittelli and Maggis [14] for a detailed treatment on quasi-convex risk measures. As pointed out in Drapeau and Kupper [8], in the case of quasi-convex risk measures the one-to-one correspondence between risk measures and acceptance sets (2) is no more true but has to be formulated in terms of acceptance sets at different levels. Differently from the cash-additive case where only the set at level 0 is relevant since all the other sets can be obtained from it by translation invariance, in the quasi-convex case the whole family of acceptance sets at different levels is needed.

Consider now a family $(\mathcal{A}_{Y,m})_{Y \in L^\infty, m \in \mathbb{R}}$ of sub-acceptance sets at different levels $m \in \mathbb{R}$, that is

- for any fixed $Y \in L^\infty$, $(\mathcal{A}_{Y,m})_{m \in \mathbb{R}}$ is a family of acceptance sets at the level m; that is, every $\mathcal{A}_{Y,m}$ is an acceptance set parametrized by $m \in \mathbb{R}$;
- for any fixed $m \in \mathbb{R}$, $(\mathcal{A}_{Y,m})_{Y \in L^\infty}$ is a monotone sub-acceptance family with respect to an acceptance set $\mathcal{A}^m$. More precisely, $\mathcal{A}_{Y,m}$ is monotone increasing in $Y \in L^\infty$.

Roughly speaking, the level $m \in \mathbb{R}$ can be seen as a degree of acceptability.

Define now

$$\Lambda_{\mathcal{A}}(X;Y) := \inf\{m \in \mathbb{R} \,|\, X \in \mathcal{A}_{Y,m}\}, \quad \text{for any } X, Y \in L^\infty. \tag{10}$$

Proposition 24 *Let $(\mathcal{A}_{Y,m})_{Y \in L^\infty, m \in \mathbb{R}}$ be a family of sub-acceptance sets at the level $m \in \mathbb{R}$.*

Then $\Lambda_{\mathcal{A}}$ defined in (10) satisfies the following properties:

(a) decreasing monotonicity in $X \in L^\infty$;
(b) decreasing monotonicity in $Y \in L^\infty$;
(c) normalization whenever $0 \in \mathcal{A}_{Y,0}$ and $0 \notin \mathcal{A}_{Y,m}$ for any $m < 0$.

Furthermore,

(d) if any $\mathcal{A}_{Y,m}$ is convex, then $\Lambda_{\mathcal{A}}$ is quasi-convex in the first variable;
(e) if $\mathcal{A}_{Y,m} \subseteq \mathcal{A}_{Y,m+c} + c$ for any $m \in \mathbb{R}$, $c \geq 0$ and $Y \in L^\infty$, then $\Lambda_{\mathcal{A}}$ is cash-subadditive in the first variable;
(f) if $\mathcal{A}_{Y,m} = \mathcal{A}_{Y,m+c} + c$ for any $m, c \in \mathbb{R}$ and $Y \in L^\infty$, then $\Lambda_{\mathcal{A}}$ is 1-cash-additive.

Proof. (a) Take any $Z \geq X$. By monotonicity of $\mathcal{A}_{Y,m}$, it follows that

$$\Lambda_{\mathcal{A}}(X;Y) = \inf\{m \in \mathbb{R} \,|\, X \in \mathcal{A}_{Y,m}\} \geq \inf\{m \in \mathbb{R} \,|\, Z \in \mathcal{A}_{Y,m}\} = \Lambda_{\mathcal{A}}(Z;Y).$$

Similarly, (b) follows by monotonicity of $\mathcal{A}_{Y,m}$ in Y.

(c) is immediate.

The proofs of (d) and (e) are similar to those in Drapeau and Kupper [8]. We include them for reader's convenience.

(d) Let $\alpha \in [0,1]$ and $X, Y, Z \in L^\infty$ be arbitrarily fixed. Assume now that $X, Z \in \mathcal{A}_{Y,\bar{m}}$ for some $\bar{m} \in \mathbb{R}$. It follows then that $\Lambda_{\mathcal{A}}(X;Y), \Lambda_{\mathcal{A}}(Z;Y) \leq \bar{m}$ and, by convexity of $\mathcal{A}_{Y,\bar{m}}$, that also $\alpha X + (1-\alpha)Z \in \mathcal{A}_{Y,\bar{m}}$. Consequently, $\Lambda_{\mathcal{A}}(\alpha X + (1-\alpha)Z;Y) \leq \bar{m}$. By a well-known result on quasi-convex functionals, it follows that $\Lambda_{\mathcal{A}}(\cdot;Y)$ is quasi-convex for any $Y \in L^\infty$.

(e) For any $m \in \mathbb{R}$, $c \geq 0$ and $X, Y \in L^\infty$ it holds that

$$\begin{aligned}
\Lambda_{\mathcal{A}}(X+c;Y) &= \inf\{m \in \mathbb{R} \,|\, X + c \in \mathcal{A}_{Y,m}\} \\
&\geq \inf\{m \in \mathbb{R} \,|\, X + c \in (\mathcal{A}_{Y,m+c} + c)\} \\
&= \inf\{m \in \mathbb{R} \,|\, X \in \mathcal{A}_{Y,m+c}\} \\
&= \inf\{m \in \mathbb{R} \,|\, X \in \mathcal{A}_{Y,m}\} - c \\
&= \Lambda_{\mathcal{A}}(X;Y) - c,
\end{aligned}$$

where the inequality above is due to the assumption $\mathcal{A}_{Y,m} \subseteq \mathcal{A}_{Y,m+c} + c$ for $c \leq 0$.

(f) can be proved similarly to item (e). ■

Notice that, thanks to the previous result, it holds that $\Lambda_{\mathcal{A}}(X+c;Y) \leq \Lambda_{\mathcal{A}}(X;Y)$ for any $c \geq 0$ and $X, Y \in L^\infty$ (by monotonicity in the first variable). Moreover, normalization and cash-subadditivity (whenever satisfied) imply that $\Lambda_{\mathcal{A}}(c;Y) \geq -c$ for any $c \geq 0$ and Y, while $\Lambda_{\mathcal{A}}(c;Y) \leq -c$ for any $c < 0$ and Y.

So far, we have defined a capital allocation rule starting from a family of sub-acceptance sets at different levels. We are now going to investigate the converse.

Consider now a capital allocation rule $\Lambda(X;Y)$ not necessarily satisfying cash-additivity.

Define now

$$\mathcal{A}_{Y,m} := \{X \,|\Lambda(X;Y) \leq m\} \quad \text{and} \quad \mathcal{A}^m := \{Y \,|\Lambda(Y;Y) \leq m\} \tag{11}$$

for any $m \in \mathbb{R}$ and $Y \in L^\infty$.

Proposition 25 *If Λ is a monotone capital allocation rule, then the corresponding $\mathcal{A}_{Y,m}$ and $\mathcal{A}^m$ defined as in (11) satisfy the following properties:*

(i) for any fixed $m \in \mathbb{R}$: $(\mathcal{A}_{Y,m})_Y \in L^\infty$, is a sub-acceptance family of $\mathcal{A}^m = \{Y : Y \in \mathcal{A}_{Y,m}\}$;
(ii) for any fixed $Y \in L^\infty$:
(a) $\mathcal{A}_{Y,m}$ is monotone for every $m \in \mathbb{R}$;
(b) $\mathcal{A}_{Y,m}$ is monotone in $m \in \mathbb{R}$ w.r.t. set inclusion;
(c) $\mathcal{A}_{Y,m}$ is convex whenever $\Lambda(X;Y)$ is quasi-convex in $X \in L^\infty$.

Proof.

(i) We start to prove the properties once $m \in \mathbb{R}$ is fixed arbitrarily. We have only to check the first statement since the second is immediate. $\mathcal{A}_{Y,m} \neq \emptyset$ follows immediately by the assumptions on Λ implying that $-m \in \mathcal{A}_{Y,m}$ for any $m \in \mathbb{R}$ (since, by 1-cash-additivity, $\Lambda(m;Y) = \Lambda(0;Y) - m = -m$). $\mathcal{A}_{Y,m} \neq L^\infty$: again by the assumptions on Λ it follows that $-\bar{m} \notin \mathcal{A}_{Y,m}$ for any $\bar{m} > m$, hence the thesis.

(ii) Let now $Y \in L^\infty$ be fixed and let $m \in \mathbb{R}$ be arbitrary.

(a) Assume that $X \in \mathcal{A}_{Y,m}$ and $Z \geq X$. By monotonicity of Λ in the first variable, it follows that $\Lambda(Z;Y) \leq \Lambda(X;Y) \leq m$. Hence, $Z \in \mathcal{A}_{Y,m}$.

(b) and (c) follow immediately by the definition of $\mathcal{A}_{Y,m}$ in (11). ■

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